Reliable solutions to the problem of periodic oscillations of an elastoplastic beam

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Abstract

We consider time-periodic oscillations of a beam with a spatially inhomogeneous Prandtl-Ishlinskii constitutive law describing the elastoplastic hysteresis. The data (mass density, Prandtl-Ishlinskii distribution, external load) are assumed to be uncertain. It is shown that a unique solution exists and is stable with respect to the data variation. Considering the total dissipated energy as a measure for the accumulated material fatigue, we identify and estimate from above the 'worst scenario' case, where the dissipation over one period is maximal within an admissible set of data obtained from inaccurate measurements.

Introduction

We consider time-periodic oscillations of an elastoplastic beam governed by the equation

$$\rho(x)\, u_{tt} + F[u_{xx}]_{xx} \;=\; g(x,t)\,, \tag{0.1}$$

subject to boundary conditions

$$u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = 0\,, \tag{0.2}$$

where $\rho(x)$ is the mass density, $g(x,t)$ is a time-periodic load, and F is a spatially inhomogeneous constitutive Prandtl-Ishlinskii operator describing the elastoplastic hysteresis.

The existence of at least one time-periodic solution to Problem (0.1), (0.2) was proved in a slightly different setting in [11]. Inspired by the paper [6] on a stationary beam bending problem with Hencky's law of plasticity and uncertain data, we admit the data of Problem (0.1), (0.2) to be uncertain, and show existence, uniqueness and stability of the solution with respect to data variation.

Note that hysteresis is the only source of energy dissipation here. It is strong enough to prevent the system from resonance under periodic loading, see a more detailed discussion in [12]. On the other hand, it was shown in [1] that the total accumulated damage during an oscillation process evaluated by the so-called *rainflow method* is mathematically related to the total dissipated energy. Thus, an estimate for the dissipated energy can be considered as a measure for the accumulated material fatigue.

Indeed, it is desirable in principle to *minimize* the total dissipation. In practice, however, we have little freedom to play with material characteristics. Moreover, hysteresis operators are known to be non-differentiable, and the problem of sensitivity with respect to small data variation in problems with hysteresis is completely open.

Following [6], we therefore propose to stay on the "safe side" by identifying and estimating from above the *worst scenario* case, where the dissipation over one period is *maximal* within an admissible set of data obtained from inaccurate measurements.

The paper is organized as follows. Section 1 is devoted to a survey of basic facts from the mathematical theory of hysteresis. In Section 2 we state the main results. A collection of auxiliary statements is put separately into Section 3. Main theorems are proved in Section 4.

1 Hysteresis operators

1.1 Stop and play operators

One of the basic elements of the theory of hysteresis operators is borrowed from continuum mechanics, more precisely, from Prandtl's rheological model for elastic-perfectly plastic constitutive laws represented on Fig. 1 as a combination in series of one linearly elastic element with elasticity modulus $E > 0$, and one dry friction element with yield point Er, see [13]. Fig. 2 shows the corresponding strain-stress diagram, where the strain e is a controlled quantity, and σ/E is the normalized stress response. In mathematical terms, it can be formally described as an input-output relation between two abstract absolutely continuous functions of time, input u (which stands for e), and output s (which stands for σ/E), satisfying the variational inequality

$$\left\{\begin{array}{ll} s(t) \in [-r, r] & \text{for every } \ t \geq 0\,, \\ (\dot{s}(t) - \dot{u}(t))(\phi - s(t)) \geq 0 & \text{for a.e. } \ t > 0 \ \text{ and every } \ \phi \in [-r, r]\,, \\ s(0) = s^0\,, & \end{array}\right. \tag{1.1}$$

where a parameter $r > 0$ and an initial condition $s^0 \in [-r, r]$ are given, and the dot denotes derivative with respect to t.

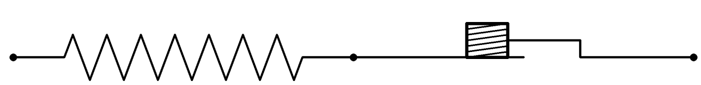

Figure 1: Rheological scheme for Prandtl's model.

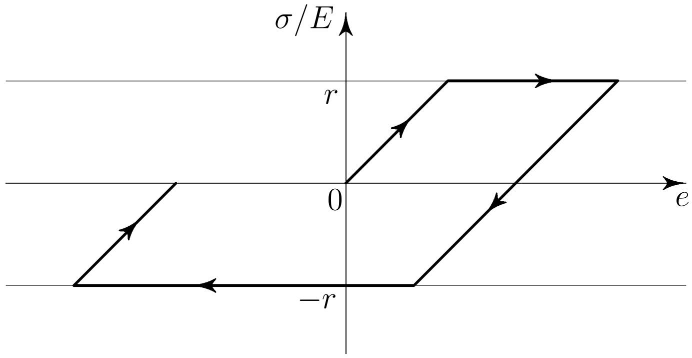

Figure 2: Strain-stress diagram for Prandtl's model.

We list below some basic well-known analytical properties of the Prandtl model and its extensions. A detailed discussion on this subject can be found in the monographs [2, 9, 12, 17]. We do not treat here its vectorial or tensorial counterparts, where the interval $[-r, r]$ is replaced by an arbitrary convex closed subset Z. The analytical properties of the model then depend substantially on the geometry of the set Z and a survey can be found in [3].

For every input $u \in W^{1,1}_{loc}(0,\infty)$, where $W^{1,1}_{loc}(0,\infty)$ is the space of absolutely continuous functions $[0,\infty[\,\to \mathbb{R}$, and every initial condition $s^0 \in [-r, r]$, problem (1.1) has a unique solution $s \in W^{1,1}_{loc}(0,\infty)$. We can therefore define the solution operator $\mathcal{S}_r : [-r, r] \times W^{1,1}_{loc}(0,\infty) \to W^{1,1}_{loc}(0,\infty)$ by the formula

$$\mathcal{S}_r[s^0, u] := s\,. \tag{1.2}$$

It is convenient to introduce also its complement

$$\mathcal{P}_r[s^0, u] := u - \mathcal{S}_r[s^0, u]\,. \tag{1.3}$$

The operators $\mathcal{S}_r$ and $\mathcal{P}_r$ are called the *stop* and *play*, respectively, with threshold r. In each interval of monotonicity $[t_0, t_1]$ of the input function u, the outputs are explicitly given by the formulas

$$\mathcal{S}_r[s^0, u](t) \;=\; \min\{r,\, \max\{-r,\, \mathcal{S}_r[s^0, u](t_0) + u(t) - u(t_0)\}\}\,, \tag{1.4}$$

$$\mathcal{P}_r[s^0, u](t) \;=\; \max\{u(t) - r,\, \min\{u(t) + r,\, \mathcal{P}_r[s^0, u](t_0)\}\}\,, \tag{1.5}$$

which have traditionally been used as alternative definitions of the stop and play on piecewise monotone inputs, see [2, 9]. The following inequalities hold, see e.g. Section II.1 of [12].

Proposition 1.1 *For* $s^0_1, s^0_2 \in [-r, r]$ *and* $u_1, u_2 \in W^{1,1}_{loc}(0,\infty)$ *put* $p_i := \mathcal{P}_r[s^0_i, u_i]$ *and* $s_i := \mathcal{S}_r[s^0_i, u_i]$, $i = 1, 2$. *Then we have*

(i) $(\dot p_1(t) - \dot p_2(t))\,(s_1(t) - s_2(t)) \;\geq\; 0$ *for a.e.* $t > 0$,

(ii) $|p_1(t) - p_2(t)| \;\leq\; \max\left\{|p_1(0) - p_2(0)|\,,\, \max_{0\le s\le t} |u_1(s) - u_2(s)|\right\} \qquad \forall t \geq 0\,.$

Part (ii) of Proposition 1.1 states that the play (and therefore also the stop) can be extended to Lipschitz continuous mappings from $[-r,r] \times C([0,\infty[\,)$ to $C([0,\infty[\,)$, where $C([0,\infty[\,)$ denotes the space of continuous functions $[0,\infty[\,\to \mathbb{R}$, endowed with a system of seminorms

$$\|u\|_{[0,t]} \;:=\; \max_{0\le s\le t} |u(s)| \qquad \text{for} \;\; t \geq 0\,. \tag{1.6}$$

To simplify the presentation, we consider special initial configurations of the stop and play operators. They consist in choosing

$$s^0 := \operatorname{sign}(u(0)) \min\{|u(0)|, r\} \tag{1.7}$$

in the variational problem (1.1). In materials sciences, this corresponds to the initially unperturbed (or *virgin*) reference state. In some applications, it is substantial to consider more general initial states, and an interested reader can find a detailed analysis in [2] or [12]. Here, as we are interested in periodic motions, the results do not depend on the choice of s^0.

