On a model for electromagnetic processes in ferromagnetic materials with hysteresis and space dependent electric conductivity

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Abstract

One-dimensional Maxwell's equations are considered in a ferromagnetic body surrounded by vacuum. Existence and uniqueness of solution for the resulting system of P.D.E.s with hysteresis on the whole real line is proved under suitable constitutive hypotheses.

Classification: 35Q60, 47J40.

Key words: electromagnetic processes, ferromagnetic materials, hysteresis.

1   Introduction

The aim of this paper is to find existence and uniqueness of a solution for the following model system

$$
\begin{cases}
\dfrac{\partial E}{\partial t} + \chi_\Omega \left(E + E_{\text{app}} \right) + (1 - \chi_\Omega)\, J_{\text{ext}} + \dfrac{\partial H}{\partial x} = 0 \\[2mm]
\dfrac{\partial B}{\partial t} + \dfrac{\partial E}{\partial x} = 0 \qquad\qquad\qquad\qquad \text{in } \mathbb{R} \times (0, T), \qquad (1.1) \\[2mm]
H = \chi_\Omega \left(\overline{\mathcal{G}}(B) + \gamma\, \dfrac{\partial B}{\partial t} \right) + (1 - \chi_\Omega)\, B
\end{cases}
$$

where Ω is an open bounded interval of the real line, χ_Ω is the characteristic function of the set Ω, $\overline{\mathcal{G}}$ is a suitable scalar hysteresis operator, γ is a given positive constant while E_{app} and J_{ext} are known functions.

This system arises in the context of electromagnetic processes. We show in the next section how it can be obtained by coupling in a suitable way the Maxwell equations, the Ohm law and a constitutive relation between the magnetic field and the magnetic induction.

A similar equation was considered in [5] in the case of prescribed boundary conditions on $\partial\Omega$. Here we consider instead the evolution in the whole space, replacing the boundary conditions by the continuity of E and H across $\partial\Omega$.

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Some details of the physical motivation for (1.1) are contained in Section 2; in Section 3 we recall some basic fact about hysteresis and hysteresis operators; Section 4 contains the statement of the main results which will be proved finally in Sections 5 and 6.

2　Physical motivation

Consider an electromagnetic process in a ferromagnetic material which occupies a Euclidean domain $Q \subset \mathbb{R}^3$ in a time interval $(0, T)$. For more details on these topics we refer for example [6]. From now on we set

$$Q_T := Q \times (0, T) \qquad \mathbb{R}^3_T := \mathbb{R}^3 \times (0, T).$$

We suppose for simplicity that the electric displacement $\vec{D}$ is proportional to the electric field $\vec{E}$, that is $\vec{D} = \epsilon \vec{E}$, where ϵ is the electric permittivity. We assume that ϵ is a scalar constant and moreover we introduce the electric conductivity σ which is supposed to vanish outside Q. We denote by $\vec{E}_{\text{app}}$ a prescribed applied electromotive force; then the Ohm law for the electric current $\vec{J}$ is given by the following relation

$$\vec{J} = \begin{cases} \sigma(\vec{E} + \vec{E}_{\text{app}}) & \text{in } Q \times (0, T) \\[2mm] \vec{J}_{\text{ext}} & \text{in } [\mathbb{R}^3 \setminus Q] \times (0, T). \end{cases} \qquad (2.1)$$

Using the characteristic function χ_Q of the set Q, i.e. $\chi_Q = 1$ inside Q and $\chi_Q = 0$ outside Q, we rewrite (2.1) as

$$\vec{J} = \chi_Q \, \sigma \, (\vec{E} + \vec{E}_{\text{app}}) + (1 - \chi_Q) \, \vec{J}_{\text{ext}} \qquad \text{in } \mathbb{R}^3_T. \qquad (2.2)$$

Now we recall the Ampère and the Faraday laws

$$c \, \nabla \times \vec{H} = 4\pi \vec{J} + \epsilon \, \frac{\partial \vec{E}}{\partial t} \qquad \text{in } \mathbb{R}^3_T \qquad (2.3)$$

$$c \, \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{in } \mathbb{R}^3_T \qquad (2.4)$$

where $\vec{H}$ is the magnetic field, $\vec{B}$ is the magnetic induction and c is the speed of light in vacuum.

Our analysis does not depend on the exact value of the constants σ, c, ϵ, π. In order to simplify the presentation, we therefore consider the system

$$\begin{cases} \nabla \times \vec{H} = \chi_Q \, (\vec{E} + \vec{E}_{\text{app}}) + (1 - \chi_Q) \, \vec{J}_{\text{ext}} + \dfrac{\partial \vec{E}}{\partial t} \\[4mm] \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \end{cases} \qquad \text{in } \mathbb{R}^3_T. \qquad (2.5)$$

We further simplify our model system (2.5) by dealing with planar waves only. More precisely, we consider an electromagnetic wave moving in a plane which we suppose for

simplicity to be orthogonal to the $x-$axis, in a domain $Q = \{(x, y, z) : x \in \Omega, (y, z) \in \mathbb{R}^2\}$, with Ω being a bounded interval of the real line.

We assume that $\vec{E}$ is parallel to the $y-$axis, i.e.

$$\vec{E}(x, t) = (0, E(x, t), 0) \qquad (x, t) \in \mathbb{R} \times (0, T).$$

This in turn implies the following restriction on $\vec{B}$

$$\vec{B}(x, t) = (0, 0, B(x, t)) \qquad (x, t) \in \mathbb{R} \times (0, T)$$

and therefore also $\vec{H}(x, t) = (0, 0, H(x, t))$ for all $(x, t) \in \mathbb{R} \times (0, T)$. Thus we have

$$\nabla \times \vec{H} = \left(0, -\frac{\partial H}{\partial x}, 0\right) \qquad \nabla \times \vec{E} = \left(0, 0, \frac{\partial E}{\partial x}\right)$$

and (2.5) reduces to a one-dimensional problem

$$\begin{cases} \dfrac{\partial E}{\partial t} + \chi_\Omega \left(E + E_{\text{app}}\right) + (1 - \chi_\Omega) J_{\text{ext}} + \dfrac{\partial H}{\partial x} = 0 \\ \dfrac{\partial B}{\partial t} + \dfrac{\partial E}{\partial x} = 0 \end{cases} \qquad \text{in } \mathbb{R} \times (0, T), \qquad (2.6)$$

where we also set $\vec{E}_{\text{app}}(x, t) = (0, E_{\text{app}}(x, t), 0)$ and $\vec{J}_{\text{ext}}(x, t) = (0, J_{\text{ext}}(x, t), 0)$.

We couple this equation with an appropriate constitutive relation. We choose to relate B and H by means of a constitutive law with hysteresis inside Ω and to set $B = H$ outside Ω.

The constitutive law between B and H inside Ω will be chosen according to the "rheological" circuit model $F - L$ as in [19, p. 54-55], where a ferromagnetic element

$$F : B^F = H^F + 4\pi M = (I + \overline{\mathcal{W}})(H^F),$$

where M is the magnetization and $\overline{\mathcal{W}}$ is a scalar Preisach operator, is coupled in series with an induction element

$$L : H^L = \gamma \frac{\partial B^L}{\partial t}.$$

The general rheological rule for series combinations yields

$$B := B^F = B^L \qquad H := H^F + H^L,$$

where B is the total induction and H is the total field. Summing up we obtain

$$H = \chi_\Omega \left(\overline{\mathcal{G}}(B) + \gamma \frac{\partial B}{\partial t}\right) + (1 - \chi_\Omega)B, \qquad (2.7)$$

where we set $\overline{\mathcal{G}} := (I + \overline{\mathcal{W}})^{-1}$. By coupling (2.6) and (2.7) we finally obtain (1.1). The case where the influence of the induction element L is negligible, corresponds to the limit as $\gamma \to 0$.

