Generalized variational inequalities

PAVEL KREJČÍ [1] [2] AND PHILIPPE LAURENÇOT [3] [4]

Abstract

We consider a rate independent evolution variational inequality with an arbitrary convex closed constraint Z in a Hilbert space X. The main results consist in proving that it is well-posed in the Young integral setting in the space of functions of essentially bounded variation for every Z and in the space of regulated functions provided 0 lies in the interior of Z.

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Introduction

We consider a real Hilbert space X endowed with a scalar product $\langle \cdot, \cdot \rangle$ and norm $|x| := \langle x, x \rangle^{1/2}$ for $x \in X$. Throughout the paper we assume that

$$Z \text{ is a convex closed subset of } X \text{ such that } 0 \in Z . \tag{0.1}$$

We mainly work with the so-called *regulated functions* (cf. [1]), that is, functions of real variable which at each point of their domain of definition admit both finite one-sided limits, see Definition 1.1 below. The space of regulated functions $[0,T] \to X$ will be denoted by $G(0,T;X)$ according to [21].

We assume that an initial condition $x_0 \in Z$ and an input $u \in G(0,T;X)$ are given, and we look for a function $\xi \in G(0,T;X)$ such that

($\mathcal{P}$) (i) $u(t) - \xi(t) \in Z \quad \forall t \in [0,T]$,

(ii) $u(0) - \xi(0) = x_0$,

(iii) for every $y \in G(0,T;Z)$, the function $\tau \mapsto u(\tau+) - \xi(\tau+) - y(\tau)$ is Young integrable on $[0,T]$ with respect to ξ according to Definition 3.1, and

$$\int_0^t \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau) \rangle \ \geq \ 0 \quad \forall t \in [0,T] .$$

[1]Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, CZ-11567 Praha 1, Czech Republic, E-Mail: krejci@math.cas.cz.

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[3]Mathématiques pour l'Industrie et la Physique, UMR CNRS 5640, Université Paul Sabatier – Toulouse 3, 118 route de Narbonne, F-31062 Toulouse Cédex 4, France, E-Mail: laurencot@mip.ups-tlse.fr.

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If u is continuous, then Problem $(\mathcal{P})$ can be stated either as a limit of classical variational inequalities with smooth inputs, see [10, 22], or in the context of the Riemann-Stieltjes integral, see [12, 5]. The solution operator $(x_0, u) \mapsto \xi$ called the *play* is one of the main building blocks of the theory of hysteresis operators and its properties have been extensively studied in various settings. If u is of bounded variation, Problem $(\mathcal{P})$ can also be interpreted as a special case of a *sweeping process*, see [16], defined as a limit of time-discrete approximations. Theorem 2.3 and Proposition 4.3 below not only illustrate this property, but also show that the time-discrete approximations in the sense of [16] coincide with the exact solutions of Problem $(\mathcal{P})$ for piecewise constant inputs.

The aim of this paper is to propose an extension of the play onto the space of regulated functions via the Young integral in the form given in [8]. An alternative, which we however do not pursue here, would be to use Kurzweil's integral introduced in [14]. Although the Kurzweil integral calculus is in general simpler, its main drawback in connection with Problem $(\mathcal{P})$ consists in the fact that one of the key lemmas (Lemma 3.3 below) which is nearly trivial for the Young integral, does not hold for the Kurzweil integral, and the analysis would have to be restricted to, say, left-continuous inputs.

Another approach can be found in [4] in the scalar case $X = \mathbb{R}$: the rate independence makes it possible to use directly the 'continuous' methods by 'filling in' the discontinuities with segments traversed with an infinite speed. In the case $\dim X > 1$, this procedure turns out to be trajectory-dependent which makes the analysis difficult even if we restrict ourselves to some canonical (the shortest, say) trajectories filling in the jumps.

As the main results of this paper (Theorems 2.3, 2.4), we show that Problem $(\mathcal{P})$ always defines an input-output mapping $\mathfrak{p}_Z : Z \times \overline{BV}(0,T\,;\,X) \to \overline{BV}(0,T\,;\,X) : (x_0, u) \mapsto \xi$ which is continuous with respect to the uniform convergence. Here, $\overline{BV}(0,T\,;\,X)$ denotes the space of functions with essentially bounded variation, see (1.7) below. Moreover, if $0 \in \text{Int}\, Z$, then the output ξ is well-defined in $\overline{BV}(0,T\,;\,X)$ for every $u \in G(0,T\,;\,X)$, and the operator $\mathfrak{p}_Z : Z \times G(0,T\,;\,X) \to \overline{BV}(0,T\,;\,X)$ is continuous with respect to the uniform convergence. It is interesting to note that inputs u_1, u_2 which are equivalent in the sense that $u_1(t-) = u_2(t-)$ for every $t \in [0,T]$, generate equivalent outputs ξ_1, ξ_2.

The paper is organized as follows. In order to fix the notation and to keep the presentation consistent, we list in Section 1 the main concepts from convex analysis and vector-valued functions that are used throughout the text. In Section 2 we state the main results. Section 3 is devoted to a self-contained extension of the Young integration theory to functions with values in a Hilbert space. Detailed proofs of statements from Section 2 are given in Section 4. In Section 5 we illustrate the connection between Problem $(\mathcal{P})$ and the concept of ε-variation introduced by Fraňková in [7].

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1 Preliminaries

The aim of this section is to recall some basic facts about the convex analysis in Hilbert spaces and vector-valued functions of a real variable. Most of the results are well-known and we refer the reader e. g. to the monographs [2, 17] for more information.

For a given convex closed set $Z \subset X$ such that $0 \in Z$ we fix the number

$$\rho := \mathrm{dist}\,(0, \partial Z) := \inf\{|z|\,;\, z \in \partial Z\} \geq 0. \tag{1.1}$$

It is clear that $\rho > 0$ if and only if $0 \in \mathrm{Int}\, Z$. In this case we have $B_\rho(0) \subset Z$, where

$$B_r(x_0) := \{x \in X\,;\, |x - x_0| \leq r\} \tag{1.2}$$

denotes the ball centered at x_0 with radius r.

We introduce in the usual way the projection $Q^Z : X \to Z$ onto Z and its complement $P^Z := I - Q^Z$ (I is the identity) by the formula

$$Q^Z x \in Z\,,\ |P^Z x| = \mathrm{dist}\,(x, Z) \quad \text{for } x \in X\,. \tag{1.3}$$

In the sequel, we call (P^Z, Q^Z) the *projection pair* associated with Z. The projection can be characterized by the variational inequality

$$q = Q^Z x \ \Leftrightarrow\ \langle x - q, q - z\rangle \geq 0 \quad \forall z \in Z\,. \tag{1.4}$$

Let now $[a, b] \subset \mathbb{R}$ be a nondegenerate closed interval. We denote by $\mathcal{D}_{a,b}$ the set of all partitions of the form

$$d = \{t_0, \ldots, t_m\}\,, \quad a = t_0 < t_1 < \ldots < t_m = b\,.$$

For a given function $g : [a, b] \to X$ and a given partition $d \in \mathcal{D}_{a,b}$ we define the *variation* $\mathcal{V}_d(g)$ of g on d by the formula

$$\mathcal{V}_d(g) := \sum_{j=1}^{m} |g(t_j) - g(t_{j-1})|$$

and the *total variation* $\mathrm{Var}_{[a,b]}\, g$ of g by

$$\mathop{\mathrm{Var}}_{[a,b]} g := \sup\{\mathcal{V}_d(g)\,;\, d \in \mathcal{D}_{a,b}\}\,.$$

In a standard way (cf. [2]) we denote the set of functions of bounded variation by

$$BV(a, b; X) := \{g : [a, b] \to X\,;\, \mathop{\mathrm{Var}}_{[a,b]} g < \infty\}\,. \tag{1.5}$$

Let us further introduce the set $S(a, b; X)$ of all *step functions* of the form

$$w(t) := \sum_{k=0}^{m} \hat{c}_k\, \chi_{\{t_k\}}(t) + \sum_{k=1}^{m} c_k\, \chi_{]t_{k-1}, t_k[}(t)\,, \quad t \in [a, b]\,, \tag{1.6}$$

where $d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b}$ is a given partition, χ_A for $A \subset [a, b]$ is the characteristic function of the set A and $\hat{c}_0, \ldots, \hat{c}_m, c_1, \ldots, c_m$ are given elements from X.

It is well-known (see e. g. the Appendix of [2]) that every function of bounded variation with values in a Banach space admits one-sided limits at each point of its domain of definition. Following [1], we separate this property from the notion of total variation and introduce the following definition.

Definition 1.1 *We say that a function* $f : [a, b] \to X$ *is* regulated *if for every* $t \in [a, b]$ *there exist both one-sided limits* $f(t+), f(t-) \in X$ *with the convention* $f(a-) = f(a)$, $f(b+) = f(b)$.

According to [21], we denote by $G(a, b\,; X)$ the set of all regulated functions $f : [a, b] \to X$.

For a given function $g \in G(a, b\,; X)$ and a given partition $d \in \mathcal{D}_{a,b}$ we define the *essential variation* $\overline{\mathcal{V}}_d(g)$ of g on d by the formula

$$\overline{\mathcal{V}}_d(g) \ := \ \sum_{j=1}^{m} |g(t_j-) - g(t_{j-1}+)| \ + \ \sum_{j=0}^{m} |g(t_j+) - g(t_j-)|$$

and the *total essential variation* $\overline{\mathrm{Var}}_{[a,b]}\, g$ of g by

$$\overline{\mathrm{Var}}_{[a,b]}\, g \ := \ \sup\{\overline{\mathcal{V}}_d(g)\,;\, d \in \mathcal{D}_{a,b}\}\,.$$

We denote the space of functions of *essentially bounded variation* by

$$\overline{BV}(a, b\,; X) \ := \ \{g : [a, b] \to X\,;\, \overline{\mathrm{Var}}_{[a,b]}\, g < \infty\}\,. \tag{1.7}$$

The terminology has been taken from [6], although we restrict ourselves a priori to regulated functions which makes the analysis easier. This however means here in particular that $\mathcal{V}_d(g)$ is defined for *every* function $g : [a, b] \to X$, but $\overline{\mathcal{V}}_d(g)$ only for a regulated function g.

We summarize some easy basic properties of the above spaces in Lemma 1.2 below the proof of which is left to the reader.

Lemma 1.2

(i) *Every regulated function is bounded.*

(ii) *For every* $g \in G(a, b\,; X)$ *we have* $\overline{\mathrm{Var}}_{[a,b]}\, g \le \mathrm{Var}_{[a,b]}\, g$.

(iii) *The sets* $S(a, b\,; X)$, $BV(a, b\,; X)$, $\overline{BV}(a, b\,; X)$, $G(a, b\,; X)$ *are vector spaces satisfying the inclusion*

$$S(a, b\,; X) \subset BV(a, b\,; X) \subset \overline{BV}(a, b\,; X) \subset G(a, b\,; X)\,.$$

(iv) *Let* $G_L(a, b\,; X)$ *be the subset of left-continuous functions in* $G(a, b\,; X)$. *Then we have* $G_L(a, b\,; X) \cap \overline{BV}(a, b\,; X) \subset BV(a, b\,; X)$.

We introduce in $G(a, b\,; X)$ a system of seminorms

$$\|f\|_{[s,t]} \ := \ \sup\{|f(\tau)|\,;\, \tau \in [s, t]\} \tag{1.8}$$

for any subinterval $[s, t] \subset [a, b]$. Indeed, $\|\cdot\|_{[a,b]}$ is a norm.

Let us note that the space $C(a, b\,; X)$ of *continuous functions* $f : [a, b] \to X$ is a closed subspace of $G(a, b\,; X)$ with respect to the norm $\|\cdot\|_{[a,b]}$.

We now list some characteristic properties of regulated functions which are needed in the sequel.

Proposition 1.3

(i) *The space* $G(a,b\,;\,X)$ *is complete with respect to the norm* $\|\cdot\|_{[a,b]}$.

(ii) *Given* $C>0$, *the set* $V_C := \{g \in \overline{BV}(a,b\,;\,X)\,;\, \overline{\mathrm{Var}}_{[a,b]}\, g \le C\}$ *is closed in* $G(a,b\,;\,X)$.