This enables us to consider the stop and play as operators from $C([0,\infty[\,)$ to $C([0,\infty[\,)$ and to write simply $\mathcal{S}_r[u]$, $\mathcal{P}_r[u]$ instead of $\mathcal{S}_r[s^0,u]$, $\mathcal{P}_r[s^0,u]$.

These operators are *odd*, that is $\mathcal{S}_r[-u] = -\mathcal{S}_r[u]$, $\mathcal{P}_r[-u] = -\mathcal{P}_r[u]$ for every input u, and homogeneous in the sense that

$$\mathcal{S}_r[zu] = z\mathcal{S}_{r/|z|}[u]\,, \quad \mathcal{P}_r[zu] = z\mathcal{P}_{r/|z|}[u] \tag{1.8}$$

for every $r > 0$, $z \neq 0$, and every input u.

Note that for all $r > 0$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_r[u] \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}_r[u] \;=\; 0 \tag{1.9}$$

whenever the derivatives exist, hence $\mathrm{d}\mathcal{P}_r[u](t)/\mathrm{d}t = \dot u(t)$, $\mathrm{d}\mathcal{S}_r[u](t)/\mathrm{d}t = 0$ or vice versa.

From Proposition 1.1 we immediately obtain the following estimate.

Corollary 1.2 *For* $u_1, u_2 \in C([0,\infty[\,)$ *put* $p_i := \mathcal{P}_r[u_i]$, $s_i := \mathcal{S}_r[u_i]$, $i = 1,2$. *Then for all* $t \geq 0$ *we have*

$$\begin{aligned} |p_1(t) - p_2(t)| \;&\leq\; \|u_1 - u_2\|_{[0,t]}\,, \\ |s_1(t) - s_2(t)| \;&\leq\; 2\,\|u_1 - u_2\|_{[0,t]}\,. \end{aligned} \tag{1.10}$$

1.2 Prandtl-Ishlinskii operators

One practical drawback of Prandtl's model in Fig. 2 consists in an instantaneous transition from the purely elastic to the purely plastic regime. In 'real' elastoplastic materials, this transition zone is smooth, see [13]. Prandtl [14] and Ishlinskii [7] therefore proposed to combine rheological elements from Fig. 1 corresponding to different values $r_1 < r_2 < \ldots < r_n < \infty$ of the yield point in parallel, as on Figure 3. The purely elastic element corresponding to $r = \infty$ accounts for the *kinematic hardening*.

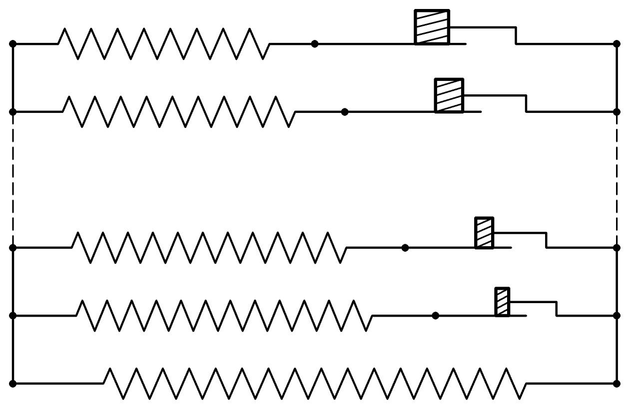

Figure 3: Rheological structure of the Prandtl-Ishlinskii model.

According to Eq. (1.2), the strain-stress law for the Prandtl-Ishlinskii model can be written in operator form as

$$\sigma \ = \ \mathcal{F}[e] \ := \ E_\infty \, e + \sum_{i=1}^{n} E_i \, \mathcal{S}_{r_i}[e] \, , \tag{1.11}$$

where E_i are given non-negative individual elasticity moduli.

In fact, there is no reason to restrict the model to finitely many yield points. For a mathematical treatment, it is more convenient to work with more general constitutive operators. This leads us to the following definition:

Definition 1.3 *Let* $\eta :]0,\infty[\to \ [0,\infty[$ *be a non-increasing function. Then the operator* $\mathcal{F}$ *defined by the Stieltjes integral*

$$\mathcal{F}[e] \ := \ \eta(\infty) \, e - \int_0^\infty \mathcal{S}_r[e] \, \mathrm{d}\eta(r) \, , \tag{1.12}$$

is called a Prandtl-Ishlinskii operator of stop type *whenever the integral exists, and* η *is called the* distribution function *of* $\mathcal{F}$ *.*

Indeed, the case (1.11) is included in the above definition; it suffices to put $r_0 := 0$, $r_{n+1} := \infty$, $E_{n+1} := E_\infty$, and

$$\eta(r) := \sum_{i=k}^{n+1} E_i \quad \text{for } r \in [r_{k-1}, r_k[\,, \quad k = 1, \dots, n+1\,. \tag{1.13}$$

In particular, constant functions η correspond to a purely elastic constitutive law.

An important practical question consists in identifying the function η from physical measurements. The usual approach is to increase monotonically the load from zero to some final value and plot the corresponding strain-stress graph called the *initial loading curve*. So, assume that $e(t)$ increases in $[0, \infty[$ from the starting value $e(0) = 0$. Then, at time t, we have by Eq. (1.4) for every $r > 0$ that

$$\mathcal{S}_r[e](t) = \min\{e(t), r\}\,,$$

and Eq. (1.12) formally yields

$$\mathcal{F}[e](t) \;=\; \eta_\infty\, e(t) - \int_0^{e(t)} r\, \mathrm{d}\eta(r) - e(t) \int_{e(t)}^{\infty} \mathrm{d}\eta(r) = \int_0^{e(t)} \eta(r)\, \mathrm{d}r\,. \tag{1.14}$$

Given an *increasing concave* experimental initial loading curve $\sigma = \varphi(e)$, Eq. (1.14) says that it determines uniquely the Prandtl-Ishlinskii operator (1.12) (the Stieltjes integral exists in this case!) through the relation

$$\eta(r) \;=\; \varphi'(r) := \frac{\mathrm{d}\varphi(r)}{\mathrm{d}r}\,. \tag{1.15}$$

In the Prandtl-Ishlinskii model, all secondary branches of the hysteresis loops have the same shape, namely $\sigma = \sigma^* + 2\varphi((e - e^*)/2)$ for an increasing branch, $\sigma = \sigma^* - 2\varphi((e^* - e)/2)$ for a decreasing branch, where (e^*, σ^*) is a turning point, cf. Fig. 4.

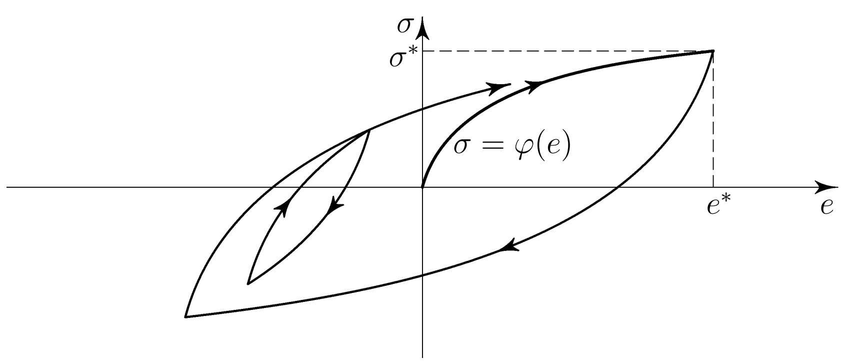

Figure 4: A diagram of the Prandtl-Ishlinskii operator.

Prandtl-Ishlinskii operators have a very specific property, namely that they are invertible and the inverse has the same structure. This result goes back to [10] in the time-periodic case. The following version can be found in [12], Corollary II.3.4.