3 Hysteresis operators

The theory of hysteresis has a long history. One hundred years ago, Madelung in [15] proposed probably the first axiomatic approach to hysteresis by defining three experimental laws of what we call nowadays return point memory hysteresis (or "wiping-out property", cf. [16]). The model for ferromagnetic hysteresis proposed by Preisach in [17] is a prominent representative that possesses the return point memory property. Only recently, Brokate and Sprekels proved (see [2, Theorem 2.7.7]) that every return point memory hysteresis operator, which admits a specific initial memory configuration, has necessarily a Preisach-type memory structure. A basic mathematical theory of hysteresis operators has been developed by M. Krasnosel'skii and his collaborators. The results of this group are summarized in the monograph [9], which constitutes until now the main source of reference on hysteresis. Our presentation here is based on more recent results from [11, 12] that are needed here, in particular the alternative one-parametric formulation of the Preisach model based on variational inequalities.

3.1 The Preisach operator

We describe the ferromagnetic behaviour using the Preisach model proposed in 1935 (see [17]). Mathematical aspects of this model were investigated by Krasnosel'skiĭ and Pokrovskiĭ (see [7], [8], and [9]). The model has been also studied in connection with partial differential equations by Visintin (see for example [18], [19]). The monograph of Mayergoyz ([16]) is mainly devoted to its modeling aspects.

Here we use the one-parametric representation of the Preisach operator which goes back to [10]. The starting point of our theory is the so called *play operator*. This operator constitutes the simplest example of continuous hysteresis operator in the space of continuous functions; it has been introduced in [9] but we can also find equivalent definitions in [2] and [19]; for its extension to less regular inputs, see also [13] and [14].

Let $r > 0$ be a given parameter. For a given input function $u \in \mathcal{C}^0([0, T])$ and initial condition $x^0 \in [-r, r]$, we define the output $\xi = \mathcal{P}_r(x^0, u) \in \mathcal{C}^0([0, T]) \cap BV(0, T)$ of the *play operator*

$$\mathcal{P}_r : [-r, r] \times \mathcal{C}^0([0, T]) \to \mathcal{C}^0([0, T]) \cap BV(0, T)$$

as the solution of the variational inequality in Stieltjes integral form

$$\begin{cases} \int_0^T (u(t) - \xi(t) - y(t)) \, d\xi(t) \geq 0 & \forall y \in \mathcal{C}^0([0, T]), \quad \max_{0 \leq t \leq T} |y(t)| \leq r, \\ |u(t) - \xi(t)| \leq r & \forall t \in [0, T], \\ \xi(0) = u(0) - x^0. \end{cases} \tag{3.1}$$

Let us consider now the whole family of play operators $\mathcal{P}_r$ parameterized by $r > 0$, which can be interpreted as a *memory variable*. Accordingly, we introduce the *hysteresis memory state space*

$$\Lambda := \{\lambda : \mathbb{R}_+ \to \mathbb{R} : |\lambda(r) - \lambda(s)| \leq |r - s| \; \forall r, s \in \mathbb{R}_+ : \lim_{r \to +\infty} \lambda(r) = 0\},$$

together with its subspaces

$$\Lambda_{\tilde{R}} = \{\lambda \in \Lambda : \lambda(r) = 0 \text{ for } r \geq \tilde{R}\}, \qquad \Lambda_\infty = \bigcup_{\tilde{R}>0} \Lambda_{\tilde{R}}. \tag{3.2}$$

For $\lambda \in \Lambda$, $u \in \mathcal{C}^0([0,T])$ and $r > 0$ we set

$$\wp_r[\lambda, u] := \mathcal{P}_r(x_r^0, u) \qquad \wp_0[\lambda, u] := u,$$

where x_r^0 is given by the formula

$$x_r^0 := \min\{r, \max\{-r, u(0) - \lambda(r)\}\}.$$

We set

$$A(r, u(0)) := u(0) - x_r^0. \tag{3.3}$$

It turns out that

$$\wp_r : \Lambda \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$$

is Lipschitz continuous in the sense that, for every $u, v \in \mathcal{C}^0([0,T])$, $\lambda, \mu \in \Lambda$ and $r > 0$ we have

$$||\wp_r[\lambda, u] - \wp_r[\mu, v]||_{\mathcal{C}^0([0,T])} \leq \max\{|\lambda(r) - \mu(r)|, ||u - v||_{\mathcal{C}^0([0,T])}\}. \tag{3.4}$$

Moreover, if $\lambda \in \Lambda_R$ and $||u||_{\mathcal{C}^0([0,T])} \leq R$, then $\wp_r[\lambda, u](t) = 0$ for all $r \geq R$ and $t \in [0,T]$. For more details, see Sections II.3, II.4 of [11].

Now we introduce the *Preisach plane* as follows

$$\mathscr{P} := \{(r,v) \in \mathbb{R}^2 : r > 0\}$$

and consider a function $\varphi \in L^1_{\text{loc}}(\mathscr{P})$ such that there exists $\beta_1 \in L^1_{\text{loc}}(0,\infty)$ with

$$0 \leq \varphi(r,v) \leq \beta_1(r) \qquad \text{for a.e. } (r,v) \in \mathscr{P}.$$

We set

$$g(r,v) := \int_0^v \varphi(r,z)\,dz \qquad \text{for } (r,v) \in \mathscr{P}$$

and for $R > 0$, we put $b_1(R) := \int_0^R \beta_1(r)\,dr$.

Then the *Preisach operator*

$$\mathcal{W} : \Lambda_\infty \times \mathcal{C}^0([0,T]) \to \mathcal{C}^0([0,T])$$

generated by the function g is defined by the formula

$$\mathcal{W}[\lambda, u](t) := \int_0^\infty g(r, \wp_r[\lambda, u](t))\,dr, \tag{3.5}$$

for any given $\lambda \in \Lambda_\infty$, $u \in \mathcal{C}^0([0,T])$ and $t \in [0,T]$. The equivalence of this definition and the classical one in [16], [19], e.g., is proved in [10].

The function $A(r, \cdot)$ in (3.3) is nondecreasing and Lipschitz continuous. Hence, the mapping

$$\mathscr{A} : \mathbb{R} \to \mathbb{R} : \qquad u(0) \mapsto \int_0^\infty g(r, A(r, u(0))) \, dr \qquad (3.6)$$

which with the initial input value $u(0)$ associates the initial output value $\mathcal{W}[\lambda, u](0)$ is nondecreasing and Lipschitz continuous as well.

Using the Lipschitz continuity (3.4) of the operator $\wp_r$, it is easy to prove that also $\mathcal{W}$ is locally Lipschitz continuous, in the sense that, for any given $R > 0$, for every $\lambda, \mu \in \Lambda_R$ and $u, v \in \mathcal{C}^0([0, T])$ with $||u||_{\mathcal{C}^0([0,T])}, ||v||_{\mathcal{C}^0([0,T])} \leq R$, we have

$$||\mathcal{W}[\lambda, u] - \mathcal{W}[\mu, v]||_{\mathcal{C}^0([0,T])} \leq \int_0^R |\lambda(r) - \mu(r)| \, \beta_1(r) \, dr + b_1(R) \, ||u - v||_{\mathcal{C}^0([0,T])}.$$

The first result on the inverse Preisach operator was proved in [3]. We make use of the following formulation proved in [11], Section II.3.