(iii) *For* $f \in G(a,b\,;\,X)$ *and* $t \in [a,b]$ *put*

$$\mathrm{osc}\; f(t) \;:=\; \max\{|f(t)-f(t-)|\,, |f(t+)-f(t)|\,,\, |f(t+)-f(t-)|\}\,.$$

Then for every $\varepsilon > 0$ *the set*

$$U_f^{\varepsilon} \;:=\; \{t \in [a,b]\,;\, \mathrm{osc}\; f(t) \ge \varepsilon\} \tag{1.9}$$

is finite, and f *is continuous except in a countable number of points.*

(iv) *Let* $f \in G(a,b\,;\,X)$ *and* $\varepsilon > 0$ *be given and let* U_f^{ε} *be the set defined by (1.9). Then there exists* $h > 0$ *such that for every* $[s,t] \subset [a,b]$ *,* $[s,t] \cap U_f^{\varepsilon} = \emptyset$ *,* $0 < t-s < h$ *we have* $|f(t)-f(s)| < \varepsilon$.

(v) *For every* $f \in G(a,b\,;\,X)$ *and* $\varepsilon > 0$ *there exists* $w \in S(a,b\,;\,X)$ *such that* $\|f-w\|_{[a,b]} \le \varepsilon$ *,* $w(t) \in \cup_{\tau\in[a,b]}\{f(\tau)\}$ *for every* $t \in [a,b]$ *,* $\mathrm{Var}_{[a,b]}\, w \le \mathrm{Var}_{[a,b]}\, f$ *and* $\overline{\mathrm{Var}}_{[a,b]}\, w \le \overline{\mathrm{Var}}_{[a,b]}\, f$.

(vi) *Let* $f \in G(a,b\,;\,X)$ *be such that*

$$\exists C > 0 \quad \forall h \in \left]0, \frac{1}{2}(b-a)\right[\;:\quad \overline{\mathrm{Var}}_{[a+h,\, b-h]}\, f \;\le\; C.$$

Then $f \in \overline{BV}(a,b\,;\,X)$ *and* $\overline{\mathrm{Var}}_{[a,b]}\, f \;\le\; C + |f(a+)-f(a)| + |f(b)-f(b-)|$.

Proof.

(i) Let $\{f_n\}$ be a Cauchy sequence in $G(a,b\,;\,X)$. For $t \in [a,b]$ put $f(t) := \lim_{n\to\infty} f_n(t)$. For every $t \in\,]a,b]$ and every sequence $t_k \nearrow t$ we have

$$|f(t_k)-f(t_\ell)| \;\le\; |f(t_k)-f_n(t_k)| + |f_n(t_k)-f_n(t_\ell)| + |f(t_\ell)-f_n(t_\ell)|\,,$$

hence $\{f(t_k)\,;\, k \in \mathbb{N}\}$ is a Cauchy sequence whose limit is independent of the choice of the sequence t_k , and we conclude that $f(t-)$ exists. In the same way we check that $f(t+)$ exists for $t \in [a,b[$.

(ii) Let $\{g_n\}$ be a sequence in V_C which converges uniformly to g in $[a,b]$. Then, for $d \in \mathcal{D}_{a,b}$, $\overline{\mathcal{V}}_d(g_n)$ converge to $\overline{\mathcal{V}}_d(g)$, hence (ii).

(iii) For every $t \in [a,b]$ there exists $\delta(t) > 0$ such that $|f(\tau)-f(t+)| < \varepsilon/4$ for $\tau \in\,]t, t+\delta(t)[\,\cap[a,b]$, $|f(\tau)-f(t-)| < \varepsilon/4$ for $\tau \in\,]t-\delta(t), t[\,\cap[a,b]$. In particular,

$$\Big(]t-\delta(t), t+\delta(t)[\setminus\{t\}\Big) \cap U_f^{\varepsilon} \;=\; \emptyset \quad \forall\, t \in [a,b]\,.$$

As $[a,b]$ is compact, we can select from the covering

$$[a,b] \;\subset\; \bigcup_{t\in[a,b]} \,]t-\delta(t), t+\delta(t)[$$

a finite covering

$$[a,b] \subset \bigcup_{j=1}^{n} \,]t_j - \delta(t_j), t_j + \delta(t_j)[\; ,$$

hence $U_f^\varepsilon \subset \{t_1, \dots, t_n\}$.

(iv) Let $U_f^\varepsilon = \{t_1, \dots, t_n\}$, $t_{j+1} > t_j$ for $j = 1, \dots, n-1$, and put $I_0 := [a, t_1[$, $I_n :=]t_n, b]$ (which might possibly be empty), $I_j :=]t_j, t_{j+1}[$ for $j = 1, \dots, n-1$. Let f_j, $j = 0, \dots, n$ be the functions $f|_{I_j}$ continuously extended to $\overline{I_j}$. The assertion holds provided we put

$$h \; := \; \inf \Big\{ |t-s| \, ; \, [s,t] \subset \overline{I_j} \, , \, |f_j(t) - f_j(s)| \geq \varepsilon \, , \, j = 0, \dots, n \Big\} \, ,$$

and we easily check that $h > 0$.

(v) Let $\varepsilon > 0$ be given and let U_f^ε, $h > 0$ and I_j, $j = 0, \dots, n$ be as in the proof of (iv). In each interval I_j we find a partition $t_j = s_j^0 < s_j^1 < \ldots < s_j^{\ell_j} = t_{j+1}$ (with the convention $t_0 = a$, $t_{n+1} = b$) such that $s_j^1, \dots, s_j^{\ell_j - 1}$ are continuity points of f, $s_j^k - s_j^{k-1} < h$ for $k = 1, \dots, \ell_j$. For $j = 0, \dots, n$ we now put

$$w(t_j) \; := \; f(t_j)$$

$$w(t) \; := \; \begin{cases} f(s_j^k) & \text{for } \; t \in]s_j^{k-1}, s_j^k] \, , \; k = 1, \dots, \ell_j - 2 \, , \\ f(s_j^{\ell_j - 1}) & \text{for } \; t \in]s_j^{\ell_j - 2}, t_{j+1}[\; . \end{cases}$$

Then $w \in S(a,b\,; X)$ and from (iii) we immediately obtain that $|f(t) - w(t)| \leq \varepsilon$ for every $t \in [a,b]$. Moreover, putting

$$d \; := \; \{a\} \cup \{b\} \cup \{t_1, \dots, t_n\} \cup \left(\bigcup_{j=0}^{n} \left\{ s_j^1, \dots, s_j^{\ell_j - 1} \right\} \right) \; \in \; \mathcal{D}_{a,b}$$

we have $\mathrm{Var}_{[a,b]} \, w \; = \; \mathcal{V}_d(f)$, $\overline{\mathrm{Var}}_{[a,b]} \, w \; \leq \; \overline{\mathcal{V}}_d(f)$, and the assertion follows.

(vi) Let $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ be an arbitrary partition. For $0 < h < \min\{t_1 - a, b - t_{m-1}\}$ we have

$$\overline{\mathcal{V}}_d(f) \; \leq \; \overline{\mathrm{Var}}_{[a+h,\, b-h]} f + |f(a+) - f(a)| + |f(a+h) - f(a+)| + |f(b) - f(b-)| + |f(b-) - f(b-h)| \, ,$$

and letting h tend to $0+$ we obtain the assertion. The proof of Proposition 1.3 is complete. ■

2 Main results

We will see in Section 3 that the integral in Problem $(\boldsymbol{\mathcal{P}})$ is meaningful if $\xi \in \overline{BV}(0,T\,; X)$. Let us denote by $\mathrm{Dom}\,(\mathcal{P}) \subset Z \times G(0,T\,; X)$ the set of all $(x_0, u) \in Z \times G(0,T\,; X)$ such that there exists a solution $\xi \in \overline{BV}(0,T\,; X)$ to Problem $(\boldsymbol{\mathcal{P}})$. We first show that the solution ξ is unique for every $(x_0, u) \in \mathrm{Dom}\,(\mathcal{P})$. In fact, we prove more, namely

Lemma 2.1 *Let $(x_0, u), (y_0, v) \in \mathrm{Dom}\,(\mathcal{P})$ be given and let $\xi, \eta \in \overline{BV}(0,T\,; X)$ be respective solutions to Problem $(\boldsymbol{\mathcal{P}})$. Then for every $t \in [a,b]$ we have*

$$|\xi(t) - \eta(t)|^2 \; \leq \; |\xi(0) - \eta(0)|^2 + 2\, \|u - v\|_{[0,t]} \, \overline{\mathrm{Var}}_{[0,t]} (\xi - \eta) \, . \tag{2.1}$$

Proof. Putting $y(\tau) := (1/2)\,(u(\tau+) + v(\tau+) - \xi(\tau+) - \eta(\tau+))$ in the inequalities $(\boldsymbol{\mathcal{P}})$ (iii) for ξ and for η and summing them up we obtain

$$\int_0^t \langle u(\tau+) - v(\tau+) - \xi(\tau+) + \eta(\tau+), d(\xi - \eta)(\tau)\rangle \;\geq\; 0\,.$$

Corollary 3.15 then yields that

$$\frac{1}{2}\left(|\xi(t) - \eta(t)|^2 - |\xi(0) - \eta(0)|^2\right) \;\leq\; \int_0^t \langle u(\tau+) - v(\tau+), d(\xi - \eta)(\tau)\rangle$$

and (2.1) follows from (3.16). ■

Lemma 2.1 enables us to define the operator

$$\mathfrak{p}_Z \;:\; \mathrm{Dom}\,(\mathcal{P}) \to \overline{BV}(0,T\,;\,X) \;:\; (x_0, u) \mapsto \xi\,, \tag{2.2}$$

where $\xi =: \mathfrak{p}_Z[x_0, u]$ is the unique solution to Problem $(\boldsymbol{\mathcal{P}})$.

We summarize the main results of this paper as Proposition 2.2 and Theorems 2.3 – 2.4 below. The proofs will be given in Section 4.

Proposition 2.2 *Let $x_0 \in Z$ be given, and for $u \in G(0,T\,;\,X)$, $t \in [0,T]$ put $u_-(t) := u(t-)$. Then $(x_0, u) \in \mathrm{Dom}\,(\mathcal{P})$ if and only if $(x_0, u_-) \in \mathrm{Dom}\,(\mathcal{P})$, and in this case we have*

$$\mathfrak{p}_Z[x_0,u](t-) = \mathfrak{p}_Z[x_0,u_-](t)\,, \quad \mathfrak{p}_Z[x_0,u](t) - \mathfrak{p}_Z[x_0,u](t-) \;=\; P^Z(u(t) - \mathfrak{p}_Z[x_0,u](t-)) \tag{2.3}$$

for every $t \in [0,T]$, with P^Z defined by (1.3).

Theorem 2.3 *For each set Z satisfying (0.1) we have $Z \times \overline{BV}(0,T\,;\,X) \subset \mathrm{Dom}\,(\mathcal{P})$ and $\overline{\mathrm{Var}}_{[0,T]}\,\mathfrak{p}_Z[x_0,u] \;\leq\; \overline{\mathrm{Var}}_{[0,T]}\,u$ for every $(x_0,u) \in Z \times \overline{BV}(0,T\,;\,X)$. Moreover, for every $(x_0,u), (y_0,v) \in Z \times \overline{BV}(0,T\,;\,X)$, $\xi = \mathfrak{p}_Z[x_0,u]$, $\eta = \mathfrak{p}_Z[y_0,v]$ and every $t \in [0,T]$ we have*

$$|\xi(t) - \eta(t)|^2 \;\leq\; |\xi(0) - \eta(0)|^2 + 2\,\|u - v\|_{[0,t]}\left(\overline{\mathrm{Var}}_{[0,t]}\,u + \overline{\mathrm{Var}}_{[0,t]}\,v\right)\,. \tag{2.4}$$

Theorem 2.4 *If $0 \in \mathrm{Int}\,Z$, then $\mathrm{Dom}\,(\mathcal{P}) = Z \times G(0,T\,;\,X)$. Moreover, if $\{(x_0^n, u_n)\,;\, n \in \mathbb{N}\}$ is a sequence in $Z \times G(0,T\,;\,X)$ such that $|x_0^n - x_0| \to 0$, $\|u_n - u\|_{[0,T]} \to 0$ as $n \to \infty$, then there exists a constant $C > 0$ independent of n such that $\overline{\mathrm{Var}}_{[0,T]}\,\mathfrak{p}_Z[x_0^n, u_n] \leq C$ and $\|\mathfrak{p}_Z[x_0^n, u_n] - \mathfrak{p}_Z[x_0, u]\|_{[0,T]} \to 0$.*

Remark 2.5 We cannot replace the variational inequality (iii) in Problem $(\boldsymbol{\mathcal{P}})$ by

$$\text{(iii)'} \quad \int_0^t \langle u(\tau) - \xi(\tau) - y(\tau), d\xi(\tau)\rangle \;\geq\; 0 \quad \forall y \in G(0,T;Z) \quad \forall t \in [0,T]$$

which might seem to be a natural extension of the continuous case in [12]. It suffices to consider the scalar case $X = \mathbb{R}$, $Z = [-r, r]$ for some $r > 0$, $x_0 = 0$, $u(\tau) = \bar{u}\,\chi_{]0,T]}(\tau)$ with some $\bar{u} > r$. Assume that there exists $\xi \in BV(0,T\,;\,X)$ satisfying (iii)', $\|u - \xi\|_{[0,T]} \leq r$, $\xi(0) = 0$. Putting $y(\tau) := r\,\chi_{\{0\}}(\tau) + (u(\tau) - \xi(\tau))\;\chi_{]0,T]}(\tau)$ we obtain from (iii)' and Proposition 3.7 that

$$0 \;\leq\; \int_0^t (u(\tau) - \xi(\tau) - r)\,\chi_{\{0\}}(\tau)\,d\xi(\tau) \;=\; -r\,\xi(0+)\,,$$

hence $\xi(0+) \leq 0$ and $u(0+) - \xi(0+) \geq \bar{u} > r$, which is a contradiction.