Theorem 1.4 *Let* $\varphi : [0,\infty[\to [0,\infty[$ *be a concave increasing function,* $\varphi(0) = 0$ *,* $\varphi(\infty) = \infty$ *, and let* $\psi = \varphi^{-1} : [0,\infty[\to [0,\infty[$ *be its inverse. Let* $\eta := \varphi'$ *,* $\zeta := \psi'$ *be their respective derivatives. Then the formula*

$$\mathcal{G}[\sigma] \;:=\; \zeta(0)\,\sigma + \int_0^\infty \mathcal{P}_r[\sigma]\,\mathrm{d}\zeta(r) \quad \text{for } \sigma \in C([0,\infty[\,) \tag{1.16}$$

defines the so-called Prandtl-Ishlinskii operator of play type $\mathcal{G} : C([0,\infty[\,) \to C([0,\infty[\,)$ with distribution function ζ *. The operator* $\mathcal{G}$ *is inverse to* $\mathcal{F}$ *given by formula (1.12), that is,* $\mathcal{F}[\mathcal{G}[u]] = \mathcal{G}[\mathcal{F}[u]] = u$ *for every* $u \in C([0,\infty[\,)$ *. Moreover, both* $\mathcal{F}$ *and* $\mathcal{G}$ *are odd operators.*

In terms of the underlying mechanical construction, we can say that the rheological models on Fig. 3 and Fig. 5 are equivalent for $\tilde{r}_k = \sum_{i=1}^{k-1} E_i\, r_i + r_k \sum_{i=k}^{n+1} E_i$.

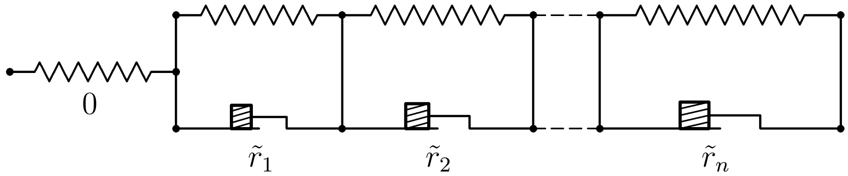

Figure 5: Rheological structure of the Prandtl-Ishlinskii operator of play type.

For every input function $\sigma \in C([0,\infty[\,)$, the function $\lambda : [0,\infty[\times [0,\infty[\to \mathbb{R}$ defined by the formula

$$\lambda(r,t) := \begin{cases} \mathcal{P}_r[\sigma](t) & \text{for } r > 0\,, \\ \sigma(t) & \text{for } r = 0\,, \end{cases} \tag{1.17}$$

represents the *memory state evolution of the system.* It has the following properties (see Proposition II.2.5 and Corollary II.2.7 of [12]).

Proposition 1.5 *Let* $\sigma \in C([0,\infty[\,)$ *and* $t \ge 0$ *be given. Then the function* $\lambda(\cdot,t)$ *is Lipschitz continuous in* $[0,\infty[$ *for every* $t \ge 0$ *with coefficient* 1*, and we have*

$$\begin{cases} \lambda(r,t) = 0 & \text{for } r \ge \|\sigma\|_{[0,t]}\,, \\ \left|\dfrac{\partial}{\partial r}\lambda(r,t)\right| = 1 & \text{for a.e. } r \in\,]0, \|\sigma\|_{[0,t]}\,[\;. \end{cases} \tag{1.18}$$

If moreover there exists $\omega > 0$ *such that* $\sigma(t+\omega) = \sigma(t)$ *for every* $t \ge 0$*, then* $\lambda(r,t+\omega) = \lambda(r,t)$ *for every* $t \ge \omega$ *.*

Prandtl-Ishlinskii operators have (as a consequence of Proposition 1.5, Corollary 1.2, and Eq. (1.9)) the following properties.

Proposition 1.6 *Let $\mathcal{G}$ be the operator (1.16), and let $\sigma_1, \sigma_2 \in C([0,\infty[)$ be given. Then for every $t \ge 0$ we have*

$$|\mathcal{G}[\sigma_1](t) - \mathcal{G}[\sigma_2](t)| \ \le \ \zeta\left(\max\{\|\sigma_1\|_{[0,t]}\,, \|\sigma_2\|_{[0,t]}\}\right)\, \|\sigma_1 - \sigma_2\|_{[0,t]}\ . \tag{1.19}$$

If moreover $\sigma \in W^{1,1}_{loc}(0,\infty)$, then for a.e. $t > 0$ we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}[\sigma](t)\right| \ \le \ \zeta(\|\sigma\|_{[0,t]})\ |\dot\sigma(t)|\ . \tag{1.20}$$

From Proposition 1.1 it follows that the operator $\mathcal{G}$ defined by (1.16) is monotone in the sense that the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{G}[\sigma_1] - \mathcal{G}[\sigma_2]\right)(\sigma_1 - \sigma_2) \ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\zeta(0)|\sigma_1 - \sigma_2|^2 + \int_0^\infty |\mathcal{P}_r[\sigma_1] - \mathcal{P}_r[\sigma_2]|^2\, d\zeta(r)\right) \tag{1.21}$$

holds a. e. for every $\sigma_1, \sigma_2 \in W^{1,1}_{loc}(0,\infty)$. In particular, if σ_1, σ_2 are ω-periodic, then $\mathcal{G}[\sigma_1], \mathcal{G}[\sigma_2]$ are ω-periodic for $t \ge \omega$, and we have

$$\int_\omega^{2\omega} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{G}[\sigma_1] - \mathcal{G}[\sigma_2]\right)(\sigma_1 - \sigma_2)\,\mathrm{d}t \ \ge \ 0\,. \tag{1.22}$$

Inequality (1.22) is strict provided ζ is strictly increasing. More precisely, the following statement is proved in Corollary II.4.11 and Proposition II.4.12 of [12].

Proposition 1.7 *Let $\sigma_1, \sigma_2 \in W^{1,1}_{loc}(0,\infty)$ be ω-periodic, and let ζ be strictly increasing. Assume that*

$$\int_\omega^{2\omega} \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{G}[\sigma_1] - \mathcal{G}[\sigma_2]\right)(\sigma_1 - \sigma_2)\,\mathrm{d}t \ = \ 0\,. \tag{1.23}$$

Then $d\sigma_1/dt = d\sigma_2/dt$, $d\mathcal{G}[\sigma_1]/dt = d\mathcal{G}[\sigma_2]/dt$ a. e.

1.3 Energy inequalities

Energy dissipation is a typical feature of hysteresis phenomena. To introduce it as a mathematical concept, we have to define an internal energy functional $U \ge 0$ corresponding to the constitutive law $\sigma = \mathcal{F}[e]$ or equivalently $e = \mathcal{G}[\sigma]$. The second principle of thermodynamics then states that the dissipation rate q satisfies

$$q := \sigma\,\dot e - \dot U \ \ge \ 0\,. \tag{1.24}$$

If we choose e as state variable (input) and $\sigma = \mathcal{F}[e]$ as state function (output), we define a continuous family of internal parameters $\sigma_r := \mathcal{S}_r[e]$ which correspond to individual stress components in the rheological construction of Fig. 3. It is assumed that no internal energy can be stored in the dry friction elements; the internal energy U of the system is then defined as the total internal energy of the individual elastic elements, that is, in operator form,

$$U = \mathcal{U}[e] := \frac{1}{2}\left(\eta(\infty)\, e^2 - \int_0^\infty \sigma_r^2 \,\mathrm{d}\eta(r)\right) = \frac{1}{2}\left(\eta(\infty)\, e^2 - \int_0^\infty (\mathcal{S}_r[e])^2 \,\mathrm{d}\eta(r)\right). \tag{1.25}$$

Conversely, if σ is the input and $e = \mathcal{G}[\sigma]$ is the output, then we choose the strain components $e_r := \mathcal{P}_r[\sigma]$ to be the internal parameters and the total internal energy has the form

$$U = \mathcal{V}[\sigma] := \frac{1}{2}\left(\zeta(0)\, \sigma^2 + \int_0^\infty e_r^2 \,\mathrm{d}\zeta(r)\right) = \frac{1}{2}\left(\zeta(0)\, \sigma^2 + \int_0^\infty (\mathcal{P}_r[\sigma])^2 \,\mathrm{d}\zeta(r)\right). \tag{1.26}$$

It can be shown using Proposition II.3.3 of [12] that formulas (1.25) and (1.26) are equivalent. A straightforward differentiation yields the formula

$$q(t) = \sigma \,\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}[\sigma] - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}[\sigma] = \int_0^\infty \mathcal{S}_r[\sigma]\,\frac{\partial}{\partial t}\mathcal{P}_r[\sigma]\,\mathrm{d}\zeta(r) \;\geq\; 0 \quad \text{a. e.} \tag{1.27}$$

for every $\sigma \in W^{1,1}_{loc}(0,\infty)$, hence the model is consistent with inequality (1.24).