Theorem 3.1. *Let* $\lambda \in \Lambda_\infty$ *and* $b > 0$ *be given. Then the operator* $b \, I + \mathcal{W}[\lambda, \cdot] : \mathcal{C}^0([0, T]) \to \mathcal{C}^0([0, T])$ *is invertible and its inverse is Lipschitz continuous. In particular the initial value mapping* $b \, I + \mathscr{A}$ *(see* (3.6)*) is increasing and bi-Lipschitz.*

As we are dealing with partial differential equations, we should consider both the input and the initial memory configuration λ that additionally depend on x. If for instance $\lambda(x, \cdot)$ belongs to Λ_∞ and $u(x, \cdot)$ belongs to $\mathcal{C}^0([0, T])$ for (almost) every x, then we define

$$\overline{\mathcal{W}}[\lambda, u](x, t) := \mathcal{W}[\lambda(x, \cdot), u(x, \cdot)](t) := \int_0^\infty g(r, \wp_r[\lambda(x, \cdot), u(x, \cdot)](t)) \, dr.$$

4   Statement of the main results

Let Ω be an open bounded interval of $\mathbb{R}$ and set $\Omega_T := \Omega \times (0, T)$; let us fix an initial memory configuration

$$\lambda \in L^2(\Omega; \Lambda_{\tilde{R}}) \qquad \text{for some } \tilde{R} > 0, \qquad (4.1)$$

where $\Lambda_{\tilde{R}}$ is introduced in (3.2).

Let $\mathcal{M}(\Omega; \mathcal{C}^0([0, T]))$ be the Fréchet space of strongly measurable functions $\Omega \to \mathcal{C}^0([0, T])$, i.e. the space of functions $v : \Omega \to \mathcal{C}^0([0, T])$ such that there exists a sequence v_n of simple functions with $v_n \to v$ in $\mathcal{C}^0([0, T])$ a.e. in Ω.

We fix a constant $b_\mathcal{F} > 0$ and introduce the operator $\overline{\mathcal{F}} : \mathcal{M}(\Omega; \mathcal{C}^0([0, T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0, T]))$ in the following way

$$\overline{\mathcal{F}}(u)(x, t) := \mathcal{F}(u(x, \cdot))(t) := b_\mathcal{F} \, u(x, t) + \mathcal{W}[\lambda(x, \cdot), u(x, \cdot)](t); \qquad (4.2)$$

here $\mathcal{W}$ is the scalar Preisach operator defined in (3.5).

Now Theorem 3.1 yields that $\mathcal{F}$ is invertible and its inverse is a Lipschitz continuous operator in $\mathcal{C}^0([0,T])$. Let us set $\mathcal{G} = \mathcal{F}^{-1}$ and let $L_{\mathcal{G}}$ be the Lipschitz constant of the operator $\mathcal{G}$.

At this point we introduce the operator

$$\overline{\mathcal{G}} : \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \to \mathcal{M}(\Omega; \mathcal{C}^0([0,T])) \qquad \overline{\mathcal{G}} := \overline{\mathcal{F}}^{-1}. \tag{4.3}$$

It turns out that

$$\overline{\mathcal{G}}(w)(x,t) := \mathcal{G}(w(x,\cdot))(t) \qquad \forall\, w \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])); \tag{4.4}$$

it follows from Theorem 3.1 that $\overline{\mathcal{G}}$ is Lipschitz continuous in the following sense

$$||\overline{\mathcal{G}}(u_1)(x,\cdot) - \overline{\mathcal{G}}(u_2)(x,\cdot)||_{\mathcal{C}^0([0,T])} \leq L_{\mathcal{G}} \, ||u_1(x,\cdot) - u_2(x,\cdot)||_{\mathcal{C}^0([0,T])}$$
$$\text{for any } u_1, u_2 \in \mathcal{M}(\Omega; \mathcal{C}^0([0,T])), \text{ a.e. in } \Omega. \tag{4.5}$$

For the sake of definiteness, we assume $\Omega = (-1,1)$, fix some $r > 1$, and set $K = (-r,r)$. The initial conditions for Problem (2.6)

$$E(x,0) := E_0(x) \qquad \text{a.e. in } \mathbb{R},$$
$$B(x,0) := B_0(x) \qquad \text{a.e. in } \mathbb{R}, \tag{4.6}$$

are assumed in the form

$$E_0(x) := \chi_\Omega E_0^1(x) + (1 - \chi_\Omega)\, E_0^2(x),$$
$$B_0(x) := \chi_\Omega B_0^1(x) + (1 - \chi_\Omega)\, B_0^2(x),$$

as in the following we will assume different regularity for the initial data inside and outside Ω.

The full P.D.E.s system for unknown functions E, B and H reads as follows

$$\begin{cases} \dfrac{\partial E}{\partial t} + \chi_\Omega\,(E + E_{\text{app}}) + (1 - \chi_\Omega)\, J_{\text{ext}} + \dfrac{\partial H}{\partial x} = 0 \\[2mm] \dfrac{\partial B}{\partial t} + \dfrac{\partial E}{\partial x} = 0 \qquad\qquad\qquad\qquad\quad \text{a.e. in } \mathbb{R} \times (0,T). \\[2mm] H = \chi_\Omega\left(\overline{\mathcal{G}}(B) + \gamma\,\dfrac{\partial B}{\partial t}\right) + (1 - \chi_\Omega)\, B \end{cases} \tag{4.7}$$

with initial conditions (4.6), where E_{app} and J_{ext} are given functions.

We first distinguish the case in which the data have compact support; in Section 5 we thus prove the following existence result.

Theorem 4.1. *Consider the following assumptions on the initial data*

$$E_0^1 \in H^2(\Omega), \quad B_0^1 \in H^1(\Omega)$$
$$\operatorname{supp} E_0^2 \subset K \setminus \overline{\Omega}, \quad \operatorname{supp} B_0^2 \subset K \setminus \overline{\Omega}, \quad E_0^2, B_0^2 \in H^1(K \setminus \overline{\Omega}) \tag{4.8}$$

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together with the following compatibility conditions

$$
\begin{cases}
\left(\mathscr{A}(B_0^1) - \gamma \dfrac{\partial E_0^1}{\partial x} \right)(-1^+) = B_0^2(-1^-), & \text{(4.9a)} \\[3mm]
\left(\mathscr{A}(B_0^1) - \gamma \dfrac{\partial E_0^1}{\partial x} \right)(1^-) = B_0^2(1^+), & \text{(4.9b)} \\[3mm]
E_0^1(-1^+) = E_0^2(-1^-), \qquad E_0^1(1^-) = E_0^2(1^+), & \text{(4.9c)}
\end{cases}
$$

where $\mathscr{A}$ is the initial value mapping associated with $\overline{\mathcal{G}}$.

Moreover assume that

$$
E_{\mathrm{app}} \in H^1(0, T; L^2(\Omega)), \qquad J_{\mathrm{ext}} \in H^1(0, T; L^2(K)).
$$

Then Problem (4.7) has a unique solution

$$
\begin{aligned}
& E \in W^{1,\infty}(0, T; L^2(\mathbb{R})) \cap L^2(0, T; H^1(\mathbb{R})) \\
& B_{|\Omega} \in H^2(0, T; L^2(\Omega)) \\
& B_{|\mathbb{R}\setminus\overline{\Omega}} \in W^{1,\infty}(0, T; L^2(\mathbb{R}\setminus\overline{\Omega})) \\
& H \in H^1(0, T; L^2(\mathbb{R})) \cap L^\infty(0, T; H^1(\mathbb{R})).
\end{aligned}
\tag{4.10}
$$

Finally in Section 6 we deal with the case of more general data, not necessarily with compact support, and the result we are able to prove is the following.