3 The Young integral

We give here a survey of those elements of the Young integral calculus that are related to Problem $(\boldsymbol{\mathcal{P}})$ using the ideas of [8, 19, 20, 21].

We fix a compact interval $[a,b] \subset \mathbb{R}$ and as in Section 1, we denote by $\mathcal{D}_{a,b}$ the set of all partitions $d = \{t_0, \dots, t_m\}$, $a = t_0 < t_1 < \dots < t_m = b$ of $[a,b]$.

We say that a partition $\hat{d}$ is a *refinement* of $d \in \mathcal{D}_{a,b}$ and write $\hat{d} \succ d$ if $\hat{d} \in \mathcal{D}_{a,b}$ and $d \subset \hat{d}$.

Let $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ be a partition. We denote by $\mathcal{B}(d)$ the set of special refinements D of d (the so-called P-*partitions*, see [19]), of the form

$$D = \{t_0, \varrho_1, t_1, \dots, t_{m-1}, \varrho_m, t_m\}, \ a = t_0 < \varrho_1 < t_1 < \varrho_2 < t_2 < \dots t_{m-1} < \varrho_m < t_m = b. \quad (3.1)$$

For given functions $f : [a,b] \to X$, $g \in G(a,b\,; X)$ and partitions $d \in \mathcal{D}_{a,b}$, $D \in \mathcal{B}(d)$ of the form (3.1) we define the integral sum $S_D(f\,\Delta g)$ by the formula

$$S_D(f\,\Delta g) = \sum_{j=1}^{m} \langle f(\varrho_j), g(t_j-) - g(t_{j-1}+)\rangle + \sum_{j=0}^{m} \langle f(t_j), g(t_j+) - g(t_j-)\rangle, \quad (3.2)$$

again with the convention $g(a-) = g(a)$, $g(b+) = g(b)$.

Definition 3.1 *We say that* $J \in \mathbb{R}$ *is the* Young integral *over* $[a,b]$ *of* f *with respect to* g *and denote*

$$J = \int_a^b \langle f(t), dg(t)\rangle, \quad (3.3)$$

if for every $\varepsilon > 0$ *there exists* $d_\varepsilon \in \mathcal{D}_{a,b}$ *such that for every* $d \succ d_\varepsilon$ *and* $D \in \mathcal{B}(d)$ *we have*

$$|J - S_D(f\,\Delta g)| \le \varepsilon. \quad (3.4)$$

It is an easy exercise to check that if the value J in Definition 3.1 exists, then it is uniquely determined. In what follows, whenever we write $\int_a^b \langle f(t), dg(t)\rangle = J$, we interpret it as *'the function* f *is Young integrable with respect to* g *in* $[a,b]$ *and the integral equals to* J*.'*

Similarly as for other integration theories (cf. e.g. [19] for the Perron-Stieltjes or Kurzweil integral), the Young integral admits the following 'Bolzano-Cauchy-type' characterization.

Lemma 3.2 *Consider* $f : [a,b] \to X$ *and* $g \in G(a,b\,; X)$*. Then* f *is Young integrable with respect to* g *in* $[a,b]$ *if and only if for every* $\varepsilon > 0$ *there exists* $d_\varepsilon \in \mathcal{D}_{a,b}$ *such that for every* $d_i \succ d_\varepsilon$ *and* $D_i \in \mathcal{B}(d_i)$, $i = 1,2$ *we have*

$$|S_{D_1}(f\,\Delta g) - S_{D_2}(f\,\Delta g)| \le \varepsilon. \quad (3.5)$$

Proof. If $\int_a^b \langle f(t), dg(t)\rangle$ exists, then (3.5) obviously holds. Conversely, let (3.5) hold. For every $n \in \mathbb{N}$ we find a partition $d_{1/n} \in \mathcal{D}_{a,b}$ such that (3.5) holds with $\varepsilon = 1/n$, and put $d_1^* := d_1$, $d_n^* := d_{n-1}^* \cup d_{1/n}$ for $n = 2,3,\dots$. For each $n \in \mathbb{N}$ we fix some $D_n \in \mathcal{B}(d_n^*)$ and put $J_n := S_{D_n}(f\,\Delta g)$. By (3.5), $\{J_n\}$ is a Cauchy sequence in $\mathbb{R}$, and putting $J := \lim_{n\to\infty} J_n$ we easily check that $J = \int_a^b \langle f(t), dg(t)\rangle$ by Definition 3.1. ∎

The reason we decided for the Young integral is its following property which is an immediate consequence of the definition and which plays an important role in our arguments. Surprisingly enough, identity (3.6) does not hold for the Kurzweil integral in general [13].

Lemma 3.3 *Consider* $f : [a,b] \to X$ *and* $g \in G(a,b;X)$, *and assume that there exists a countable set* $A \subset [a,b]$ *such that* $g(t) = 0$ *for every* $t \in [a,b] \setminus A$. *Then we have*

$$\int_a^b \langle f(t), dg(t)\rangle \;=\; \langle f(b), g(b)\rangle \;-\; \langle f(a), g(a)\rangle\,. \tag{3.6}$$

The Young integral is linear with respect to both functions f and g. For the sake of completeness, we state this result explicitly.

Proposition 3.4

(i) *Let* $\int_a^b \langle f_1(t), dg(t)\rangle$, $\int_a^b \langle f_2(t), dg(t)\rangle$ *exist. Then we have*

$$\int_a^b \langle (f_1+f_2)(t), dg(t)\rangle \;=\; \int_a^b \langle f_1(t), dg(t)\rangle \;+\; \int_a^b \langle f_2(t), dg(t)\rangle\,. \tag{3.7}$$

(ii) *Let* $\int_a^b \langle f(t), dg_1(t)\rangle$, $\int_a^b \langle f(t), dg_2(t)\rangle$ *exist. Then we have*

$$\int_a^b \langle f(t), d(g_1+g_2)(t)\rangle \;=\; \int_a^b \langle f(t), dg_1(t)\rangle \;+\; \int_a^b \langle f(t), dg_2(t)\rangle\,. \tag{3.8}$$

(iii) *Let* $\int_a^b \langle f(t), dg(t)\rangle$ *exist. Then for every constant* $\lambda \in \mathbb{R}$ *we have*

$$\int_a^b \langle \lambda f(t), dg(t)\rangle \;=\; \int_a^b \langle f(t), d(\lambda g)(t)\rangle \;=\; \lambda \int_a^b \langle f(t), dg(t)\rangle\,. \tag{3.9}$$

Proof. (i) Let $\varepsilon > 0$ be given. We find $d^1_{\varepsilon/2}$, $d^2_{\varepsilon/2} \in \mathcal{D}_{a,b}$ such that for all $d_i \succ d^i_{\varepsilon/2}$, $D_i \in \mathcal{B}(d_i)$, $i = 1,2$ we have

$$\left| \int_a^b \langle f_i(t), dg(t)\rangle \;-\; S_{D_i}(f_i\,\Delta g)\right| \;<\; \frac{\varepsilon}{2}\,. \tag{3.10}$$

Putting $d_\varepsilon := d^1_{\varepsilon/2} \cup d^2_{\varepsilon/2}$ we obtain (3.7) immediately from (3.10). The same argument applies to the case (ii), while (iii) is obvious. ∎

The Young integral behaves in the following way with respect to the variation of the integration domain.

Proposition 3.5 *Let* $f : [a,b] \to X$, $g \in G(a,b;X)$ *be given functions and let* $[r,s] \subset [a,b]$ *be a nondegenerate interval.*

(i) *Assume that* $\int_a^b \langle f(t), dg(t)\rangle$ *exists. Then* $\int_r^s \langle f(t), dg(t)\rangle$ *exists.*

(ii) *Assume that* $\int_r^s \langle f(t), dg(t)\rangle$ *exists. Then we have*

$$\int_a^b \Big\langle \Big(f\,\chi_{]r,s[}\Big)(t), dg(t)\Big\rangle \;=\; \int_r^s \langle f(t), dg(t)\rangle \;-\; \langle f(r), g(r+)-g(r)\rangle \;-\; \langle f(s), g(s)-g(s-)\rangle\,. \tag{3.11}$$

Remark 3.6 Proposition 3.5 needs some comment. Here and in the sequel, whenever we integrate functions f, g defined in $[a,b]$ over an interval $[r,s] \subset [a,b]$, we implicitly consider their restrictions $f|_{[r,s]}, g|_{[r,s]}$. In particular, we have e. g. $f|_{[r,s]}(s+) = f(s)$, $f|_{[r,s]}(r-) = f(r)$, similarly as in Definition 1.1.

Proof of Proposition 3.5.

(i) Let $\varepsilon > 0$ be given. By Lemma 3.2 we find $d_\varepsilon \in \mathcal{D}_{a,b}$ such that for every $d_i \succ d_\varepsilon$ and $D_i \in \mathcal{B}(d_i)$, $i = 1,2$ we have

$$|S_{D_1}(f\,\Delta g) \;-\; S_{D_2}(f\,\Delta g)| \;<\; \varepsilon\,,$$

and put $d_\varepsilon^* := (d_\varepsilon \cap]r,s[\,) \cup \{r\} \cup \{s\}$. Then $d_\varepsilon^* \in \mathcal{D}_{r,s}$, and we arbitrarily fix $d_i \succ d_\varepsilon^*$ and $D_i \in \mathcal{B}(d_i)$, $i = 1,2$. Put $\hat{d}_i := d_i \cup d_\varepsilon$. Then $\hat{d}_1, \hat{d}_2$ may be written in the form

$$\begin{aligned} \hat{d}_1 &= \{a = t_0 < t_1 < \ldots < t_k = r < t^1_{k+1} < \ldots < s = t^1_{m_1-\ell} < \ldots < t^1_{m_1} = b\}\,, \\ \hat{d}_2 &= \{a = t_0 < t_1 < \ldots < t_k = r < t^2_{k+1} < \ldots < s = t^2_{m_2-\ell} < \ldots < t^2_{m_2} = b\}\,, \end{aligned}$$

where $t^1_{m_1-j} = t^2_{m_2-j}$ for $j = 0,\ldots,\ell$. We now fix arbitrary $\varrho_i \in]t_{i-1}, t_i[$ for $i = 1,\ldots,k$ and $\hat{\varrho}_j \in]t^1_{m_1-j}, t^1_{m_1-j+1}[$ for $j = 1,\ldots,\ell$, and put

$$\begin{aligned} \hat{D}_1 &= \left\{t_0, \varrho_1, t_1, \ldots, t_{k-1}, \varrho_k, D_1, \hat{\varrho}_\ell, t^1_{m_1-\ell+1}, \ldots, t^1_{m_1-1}, \hat{\varrho}_1, t^1_{m_1}\right\}\,, \\ \hat{D}_2 &= \left\{t_0, \varrho_1, t_1, \ldots, t_{k-1}, \varrho_k, D_2, \hat{\varrho}_\ell, t^2_{m_2-\ell+1}, \ldots, t^2_{m_2-1}, \hat{\varrho}_1, t^2_{m_2}\right\}\,. \end{aligned}$$

Then $\hat{D}_i \in \mathcal{B}(\hat{d}_i)$, $i = 1,2$, and we have

$$S_{D_1}(f\,\Delta g) \;-\; S_{D_2}(f\,\Delta g) = S_{\hat{D}_1}(f\,\Delta g) \;-\; S_{\hat{D}_2}(f\,\Delta g)\,,$$

hence the assertion follows from Lemma 3.2.