Hysteresis operators admit a *second order energy inequality* which is related to the convexity of hysteresis loops. A detailed discussion on this subject can be found in Section II.4 of [12]. We need here the following consequence of Proposition II.4.21 and Corollary II.4.23 of [12].

Theorem 1.8 *Let $\mathcal{G}$ be defined by (1.16), and for $r > 0$ put*

$$\gamma(r) := \inf\left\{\frac{\zeta(b) - \zeta(a)}{b - a}\,;\; 0 < a < b \leq r\right\}.$$

Let $\sigma \in W^{2,1}_{loc}(0,\infty)$ be a given ω-periodic function. Then $\mathcal{G}[\sigma] \in W^{1,\infty}(0,\infty)$, and we have

$$-\int_\omega^{2\omega} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}[\sigma](t)\,\ddot\sigma(t)\,\mathrm{d}t \;\geq\; \frac{1}{4}\,\gamma(\|\sigma\|_{[0,\omega]})\int_\omega^{2\omega} |\dot\sigma|^3\,\mathrm{d}t\,. \tag{1.28}$$

1.4 Parameter dependent hysteresis operators

We will consider Prandtl-Ishlinskii operators whose distribution functions depend on a parameter a in a Lebesgue measurable parameter set $A \subset \mathbb{R}^N$. Given an input function $e : A \times [0,\infty[\to \mathbb{R}$, we define the output σ by the formula

$$\sigma(a,t) = \mathcal{F}[e](a,t) := \eta(a,\infty)\, e(a,t) \;-\; \int_0^\infty \mathcal{S}_r[e(a,\cdot)](t)\,\mathrm{d}_r\eta(a,r) \tag{1.29}$$

for $(a,t) \in A \times [0,\infty[$. It makes sense for every $a \in A$, for which the function $t \mapsto e(a,t)$ is continuous, and the function $r \mapsto \eta(a,r)$ is non-increasing and non-negative for a. e. $a \in A$. For an input function σ, we similarly define the output of the inverse operator

$$e(a,t) := \mathcal{G}[\sigma](a,t) = \zeta(a,0)\,\sigma(a,t) \;+\; \int_0^\infty \mathcal{P}_r[\sigma(a,\cdot)](t)\,\mathrm{d}_r\zeta(a,r)\,, \tag{1.30}$$

where $\zeta(a,\cdot))$ is associated with $\eta(a,\cdot)$ as in Theorem 1.4. Typically, a may represent the space variable. This enables us to consider also spatially inhomogeneous materials.

The following weak continuity result is a modification of Proposition 2.12 in [4] and will play a substantial role in Section 4.

Proposition 1.9 *Let $\{\sigma_n\,;\, n\in\mathbb{N}\}$, be a sequence in $L^\infty(A\times\,]0,\infty[\,)$ such that $\sigma_n(a,t)\le R$ a. e., $\sigma_n(a,\cdot)\in C([0,\infty[\,)$ for a.e. $a\in A$, and $\lim_{n\to\infty}\int_A \|(\sigma_n-\sigma)(a,\cdot)\|_{[0,t]}\,\mathrm{d}a = 0$.*

Let ζ, $\{\zeta_n\,;\, n\in\mathbb{N}\}$, be functions in $L^\infty_{loc}(A\times\,]0,\infty[\,)$ such that $\zeta(a,\cdot)$, $\zeta_n(a,\cdot)$ are non-decreasing and non-negative for every $n\in\mathbb{N}$ and a. e. $a\in A$, and let $\zeta_n|_{(A\times\,]0,R[\,)}$ converge to $\zeta|_{(A\times\,]0,R[\,)}$ in $L^\infty(A\times\,]0,R[\,)$ weakly-star as $n\to\infty$.

Let $\mathcal{G}_n$, $\mathcal{G}$ be the operators corresponding to ζ_n,ζ, respectively, according to Eq. (1.30).

Then $\mathcal{G}_n[\sigma_n](\cdot,t)$ converge to $\mathcal{G}[\sigma](\cdot,t)$ for every $t\ge 0$ in $L^\infty(A)$ weakly-star as $n\to\infty$.

Proof. Integrating by parts in the Stieltjes integral and using Proposition 1.5, we obtain for a. e. $a\in A$ and every $t>0$ that

$$
\begin{aligned}
(\mathcal{G}_n[\sigma_n]-\mathcal{G}[\sigma])(a,t) &= \zeta_n(a,0)(\sigma_n-\sigma)(a,t) \\
&+ \int_0^R (\mathcal{P}_r[\sigma_n(a,\cdot)]-\mathcal{P}_r[\sigma(a,\cdot)])(t)\,\mathrm{d}_r\zeta_n(a,r) \\
&- \int_0^R \frac{\partial}{\partial r}\mathcal{P}_r[\sigma(a,\cdot)](t)\,(\zeta_n-\zeta)(a,r)\,\mathrm{d}r\,.
\end{aligned}
\tag{1.31}
$$

A weakly-star convergent sequence is bounded. We can therefore estimate the first two terms on the right-hand side of (1.31) using Corollary 1.2 by

$$
\left|\zeta_n(a,0)(\sigma_n-\sigma)(a,t)+\int_0^R (\mathcal{P}_r[\sigma_n(a,\cdot)]-\mathcal{P}_r[\sigma(a,\cdot)])(t)\,\mathrm{d}_r\zeta_n(a,r)\right| \le C\,\|(\sigma_n-\sigma)(a,\cdot)\|_{[0,t]}
\tag{1.32}
$$

a. e. with a constant C independent of a and n. The third term converges weakly-star by Proposition 1.5, and the assertion follows. ■

2 Main results

2.1 Derivation of the model

Let us consider a cylindrical beam of length π and constant cross section S. In the referential state, the beam is oriented in such a way that the x-axis coincides with its longitudinal axis. The cross section is assumed to be a plane figure which is symmetric with respect to both coordinate axes. We further assume that the motion takes place only in the xz-plane, and neglect longitudinal displacements. The transversal displacement denoted by u is assumed to depend only on $x \in [0, \pi]$ and the time $t \geq 0$.

The motion is governed by the equation (see §21 of [15] or §87 of [8])

$$\rho(x) \frac{\partial^2 u}{\partial t^2} + \frac{1}{|S|} \frac{\partial^2 M}{\partial x^2} = g(x,t), \tag{2.1}$$

where $|S|$ is the two-dimensional area of the cross section, $g(x,t)$ is the load density, and M is the y-component of the bending moment.

In the next step, we neglect all components e_{ij}, σ_{ij} of the strain and stress tensor except for $e = e_{11}$, $\sigma = \sigma_{11}$. Under the small deformation hypothesis, the strain e has the form

$$e(x,z,t) = -z \frac{\partial^2 u}{\partial x^2}(x,t). \tag{2.2}$$

The bending moment satisfies the equation

$$M(x,t) = -\iint_S z\,\sigma(x,z,t)\,\mathrm{d}y\,\mathrm{d}z. \tag{2.3}$$

Let σ be related to e by a Prandtl-Ishlinskii relation (1.29) with parameter $a = (x,z)$, that is,

$$\sigma(x,z,t) = \eta(x,z,\infty)\,e(x,z,t) - \int_0^\infty \mathcal{S}_r[e(x,z,\cdot)](t)\,\mathrm{d}_r\eta(x,z,r). \tag{2.4}$$

By (1.8), we have

$$z\,\mathcal{S}_r[-z\,u_{xx}(x,\cdot)](t) = -z^2\,\mathcal{S}_{r/|z|}[u_{xx}(x,\cdot)](t) \tag{2.5}$$

for every $r > 0$ and $z \neq 0$. Combining (2.3) with (2.4) and (2.5) we obtain

$$M(x,t) = \hat{\mathcal{F}}[u_{xx}](x,t) := \hat{\eta}(x,\infty)\,u_{xx}(x,t) - \int_0^\infty \mathcal{S}_r[u_{xx}(x,\cdot)](t)\,\mathrm{d}_r\hat{\eta}(x,r), \tag{2.6}$$

where $\hat{\eta}(x,r) = \iint_S z^2\,\eta(x,z,|z|r)\,\mathrm{d}y\,\mathrm{d}z$. This enables us to rewrite the equation of motion (2.1) in the form

$$\rho(x)\,u_{tt} + \frac{1}{|S|}\,\hat{\mathcal{F}}[u_{xx}]_{xx} = g(x,t), \tag{2.7}$$

which is (up to a change of notation) nothing but (0.1).