Theorem 4.2. *Consider the following assumptions on the initial data*

$$
\begin{aligned}
E_0^1 &\in H^2(\Omega), & B_0^1 &\in H^1(\Omega) \\
E_0^2 &\in H^1_{\mathrm{loc}}(\mathbb{R}\setminus\overline{\Omega}), & B_0^2 &\in H^1_{\mathrm{loc}}(\mathbb{R}\setminus\overline{\Omega}),
\end{aligned}
$$

together with the compatibility conditions (4.9a), (4.9b) and (4.9c); moreover assume that

$$
E_{\mathrm{app}} \in H^1(0, T; L^2(\Omega)), \qquad J_{\mathrm{ext}} \in H^1(0, T; L^2_{\mathrm{loc}}(\mathbb{R})).
$$

Then Problem (4.7) has a unique solution such that

$$
\begin{aligned}
& E \in W^{1,\infty}(0, T; L^2_{\mathrm{loc}}(\mathbb{R})) \cap L^2(0, T; H^1_{\mathrm{loc}}(\mathbb{R})) \\
& B_{|\Omega} \in H^2(0, T; L^2(\Omega)) \\
& B_{|\mathbb{R}\setminus\overline{\Omega}} \in W^{1,\infty}(0, T; L^2_{\mathrm{loc}}(\mathbb{R}\setminus\overline{\Omega})) \\
& H \in H^1(0, T; L^2_{\mathrm{loc}}(\mathbb{R})) \cap L^\infty(0, T; H^1_{\mathrm{loc}}(\mathbb{R})).
\end{aligned}
\tag{4.11}
$$

5 Proof of Theorem 4.1

The main idea in proving Theorem 4.1 is to use a space discretization scheme together with a fixed point argument.

Let us fix $R > r + T$; for simplicity (this will be useful in the space discretization procedure) we take $R \in \mathbb{N}$. We prescribe the following "boundary conditions"

$$H(-R, t) = 0, \qquad E(R, t) = 0. \tag{5.1}$$

• STEP 1: FREEZING. First of all we fix some $Z \in H^1(0, T; L^2(\Omega))$ with $Z(0, x) = B_0^1(x)$ a.e. in Ω, and consider (4.7) with $\overline{\mathcal{G}}(B)$ replaced by $\overline{\mathcal{G}}(Z)$, i.e. we look for three functions E, B, H with the regularity outlined in (4.10) and initial conditions (4.6) such that the following holds

$$\begin{cases} \dfrac{\partial E}{\partial t} + \chi_\Omega \left(E + E_{\text{app}} \right) + (1 - \chi_\Omega) \, J_{\text{ext}} + \dfrac{\partial H}{\partial x} = 0 \\[2mm] \dfrac{\partial B}{\partial t} + \dfrac{\partial E}{\partial x} = 0 \qquad\qquad\qquad \text{a.e. in } \mathbb{R} \times (0, T). \\[2mm] H = \chi_\Omega \left(\overline{\mathcal{G}}(Z) + \gamma \, \dfrac{\partial B}{\partial t} \right) + (1 - \chi_\Omega) \, B \end{cases} \tag{5.2}$$

• STEP 2: EXISTENCE OF SOLUTIONS FOR (5.2): SPACE DISCRETIZATION SCHEME.

We now fix $n \in \mathbb{N}$ and consider the equidistant partition of the interval $[-R, R]$

$$x_k := \frac{k}{n}, \qquad k = -Rn, -Rn + 1, \dots, Rn.$$

The characteristic function χ_Ω reduces to the following

$$\chi_k = \begin{cases} 1 & k \in \{-n + 1, \dots, n\} \\ 0 & \text{else.} \end{cases}$$

We introduce the function

$$F(x, t) := \chi_\Omega \, E_{\text{app}}(x, t) + (1 - \chi_\Omega) \, J_{\text{ext}}(x, t).$$

It turns out that

$$F \in H^1(0, T; L^2(\mathbb{R})) \subset L^2(\mathbb{R}; \mathcal{C}^0([0, T])). \tag{5.3}$$

Now we set

$$F_k(t) = n \int_{(k-1)/n}^{k/n} F(x, t) \, dx \qquad\qquad k = -Rn + 1, \dots, Rn,$$

$$G_k(t) = n \int_{(k-1)/n}^{k/n} \overline{\mathcal{G}}(Z)(x, t) \, dx, \qquad k = -n + 2, \dots, n$$

and also

$$\mathcal{D}_n := \{-Rn + 1, \dots, -n\} \cup \{n + 1, \dots, Rn - 1\}.$$

We now approximate (5.2) by a system of O.D.E.s, where the dot indicates the time derivative.

Our aim is to find unknown functions E_k, B_k, H_k such that the following holds, for $k = -Rn + 1, \ldots, Rn - 1$

$$
\begin{cases}
\dot{E}_k + \chi_k\, E_k + n\left(H_k - H_{k-1}\right) + F_k = 0 & \text{(5.4a)} \\[2ex]
\dot{B}_k + n\left(E_{k+1} - E_k\right) = 0 & \text{(5.4b)} \\[2ex]
H_k = \chi_k\left[G_k + \gamma \dot{B}_k\right] + (1 - \chi_k)B_k = 0. & \text{(5.4c)}
\end{cases}
$$

This is coupled with the following boundary conditions

$$
H_{-Rn}(t) = 0, \qquad E_{Rn}(t) = 0 \tag{5.5}
$$

and the following initial conditions

$$
\begin{aligned}
E_k(0) &= E_0^k := E_0\left(\frac{k-1}{n}\right) \\
B_k(0) &= B_0^k := n \int_{(k-1)/n}^{k/n} B_0(x)\,dx.
\end{aligned}
\tag{5.6}
$$

We choose the averages for $B_k(0)$ in order to avoid difficulties related to the fact that B_0 may be discontinuous.

Eliminating H_k, H_{k-1} from (5.4a), we rewrite (5.4a)-(5.4b) as a system of $2(2Rn - 1)$ equations and $2(2Rn - 1)$ unknown functions

$$
\dot{V} = \Phi V + \tilde{F} \qquad V = (E_{-Rn+1}, \ldots, E_{Rn-1}, B_{-Rn+1}, \ldots, B_{Rn-1}),
$$

where Φ is a matrix and $\tilde{F} \in W^{1,2}(0, T; \mathbb{R}^{2(2Rn-1)})$. This is enough to conclude that the system, coupled with (5.5) and (5.6) admits a unique solution.

In the following, for the sake of simplicity, we denote by $C_1, C_2, \ldots$ any constant depending possibly on the data but independent on the discretization parameter n.

We now differentiate (5.4a), (5.4b) and (5.4c) in time, getting

$$
\begin{cases}
\ddot{E}_k + \chi_k\, \dot{E}_k + n\left(\dot{H}_k - \dot{H}_{k-1}\right) + \dot{F}_k = 0 & \text{(5.7a)} \\[2ex]
\ddot{B}_k + n\left(\dot{E}_{k+1} - \dot{E}_k\right) = 0 & \text{(5.7b)} \\[2ex]
\dot{H}_k = \chi_k\left[\dot{G}_k + \gamma \ddot{B}_k\right] + (1 - \chi_k)\dot{B}_k = 0. & \text{(5.7c)}
\end{cases}
$$

Now we test (5.7a) by $\dot{E}_k$ and (5.7b) by $\dot{H}_k$, sum the result and divide by n. We have

$$
\frac{1}{2n}\frac{d}{dt}\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k|^2 + \frac{1}{n}\sum_{k=-n+1}^{n} |\dot{E}_k|^2 + \sum_{k=-Rn+1}^{Rn-1} \left(\dot{H}_k - \dot{H}_{k-1}\right)\dot{E}_k
$$

$$
+ \frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} \dot{F}_k\, \dot{E}_k + \frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} \ddot{B}_k\, \dot{H}_k + \sum_{k=-Rn+1}^{Rn-1} \left(\dot{E}_{k+1} - \dot{E}_k\right)\dot{H}_k = 0.
$$