(ii) Let $\varepsilon > 0$ be given and let $d_\varepsilon = \{t_0, \ldots, t_m\} \in \mathcal{D}_{r,s}$ be such that for every $d \succ d_\varepsilon$ and $D \in \mathcal{B}(d)$ we have

$$\left|\int_r^s \langle f(t), dg(t)\rangle \;-\; S_D(f\,\Delta g)\right| \;<\; \varepsilon\,. \tag{3.12}$$

Put $d_\varepsilon^* := d_\varepsilon \cup \{a,b\}$. Then $d_\varepsilon^* \in \mathcal{D}_{a,b}$, and every $D^* \in \mathcal{B}(d^*)$ with $d^* \succ d_\varepsilon^*$ is of the form

$$D^* \;=\; \{t_0^*, \varrho_1^*, t_1^*, \ldots, t^*_{m^*-1}, \varrho^*_{m^*}, t^*_{m^*}\}\,,$$

where $r = t_i^*$, $s = t_k^*$ for some $0 \le i < k \le m^*$. By construction, $d := \{t_i^*, t_{i+1}^*, \ldots, t_k^*\}$ is a refinement of d_ε and $D := \{t_i^*, \varrho_{i+1}^*, t_{i+1}^*, \ldots, \varrho_k^*, t_k^*\}$ belongs to $\mathcal{B}(d)$. On the other hand, for $j \le i$ and $j > k$ we have $f\,\chi_{]r,s[}\,(\varrho_j^*) = 0$, hence

$$S_{D^*}\Big(\big(f\,\chi_{]r,s[}\,\big)\,\Delta g\Big) \;=\; S_D(f\,\Delta g) \;-\; \langle f(r), g(r+) - g(r)\rangle \;-\; \langle f(s), g(s) - g(s-)\rangle\,,$$

and the assertion follows. ∎

We next investigate some typical cases.

Proposition 3.7 *For every $f : [a,b] \to X$, $g \in G(a,b;X)$, $a \le r \le b$ and $v \in X$ we have*

(i) $\displaystyle\int_a^b \Big\langle v\, \chi_{\{r\}}(t), dg(t)\Big\rangle \;=\; \langle v, g(r+) - g(r-)\rangle \;,$

(ii) $\displaystyle\int_a^b \Big\langle f(t), d\Big(v\, \chi_{\{r\}}\Big)(t)\Big\rangle \;=\; \begin{cases} 0 & \textit{if } \; r \in \,]a,b[\;, \\ -\langle f(a), v\rangle & \textit{if } \; r = a\,, \\ \langle f(b), v\rangle & \textit{if } \; r = b\,, \end{cases}$

(iii) $\displaystyle\int_a^b \Big\langle v\, \chi_{]r,s[}(t), dg(t)\Big\rangle \;=\; \langle v, g(s-) - g(r+)\rangle \qquad \forall\, s \in \,]r,b] \;,$

(iv) $\displaystyle\int_a^b \Big\langle f(t), d\Big(v\, \chi_{]r,s[}\Big)(t)\Big\rangle \;=\; \langle f(r) - f(s), v\rangle \qquad \forall\, s \in \,]r,b] \;.$

Proof. Identity (iii) follows immediately from Proposition 3.5 for $f(t) \equiv v$ and from the obvious fact (cf. Remark 3.6) that $\int_r^s \langle v, dg(t)\rangle = \langle v, g(s) - g(r)\rangle$. For $a < r < b$ we obtain (i) from (iii) and from Proposition 3.4 using the formula $\chi_{\{r\}} = \chi_{]a,b[} - \chi_{]a,r[} - \chi_{]r,b[}$. To treat the case $r = a$, we just note that for each $d \in \mathcal{D}_{a,b}$ and $D \in \mathcal{B}(d)$, the integral sum $S_D(v\, \chi_{\{a\}}\, \Delta g)$ contains only one nonzero term, namely $\langle v, g(a+) - g(a)\rangle$. For $r = b$ we argue analogously. Statement (ii) is a trivial consequence of Lemma 3.3.

To prove (iv), we set $d_0 := \{a, r, s, b\}$. Then for every $d \succ d_0$ and $D \in \mathcal{B}(d)$ of the form (3.1) we have $r = t_i$, $s = t_k$ for some $0 \le i < k \le m$, where $\chi_{]r,s[}(t_j-) = \chi_{]r,s[}(t_{j-1}+)$ for every $j = 1, \ldots, m$, $\chi_{]r,s[}(t_j-) = \chi_{]r,s[}(t_j+)$ for every $j \ne i, k$, $\chi_{]r,s[}(t_i+) - \chi_{]r,s[}(t_i-) = 1$, $\chi_{]r,s[}(t_k+) - \chi_{]r,s[}(t_k-) = -1$, hence $S_D\Big(f \Delta \Big(v\, \chi_{]r,s[}\Big)\Big) = \langle f(r) - f(s), v\rangle$, and Proposition 3.7 is proved. ∎

Corollary 3.8 *Let $f : [a,b] \to X$, $g \in G(a,b\,; X)$ and $s \in \,]a,b[$ be such that $\int_a^s \langle f(t), dg(t)\rangle$, $\int_s^b \langle f(t), dg(t)\rangle$ exist. Then we have*

$$\int_a^b \langle f(t), dg(t)\rangle \;=\; \int_a^s \langle f(t), dg(t)\rangle \;+\; \int_s^b \langle f(t), dg(t)\rangle \;. \tag{3.13}$$

Proof. We clearly have $f = f\, \chi_{]a,s[} + f\, \chi_{]s,b[} + f\, \chi_{\{a\}} + f\, \chi_{\{s\}} + f\, \chi_{\{b\}}$. By Propositions 3.5 and 3.7 (i), each of these five functions is Young integrable with respect to g in $[a,b]$. Owing to Proposition 3.4, f is Young integrable with respect to g in $[a,b]$, and (3.13) readily follows from (3.11), Proposition 3.7 (i), and the identity

$$\begin{aligned} \int_a^s \langle f(t), dg(t)\rangle \;+\; \int_s^b \langle f(t), dg(t)\rangle \;&=\; \int_a^b \Big\langle \Big(f\, \Big(\chi_{]a,s[} + \chi_{]s,b[}\Big)\Big)(t), dg(t)\Big\rangle \\ &\quad +\; \langle f(a), g(a+) - g(a)\rangle \;+\; \langle f(s), g(s+) - g(s-)\rangle \;+\; \langle f(b), g(b) - g(b-)\rangle \\ &=\; \int_a^b \Big\langle \Big(f\, \Big(\chi_{]a,s[} + \chi_{]s,b[} + \chi_{\{a\}} + \chi_{\{s\}} + \chi_{\{b\}}\Big)\Big)(t), dg(t)\Big\rangle \;=\; \int_a^b \langle f(t), dg(t)\rangle \,. \end{aligned}$$

∎

In order to preserve the consistency of (3.13) also in the limit cases $s = a$ and $s = b$, we set

$$\int_s^s \langle f(t), dg(t)\rangle \;=\; 0 \qquad \forall s \in [a,b]\,,\; \forall\, f, g : [a,b] \to X\,. \tag{3.14}$$

Propositions 3.4 and 3.7 enable us to evaluate the integral $\int_a^b \langle f(t), dg(t)\rangle$ provided one of the functions f, g belongs to $S(a,b;X)$. The next strategy consists in exploiting the density of $S(a,b;X)$ in $G(a,b;X)$ stated in Proposition 1.3 (iv). We first notice that for all functions $f,g:[a,b]\to X$ and every P-partition D of the form (3.1) we have

$$\begin{aligned} S_D(f\,\Delta g) &= \sum_{j=1}^{m}\langle f(\varrho_j), g(t_j-)-g(t_{j-1}+)\rangle + \sum_{j=0}^{m}\langle f(t_j), g(t_j+)-g(t_j-)\rangle \\ &= \langle f(b), g(b)\rangle - \langle f(a), g(a)\rangle \\ &\quad + \sum_{j=1}^{m}\langle f(\varrho_j)-f(t_j), g(t_j-)\rangle - \sum_{j=0}^{m-1}\langle f(\varrho_{j+1})-f(t_j), g(t_j+)\rangle\ , \end{aligned}$$

hence

$$|S_D(f\,\Delta g)| \;\le\; \min\left\{\|f\|_{[a,b]}\,\overline{\mathcal{V}}_d(g)\,, (|f(a)|+|f(b)|+\mathcal{V}_d(f))\;\|g\|_{[a,b]}\right\}. \tag{3.15}$$

The extension of the Young integral to $G(a,b;X)$ is based on Theorem 3.9 below.

Theorem 3.9 *Consider $f, f_n:[a,b]\to X$, $n\in\mathbb{N}$ such that $\lim_{n\to\infty}\|f-f_n\|_{[a,b]}=0$. Then the following implications hold.*

(i) *If $g\in\overline{BV}(a,b;X)$ and $\int_a^b\langle f_n(t), dg(t)\rangle$ exists for each $n\in\mathbb{N}$, then $\int_a^b\langle f(t), dg(t)\rangle$ exists and*

$$\int_a^b\langle f(t), dg(t)\rangle \;=\; \lim_{n\to\infty}\int_a^b\langle f_n(t), dg(t)\rangle\ .$$

(ii) *If $g\in BV(a,b;X)$ and $\int_a^b\langle g(t), df_n(t)\rangle$ exists for each $n\in\mathbb{N}$, then $\int_a^b\langle g(t), df(t)\rangle$ exists and*

$$\int_a^b\langle g(t), df(t)\rangle \;=\; \lim_{n\to\infty}\int_a^b\langle g(t), df_n(t)\rangle\ .$$

Proof.

(i) For $n\in\mathbb{N}$ put $J_n := \int_a^b\langle f_n(t), dg(t)\rangle$. For each n we find $d_n\in\mathcal{D}_{a,b}$ such that for every $d\succ d_n$ and $D\in\mathcal{B}(d)$ we have

$$|S_D(f_n\,\Delta g)\;-\;J_n| \;<\; \frac{1}{n}\,.$$

For $m,n\in\mathbb{N}$ put $d_{mn} := d_n\cup d_m$. For every $d\succ d_{mn}$ and $D\in\mathcal{B}(d)$ we then have

$$|S_D(f_n\,\Delta g)\;-\;J_n| \;<\; \frac{1}{n}\,,\quad |S_D(f_m\,\Delta g)\;-\;J_m| \;<\; \frac{1}{m}\,,$$

and (3.15) with $f:=f_n-f_m$, $g:=g$ implies that

$$\begin{aligned} |J_n-J_m| &\le |S_D(f_n\,\Delta g)\;-\;J_n\;-\;S_D(f_m\,\Delta g)\;+\;J_m|\;+\;|S_D((f_n-f_m)\,\Delta g)| \\ &\le \frac{1}{m}+\frac{1}{n}+\|f_n-f_m\|_{[a,b]}\,\overline{\operatorname*{Var}_{[a,b]}}\,g\,, \end{aligned}$$

hence $\{J_n\}$ is a Cauchy sequence and we may put $J := \lim_{n\to\infty} J_n$. For each $d \succ d_n$ and $D \in \mathcal{B}(d)$ we obtain that

$$\begin{aligned} |S_D(f\,\Delta g) - J| &\leq |S_D((f-f_n)\,\Delta g)| + |S_D(f_n\,\Delta g) - J_n| + |J_n - J| \\ &\leq \|f - f_n\|_{[a,b]} \overline{\operatorname*{Var}_{[a,b]}}\, g + 1/n + |J_n - J|\,, \end{aligned}$$

hence $\int_a^b \langle f(t), dg(t)\rangle = J$ and **(i)** is proved.

The same argument based on (3.15) with $f := g$, $g := f_n - f_m$ yields **(ii)**. ■

From Propositions 3.4 and 3.7 it follows that the integral $\int_a^b \langle f(t), dg(t)\rangle$ exists whenever one of the functions f, g belongs to $S(a,b\,;\,X)$. Using the fact that every regulated function can be uniformly approximated by step functions (cf. Proposition 1.3 (v)), we obtain the following result as an immediate consequence of Theorem 3.9 and inequality (3.15).

Corollary 3.10 *If either* $f \in G(a,b\,;\,X)$ *and* $g \in \overline{BV}(a,b\,;\,X)$, *or* $f \in BV(a,b\,;\,X)$ *and* $g \in G(a,b\,;\,X)$, *then* $\int_a^b \langle f(t), dg(t)\rangle$ *exists and satisfies the estimate*

$$\left|\int_a^b \langle f(t), dg(t)\rangle\right| \leq \min\left\{\|f\|_{[a,b]} \overline{\operatorname*{Var}_{[a,b]}}\, g\,, \left(|f(a)| + |f(b)| + \operatorname*{Var}_{[a,b]} f\right) \|g\|_{[a,b]}\right\}. \tag{3.16}$$

The estimate (3.16) is optimal in the following sense.