2.2 Governing equations

Using the inversion formula (1.16) and introducing new variables $w = \mathcal{F}[u_{xx}]$, $v = u_t$, we rewrite the system (0.1), (0.2) in the form

$$\rho(x)\,v_t + w_{xx} = g(x,t), \tag{2.8}$$

$$\mathcal{G}[w]_t - v_{xx} = 0, \tag{2.9}$$

with boundary conditions

$$v(0,t) = v(\pi,t) = w(0,t) = w(\pi,t) = 0\,, \tag{2.10}$$

where $\mathcal{G}$ is an operator of the form

$$\mathcal{G}[w](x,t) = \zeta(x,0)\,w(x,t) \;+\; \int_0^\infty \mathcal{P}_r[w(x,\cdot)](t)\,\mathrm{d}_r\zeta(x,r)\,. \tag{2.11}$$

We introduce the following notation.

Notation 2.1 We fix $\omega > 0$, set $J :=]0,\pi[$, and define the spaces

(i) $L^p_\omega(J)$ of functions $u \in L^p_{loc}(J\times\,]0,\infty[\,)$ such that $u(x,t+\omega) = u(x,t)$ for every $t > 0$, endowed with the norm

$$\|u\|_p = \begin{cases} \left(\int_\omega^{2\omega}\int_J |u(x,t)|^p\,\mathrm{d}x\,\mathrm{d}t\right)^{1/p} & \text{if } 1 \le p < \infty\,,\\ \operatorname{ess\,sup}\{|u(x,t)|\,;\,(x,t)\in J\times\,]\omega,2\omega[\,\} & \text{if } p = \infty\,,\end{cases}$$

(ii) $C_\omega(\bar J)$ of continuous functions $u : \bar J \to \mathbb{R}$ such that $u(x,t+\omega) = u(x,t)$ for every $t \ge 0$, endowed with the norm $\|u\|_\infty$,

(iii) $Z = \{(v,w) \in C_\omega(\bar J) \times C_\omega(\bar J)\,;\, v_t, w_{xx} \in L^2_\omega(J)\,, w_t, v_{xx} \in L^3_\omega(J)\,, \text{and (2.10) holds}\}$, endowed with the norm

$$\|(v,w)\|_Z := \|v_t\|_2 + \|w_{xx}\|_2 + \|w_t\|_3 + \|v_{xx}\|_3\,.$$

Note that we have the compact embedding

$$Z \hookrightarrow\hookrightarrow C_\omega(\bar J) \times C_\omega(\bar J)\,. \tag{2.12}$$

We make the following hypotheses.

Hypothesis 2.2 There exists constants $\bar g, \bar\rho_0, \bar\rho_1, \alpha, \beta, \bar\gamma_0, \bar\gamma_1$, satisfying the inequalities $\bar g > 0$, $\bar\rho_1 > \bar\rho_0 > 0$, $\alpha \le 1$, $\beta > 0$, $\bar\gamma_1 > \bar\gamma_0(\beta+1)/\beta > 0$, $\alpha \le \beta$, $3\beta - 2\alpha < 2$, such that

(i) $g, g_t \in L^2_\omega(J)$, $\|g\|_2^2 + \|g_t\|_2^2 \le \bar g^2$,

(ii) $\rho \in L^\infty(J)$, $\quad \bar\rho_0 \le \rho(x) \le \bar\rho_1$ a.e. ,

(iii) $\zeta \in L^\infty_{loc}(J\times\,]0,\infty[\,)$, $\zeta(x,\cdot)$ is increasing for a. e. $x \in J$, and the inequalities

$$0 \;\le\; \zeta(x,r) \;\le\; \bar\Gamma(r) := \bar\gamma_1\,\max\{1, r^\beta\}\,, \tag{2.13}$$

$$\frac{\zeta(x,r)-\zeta(x,s)}{r-s} \;\ge\; \bar\gamma(r) := \bar\gamma_0\,\min\{1, r^{\alpha-1}\} \tag{2.14}$$

hold for a. e. $x \in J$ and for every $r > s > 0$.

The set of all $\Sigma = (g, \rho, \zeta) \in L^2_\omega(J) \times L^\infty(J) \times L^\infty_{loc}(J \times]0, \infty[)$ satisfying Hypothesis 2.2 will be denoted by $\mathbf{\Sigma}$ in the sequel. Note that in Hypothesis 2.2 we have $\bar\Gamma(r) > \int_0^r \bar\gamma(s)\,\mathrm{d}s + \bar\gamma_0/\beta$ for every $r > 0$, hence the set $\mathbf{\Sigma}$ is non-empty.

Actually, the approximations will live in larger sets $\mathbf{\Sigma}_c$ (parametrized by $c > 0$) of all $\Sigma = (g, \rho, \zeta)$ satisfying Hypothesis 2.2 with (2.13) replaced by

$$0 \le \zeta(x, r) \le c + \bar\Gamma(r)\,. \tag{2.15}$$

2.3 Main theorems

We now state the main results. Proofs will be given in Section 4. Let us first establish the existence, uniqueness, and continuous dependence of solutions to Eqs. (2.8) – (2.10).

Theorem 2.3 *For every $\Sigma \in \mathbf{\Sigma}$ there exists a unique $(v, w) \in Z$ such that Eqs. (2.8) – (2.10) are satisfied almost everywhere in $J \times]\omega, \infty[$.*

Theorem 2.3 enables us to define the solution operator

$$S : \mathbf{\Sigma} \to Z : \Sigma \mapsto (v, w)\,. \tag{2.16}$$

This mapping is weakly compact in the following sense.

Theorem 2.4 *Let $\{\Sigma^{(n)}\,;\, n \in \mathbb{N}\}$, $\Sigma^{(n)} = (g^{(n)}, \rho^{(n)}, \zeta^{(n)})$ be any sequence in $\mathbf{\Sigma}$. Then there exists a subsequence $\{\Sigma^{(n_k)}\}$ of $\{\Sigma^{(n)}\}$ and an element $\Sigma \in \mathbf{\Sigma}$ such that $S(\Sigma^{(n_k)}) \to S(\Sigma)$ as $k \to \infty$ weakly in Z and uniformly in $C_\omega(\bar J) \times C_\omega(\bar J)$.*

With each $\Sigma \in \mathbf{\Sigma}$ we associate the total dissipation over one period of the process defined by Eqs. (2.8) – (2.10) using the formula (1.27), that is,

$$D(\Sigma) := \int_\omega^{2\omega} \int_J \mathcal{G}[w]_t\, w \,\mathrm{d}x\,\mathrm{d}t\,, \tag{2.17}$$

where $(v, w) = S(\Sigma)$. Testing Eq. (2.8) by v and (2.9) by w, we obtain equivalently

$$D(\Sigma) = \int_\omega^{2\omega} \int_J g\, v \,\mathrm{d}x\,\mathrm{d}t\,. \tag{2.18}$$

The worst scenario result mentioned in the introduction reads as follows.

Theorem 2.5 *There exists $\Sigma^* \in \mathbf{\Sigma}$ such that for every $\Sigma \in \mathbf{\Sigma}$ we have $D(\Sigma) \le D(\Sigma^*)$. Moreover, the value of $D(\Sigma^*)$ can be found as a limit of finite-dimensional approximations.*

3 Auxiliary results

3.1 Weak compactness

We first show that the sets $\mathbf{\Sigma}_c$ themselves are weakly compact in the following sense.