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We remark that

$$\sum_{k=-Rn+1}^{Rn-1} [(\dot{H}_k - \dot{H}_{k-1})\,\dot{E}_k + (\dot{E}_{k+1} - \dot{E}_k)\,\dot{H}_k]$$

$$= \sum_{k=-Rn+1}^{Rn-1} [\dot{E}_{k+1}\,\dot{H}_k - \dot{H}_{k-1}\,\dot{E}_k] = [\dot{E}_{Rn}\,\dot{H}_{Rn-1} - \dot{H}_{-Rn}\,\dot{E}_{-Rn+1}] \overset{(5.5)}{=} 0.$$

Therefore, using (5.7c), we deduce

$$\frac{1}{2n}\frac{d}{dt}\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k|^2 + \frac{1}{n}\sum_{k=-n+1}^{n} |\dot{E}_k|^2 + \frac{\gamma}{n}\sum_{k=-n+1}^{n} |\ddot{B}_k|^2 + \frac{1}{2n}\frac{d}{dt}\sum_{\mathcal{D}_n} |\dot{B}_k|^2$$

$$\leq \frac{1}{n}\sum_{k=-n+1}^{n} \ddot{B}_k\,\dot{G}_k - \frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} \dot{F}_k\,\dot{E}_k \leq \frac{\gamma}{2n}\sum_{k=-n+1}^{n} |\ddot{B}_k|^2$$

$$+ \frac{1}{2\gamma n}\sum_{k=-n+1}^{n} |\dot{G}_k|^2 + \frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} |\dot{F}_k|^2 + \frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k|^2.$$

Now, using (5.3) we deduce in particular that

$$\frac{1}{2n}\frac{d}{dt}\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k|^2 + \frac{1}{2n}\frac{d}{dt}\sum_{\mathcal{D}_n} |\dot{B}_k|^2$$

$$\leq \int_{-R}^{R} F^2(x,t)\,dx + \frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k|^2 + \frac{1}{n}\sum_{\mathcal{D}_n} |\dot{B}_k|^2.$$

The Gronwall lemma then yields

$$\frac{1}{n}\left(\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k|^2 + \sum_{\mathcal{D}_n} |\dot{B}_k|^2\right) \leq \frac{C_1}{n}\left[\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k(0)|^2 + \sum_{\mathcal{D}_n} |\dot{B}_k(0)|^2\right] + C_2.$$

We have now to show that, due to our assumptions on the data, the term

$$\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k(0)|^2 + \frac{1}{n}\sum_{\mathcal{D}_n} |\dot{B}_k(0)|^2$$

can be controlled by a constant independent of the discretization parameter n. First of all, by comparison and using (5.4a) we have that

$$\frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k(0)|^2 \leq \frac{1}{n}\sum_{k=-n+1}^{n} |E_k(0)|^2 + n\sum_{k=-Rn+1}^{Rn-1} |H_k(0) - H_{k-1}(0)|^2$$

$$+ \frac{1}{n}\sum_{k=-Rn+1}^{Rn-1} |F_k(0)|^2.$$

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As $E_0^1 \in H^2(\Omega)$, the first term on the right-hand side of the previous inequality can be controlled while the third one is controlled due to (5.3).

To estimate the second term, we first notice that for $k \in \mathcal{D}_n$, we have

$$H_k(0) = B_k(0),$$

while for $k' \in \{-n+1, \ldots, n\}$, equation (5.4c) yields

$$H_{k'}(0) = G_{k'}(0) + \gamma \dot{B}_{k'}(0) = G_{k'}(0) - \gamma \, n \, (E_{k'+1}(0) - E_{k'}(0)).$$

We have by (3.3) for $x \in \Omega$ that

$$\overline{\mathcal{G}}(Z)(x, 0) = \mathscr{A}(B_0^1(x)),$$

hence

$$G_{k'}(0) = n \int_{(k'-1)/n}^{k'/n} \mathscr{A}(B_0^1(x)) \, dx \qquad k' \in \{-n+2, \ldots, n\}.$$

This yields

$$n \sum_{k=-Rn+1}^{Rn-1} |H_k(0) - H_{k-1}(0)|^2 = n \sum_{k=-Rn+1}^{-n} |B_k(0) - B_{k-1}(0)|^2 + n \, |H_{-n+1}(0) - H_{-n}(0)|^2$$

$$+ n \sum_{k=-n+2}^{n} |G_k(0) - G_{k-1}(0) - \gamma \, n \, (E_{k+1}(0) - 2 \, E_k(0) + E_{k-1}(0))|^2$$

$$+ n \, |H_{n+1}(0) - H_n(0)|^2 + n \sum_{k=n+2}^{Rn-1} |B_k(0) - B_{k-1}(0)|^2.$$

Two terms deserve special attention. First

$$H_{-n+1}(0) - H_{-n}(0) = G_{-n+1}(0) - \gamma \, n \, (E_{-n+2}(0) - E_{-n+1}(0)) - B_{-n}(0)$$

$$= n \int_{-1}^{-1+\frac{1}{n}} \mathscr{A}(B_0^1(x)) \, dx - \gamma \, n \, \left(E_0^1 \left(-1 + \frac{1}{n} \right) - E_0^1(-1) \right) - n \int_{-1-\frac{1}{n}}^{-1} B_0^2(x) \, dx$$

$$= n \int_{-1}^{-1+\frac{1}{n}} \mathscr{A}(B_0^1(x)) \, dx - \gamma \, n \int_{-1}^{-1+\frac{1}{n}} \frac{\partial E_0^1}{\partial \xi}(\xi) \, d\xi - n \int_{-1-\frac{1}{n}}^{-1} B_0^2(x) \, dx$$

$$= n \int_{-1}^{-1+\frac{1}{n}} [\mathscr{A}(B_0^1(x)) - \mathscr{A}(B_0^1(-1^+))] \, dx - \gamma \, n \int_{-1}^{-1+\frac{1}{n}} \left[\frac{\partial E_0^1}{\partial \xi}(\xi) - \frac{\partial E_0^1}{\partial \xi}(-1^+) \right] \, d\xi$$

$$- n \int_{-1-\frac{1}{n}}^{-1} [B_0^2(x) - B_0^2(-1^-)] \, dx + \mathscr{A}(B_0^1(-1^+)) - \gamma \frac{\partial E_0^1}{\partial \xi}(-1^+) - B_0^2(-1^-).$$

Using the compatibility condition (4.9a), we obtain

$$|H_{-n+1}(0) - H_{-n}(0)| \leq n \int_{-1}^{-1+\frac{1}{n}} |\mathscr{A}(B_0^1(x)) - \mathscr{A}(B_0^1(-1^+))| \, dx$$

$$+ \gamma \, n \int_{-1}^{-1+\frac{1}{n}} \left| \frac{\partial E_0^1}{\partial \xi}(\xi) - \frac{\partial E_0^1}{\partial \xi}(-1^+) \right| \, d\xi \, dx + n \int_{-1-\frac{1}{n}}^{-1} |B_0^2(x) - B_0^2(-1^-)| \, dx.$$

The initial value mapping $\mathscr{A}$ is Lipschitz continuous. Hence,

$$n \int_{-1}^{-1+\frac{1}{n}} |\mathscr{A}(B_0^1(x)) - \mathscr{A}(B_0^1(-1^+))|\, dx \leq C_3\, n \int_{-1}^{-1+\frac{1}{n}} \int_{-1}^{x} \left| \frac{\partial B_0^1}{\partial \xi}(\xi) \right| d\xi\, dx$$

$$= C_3\, n \int_{-1}^{-1+\frac{1}{n}} \left| \frac{\partial B_0^1}{\partial \xi}(\xi) \right| \left(\int_{\xi}^{-1+\frac{1}{n}} dx \right) d\xi$$

$$\leq C_3 \int_{-1}^{-1+\frac{1}{n}} \left| \frac{\partial B_0^1}{\partial \xi}(\xi) \right| d\xi \leq \frac{C_3}{\sqrt{n}} \left(\int_{-1}^{-1+\frac{1}{n}} \left| \frac{\partial B_0^1}{\partial x} \right|^2 dx \right)^{1/2}.$$

We can similarly estimate the term $n\, |H_{n+1}(0) - H_n(0)|^2$, using this time the compatibility condition (4.9b).