Theorem 3.11 *For every* $g \in BV(a,b\,;\,X)$ *we have*

$$|g(a)| + |g(b)| + \operatorname*{Var}_{[a,b]} g = \sup\left\{\int_a^b \langle g(t), df(t)\rangle\ ;\ f \in S(a,b\,;\,B_1(0))\right\}. \tag{3.17}$$

For every $g \in \overline{BV}(a,b\,;\,X)$ *we have*

$$\overline{\operatorname*{Var}_{[a,b]}}\, g = \sup\left\{\int_a^b \langle f(t), dg(t)\rangle\ ;\ f \in S(a,b\,;\,B_1(0))\right\}. \tag{3.18}$$

Proof. To prove (3.17), we fix $\varepsilon > 0$ and find a partition $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ such that

$$\sum_{j=1}^m |g(t_j) - g(t_{j-1})| \geq \operatorname*{Var}_{[a,b]} g - \varepsilon\,. \tag{3.19}$$

Let $\sigma : X \to X$ be the function

$$\sigma(x) = x/|x| \quad \text{for } x \neq 0\,, \quad \sigma(0) = 0\,,$$

and for $t \in [a,b]$ put

$$f_1(t) := \sigma(g(b))\,\chi_{\{b\}}(t) - \sigma(g(a))\,\chi_{\{a\}}(t) - \sum_{j=1}^m \sigma(g(t_j) - g(t_{j-1}))\,\chi_{]t_{j-1},t_j[}(t)\,. \tag{3.20}$$

We then infer from (3.19), Proposition 3.7 (ii), (iv) and Proposition 3.4 (ii) that

$$\int_a^b \langle g(t), df_1(t)\rangle \;=\; |g(a)| + |g(b)| + \sum_{j=1}^{m} |g(t_j) - g(t_{j-1})| \;\geq\; |g(a)| + |g(b)| + \operatorname*{Var}_{[a,b]} g \,-\, \varepsilon\,, \quad (3.21)$$

which, together with Corollary 3.10, yields (3.17).

We now turn to the proof of (3.18) and consider $g \in \overline{BV}(a,b\,;\,X)$ and $\varepsilon > 0$. We find again a partition $d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b}$ such that

$$\sum_{j=1}^{m} |g(t_j-) - g(t_{j-1}+)| \,+\, \sum_{j=0}^{m} |g(t_j+) - g(t_j-)| \;\geq\; \overline{\operatorname*{Var}_{[a,b]}}\, g \,-\, \varepsilon\,, \quad (3.22)$$

and put

$$f_2(t) \;:=\; \sum_{j=1}^{m} \sigma(g(t_j-) - g(t_{j-1}+))\, \chi_{\,]t_{j-1},t_j[}\,(t) \,+\, \sum_{j=0}^{m} \sigma(g(t_j+) - g(t_j-))\, \chi_{\{t_j\}}(t)\,. \quad (3.23)$$

Then $f_2 \in S(a,b\,;\,B_1(0))$ and it follows from Propositions 3.4 and 3.7 (i), (iii) that

$$\int_a^b \langle f_2(t), dg(t)\rangle \;=\; \sum_{j=1}^{m} |g(t_j-) - g(t_{j-1}+)| \,+\, \sum_{j=0}^{m} |g(t_j+) - g(t_j-)|\,. \quad (3.24)$$

The assertion (3.18) is then a consequence of (3.22), (3.24), and Corollary 3.10. ■

As an easy extension of Theorem 3.9 we have the following convergence result.

Proposition 3.12 *Consider* $f, f_n \in G(a,b\,;\,X)$, $g, g_n \in \overline{BV}(a,b\,;\,X)$, $n \in \mathbb{N}$ *such that*

$$\lim_{n\to\infty} \|f - f_n\|_{[a,b]} = 0\,, \quad \lim_{n\to\infty} \|g - g_n\|_{[a,b]} = 0\,, \quad \sup_{n\in\mathbb{N}} \overline{\operatorname*{Var}_{[a,b]}}\, g_n \;=\; C \;<\; \infty\,.$$

Then

$$\int_a^b \langle f(t), dg(t)\rangle \;=\; \lim_{n\to\infty} \int_a^b \langle f_n(t), dg_n(t)\rangle\;. \quad (3.25)$$

Proof. For any $w \in S(a,b\,;\,X)$ we have by Corollary 3.10 that

$$\begin{aligned}\left|\int_a^b \langle f(t), dg(t)\rangle \;-\; \int_a^b \langle f_n(t), dg_n(t)\rangle\right| \;&\leq\; \left|\int_a^b \langle (f-f_n)(t), dg_n(t)\rangle\right| \\ &+\; \left|\int_a^b \langle (f-w)(t), d(g-g_n)(t)\rangle\right| \;+\; \left|\int_a^b \langle w(t), d(g-g_n)(t)\rangle\right| \\ &\leq\; C\,\|f - f_n\|_{[a,b]} \,+\, 2\,C\,\|f - w\|_{[a,b]} \,+\, \left(2\,\|w\|_{[a,b]} + \operatorname*{Var}_{[a,b]} w\right)\|g - g_n\|_{[a,b]}\end{aligned}$$

and the assertion follows from Proposition 1.3 (v). ■

Example 3.13

(i) Notice that the pointwise convergence $g_n(t) \to g(t)$ for every $t \in [a,b]$ is not sufficient in Proposition 3.12 as in the case of the Riemann-Stieltjes integral. In the example $X = \mathbb{R}$,

$$f_n(t) = f(t) = \chi_{\{0\}}(t), \quad g(t) \equiv 0, \quad g_n(t) = \chi_{]0,1/n[}(t) \qquad \text{for } t \in [0,1] \tag{3.26}$$

we have $\int_0^1 f_n(t)\, dg_n(t) = 1$ for every $n \in \mathbb{N}$, $\int_0^1 f(t)\, dg(t) = 0$, hence the assertion of Proposition 3.12 does not hold.

(ii) Similarly, the pointwise convergence of $\{f_n\}$ is not sufficient for Proposition 3.12 to hold. Indeed, putting

$$f_n(t) := \sum_{k=1}^{n} (-1)^{k-1} \chi_{\{k/n^2\}}(t) \qquad \text{for } t \in [0,1], \tag{3.27}$$

$$g_n(t) := \begin{cases} \frac{1}{2n}\Big((-1)^k + 1\Big) & \text{for } t \in \Big[\frac{k-1}{n^2}, \frac{k}{n^2}\Big[\,, \quad k = 1, \dots, n, \\ 0 & \text{for } t \in \Big[\frac{1}{n}, 1\Big]\,, \end{cases} \tag{3.28}$$

for $n \in \mathbb{N}$, we see that $f_n, g_n \in S(a,b\,;X)$, $\|f_n\|_{[a,b]} \le 1$, $\operatorname{Var}_{[a,b]} g_n \le 1$, $\|g_n\|_{[a,b]} \to 0$ and $f_n(t) \to 0$ for every $t \in [a,b]$ as $n \to \infty$, while $\int_a^b f_n(t)\, dg_n(t) \to 1$.

The uniform convergence of f_n towards f in Proposition 3.12 can however be relaxed, as we will see in Section 5, Proposition 5.4. To conclude this section, we derive two integration-by-parts formulas.

Theorem 3.14 *For every $f \in G(a,b\,;X)$, $g \in BV(a,b\,;X)$ we have*

$$\begin{aligned} \int_a^b \langle f(t), dg(t)\rangle + \int_a^b \langle g(t), df(t)\rangle \;&=\; \langle f(b), g(b)\rangle - \langle f(a), g(a)\rangle \\ &+ \sum_{t\in[a,b]} \Big(\langle f(t) - f(t-)\,,\, g(t) - g(t-)\rangle - \langle f(t+) - f(t)\,,\, g(t+) - g(t)\rangle \Big). \end{aligned} \tag{3.29}$$

Proof. From Proposition 1.3 (ii) it follows that the sum on the right-hand side of (3.29) is at most countable, hence the formula is meaningful. Using Proposition 3.7 we check in a straightforward way that (3.29) holds for every $g \in BV(a,b\,;X)$ whenever f is of the form $v\,\chi_{\{r\}}$ or $v\,\chi_{]r,s[}$, hence also for every $f \in S(a,b\,;X)$ by Proposition 3.4. For $f \in G(a,b\,;X)$ and $n \in \mathbb{N}$ we find $f_n \in S(a,b\,;X)$ such that $\|f - f_n\|_{[a,b]} \to 0$ as $n \to \infty$ and pass to the limit using Theorem 3.9 and the obvious inequality

$$\sum_{t\in[a,b]} \Big(|g(t) - g(t-)| + |g(t+) - g(t)| \Big) \;\le\; \operatorname*{Var}_{[a,b]} g\,.$$

∎

Corollary 3.15 *For every $g \in \overline{BV}(a,b\,;X)$ we have*

$$\int_a^b \langle g(t+), dg(t)\rangle \;=\; \frac{1}{2}\Big(|g(b)|^2 - |g(a)|^2\Big) + \frac{1}{2}\sum_{t\in[a,b]} |g(t+) - g(t-)|^2\,. \tag{3.30}$$

Proof. The function $g_+(t) := g(t+)$ satisfies $g_+(t+) = g(t+) = g_+(t)$ for every $t \in [a,b]$, $g_+(t-) = g(t-)$ for every $t \in]a,b]$, and belongs to $BV(a,b;X)$. By Theorem 3.14 we have

$$\int_a^b \langle g_+(t), dg_+(t)\rangle \;=\; \frac{1}{2}\left(|g(b)|^2 - |g(a+)|^2\right) + \frac{1}{2}\sum_{t\in]a,b]} |g(t+) - g(t-)|^2$$

(note that the sum is taken over the semi-open interval $]a,b]$), while (3.6) yields that

$$\int_a^b \langle g_+(t), d(g-g_+)(t)\rangle \;=\; \langle g(a+), g(a+) - g(a)\rangle \;.$$

Combining the above identities we obtain the assertion. ■

4 Proofs of main results

We first investigate local properties of the mapping $\mathfrak{p}_Z$ introduced in (2.2) that will enable us to treat the general case in Theorems 2.3 – 2.4. With the notation from Section 2, we establish the following lemma.

Lemma 4.1 *Consider $(x_0,u) \in \mathrm{Dom}\,(\mathcal{P})$ and put $\xi := \mathfrak{p}_Z[x_0,u]$. Then for every $t \in [0,T]$ we have*

$$\xi(t) - \xi(t-) \;=\; P^Z(u(t) - \xi(t-))\,, \quad \xi(t+) - \xi(t-) \;=\; P^Z(u(t+) - \xi(t-))\,, \tag{4.1}$$

where (P^Z, Q^Z) is the projection pair introduced in (1.3). In particular, the inequalities

$$|\xi(t) - \xi(t-)| \;\le\; |u(t) - u(t-)|\,, \quad |\xi(t+) - \xi(t)| \;\le\; |u(t+) - u(t)| \tag{4.2}$$

hold for every $t \in [0,T]$.

Proof. For a given $t \in [0,T]$ and $z \in Z$ put

$$y(\tau) \;:=\; z\,\chi_{\{t\}}(\tau) + \big(u(\tau+) - \xi(\tau+)\big)\big(\chi_{[0,t[}(\tau) + \chi_{]t,T]}(\tau)\big)$$

for $\tau \in [0,T]$. Then Proposition 3.7 (i) yields (cf. Remark 3.6) that

$$\begin{aligned} 0 \;&\le\; \int_0^t \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau)\rangle \;=\; \int_0^t \Big\langle (u(t) - \xi(t) - z)\chi_{\{t\}}(\tau), d\xi(\tau)\Big\rangle \\ &=\; \langle u(t) - \xi(t) - z,\, \xi(t) - \xi(t-)\rangle \;. \end{aligned} \tag{4.3}$$

If moreover $T > t$, then

$$\begin{aligned} 0 \;&\le\; \int_0^T \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau)\rangle = \int_0^T \Big\langle (u(t+) - \xi(t+) - z)\chi_{\{t\}}(\tau), d\xi(\tau)\Big\rangle \\ &=\; \langle u(t+) - \xi(t+) - z,\, \xi(t+) - \xi(t-)\rangle \;. \end{aligned} \tag{4.4}$$

Since $z \in Z$ is arbitrary, we obtain from (1.4), (4.3), (4.4) that $u(t) - \xi(t) \;=\; Q^Z(u(t) - \xi(t-))$, $u(t+) - \xi(t+) \;=\; Q^Z(u(t+) - \xi(t-))$, and the assertion follows. The inequalities (4.2) are obvious: the first one follows from (4.3) by putting $z := u(t-) - \xi(t-)$, to prove the second one we put $z := u(t+) - \xi(t+)$ in (4.3), $z := u(t) - \xi(t)$ in (4.4) and sum up both inequalities. ■

As a consequence of Lemma 4.1, we see that $\xi := \mathfrak{p}_Z[x_0, u]$ is left- (right-) continuous if u is left- (right-) continuous, respectively. We now pass to the proof of Proposition 2.2 which shows that we can reduce the analysis to the left-continuous case.