Lemma 3.1 *Let* $\{c_n\}$ *be a sequence of real numbers such that* $\liminf_{n\to\infty} c_n = c \geq 0$, *let* $\Sigma^{(n)} = (g^{(n)}, \rho^{(n)}, \zeta^{(n)})$ *be an arbitrary sequence in* $\mathbf{\Sigma}_{c_n}$, *and let* $R > 1$ *be a constant. Then there exists a subsequence* $\Sigma^{(n_k)}$ *of* $\Sigma^{(n)}$ *and* $\Sigma = (g, \rho, \zeta) \in \mathbf{\Sigma}_c$ *such that*

$$\begin{cases} g^{(n_k)} \to g\,,\ g_t^{(n_k)} \to g_t & \text{weakly in } L^2_\omega(J)\,, \\ \rho^{(n_k)} \to \rho & \text{weakly-star in } L^\infty(J)\,, \\ \zeta^{(n_k)}\Big|_{J\times\,]0,R[} \to \zeta\Big|_{J\times\,]0,R[} & \text{weakly-star in } L^\infty(J\times\,]0,R[\,)\,. \end{cases} \tag{3.1}$$

Proof. The existence of convergent subsequences is obvious. The only fact one has to prove is that the weak-star limit of $\zeta^{(n_k)}|_{J\times\,]0,R[}$ is a restriction of a function ζ satisfying Hypothesis 2.2 (iii). This can be done by the standard diagonalization procedure. Let $\{r_j\,;\, j \in \mathbb{N}\}$ be a dense countable subset of $[0, R]$. We successively select a subsequence $\zeta^{(n_k)}$ in such a way that $\zeta^{(n_k)}(\cdot, r_j)$ converge to some ζ_j weakly-star in $L^\infty(J)$ for every $j \in \mathbb{N}$. Each function ζ_j is defined in a set $A_j \subset J$ of full measure, therefore also $A = \cap_{j=1}^\infty A_j$ has full measure. For $x \in A$ and $r \geq 0$ we now put

$$\zeta(x, r) \ := \ \begin{cases} \zeta_j(x) & \text{if } r = r_j\,,\ j \in \mathbb{N}\,, \\ \inf\{\zeta_j(x)\,;\, r_j > r\} & \text{if } r \in [0, R[\,\setminus \bigcup_{j=1}^\infty \{r_j\}\,, \\ \bar\Gamma(r) & \text{if } r \geq R\,. \end{cases} \tag{3.2}$$

Then ζ satisfies Hypothesis 2.2 (iii). Moreover, for every test function $q \in L^1(J)$, the functions $z_k(r) := \int_J \zeta^{(n_k)}(x, r)\, q(x)\,\mathrm{d}x$, $z(r) := \int_J \zeta(x, r)\, q(x)\,\mathrm{d}x$ are non-decreasing and $z_k(r_j) \to z(r_j)$ for every $j \in \mathbb{N}$, hence $z_k(r) \to z(r)$ at every point $r \in [0, R]$ of continuity of z. In particular, $\zeta^{(n_k)}|_{J\times\,]0,R[}$ converge weakly-star to $\zeta|_{J\times\,]0,R[}$ in $L^\infty(J\times\,]0, R[\,)$, and the proof is complete. ■

3.2 Finite-dimensional approximations

Approximate solutions will be constructed by the Fourier method, see [16]. We choose in $L^2_\omega(J)$ the orthonormal basis $\{e_{jk}\,;\, j \in \mathbb{Z}\,,\ n \in \mathbb{N}\}$ given by

$$e_{jk}(x, t) \ = \ \begin{cases} \frac{2}{\sqrt{\omega\pi}} \cos \frac{2\pi j}{\omega}\, t\, \sin kx & \text{for } j < 0\,, \\ \sqrt{\frac{2}{\omega\pi}}\, \sin kx & \text{for } j = 0\,, \\ \frac{2}{\sqrt{\omega\pi}} \sin \frac{2\pi j}{\omega}\, t\, \sin kx & \text{for } j > 0\,. \end{cases} \tag{3.3}$$

We have the identities

$$(e_{jk})_t = \frac{2\pi j}{\omega}\, e_{-jk}\,, \quad (e_{jk})_{xx} = -k^2\, e_{jk}\,. \tag{3.4}$$

Instead of (2.8) – (2.10), we fix some $n \in \mathbb{N}$, and solve the approximate system

$$\int_\omega^{2\omega}\int_J \rho^{(n)}(x)\, v_t^{(n)}\, e_{jk}\,\mathrm{d}x\,\mathrm{d}t \;-\; k^2\, w_{jk} \;=\; \int_\omega^{2\omega}\int_J g^{(n)}\, e_{jk}\,\mathrm{d}x\,\mathrm{d}t\,, \tag{3.5}$$

$$\int_\omega^{2\omega}\int_J \mathcal{G}^{(n)}[w^{(n)}]_t\, e_{jk}\,\mathrm{d}x\,\mathrm{d}t \;+\; k^2\, v_{jk} \;=\; 0\,, \tag{3.6}$$

with unknowns $v_{jk}, w_{jk}\,,\ j = -n, \dots, n\,,\ k = 1, \dots, n$, where

$$\begin{aligned} v^{(n)}(x,t) &= \textstyle\sum_{j=-n}^{n}\sum_{k=1}^{n} v_{jk}\, e_{jk}(x,t)\,,\\ w^{(n)}(x,t) &= \textstyle\sum_{j=-n}^{n}\sum_{k=1}^{n} w_{jk}\, e_{jk}(x,t)\,, \end{aligned} \tag{3.7}$$

$$\mathcal{G}^{(n)}[w^{(n)}](x,t) \;=\; \zeta^{(n)}(x,0)\, w^{(n)}(x,t) \;+\; \int_0^\infty \mathcal{P}_r[w^{(n)}(x,\cdot)](t)\,\mathrm{d}_r\zeta^{(n)}(x,r)\,, \tag{3.8}$$

with suitably chosen $\Sigma^{(n)} = (g^{(n)}, \rho^{(n)}, \zeta^{(n)}) \in \mathbf{\Sigma}_1$.

The following a priori estimate will play a crucial role throughout the paper.

Lemma 3.2 *There exists a constant $R > 1$ independent of n and of $\Sigma^{(n)} \in \mathbf{\Sigma}_1$ such that every solution $(v^{(n)}, w^{(n)})$ to (3.5), (3.6) satisfies the estimate*

$$\|(v^{(n)}, w^{(n)})\|_Z \;\le\; R\,, \quad \|v^{(n)}\|_\infty \;\le\; R\,, \quad \|w^{(n)}\|_\infty \;\le\; R\,. \tag{3.9}$$

Proof. We multiply Eq. (3.5) by $(2\pi j/\omega)^2\, v_{jk}$, Eq. (3.6) by $(2\pi j/\omega)^2\, w_{jk}$, and sum up. This yields that

$$-\int_\omega^{2\omega}\int_J \mathcal{G}^{(n)}[w^{(n)}]_t\, w_{tt}^{(n)}\,\mathrm{d}x\,\mathrm{d}t \;=\; \int_\omega^{2\omega}\int_J g_t^{(n)}\, v_t^{(n)}\,\mathrm{d}x\,\mathrm{d}t\,. \tag{3.10}$$

From Theorem 1.8 it follows that

$$\frac{1}{4}\,\bar\gamma(\|w^{(n)}\|_\infty)\,\|w_t^{(n)}\|_3^3 \;\le\; \bar g\,\|v_t^{(n)}\|_2\,. \tag{3.11}$$

We next multiply Eq. (3.5) by $-(2\pi j/\omega)\, v_{-jk}$, Eq. (3.6) by $(2\pi j/\omega)\, w_{-jk}$, and sum up. For every $k = 1, \dots, n$ we have

$$\sum_{j=-n}^{n} j\,(w_{jk}\, v_{-jk} + v_{jk}\, w_{-jk}) \;=\; 0\,,$$

and Ineq. (1.20) implies that

$$\bar\rho_0\,\|v_t^{(n)}\|_2^2 \;\le\; (1 + \bar\Gamma(\|w^{(n)}\|_\infty))\,\|w_t^{(n)}\|_2^2 + \bar g\,\|v_t^{(n)}\|_2\,. \tag{3.12}$$

Finally, Eqs. (3.5), (3.6) directly yield that

$$\|w_{xx}^{(n)}\|_2 \;\leq\; \bar\rho_1\,\|v_t^{(n)}\|_2 + \bar g\,, \tag{3.13}$$

$$\|v_{xx}^{(n)}\|_3 \;\leq\; (1+\bar\Gamma(\|w^{(n)}\|_\infty)\,\|w_t^{(n)}\|_3\,. \tag{3.14}$$