The remaining terms can be estimated in a more standard way using the regularity of the data (4.8).

On the other hand, by comparison, using (5.4b)

$$\frac{1}{n} \sum_{\mathcal{D}_n} |\dot{B}_k(0)|^2 \leq n \sum_{\mathcal{D}_n} |E_{k+1}(0) - E_k(0)|^2$$

and this can be controlled using the fact that $E_0^2 \in H^1(K)$. Therefore, summing up we deduce

$$\max_{0 \leq t \leq T} \left[\frac{1}{n} \sum_{k=-Rn+1}^{Rn-1} |\dot{E}_k(t)|^2 + \frac{1}{n} \sum_{\mathcal{D}_n} |\dot{B}_k(t)|^2 \right]$$
$$+ \int_0^T \left(\frac{1}{n} \sum_{k=-n+1}^{n} |\dot{E}_k(t)|^2 \right) dt + \frac{\gamma}{n} \int_0^T \left(\sum_{k=-n+1}^{n} |\ddot{B}_k(t)|^2 \right) dt \leq C_4. \tag{5.8}$$

At this point, with the intention to let n tend to ∞, we define the following interpolates

$$E^{(n)}(x,t) = E_{k-1}(x) + n\left(x - \frac{(k-1)}{n} \right) [E_k(x) - E_{k-1}(x)]$$
$$H^{(n)}(x,t) = H_{k-1}(x) + n\left(x - \frac{(k-1)}{n} \right) [H_k(x) - H_{k-1}(x)]$$
$$\bar{E}^{(n)}(x,t) = E_k(t)$$
$$\bar{H}^{(n)}(x,t) = H_k(t)$$
$$\bar{B}^{(n)}(x,t) = B_k(t)$$
$$\bar{G}^{(n)}(x,t) = G_k(t)$$
$$\bar{\chi}^{(n)}(x) = \chi_k$$
$$\bar{F}^{(n)}(x,t) = F_k(t)$$

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for $x \in ((k-1)n, kn]$, $k = -Rn+1, \ldots, Rn$ and $t \in [0, T]$. Therefore Problem (5.2) can be rewritten as

$$\begin{cases} \dfrac{\partial \bar{E}^{(n)}}{\partial t} + \bar{\chi}^{(n)} \bar{E}^{(n)} + \dfrac{\partial H^{(n)}}{\partial x} + \bar{F}^{(n)} = 0 & \text{(5.9a)} \\[2ex] \dfrac{\partial E^{(n)}}{\partial x} + \dfrac{\partial \bar{B}^{(n)}}{\partial t} = 0 & \text{(5.9b)} \\[2ex] \bar{H}^{(n)} = \bar{\chi}^{(n)} \left(\bar{G}^{(n)} + \gamma \dfrac{\partial \bar{B}^{(n)}}{\partial t} \right) + (1 - \bar{\chi}^{(n)}) \bar{B}^{(n)}. & \text{(5.9c)} \end{cases}$$

The a priori estimate (5.8) gives

$$\max_{0 \le t \le T} \left(\left\| \frac{\partial \bar{E}^{(n)}}{\partial t}(t) \right\|^2_{L^2(\mathbb{R})} + \left\| \frac{\partial \bar{B}^{(n)}}{\partial t}(t) \right\|^2_{L^2(\mathbb{R} \setminus \Omega)} \right) + \left\| \frac{\partial^2 \bar{B}^{(n)}}{\partial t^2}(t) \right\|^2_{L^2(0,T;L^2(\Omega))} \le C_4. \tag{5.10}$$

At this point, by comparison, (5.3), (5.9a) and (5.10) give

$$\max_{0 \le t \le T} \left\| \frac{\partial H^{(n)}}{\partial x}(t) \right\|^2_{L^2(\mathbb{R})} \le C_5.$$

On the other hand (5.10) entails

$$\left\| \frac{\partial \bar{B}^{(n)}}{\partial t}(t) \right\|^2_{L^2(0,T;L^2(\mathbb{R}))} \le C_6,$$

and, by comparison, using (5.9b) we deduce

$$\left\| \frac{\partial E^{(n)}}{\partial x}(t) \right\|^2_{L^2(0,T;L^2(\mathbb{R}))} \le C_7.$$

Combining the above estimates and possible selecting a suitable subsequence of $n \to \infty$, we find that there exist functions E, B, H in the appropriate Sobolev spaces such that the following convergences take place:

$$\left. \frac{\partial \bar{E}^{(n)}}{\partial t} \to \frac{\partial E}{\partial t}, \quad \frac{\partial H^{(n)}}{\partial x} \to \frac{\partial H}{\partial x} \right\} \quad \text{weakly-star in } L^\infty(0, T; L^2(\mathbb{R})),$$

$$\left. \frac{\partial^2 \bar{B}^{(n)}}{\partial t^2} \to \frac{\partial^2 B}{\partial t^2} \right\} \quad \text{weakly in } L^2(0, T; L^2(\Omega)),$$

$$\left. \frac{\partial \bar{B}^{(n)}}{\partial t} \to \frac{\partial B}{\partial t}, \quad \frac{\partial E^{(n)}}{\partial x} \to \frac{\partial E}{\partial x} \right\} \quad \text{weakly in } L^2(0, T; L^2(\mathbb{R})).$$

Therefore we can pass to the limit in (5.9a)-(5.9c) to see that (5.2) is satisfied a.e. in $\mathbb{R} \times (0, T)$ and the solution has the regularity outlined in (4.10).
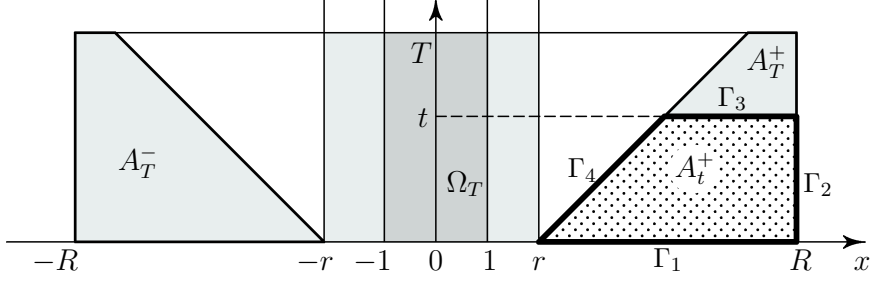
• STEP 3: FINITE SPEED OF PROPAGATION.

Figure 1: Finite speed of propagation outside Ω.

In order to find solutions to (5.2), we used a space discretization scheme; by doing that, we actually solved a boundary value problem in the domain $(-R, R) \times (0, T)$. To complete our existence result we have actually to show that the solution we found solves also our original model problem.