Proof of Proposition 2.2. Let $u \in G(0,T\,;\,X)$, $y \in G(0,T\,;\,Z)$, $x_0 \in Z$ and $\xi \in \overline{BV}(0,T\,;\,X)$ be given. For $t \in [0,T]$ put $u_-(t) := u(t-)$, $\xi_-(t) := \xi(t-)$, $\xi^*(t) := \xi(t) - \xi_-(t)$. For all $t \in [0,T]$ and $\tau \in [0,t[$ we have $u_-(\tau+) - \xi_-(\tau+) = u(\tau+) - \xi(\tau+)$, and from Proposition 3.7 (i), Lemma 3.3 and property (1.4) it follows that

$$\begin{aligned}\int_0^t \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau)\rangle \;&=\; \int_0^t \Big\langle (u(t) - \xi(t) - u_-(t) + \xi_-(t))\chi_{\{t\}}(\tau), d\xi_-(\tau)\Big\rangle \qquad (4.5)\\ &+\; \int_0^t \langle u_-(\tau+) - \xi_-(\tau+) - y(\tau), d\xi_-(\tau)\rangle \;+\; \int_0^t \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi^*(\tau)\rangle\\ &=\; \int_0^t \langle u_-(\tau+) - \xi_-(\tau+) - y(\tau), d\xi_-(\tau)\rangle \;+\; \langle u(t) - \xi(t) - y(t), \xi(t) - \xi(t-)\rangle\,.\end{aligned}$$

Assume first that $(x_0, u_-) \in \mathrm{Dom}\,(\mathcal{P})$, and put $\eta := \mathfrak{p}_Z[x_0, u_-]$. By Lemma 4.1, η is left-continuous, and putting

$$\xi(t) := \eta(t) + P^Z(u(t) - \eta(t)) \qquad (4.6)$$

we obtain $u(t) - \xi(t) \in Z$ for every $t \in [0,T]$, $\xi(t-) = \eta(t) + P^Z(u_-(t) - \eta(t)) = \eta(t)$, hence $u(0) - \xi(0) = u_-(0) - \eta(0) = x_0$. Moreover, the first term on the rightmost side of (4.5) is non-negative by hypothesis, the second term is non-negative by (4.6), hence $\xi = \mathfrak{p}_Z[x_0, u]$.

Conversely, assume that $(x_0, u) \in \mathrm{Dom}\,(\mathcal{P})$, and for a fixed $y \in G(0,T\,;\,Z)$ and $t \in [0,T]$ put $y^*(\tau) := y(\tau) + \chi_{\{t\}}(\tau)\,(u(t) - \xi(t) - y(t))$ for $\tau \in [0,T]$. Then $y^* \in G(0,T\,;\,Z)$ and identity (4.5) with y replaced by y^* yields

$$\begin{aligned}0 \;&\le\; \int_0^t \langle u(\tau+) - \xi(\tau+) - y^*(\tau), d\xi(\tau)\rangle\\ &=\; \int_0^t \Big\langle u_-(\tau+) - \xi_-(\tau+) - y(\tau) + \chi_{\{t\}}(\tau)\,(y(t) - y^*(t)), d\xi_-(\tau)\Big\rangle\\ &=\; \int_0^t \langle u_-(\tau+) - \xi_-(\tau+) - y(\tau), d\xi_-(\tau)\rangle\;,\end{aligned}$$

where we used Proposition 3.7 (i) and the left-continuity of ξ_-. We have indeed $u_-(t) - \xi_-(t) \in Z$ for every t, $u_-(0) - \xi_-(0) = u(0) - \xi(0)$, and Proposition 2.2 is proved. ■

In the sequel, we denote by $BV_L(0,T\,;\,X)$ the space of functions from $BV(0,T\,;\,X)$ which are left-continuous. Besides the fact that $\mathrm{Var}_{[a,b]}\, f = \overline{\mathrm{Var}}_{[a,b]}\, f$ for every $f \in G_L(0,T\,;\,X)$ and every $[a,b] \subset [0,T]$, the restriction to left-continuous functions has the following advantage.

Lemma 4.2 *Let $u \in G(0,T\,;\,X)$ and $\xi \in BV_L(0,T\,;\,X)$ be such that $u(t) - \xi(t) \in Z$ for every $t \in [0,T]$, $u(0) - \xi(0) = x_0$. Assume that*

$$\int_0^T \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau)\rangle \;\ge\; 0 \quad \forall y \in G(0,T;Z)\,. \qquad (4.7)$$

Then for every $0 \le s < t \le T$ *we have*

$$\int_s^t \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau) \rangle \;\ge\; 0 \quad \forall y \in G(0,T;Z)\,, \tag{4.8}$$

in particular $\xi = \mathfrak{p}_Z[x_0, u]$.

Proof. For $y \in G(0,T;Z)$ and $\tau \in [0,T]$ put

$$\hat{y}(\tau) := (u(\tau+) - \xi(\tau+))\Big(\chi_{[0,s[}(\tau) + \chi_{[t,T]}(\tau)\Big) + y(\tau)\,\chi_{[s,t[}(\tau)\,.$$

Then $\hat{y} \in G(0,T;Z)$ and combining (4.7) with Propositions 3.5, 3.7 (i) we obtain using the left-continuity of ξ that

$$\begin{aligned} 0 \;&\le\; \int_0^T \langle u(\tau+) - \xi(\tau+) - \hat{y}(\tau), d\xi(\tau) \rangle \;=\; \int_0^T \Big\langle (u(\tau+) - \xi(\tau+) - y(\tau))\chi_{[s,t[}(\tau), d\xi(\tau) \Big\rangle \\ &=\; \int_s^t \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau) \rangle \end{aligned}$$

and Lemma 4.2 is proved. ■

Lemma 4.2 and Proposition 2.2 enable us to construct easily $\xi = \mathfrak{p}_Z[x_0, u] \in S(0,T\,;\,X)$ whenever $u \in S(0,T\,;\,X)$. As it has already been mentioned, the explicit formula coincides with the time-discrete scheme of [16, 12].

Proposition 4.3 *Let* $x_0 \in Z$ *be given and let* $u \in S(0,T\,;\,X)$ *be of the form*

$$u(t) \;=\; u_0\,\chi_{\{0\}}(t) + \sum_{k=1}^m u_k\,\chi_{]t_{k-1},t_k]}(t) \qquad \text{for } t \in [0,T]\,, \tag{4.9}$$

with $\{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$. *For* $k = 1, \dots, m$ *put* $\xi_0 := u_0 - x_0$, $\xi_k := \xi_{k-1} + P^Z(u_k - \xi_{k-1})$. *Then* $\xi = \mathfrak{p}_Z[x_0, u]$ *has the form*

$$\xi(t) \;=\; \xi_0\,\chi_{\{0\}}(t) + \sum_{k=1}^m \xi_k\,\chi_{]t_{k-1},t_k]}(t) \qquad \text{for } t \in [0,T] \tag{4.10}$$

and we have $\overline{\mathrm{Var}}_{[0,T]}\,\xi = \mathrm{Var}_{[0,T]}\,\xi \le \mathrm{Var}_{[0,T]}\,u = \overline{\mathrm{Var}}_{[0,T]}\,u$.

Proof. For every $k = 1, \dots, m$ we have $u_k - \xi_k = Q^Z(u_k - \xi_{k-1}) \in Z$, hence $u(t) - \xi(t) \in Z$ for every $t \in [0,T]$. Moreover, by (1.4) and Proposition 3.7, the function ξ given by (4.10) satisfies the inequality

$$\int_0^T \langle u(\tau+) - \xi(\tau+) - y(\tau), d\xi(\tau) \rangle \;=\; \sum_{k=1}^m \langle \xi_k - \xi_{k-1}, u_k - \xi_k - y(t_{k-1}) \rangle \;\ge\; 0$$

for each $y \in G(0,T\,;\,Z)$, hence $\xi = \mathfrak{p}_Z[x_0, u]$ by Lemma 4.2. To complete the proof, we use again (1.4) which entails that $\langle \xi_k - \xi_{k-1}, u_k - \xi_k - u_{k-1} + \xi_{k-1} \rangle \ge 0$, hence $|\xi_k - \xi_{k-1}| \le |u_k - u_{k-1}|$ for every $k = 1, \dots, m$ and $\mathrm{Var}_{[0,T]}\,\xi \le \mathrm{Var}_{[0,T]}\,u$. ■

We are now ready to prove Theorem 2.3.

Proof of Theorem 2.3. From Propositions 4.3 and 2.2 it follows that for each function

$$u(t) \;=\; \sum_{k=0}^{m} \hat{u}_k\, \chi_{\{t_k\}}(t) \;+\; \sum_{k=1}^{m} u_k\, \chi_{]t_{k-1},t_k[}(t)$$

we have $\xi = \mathfrak{p}_Z[x_0, u]$ if and only if

$$\xi(t) \;=\; \sum_{k=0}^{m} \hat{\xi}_k\, \chi_{\{t_k\}}(t) \;+\; \sum_{k=1}^{m} \xi_k\, \chi_{]t_{k-1},t_k[}(t)$$

for $t \in [a,b]$ with $\hat{\xi}_0 = \hat{u}_0 - x_0$, $\xi_k - \xi_{k-1} = P^Z(u_k - \xi_{k-1})$, $\hat{\xi}_k - \xi_k = P^Z(\hat{u}_k - \xi_k)$ for $k = 1, \dots, m$. By Lemma 4.1 we have $|\xi_k - \xi_{k-1}| \le |u_k - u_{k-1}|$, hence $\overline{\mathrm{Var}}_{[0,T]}\, \xi \le \overline{\mathrm{Var}}_{[0,T]}\, u$.

Consider now $u \in \overline{BV}(0,T\,;\,X)$ and $x_0 \in Z$. By Proposition 1.3 (iv) we find $u_n \in S(0,T\,;\,X)$ such that $\|u - u_n\|_{[0,T]} \to 0$ as $n \to \infty$, $\overline{\mathrm{Var}}_{[0,T]}\, u_n \le \overline{\mathrm{Var}}_{[0,T]}\, u$. Put $\xi_n := \mathfrak{p}_Z[x_0, u_n]$. By Lemma 2.1 we have for $n, m \in \mathbb{N}$ that

$$\begin{aligned} |\xi_n(t) - \xi_m(t)|^2 \;&\le\; |\xi_n(0) - \xi_m(0)|^2 + 2\, \|u_n - u_m\|_{[0,t]} \left(\overline{\mathop{\mathrm{Var}}_{[0,t]}}\,(\xi_n) \;+\; \overline{\mathop{\mathrm{Var}}_{[0,t]}}\,(\xi_m) \right) \qquad (4.11)\\ &\le\; |u_n(0) - u_m(0)|^2 + 4\, \|u_n - u_m\|_{[0,t]}\; \overline{\mathop{\mathrm{Var}}_{[0,t]}}\; u\,, \end{aligned}$$

hence $\{\xi_n\}$ is a Cauchy sequence in $G(0,T\,;\,X)$, $\overline{\mathrm{Var}}_{[0,T]}\, \xi_n \le \overline{\mathrm{Var}}_{[0,T]}\, u$ for every $n \in \mathbb{N}$. Let $\xi \in \overline{BV}(0,T\,;\,X)$ be its limit, $\overline{\mathrm{Var}}_{[0,T]}\, \xi \le \overline{\mathrm{Var}}_{[0,T]}\, u$. From Proposition 3.12 we infer that $\xi = \mathfrak{p}_Z[x_0, u]$, and the estimate (2.4) follows immediately from Lemma 2.1. Theorem 2.3 is proved. ■

For the proof of Theorem 2.4 we need the following crucial Lemma the idea of which (for bounded domains Z) goes back to A. Vladimirov, see Sect. 19 of [10], cf. also Chapter 2 of [15].

Lemma 4.4 *Let* $0 \in \mathrm{Int}\, Z$ *and* $\{(x_0^n, u_n)\,;\, n \in \mathbb{N}\}$ *be an arbitrary sequence in* $\mathrm{Dom}\,(\mathcal{P}) \cap (Z \times G_L(0,T\,;\,X))$ *such that* $|x_0^n - x_0| \to 0$, $\|u_n - u\|_{[0,T]} \to 0$ *as* $n \to \infty$. *Then there exists a constant* $C > 0$ *independent of* n *such that* $\mathrm{Var}_{[0,T]}\, \mathfrak{p}_Z[x_0^n, u_n] \le C$.