In the rest of the proof, we denote by $K_1, K_2, \dots$ any positive constant independent of n and of $\Sigma^{(n)} \in \mathbf{\Sigma}_1$. Eliminating $v_t^{(n)}$ from Eqs. (3.11), (3.12) we obtain

$$\bar\gamma(\|w^{(n)}\|_\infty)\,\|w_t^{(n)}\|_3^3 \;\leq\; K_1\,\left(1+\bar\Gamma^{1/2}(\|w^{(n)}\|_\infty)\,\|w_t^{(n)}\|_3\right), \tag{3.15}$$

hence

$$\|w_t^{(n)}\|_3 \;\leq\; K_2\left(1+\frac{\bar\Gamma^{1/4}(\|w^{(n)}\|_\infty)}{\bar\gamma^{1/2}(\|w^{(n)}\|_\infty)}\right), \tag{3.16}$$

and using Eqs. (3.13), (3.12), we derive the estimates

$$\|w_t^{(n)}\|_2 + \|w_{xx}^{(n)}\|_2 \;\leq\; K_3\left(1+\frac{\bar\Gamma^{3/4}(\|w^{(n)}\|_\infty)}{\bar\gamma^{1/2}(\|w^{(n)}\|_\infty)}\right), \tag{3.17}$$

where for all $r>0$ we have

$$\frac{\bar\Gamma^{3/4}(r)}{\bar\gamma^{1/2}(r)} \;\leq\; K_4\,(1+r^\kappa)\,,\quad \kappa=\frac{1}{4}(3\beta-2\alpha+2)<1\,. \tag{3.18}$$

Combining Eqs. (3.17) – (3.18) with the embedding inequality

$$\|w^{(n)}\|_\infty \;\leq\; K_5\left(\|w_t^{(n)}\|_3 + \|w_{xx}^{(n)}\|_2\right) \tag{3.19}$$

we thus obtain

$$\|w^{(n)}\|_\infty \;\leq\; K_6\,\left(1+\|w^{(n)}\|_\infty^\kappa\right),$$

hence $\|w^{(n)}\|_\infty \leq K_7$ and the assertion follows from Eqs. (3.14), (3.16), and (3.17). ∎

Lemma 3.3 *For each $n \in \mathbb{N}$ and $\Sigma^{(n)} \in \mathbf{\Sigma}_1$ there exists a unique solution $\{v_{jk}, w_{jk}\,;\, j = -n,\dots,n\,,\ k=1,\dots,n\}$ to the system (3.5), (3.6).*

Proof. System (3.5), (3.6) is of the form $\Phi(W)=G$, where $W=(v_{jk},w_{jk}) \in X = \mathbb{R}^{2n(2n+1)}$ is the unknown vector, $G \in X$ is a datum, and $\Phi : X \to X$ is the mapping given by the left-hand side of (3.5), (3.6). From Theorem 1.4 and Proposition 1.6 it follows that Φ is continuous and odd. By Lemma 3.2, the equation $\Phi(X) = \delta\,G$ has no solution for any $\delta \in [0,1]$ outside a sufficiently large ball in X. The homotopy argument in the topological degree theory (cf. e.g. [5]) then entails that at least one solution exists inside the ball.

To prove the uniqueness, we consider two solutions (v_{jk}, w_{jk}), (v'_{jk}, w'_{jk}), and denote $\bar{v}_{jk} = v_{jk} - v'_{jk}$, $\bar{w}_{jk} = w_{jk} - w'_{jk}$. Then

$$\int_{\omega}^{2\omega} \int_J \rho^{(n)}(x)\, \bar{v}_t^{(n)}\, e_{jk}\, \mathrm{d}x\, \mathrm{d}t \;-\; k^2\, \bar{w}_{jk} \;=\; 0\,, \tag{3.20}$$

$$\int_{\omega}^{2\omega} \int_J \Big(\mathcal{G}^{(n)}[w^{(n)}] - \mathcal{G}^{(n)}[w'^{(n)}]\Big)_t\, e_{jk}\, \mathrm{d}x\, \mathrm{d}t \;+\; k^2\, \bar{v}_{jk} \;=\; 0\,. \tag{3.21}$$

We multiply Eq. (3.20) by $\bar{v}_{jk}$, (3.21) by $\bar{w}_{jk}$, and sum up. This yields

$$\int_{\omega}^{2\omega} \int_J \Big(\mathcal{G}^{(n)}[w^{(n)}] - \mathcal{G}^{(n)}[w'^{(n)}]\Big)_t\, (w^{(n)} - w'^{(n)})\, \mathrm{d}x\, \mathrm{d}t \;=\; 0\,. \tag{3.22}$$

The function $\zeta^{(n)}$ is strictly increasing by Hypothesis 2.2 (iii). Proposition 1.7 enables us to conclude that $w_t^{(n)} = w'^{(n)}_t$, $\mathcal{G}^{(n)}[w^{(n)}]_t = \mathcal{G}^{(n)}[w'^{(n)}]_t$ a. e., and the assertion follows. ■

4 Proofs of main results

4.1 Proof of Theorem 2.3

Let $\Sigma \in \mathbf{\Sigma}$ be given. For $n \in \mathbb{N}$ we use Lemma 3.3 to construct the solution $(v^{(n)}, w^{(n)})$ to the system (3.5), (3.6), with $\Sigma^{(n)} = \Sigma$. The estimate in Lemma 3.2 and the compact embedding (2.12) allow us to select a subsequence $(v^{(n_k)}, w^{(n_k)})$ such that $(v^{(n_k)}, w^{(n_k)}) \to (v, w)$ weakly in Z and uniformly in $C_\omega(\bar{J}) \times C_\omega(\bar{J})$. Proposition 1.6 yields that $\mathcal{G}[w^{(n_k)}]$ converge to $\mathcal{G}[w]$ uniformly, hence $(v, w) \in Z$ is a solution to Eqs. (2.8) – (2.10). The uniqueness follows from the same argument as in the proof of Lemma 3.3. Theorem 2.3 is proved.

4.2 Proof of Theorem 2.4

The solutions $(v^{(n)}, w^{(n)}) = S(\Sigma^{(n)})$ constructed as limits of finite-dimensional approximations following the argument of the proof of Theorem 2.3 satisfy the estimates (3.9). By Lemma 3.1, we can select a subsequence $\{\Sigma^{(n_k)}\}$ satisfying (3.1) and such that $(v^{(n_k)}, w^{(n_k)}) \to (v, w)$ weakly in Z and uniformly in $C_\omega(\bar{J}) \times C_\omega(\bar{J})$. Using Proposition 1.9 we can pass to the limit in (2.8), (2.9). Indeed, $\rho^{(n_k)}$ are independent of t, hence $\rho^{(n_k)}\, v_t^{(n_k)}$ converge in $L^2_\omega(J)$ weakly to $\rho\, v_t$, and the assertion follows.

4.3 Proof of Theorem 2.5

The existence of Σ^* is obvious. Indeed, the mapping $D : \mathbf{\Sigma} \to \mathbb{R}$ is bounded, and we may put $D^* := \sup\{D(\Sigma)\,;\, \Sigma \in \mathbf{\Sigma}\}$. Choosing any maximizing sequence $\{\Sigma^{(n)}\}$ in $\mathbf{\Sigma}$, we use Theorem 2.4 to construct a convergent subsequence which enables us to pass to the limit in (2.18).

The finite-dimensional approximation of D^* is interesting for practical applications. It can be done in the following way. For $n \in \mathbb{N}$, we denote by $\mathbf{\Sigma}^{(n)}$ the set of all $\Sigma = (g, \rho, \zeta)$ such that the Fourier expansion of g is finite, e. g.

$$g(x,t) \;=\; \sum_{j=-n}^{n} \sum_{k=1}^{n} g_{jk}\, e_{jk}(x,t)\,, \tag{4.1}$$

and ρ, ζ have a piecewise constant/piecewise linear form

$$\rho(x) \;=\; \rho_\ell\,, \tag{4.2}$$

$$\zeta(x,r) \;=\; \zeta_{\ell(m-1)} + \frac{n}{R}\left(r - \frac{m-1}{n}\,R\right)(\zeta_{\ell m} - \zeta_{\ell(m-1)}) \tag{4.3}$$

for $x \in [(\ell-1)\pi/n, \ell\pi/n[$, $r \in [(m-1)R/n, mR/n[$, $\ell = 1, \ldots, n$, $m = 1, \ldots, n$, continuously extended to $x = \pi$, and

$$\zeta(x,r) \;=\; c_n + \bar\Gamma(r) \qquad \text{for } \; r \geq R\,, \tag{4.4}$$

with $R > 1$ from Lemma 3.2 and with $c_n = \bar\gamma_0 R/n$. The real numbers g_{jk}, ρ_ℓ, $\zeta_{\ell m}$ are subject to the following restrictions.