Our argument is illustrated on Figure 1. We define the sets

$$A_T^+ = \{(x, t) \in \mathbb{R} \times (0, T); \ t \in (0, T), \ r + t < x < R\}$$

$$A_T^- = \{(x, t) \in \mathbb{R} \times (0, T); \ t \in (0, T), \ -R < x < -r - t\}$$

with the intention to prove that $E \equiv H \equiv 0$ in $A_T^+ \cup A_T^-$. In A_T^+, E and H are solutions of the linear wave equation

$$\begin{cases} \dfrac{\partial E}{\partial t} + \dfrac{\partial H}{\partial x} = 0 \\ \dfrac{\partial H}{\partial t} + \dfrac{\partial E}{\partial x} = 0 \end{cases} \tag{5.11}$$

hence satisfy the energy balance

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\left(E^2 + H^2\right)\right) + \frac{\partial}{\partial x}(E\,H) = 0. \tag{5.12}$$

We now proceed as in [4] and integrate (5.12) over the set $A_t^+ = A_T^+ \cap (\mathbb{R} \times (0, t))$ with some fixed $t \in (0, T)$. Using Green's formula, we obtain

$$\int_{A_t^+} \frac{\partial}{\partial t}\left(\frac{1}{2}\left(E^2 + H^2\right)\right) + \frac{\partial}{\partial x}(EH)\, dx\, dt = \int_{\partial A_t^+}\left(E\,H\,n_1 + \frac{1}{2}\left(E^2 + H^2\right)n_2\right)\, ds, \tag{5.13}$$

where $\vec{n} = (n_1, n_2)$ is the unit outward normal vector to ∂A_t^+. The boundary of A_t^+ consists of four parts: $\partial A_t^+ = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$, with $\vec{n} = (0, -1)$ on Γ_1, $\vec{n} = (1, 0)$

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on Γ_2, $\vec{n} = (0, 1)$ on Γ_3, and $\vec{n} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ on Γ_4. We have

$$\int_{\Gamma_1} \left(E\,H\,n_1 + \frac{1}{2}\left(E^2 + H^2\right)n_2\right) ds = -\int_r^R \frac{1}{2}\left[(E_0^2)^2(x) + (B_0^2)^2(x)\right] dx = 0$$

$$\int_{\Gamma_2} \left(E\,H\,n_1 + \frac{1}{2}\left(E^2 + H^2\right)n_2\right) ds = \int_0^t E(R, \tau)\,H(R, \tau)\,d\tau = 0$$

$$\int_{\Gamma_3} \left(E\,H\,n_1 + \frac{1}{2}\left(E^2 + H^2\right)n_2\right) ds = \int_{r+t}^R \frac{1}{2}\left(E^2(x, t) + H^2(x, t)\right) dx \geq 0$$

$$\int_{\Gamma_4} \left(E\,H\,n_1 + \frac{1}{2}\left(E^2 + H^2\right)n_2\right) ds = \frac{1}{2\sqrt{2}}\int_0^t (E - H)^2(r + \tau, \tau)\,d\tau \geq 0.$$

Comparing these identities with (5.12) and (5.13), we see that $E = H = 0$ on A_t^+. As $t \in (0, T)$ has been arbitrary, we obtain the desired result $E = H = 0$ in A_T^+.

The argument in A_T^- is fully analogous.

● STEP 4: FIXED POINT.

Let us come back to our original problem. For any $Z \in H^1(0, T; L^2(\Omega))$ we found a unique solution (E, B, H) to (5.2) with the regularity outlined in (4.10). Thus we can introduce the following closed subspace S of $H^1(0, T; L^2(\Omega))$

$$S := \{Z \in H^1(0, T; L^2(\Omega)) : Z(0, x) = B_0^1(x)\},$$

and an operator $J : S \to S$ which associates with every fixed datum $Z \in S$ the quantity $B \in S$ of the corresponding solution (E, B, H). At this point we take two different data $Z_1, Z_2 \in S$ and consider the corresponding solutions to (5.2), (E_1, B_1, H_1) and (E_2, B_2, H_2), respectively associated with Z_1 and Z_2. We have $B_1 := J(Z_1)$ and $B_2 := J(Z_2)$ and moreover we obtain

$$\begin{cases} \dfrac{\partial}{\partial t}(E_1 - E_2) + \chi_\Omega\,(E_1 - E_2) + \dfrac{\partial}{\partial x}(H_1 - H_2) = 0 & \text{(5.14a)} \\[2mm] \dfrac{\partial}{\partial t}(B_1 - B_2) + \dfrac{\partial}{\partial x}(E_1 - E_2) = 0 & \text{(5.14b)} \\[2mm] H_1 - H_2 = \chi_\Omega\left[\overline{\mathcal{G}}(Z_1) - \overline{\mathcal{G}}(Z_2) + \gamma\dfrac{\partial}{\partial t}(B_1 - B_2)\right] + (1 - \chi_\Omega)\,(B_1 - B_2). & \text{(5.14c)} \end{cases}$$

We test (5.14a) by $(E_1 - E_2)$ and (5.14b) by $(H_1 - H_2)$ and then sum the result; taking into account that the terms

$$\int_{\mathbb{R}} \frac{\partial}{\partial x}(H_1 - H_2)\,(E_1 - E_2)\,dx \qquad \int_{\mathbb{R}} \frac{\partial}{\partial x}(E_1 - E_2)\,(H_1 - H_2)\,dx$$

cancel out, we deduce, using (5.14c)

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} |E_1 - E_2|^2\,dx + \int_\Omega |E_1 - E_2|^2\,dx + \frac{3}{4}\gamma\int_\Omega \left|\frac{\partial}{\partial t}(B_1 - B_2)\right|^2 dx$$

$$+ \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}\backslash\Omega} |B_1 - B_2|^2\,dx \leq \frac{1}{\gamma}\int_\Omega |\overline{\mathcal{G}}(Z_1) - \overline{\mathcal{G}}(Z_2)|^2\,dx.$$

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We remark that this last estimate together with the causality of the operator $\overline{\mathcal{G}}$ entails that (5.2) admits a unique solution, for any fixed $Z \in H^1(0, T; L^2(\Omega))$.

We now set

$$D(t) := \frac{1}{2} \left[\int_{\mathbb{R}} |E_1 - E_2|^2 \, dx + \int_{\mathbb{R} \setminus \Omega} |B_1 - B_2|^2 \, dx \right]$$

and we remark that $D(0) = 0$. Thus in particular we deduce

$$\frac{dD}{dt} + \frac{3}{4} \gamma \int_{\Omega} \left| \frac{\partial}{\partial t}(B_1 - B_2) \right|^2 dx \leq \frac{1}{\gamma} \int_{\Omega} |\overline{\mathcal{G}}(Z_1) - \overline{\mathcal{G}}(Z_2)|^2 \, dx. \qquad (5.15)$$

On the other hand

$$\int_{\Omega} [\overline{\mathcal{G}}(Z_1)(x,t) - \overline{\mathcal{G}}(Z_2)(x,t)]^2 \, dx \overset{(4.5)}{\leq} L_{\mathcal{G}}^2 \int_{\Omega} ||Z_1(x, \cdot) - Z_2(x, \cdot)||^2_{\mathcal{C}^0([0,t])} \, dx$$

$$\leq L_{\mathcal{G}}^2 \int_{\Omega} \left(\int_0^t \left| \frac{\partial}{\partial \tau}(Z_1 - Z_2) \right| (x, \tau) \, d\tau \right)^2 dx \quad (5.16)$$

$$\leq L_{\mathcal{G}}^2 \, t \int_0^t \int_{\Omega} \left| \frac{\partial}{\partial \tau}(Z_1 - Z_2) \right|^2 (x, \tau) \, dx \, d\tau,$$

where we used the fact that $Z_1(0, x) = Z_2(0, x)$, a.e. in Ω, as $Z_1, Z_2 \in S$.