Proof. Notice first that the uniform convergence of $\{u_n\}$ and Proposition 1.3 (i) guarantee that $u \in G_L(0,T;X)$. Next, let $\rho > 0$ be as in (1.1) and let us denote $U_\rho := \{t \in [0,T]\,;\, |u(t+) - u(t)| \ge \rho/6\}$. By Proposition 1.3 (iii), (iv), the set U_ρ is finite and there exists $h > 0$ such that for every $[s,t] \subset [0,T]$ with $|t - s| < h$ and $[s,t] \cap U_\rho = \emptyset$, we have $|u(t) - u(s)| \le \rho/6$. We fix a partition $d = \{t_0, \dots, t_m\}$ such that $U_\rho \subset d$, $t_k - t_{k-1} < h$ for $k = 1, \dots, m$, and a number $n_0 \in \mathbb{N}$ such that $\|u_n - u\|_{[0,T]} \le \rho/6$ for $n \ge n_0$.

For $\xi_n = \mathfrak{p}_Z[x_0^n, u_n]$ and $k \in \{0, \dots, m\}$ put

$$x_n^k \;:=\; u_n(t_k+) - \xi_n(t_k+) \;\in\; Z\,. \qquad (4.12)$$

Consider now a fixed $k \in \{1, \dots, m\}$. By Lemma 4.2 we have for every $y \in G(0,T\,;\,Z)$, $n \in \mathbb{N}$ and $\varepsilon \in]0, (t_k - t_{k-1})/2[$ that

$$\int_{t_{k-1}+\varepsilon}^{t_k-\varepsilon} \langle u_n(\tau+) - \xi_n(\tau+) - y(\tau), d\xi_n(\tau) \rangle \;\ge\; 0\,. \qquad (4.13)$$

In particular, for $n \geq n_0$ we may put in (4.13)

$$y(\tau) \;=\; \left(u_n(\tau+) - u_n(t_{k-1}+) + \frac{\rho}{2}\, w(\tau)\right)\; \chi_{]t_{k-1},t_k[}\,(\tau)$$

for an arbitrary $w \in S(0,T\,;\,B_1(0))$, since for $\tau \in]t_{k-1}, t_k[$ we have $|u_n(\tau+) - u_n(t_{k-1}+)| \leq \rho/2$, hence $y(\tau) \in B_\rho(0) \subset Z$. Then (4.13) yields

$$\frac{\rho}{2} \int_{t_{k-1}+\varepsilon}^{t_k-\varepsilon} \langle w(\tau), d\xi_n(\tau)\rangle \;\leq\; \int_{t_{k-1}+\varepsilon}^{t_k-\varepsilon} \langle u_n(t_{k-1}+) - \xi_n(\tau+), d\xi_n(\tau)\rangle$$

and from Theorem 3.11 and Corollary 3.15 it follows that

$$\frac{\rho}{2} \overline{\operatorname*{Var}_{[t_{k-1}+\varepsilon,t_k-\varepsilon]}} \xi_n \;\leq\; \frac{1}{2}\left(|u_n(t_{k-1}+) - \xi_n(t_{k-1}+\varepsilon)|^2 - |u_n(t_{k-1}+) - \xi_n(t_k-\varepsilon)|^2\right).$$

Letting $\varepsilon \to 0$ (note that ξ_n is left-continuous), we obtain on the one hand that

$$|u_n(t_{k-1}+) - \xi_n(t_{k-1}+)| \;\geq\; |u_n(t_{k-1}+) - \xi_n(t_k)|\,, \tag{4.14}$$

and, on the other hand, Proposition 1.3 (vi) yields that

$$\begin{aligned} \rho \operatorname*{Var}_{[t_{k-1},t_k]} \xi_n \;&=\; \rho \overline{\operatorname*{Var}_{[t_{k-1},t_k]}} \xi_n \;\leq\; \rho\,|\xi_n(t_{k-1}+) - \xi_n(t_{k-1})| \\ &\quad + |u_n(t_{k-1}+) - \xi_n(t_{k-1}+)|^2 - |u_n(t_{k-1}+) - \xi_n(t_k)|^2\,. \end{aligned} \tag{4.15}$$

Lemma 4.1, inequality (4.14) and the triangle inequality imply that

$$\begin{aligned} |x_n^k| \;&\leq\; |u_n(t_{k-1}+) - \xi_n(t_k)| + |u_n(t_k+) - u_n(t_{k-1}+)| + |\xi_n(t_k+) - \xi_n(t_k)| \\ &\leq\; |x_n^{k-1}| + 2\,|u_n(t_k+) - u_n(t_k)| + \rho/2 \\ &\leq\; |x_n^{k-1}| + 2\,|u(t_k+) - u(t_k)| + 3\rho/2\,. \end{aligned}$$

Now consider $\ell \in \{1,\ldots,m\}$. Summing up the above inequalities from $k=1$ to $k=\ell$, we obtain that

$$|x_n^\ell| \;\leq\; |x_n^0| + 2\sum_{k=1}^m |u(t_k+) - u(t_k)| + 3m\,\rho/2\,.$$

Observe further that Lemma 4.1 and the triangle inequality ensure that

$$|x_n^0| \;\leq\; |x_0^n| + 2\;|u_n(0+) - u_n(0)| \leq \sup_n\,\{|x_0^n|\} + 2\;|u(0+) - u(0)| + \rho/2\,.$$

Consequently, for $\ell \in \{0,\ldots,m\}$,

$$|x_n^\ell| \;\leq\; \sup_n\,\{|x_0^n|\} + 2\sum_{k=0}^m |u(t_k+) - u(t_k)| + (3m+1)\,\rho/2 \;=:\; C_1\,. \tag{4.16}$$

From (4.15), (4.16) and Lemma 4.1, we thus obtain that

$$\rho \operatorname*{Var}_{[t_{k-1},t_k]} \xi_n \;\leq\; \rho\,|u_n(t_{k-1}+) - u_n(t_{k-1})| + |x_n^{k-1}|^2 \;\leq\; \rho^2/2 + C_1^2 + \rho\,|u(t_{k-1}+) - u(t_{k-1})|\,,$$

hence $\operatorname{Var}_{[0,T]} \xi_n \leq m\rho/2 + mC_1^2/\rho + C_1/2$, and Lemma 4.4 is proved. ∎

Proof of Theorem 2.4. For an arbitrary $(x_0, u) \in Z \times G(0,T\,;\,X)$ we find a sequence $\{u_n\}_{n=1}^{\infty}$ of step functions such that $\|u_n - u\|_{[0,T]} \to 0$ as $n \to \infty$. For $t \in [0,T]$ and $n \in \mathbb{N}$ put $u_-^n(t) := u_n(t-)$, $u_-(t) := u(t-)$, $\xi_n(t) := \mathfrak{p}_Z[x_0^n, u_n](t)$, $\xi_-^n(t) := \mathfrak{p}_Z[x_0^n, u_-^n](t)$. From Proposition 2.2 it follows that $\xi_n(t-) = \xi_-^n(t)$ for every $t \in [0,T]$ and $n \in \mathbb{N}$. Using Lemma 4.4 we find a constant $C > 0$ such that $\operatorname{Var}_{[0,T]} \xi_-^n \le C$, and Lemma 4.1 yields

$$\overline{\operatorname{Var}}_{[0,T]} \xi_n \;\le\; C + |\xi_n(T) - \xi_n(T-)| \;\le\; C + |u_n(T) - u_n(T-)| \;\le\; \bar{C}$$

for some constant $\bar{C}$. The same argument as in (4.11) implies that $\{\xi_n\}$ is a Cauchy sequence in $G(0,T\,;\,X)$. Denoting its limit by ξ, we obtain from Proposition 3.12 that $\xi := \mathfrak{p}_Z[x_0, u]$. Repeating the same argument for an arbitrary sequence $\|u_n - u\|_{[0,T]} \to 0$, $u_n \in G(0,T\,;\,X)$, we complete the proof. ∎

5 Functions of bounded ε-variation

In this section, we give an extension of Proposition 3.12 to the case where the sequence $\{f_n\}$ does not converge uniformly. The idea is based on the following concept introduced in [7], Def. 3.3.

Definition 5.1 *We say that a set $\mathcal{A} \subset G(a,b\,;\,X)$ has* uniformly bounded ε-variation*, if*

$$\forall\, \varepsilon > 0 \;\; \exists\, L_\varepsilon > 0 \;\; \forall\, f \in \mathcal{A} : \quad \inf \left\{ \operatorname{Var}_{[a,b]} \psi\,;\, \psi \in BV(a,b\,;\,X)\,,\; \|f - \psi\|_{[a,b]} < \varepsilon \right\} \;\le\; L_\varepsilon\,.$$

We will see in Proposition 5.6 below that every uniformly convergent sequence in $G(a,b\,;\,X)$ has uniformly bounded ε-variation. The converse is obviously false, as we can see from the example $f_n(t) = \chi_{[0,1/n]}(t)$ for $t \in [0,1]$. On the other hand, we prove the following generalization of Helly's Selection Principle as an extension of Theorem 3.8 of [7] to the infinite dimensional case.

Theorem 5.2 *Let X be a real separable Hilbert space and let $\{f_n\,;\, n \in \mathbb{N}\}$ be a bounded sequence of functions from $G(a,b\,;\,X)$ which has uniformly bounded ε-variation. Then there exist $f \in G(a,b\,;\,X)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(t)$ converges weakly to $f(t)$ as $k \to \infty$ for every $t \in [a,b]$.*

The proof of Theorem 5.2 consists in a gradual selection of subsequences similar to the proof of the classical Helly Selection Principle (see e. g. [9], pp. 372 – 374). In order to make the diagonalization argument more transparent, we introduce the following notation.

By $\mathcal{G}(\mathbb{N})$ we denote the set of all infinite subsets $M \subset \mathbb{N}$. We say that a sequence $\{x_n\,;\, n \in \mathbb{N}\}$ of elements of a topological space *M-converges to x* if for every neighborhood $\mathcal{U}(x)$ of x there exists n_0 such that $x_n \in \mathcal{U}(x)$ for every $n \in M$, $n \ge n_0$.

We start with the following Lemma as the Hilbert-space version of [3], Theorem I.3.5.

Lemma 5.3 *Let $\{\psi_n\,;\, n \in \mathbb{N}\}$ be a bounded sequence in $BV(a,b\,;\,X)$ such that* $\operatorname{Var}_{[a,b]} \psi_n \le C$ *for every $n \in \mathbb{N}$. Then there exist $\psi \in BV(a,b\,;\,X)$ and a set $M \in \mathcal{G}(\mathbb{N})$ such that* $\operatorname{Var}_{[a,b]} \psi \le C$ *and the sequence $\psi_n(t)$ weakly M-converges in X to $\psi(t)$ for every $t \in [a,b]$.*

Proof. Let $\{w_j\,;\,j \in \mathbb{N}\}$ be a countable dense subset of X. The functions $t \mapsto \langle \psi_n(t), w_1\rangle$ have uniformly bounded variation, and according to the one-dimensional Helly Selection Principle we find $N_1 \in \mathcal{G}(\mathbb{N})$ such that the sequence $\{\langle \psi_n(t), w_1\rangle\}$ N_1-converges to a limit $v_1(t)$ for every $t \in [a,b]$. By induction we construct a sequence $\{N_k\,;\,k \in \mathbb{N}\}$ of sets in $\mathcal{G}(\mathbb{N})$, $N_1 \supset N_2 \supset \ldots$, such that the sequence $\{\langle \psi_n(t), w_j\rangle\}$ N_j-converges to a limit $v_j(t)$ for every $t \in [a,b]$. We now put $n_1 := \min N_1$, $n_k := \min\{n \in N_k\,;\,n > n_{k-1}\}$ for $k = 2,3,\ldots$, and define the set $M := \{n_k\,;\,k \in \mathbb{N}\} \in \mathcal{G}(\mathbb{N})$. By construction, every N_j-convergent sequence is M-convergent, hence $\{\langle \psi_n(t), w_j\rangle\}$ M-converges to $v_j(t)$ for every $t \in [a,b]$ and $j \in \mathbb{N}$.

For a fixed $t \in [a,b]$, the mapping $w_j \mapsto v_j(t)$ can be extended in a unique way to a bounded linear functional on X. By the Riesz Representation Theorem, there exists an element $\psi(t) \in X$ such that $v_j(t) = \langle \psi(t), w_j\rangle$ for every $j \in \mathbb{N}$. Since the system $\{w_j\}$ is dense in X, we obtain that

$$\lim_{k\to\infty} \langle \psi_{n_k}(t), w\rangle \;=\; \langle \psi(t), w\rangle$$

for every $w \in X$ and $t \in [a,b]$. Moreover, for a fixed partition $a = t_0 < t_1 < \ldots < t_m = b$ we have

$$\sum_{i=1}^{m} |\psi(t_i) - \psi(t_{i-1})| \;\le\; \liminf_{k\to\infty} \sum_{i=1}^{m} |\psi_{n_k}(t_i) - \psi_{n_k}(t_{i-1})| \;\le\; C\,,$$

and the assertion follows. ∎

We now use Lemma 5.3 to prove Theorem 5.2 by an argument similar to the one used in [7] in the case $\dim X < \infty$.