$$\sum_{j=-n}^{n} \sum_{k=1}^{n} (1+j^2) g_{jk}^2 \;\leq\; \bar g^2\,, \tag{4.5}$$

$$\bar\rho_0 \;\leq\; \rho_\ell \;\leq\; \bar\rho_1 \quad \forall \ell = 1, \ldots, n\,, \tag{4.6}$$

$$\left.\begin{array}{rcl} \zeta_{\ell m} & \leq & c_n + \bar\Gamma\left(\frac{m-1}{n} R\right) \\ \zeta_{\ell m} - \zeta_{\ell(m-1)} & \geq & \frac{R}{n}\,\bar\gamma\left(\frac{m-1}{n} R\right) \end{array}\right\} \quad \forall \ell, m = 1, \ldots, n\,, \tag{4.7}$$

with $\zeta_{\ell 0} \geq 0$. We easily check that by (4.5) – (4.7), $\mathbf{\Sigma}^{(n)}$ is a non-empty compact subset of $\mathbf{\Sigma}_{c_n}$ for every $n \in \mathbb{N}$. By Lemma 3.3, for each $\Sigma \in \mathbf{\Sigma}^{(n)}$ and $n \geq \bar\gamma_0 R$, there exists a unique solution $\{(v_{jk}, w_{jk})\,;\, j = -n, \ldots, n\,,\ k = 1, \ldots, n\}$ to the system (3.5), (3.6). Repeating the argument of the proof of Theorem 2.4, we see that the mapping $S_n : \mathbf{\Sigma}^{(n)} \to \mathbb{R}^{2n(2n+1)} : \Sigma \mapsto \{(v_{jk}, w_{jk})\}$ is continuous. In particular, the corresponding 'discrete dissipation' mapping $D_n : \mathbf{\Sigma}^{(n)} \to \mathbb{R}$ given by

$$D_n(\Sigma) \;:=\; \sum_{j=-n}^{n} \sum_{k=1}^{n} g_{jk}\, v_{jk}$$

is continuous, hence it attains its maximum on $\mathbf{\Sigma}^{(n)}$ at some point $\hat\Sigma^{(n)} \in \mathbf{\Sigma}^{(n)}$. We denote

$$D_+ := \limsup_{n\to\infty} D_n(\hat\Sigma^{(n)})\,, \quad D_- := \liminf_{n\to\infty} D_n(\hat\Sigma^{(n)})\,, \tag{4.8}$$

with the intention to prove that $D_+ = D_- = D^*$.

By Lemma 3.1, we find a subsequence $\hat\Sigma^{(n_k)}$ of $\hat\Sigma^{(n)}$ such that

$$\lim_{k\to\infty} D_{n_k}(\hat\Sigma^{(n_k)}) = D_+\,,$$

and the convergence (3.1) holds for $\hat{\Sigma}^{(n_k)}$ with some $\Sigma = (g, \rho, \zeta) \in \mathbf{\Sigma}$, and with R from Lemma 3.2. Selecting again a subsequence, if necessary, we may argue as in the proof of Theorem 2.4 and assume that the corresponding solutions $(\hat{v}^{(n_k)}, \hat{w}^{(n_k)})$ to (3.5), (3.6) converge weakly in Z and uniformly in $C_\omega(\bar{J}) \times C_\omega(\bar{J})$ to $(v, w) = S(\Sigma) \in Z$. Passing to the limit as $k \to \infty$ we obtain $D(\Sigma) = D_+$, hence $D_+ \leq D^*$.

The proof will be complete if we check that $D(\Sigma) \leq D_-$ for every $\Sigma \in \mathbf{\Sigma}$. To this end, consider an arbitrary $\Sigma = (g, \rho, \zeta) \in \mathbf{\Sigma}$ and for $n \in \mathbb{N}$, $j = -n, \dots, n$, $k = 1, \dots, n$, $\ell, m = 1, \dots, n$ put

$$
\begin{aligned}
g_{jk} &:= \int_\omega^{2\omega} \int_J g(x,t)\, e_{jk}(x,t)\, \mathrm{d}x\, \mathrm{d}t\,, \\
\rho_\ell &:= \frac{n}{\pi} \int_{(\ell-1)\pi/n}^{\ell\pi/n} \rho(x)\, \mathrm{d}x\,, \\
\zeta_{\ell m} &:= c_n + \frac{n}{\pi} \int_{(\ell-1)\pi/n}^{\ell\pi/n} \zeta\left(x, \frac{m-1}{n} R\right) \mathrm{d}x\,, \\
\zeta_{\ell 0} &:= \zeta_{\ell 1} - c_n\,.
\end{aligned}
$$

Then the conditions (4.5) – (4.7) are fulfilled, and we may define $g^{(n)}$, $\rho^{(n)}$, $\zeta^{(n)}$ as in (4.1) – (4.3). For $n \geq \bar{\gamma}_0 R$ we have $\Sigma^{(n)} := (g^{(n)}, \rho^{(n)}, \zeta^{(n)}) \in \mathbf{\Sigma}_{c_n} \subset \mathbf{\Sigma}_1$, hence $D_n(\Sigma^{(n)}) \leq D_n(\hat{\Sigma}^{(n)})$. We choose a subsequence such that $D_{n_k}(\hat{\Sigma}^{(n_k)}) \to D_-$. By construction, Eqs. (3.1) hold, and passing to the limit as $k \to \infty$ we obtain $D(\Sigma) \leq D_-$. Theorem 2.5 is proved.

References

[1] M. Brokate, K. Dreßler and P. Krejčí, Rainflow counting and energy dissipation for hysteresis models in elastoplasticity. *Eur. J. Mech. A/Solids* **15**, 705–735 (1996).

[2] M. Brokate and J. Sprekels, *Hysteresis and phase transitions*, Appl. Math. Sci. Vol. 121, Springer-Verlag, New York (1996).

[3] P. Drábek, P. Krejčí, P. Takáč, *Nonlinear differential equations*, Research Notes in Mathematics, Vol. 404, Chapman & Hall/CRC, London (1999).

[4] J. Franců and P. Krejčí, Homogenization of scalar wave equations with hysteresis. *Cont. Mech. & Ther.* **11**, 371–391 (1999).

[5] S. Fučík and A. Kufner, *Nonlinear differential equations*, Studies in Applied Mechanics 2, Elsevier, Amsterdam, Oxford, New York (1980).

[6] I. Hlaváček, Reliable solution of an elasto-plastic Reissner-Mindlin beam for Hencky's model with uncertain yield function. *Appl. Math.* **43**, 223–237 (1998).

[7] A. Yu. Ishlinskii, Some applications of statistical methods to describing deformations of bodies (Russian). *Izv. Akad. Nauk SSSR, Techn. Ser.*, No. 9 580–590 (1944).

[8] H. Kauderer, *Nichtlineare Mechanik* (German). Springer-Verlag, Berlin-Göttingen-Heidelberg (1958).

[9] M. A. Krasnosel'skii and A. V. Pokrovskii, *Systems with hysteresis.* Springer, Berlin (1989).

[10] P. Krejčí, Hysteresis and periodic solutions of semilinear and quasilinear wave equations. *Math. Z.* **193**, 247–264 (1986).

[11] P. Krejčí, Periodic oscillations of an elastoplastic rod (Russian). In: *Functional and numerical methods of mathematical physics.* Naukova Dumka, Kiev, 117–120 (1988).

[12] P. Krejčí, *Hysteresis, convexity and dissipation in hyperbolic equations*, Gakuto Int. Series Math. Sci. & Appl., Vol. 8, Gakkōtosho, Tokyo (1996).

[13] J. Lemaitre, J.-L. Chaboche, *Mechanics of solid materials*, Cambridge University Press, Cambridge (1990).

[14] L. Prandtl, Ein Gedankenmodell zur kinetischen Theorie der festen Körper (German). *Z. Angew. Math. Mech.*, **8**, 85–106 (1928).

[15] S. Timoshenko, *Strength of materials, Part I, Elementary theory and problems*, Second Edition. D. van Nostrand, New York (1940).

[16] O. Vejvoda et al., *Partial differential equations: time-periodic solutions*, Sijthoff & Noordhoff, Alphen aan den Rijn (1981).

[17] A. Visintin, *Differential models of hysteresis.* Springer, Berlin – Heidelberg (1994).