Proof of Theorem 5.2. We fix a sequence $\varepsilon_i \to 0$ and for every $n, i \in \mathbb{N}$ we find $\psi_n^i \in BV(a,b\,;X)$ such that $\|\psi_n^i - f_n\|_{[a,b]} < \varepsilon_i$, $\mathrm{Var}_{[a,b]}\,\psi_n^i \le L_{\varepsilon_i} + 1$. We now apply Lemma 5.3 to find $M_1 \in \mathcal{G}(\mathbb{N})$ and $\psi^1 \in BV(a,b\,;X)$ such that $\mathrm{Var}_{[a,b]}\,\psi^1 \le L_{\varepsilon_1} + 1$ and $\psi_n^1(t)$ weakly M_1-converges to $\psi^1(t)$ for every $t \in [a,b]$. We continue by induction and construct a sequence $\{M_i\}$ of sets in $\mathcal{G}(\mathbb{N})$, $M_1 \supset M_2 \supset \ldots$, such that the sequence $\{\psi_n^i(t)\}$ weakly M_i-converges to $\psi^i(t)$ for every $t \in [a,b]$ and $i \in \mathbb{N}$, $\psi^i \in BV(a,b\,;X)$, $\mathrm{Var}_{[a,b]}\,\psi^i \le L_{\varepsilon_i} + 1$. Putting $n_1 := \min M_1$, $n_k := \min\{n \in M_k\,;\,n > n_{k-1}\}$ for $k = 2,3,\ldots$, $M^* := \{n_k\,;\,k \in \mathbb{N}\}$ we argue as in the proof of Lemma 5.3 to obtain that $\psi_n^i(t)$ weakly M^*-converges to $\psi^i(t)$ for every $t \in [a,b]$ and $i \in \mathbb{N}$.

We now check that $\{\psi^i\}$ is a Cauchy sequence in $G(a,b\,;X)$. For $i, j, n \in \mathbb{N}$ we have

$$\left\|\psi_n^i - \psi_n^j\right\|_{[a,b]} \le \left\|\psi_n^i - f_n\right\|_{[a,b]} + \left\|f_n - \psi_n^j\right\|_{[a,b]} \le \varepsilon_i + \varepsilon_j.$$

Consequently we have for $t \in [a,b]$,

$$\left|\psi^i(t) - \psi^j(t)\right| \le \liminf_{k\to+\infty} \left|\psi_{n_k}^i(t) - \psi_{n_k}^j(t)\right| \le \varepsilon_i + \varepsilon_j,$$

from which readily follows that $\{\psi^i\}$ is a Cauchy sequence in $G(a,b\,;X)$. We denote by $f \in G(a,b\,;X)$ its limit. For each $t \in [a,b]$, $w \in X$ and $k \in \mathbb{N}$ we then have

$$\langle f(t) - f_{n_k}(t), w\rangle \;=\; \left\langle f(t) - \psi^i(t), w\right\rangle + \left\langle \psi^i(t) - \psi_{n_k}^i(t), w\right\rangle + \left\langle \psi_{n_k}^i(t) - f_{n_k}(t), w\right\rangle$$

for a suitably chosen i, and we easily conclude that $f_{n_k}(t)$ weakly converges to $f(t)$ for every $t \in [a,b]$. Theorem 5.2 is proved. ∎

As a complement to Theorem 5.2, the following extension of Proposition 3.12 holds true.

Proposition 5.4 *Let $f, f_n \in G(a,b\,;X)$, $g, g_n \in \overline{BV}(a,b\,;X)$ for $n \in \mathbb{N}$ be such that the sequence $\{f_n\}$ has uniformly bounded ε-variation and*

$$f_n(t) \to f(t) \qquad \text{weakly for every } \; t \in [a,b]\,,$$

$$\lim_{n\to\infty} \|g - g_n\|_{[a,b]} = 0\,, \quad \sup_{n\in\mathbb{N}} \overline{\mathrm{Var}}_{[a,b]}\, g_n \;=\; C \;<\; \infty\,.$$

Then (3.25) holds.

For the proof of Proposition 5.4 we need the following lemma.

Lemma 5.5 *Consider $w \in S(a,b\,;X)$ and $\tilde{f}_n : [a,b] \to X$, $\tilde{f}_n(t) \to 0$ weakly for every $t \in [a,b]$. Then*

$$\lim_{n\to\infty} \int_a^b \left\langle \tilde{f}_n(t), dw(t) \right\rangle \;=\; 0\,.$$

Proof of Lemma 5.5. For a function w of the form (1.6) we have by Proposition 3.7

$$\int_a^b \left\langle \tilde{f}_n(t), dw(t) \right\rangle \;=\; \sum_{k=0}^{m} \left\langle \tilde{f}_n(t_k), c_{k+1} - c_k \right\rangle\,,$$

where we put $c_0 := \hat{c}_0$, $c_{m+1} := \hat{c}_m$, and it suffices to pass to the limit as $n \to \infty$. ∎

Proof of Proposition 5.4. For each $\varepsilon > 0$ and $n \in \mathbb{N}$ we find $\{\psi^\varepsilon\}$, $\{\psi^\varepsilon_n\}$ in $BV(a,b\,;X)$ such that $\|f_n - \psi^\varepsilon_n\|_{[a,b]} \;\le\; \varepsilon$, $\|f - \psi^\varepsilon\|_{[a,b]} \;\le\; \varepsilon$, $\mathrm{Var}_{[a,b]}\, \psi^\varepsilon_n \;\le\; L_\varepsilon + 1$, and put $\hat{L}_\varepsilon := \max\{\mathrm{Var}_{[a,b]}\, \psi^\varepsilon, L_\varepsilon + 1\}$.

The sequence $\{f_n\}$ is obviously bounded in $G(a,b\,;X)$. Indeed, as $\{f_n(a)\}$ is weakly convergent, it is necessarily bounded and we have for every n and t that

$$|f_n(t)| \;\le\; |f_n(t) - \psi^\varepsilon_n(t)| + |\psi^\varepsilon_n(t) - \psi^\varepsilon_n(a)| + |f_n(a) - \psi^\varepsilon_n(a)| + |f_n(a)| \;\le\; 2\,\varepsilon + \hat{L}_\varepsilon + |f_n(a)|$$

and taking the infimum over ε we obtain an upper bound for $\|f_n\|_{[a,b]}$ independent of n and ε, say

$$\|f_n\|_{[a,b]} \;\le\; R\,.$$

Let now $\varepsilon > 0$ be fixed. By Proposition 1.3 (v), there exists a step function $w \in S(a,b\,;X)$ such that $\|g - w\|_{[a,b]} \le \varepsilon/\hat{L}_\varepsilon$, $\overline{\mathrm{Var}}_{[a,b]}\, w \le C$. Using Lemma 5.5 and the uniform convergence of $\{g_n\}$, we find n_0 such that for $n \ge n_0$ we have $|\int_a^b \langle (f - f_n)(t), dw(t)\rangle| \le \varepsilon$, $\|g - g_n\|_{[a,b]} \le \varepsilon/\hat{L}_\varepsilon$. Then Corollary 3.10 yields

$$\begin{aligned}
\left| \int_a^b \langle f(t), dg(t)\rangle \;-\; \int_a^b \langle f_n(t), dg_n(t)\rangle \right| \;\le\; & \left| \int_a^b \langle (f - \psi^\varepsilon - f_n + \psi^\varepsilon_n)(t), d(g - w)(t)\rangle \right| \\
& + \left| \int_a^b \langle (f - f_n)(t), dw(t)\rangle \right| + \left| \int_a^b \langle (\psi^\varepsilon - \psi^\varepsilon_n)(t), d(g - w)(t)\rangle \right| \\
& + \left| \int_a^b \langle (f_n - \psi^\varepsilon_n)(t), d(g - g_n)(t)\rangle \right| \;+\; \left| \int_a^b \langle \psi^\varepsilon_n(t), d(g - g_n)(t)\rangle \right| \\
\le\; & 2\,C\, \|f - \psi^\varepsilon - f_n + \psi^\varepsilon_n\|_{[a,b]} + \varepsilon + \;(4\,(R+\varepsilon) + 2\,\hat{L}_\varepsilon)\; \|g - w\|_{[a,b]} \\
& + 2\,C\, \|f_n - \psi^\varepsilon_n\|_{[a,b]} \;+\; (2\,(R+\varepsilon) + \hat{L}_\varepsilon)\; \|g - g_n\|_{[a,b]} \\
\le\; & M\,\varepsilon
\end{aligned}$$

for $n \geq n_0$, where M is a constant independent of n and ε, hence (3.25) holds. ∎

To conclude the paper, we show how Theorem 2.4 can be used to prove directly the following link between Propositions 3.12 and 5.4.

Proposition 5.6 *Let* $\{u^n\}$ *be a sequence in* $G(0,T\,;\,X)$, $\|u^n - u\|_{[0,T]} \to 0$ *as* $n \to \infty$. *Then* $\{u^n\}$ *has uniformly bounded* ε*-variation.*

Proof. Let $\varepsilon > 0$ be given. For $t \in [0,T]$ and $n \in \mathbb{N}$ put $u_-(t) := u(t-)$, $u^n_-(t) := u^n(t-)$, $\xi^n_-(t) := \mathfrak{p}_Z[0, u^n_-](t)$ with $Z = B_{\varepsilon/2}(0)$. By Theorem 2.4 there exists $C_\varepsilon > 0$ such that $\mathrm{Var}_{[0,T]}\, \xi^n_- \leq C_\varepsilon$ independently of n, $\left\| u^n_- - \xi^n_- \right\|_{[0,T]} \leq \varepsilon/2$.

Let $U_\varepsilon \subset [0,T]$ be the finite set of those t for which $|u(t) - u_-(t)| \geq \varepsilon/4$. For $t \in [0,T]$ and $n \in \mathbb{N}$ we now put

$$u^n_\varepsilon(t) := u^n_-(t) + (u^n(t) - u^n_-(t))\, \chi_{U_\varepsilon}(t)\,, \quad \xi^n_\varepsilon(t) := \mathfrak{p}_Z[0, u^n_\varepsilon](t)\,. \tag{5.1}$$

Clearly, $u^n_\varepsilon(t-) = u^n_-(t)$ for $t \in [0,T]$ and we infer from Proposition 2.2 that $\xi^n_\varepsilon(t-) = \xi^n_-(t)$ for $t \in [0,T]$. Consequently, $\xi^n_\varepsilon(t) = \xi^n_\varepsilon(t-) = \xi^n_-(t)$ for every $t \in [0,T] \setminus U_\varepsilon$, while $|\xi^n_\varepsilon(t) - \xi^n_\varepsilon(t-)| \leq |u^n(t) - u^n(t-)|$ for $t \in U_\varepsilon$ by Lemma 4.1. Then $\xi^n_\varepsilon \in BV(0,T\,;\,X)$ with $\mathrm{Var}_{[0,T]}\, \xi^n_\varepsilon \leq C_\varepsilon + 2 \sum_{t \in U_\varepsilon} |u^n(t) - u^n_-(t)|$, where $|u^n(t) - u^n_-(t)| \leq 2\, \|u^n - u\|_{[0,T]} + |u(t) - u_-(t)|$, and we may put $L_\varepsilon := \sup_n \mathrm{Var}_{[0,T]}\, \xi^n_\varepsilon < +\infty$. For every $t \in [0,T]$ and $n \in \mathbb{N}$ we have by (5.1) that $|u^n_\varepsilon(t) - \xi^n_\varepsilon(t)| \leq \varepsilon/2$ and, on the other hand,

$$\begin{aligned} |u^n(t) - u^n_\varepsilon(t)| &= \left| \left(u^n(t) - u^n_-(t)\right) \left(1 - \chi_{U_\varepsilon}(t)\right) \right| \\ &\leq \left| (u(t) - u_-(t)) \left(1 - \chi_{U_\varepsilon}(t)\right) \right| + 2\, \|u^n - u\|_{[0,T]} \\ &\leq \varepsilon/4 + 2\, \|u^n - u\|_{[0,T]} \ \leq\ \varepsilon/2 \end{aligned}$$

for $n \geq n_0$ with n_0 sufficiently large. This yields that $\|u^n - \xi^n_\varepsilon\|_{[0,T]} \leq \varepsilon$ for $n \geq n_0$. For $n < n_0$ we approximate the functions u^n for instance by step functions and taking a larger L_ε if necessary, we complete the proof. ∎

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