We now introduce an equivalent norm on $H^1(0, T; L^2(\Omega))$: for all $z \in H^1(0, T; L^2(\Omega))$

$$|||z||| := \left(||z(0)||^2_{L^2(\Omega)} + \int_0^T \exp\left(-\frac{2 L_{\mathcal{G}}^2 t^2}{\gamma^2} \right) \left\| \frac{\partial z}{\partial t} \right\|^2_{L^2(\Omega)} (t) \, dt \right)^{1/2}.$$

At this point we divide (5.15) by γ, multiply the result by $\exp\left(-\dfrac{2 L_{\mathcal{G}}^2 t^2}{\gamma^2} \right)$ and integrate in time, for $t \in (0, T)$. We first remark that

$$\int_0^T \exp\left(-\frac{2 L_{\mathcal{G}}^2 t^2}{\gamma^2} \right) \frac{dD}{dt}(t) \, dt = \exp\left(-\frac{2 L_{\mathcal{G}}^2 T^2}{\gamma^2} \right) D(T)$$

$$+ \int_0^T D(t) \exp\left(-\frac{2 L_{\mathcal{G}}^2 t^2}{\gamma^2} \right) \frac{4 L_{\mathcal{G}}^2 t}{\gamma^2} \, dt \geq 0$$

and therefore we have

$$\frac{3}{4} |||J(Z_1) - J(Z_2)|||^2 \leq \int_0^T \exp\left(-\frac{2 L_{\mathcal{G}}^2 t^2}{\gamma^2} \right) \frac{L_{\mathcal{G}}^2 t}{\gamma^2} \int_0^t \int_{\Omega} \left| \frac{\partial}{\partial \tau}(Z_1 - Z_2) \right|^2 (x, \tau) \, dx \, d\tau$$

$$= -\frac{1}{4} \exp\left(-\frac{2 L_{\mathcal{G}}^2 T^2}{\gamma^2} \right) \int_0^T \int_{\Omega} \left| \frac{\partial}{\partial t}(Z_1 - Z_2) \right|^2 (x, t) \, dx \, dt$$

$$+ \frac{1}{4} \int_0^T \exp\left(-\frac{2 L_{\mathcal{G}}^2 t^2}{\gamma^2} \right) \int_{\Omega} \left| \frac{\partial}{\partial t}(Z_1 - Z_2) \right|^2 (x, t) \, dx \, dt$$

which in turn gives

$$|||J(Z_1) - J(Z_2)|||^2 \leq \frac{1}{3} |||Z_1 - Z_2|||^2.$$

Hence J is a contraction on the closed subset S of $H^1(0, T; L^2(\Omega))$, which finishes the proof. $\square$

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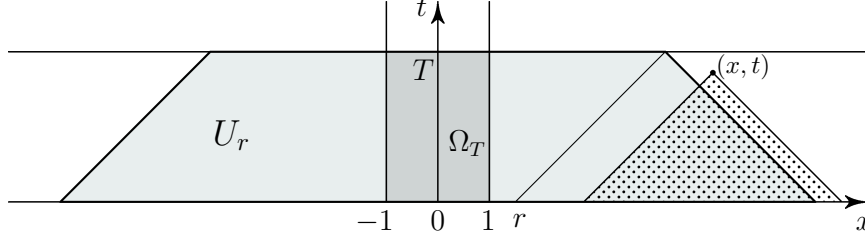# 6   Proof of Theorem 4.2

We now deal with the general case.

Figure 2: Set U_r and domain of dependence for (x,t) outside U_r.

For data as in Theorem 4.2, we define sequences indexed by $n \in \mathbb{N}$ of truncated data with supports in $(-2n, 2n)$ for $n \to \infty$ as

$$J_{\text{ext}}^n(x,t) = \chi_{(-n,n)}(x)\, J_{\text{ext}}(x,t),$$

$$(E_0^2)^n(x) = \begin{cases} E_0^2(x) & \text{for } x \in (-n,n) \\ \left(2 - \dfrac{x}{n}\right) E_0^2(n) & \text{for } x \in [n, 2n) \\ \left(2 + \dfrac{x}{n}\right) E_0^2(-n) & \text{for } x \in (-2n, -n] \\ 0 & \text{for } |x| \geq 2n, \end{cases}$$

$$(B_0^2)^n(x) = \begin{cases} B_0^2(x) & \text{for } x \in (-n,n) \\ \left(2 - \dfrac{x}{n}\right) B_0^2(n) & \text{for } x \in [n, 2n) \\ \left(2 + \dfrac{x}{n}\right) B_0^2(-n) & \text{for } x \in (-2n, -n] \\ 0 & \text{for } |x| \geq 2n, \end{cases}$$

By Theorem 4.1, there exists a sequence of solutions $\{(E^n, B^n, H^n)\}_{n \in \mathbb{N}}$ associated with these data. We now refer to the Courant-Hilbert trick and show that on every domain

$$U_r = \{(x,t) \in \mathbb{R} \times (0,T);\ 0 \leq t \leq T,\ -r - 2T + t \leq x \leq r + 2T - t\}$$

(see Fig. 2) with an arbitrarily fixed parameter $r > 1$, all these solutions with $n > r + 2T$ coincide.

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Let us take two solutions (E^j, B^j, H^j) and (E^k, B^k, H^k), with $j \neq k$. We set $\bar{E} := E^j - E^k$, $\bar{B} := B^j - B^k$, $\bar{H} := H^j - H^k$, $\bar{G} := \bar{\mathcal{G}}(B^j) - \bar{\mathcal{G}}(B^k)$. Then we obtain

$$
\begin{cases}
\dfrac{\partial \bar{E}}{\partial t} + \chi_\Omega \, \bar{E} + \dfrac{\partial \bar{H}}{\partial x} = 0 & \text{(6.1a)} \\[3mm]
\dfrac{\partial \bar{B}}{\partial t} + \dfrac{\partial \bar{E}}{\partial x} = 0 & \text{(6.1b)} \\[3mm]
\bar{H} = \chi_\Omega \left(\bar{G} + \gamma \dfrac{\partial \bar{B}}{\partial t} \right) + (1 - \chi_\Omega) \, \bar{B}. & \text{(6.1c)}
\end{cases}
$$

We test (6.1a) by $\bar{E}$, (6.1b) by $\bar{H}$ and integrate over U_r. We obtain

$$
\int_{\Omega_T} \left(\bar{E}^2 + \gamma \left(\frac{\partial \bar{B}}{\partial t} \right)^2 + \bar{G} \frac{\partial \bar{B}}{\partial t} \right) dx \, dt
$$

$$
+ \int_{U_r} \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \left(\bar{E}^2 + (1 - \chi_\Omega) \, \bar{B}^2 \right) \right) + \frac{\partial}{\partial x} (\bar{E} \, \bar{H}) \right] dx \, dt = 0. \tag{6.2}
$$

The set ∂U_r consists of four straight segments, $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3, \tilde{\Gamma}_4$, with endpoints respectively $(-r-2T, 0)$ and $(r+2T, 0)$, $(r+2T, 0)$ and $(r+T, T)$, $(r+T, T)$ and $(-r-T, T)$, $(-r-T, T)$ and $(-r-2T, 0)$, see Fig. 2.

Now, the Green formula yields

$$
\int_{U_r} \left[\frac{\partial}{\partial t} \left(\frac{1}{2} \left(\bar{E}^2 + (1 - \chi_\Omega) \, \bar{B}^2 \right) \right) + \frac{\partial}{\partial x} (\bar{E} \, \bar{H}) \right] dx \, dt = - \int_{\tilde{\Gamma}_1} \frac{1}{2} \left(\bar{E}^2 + (1 - \chi_\Omega) \, \bar{B}^2 \right) ds
$$

$$
+ \int_{\tilde{\Gamma}_2} \left[\frac{1}{\sqrt{2}} (\bar{E} \, \bar{H}) + \frac{1}{2\sqrt{2}} \left(\bar{E}^2 + (1 - \chi_\Omega) \, \bar{B}^2 \right) \right] ds + \int_{\tilde{\Gamma}_3} \frac{1}{2} \left(\bar{E}^2 + (1 - \chi_\Omega) \, \bar{B}^2 \right) ds
$$

$$
+ \int_{\tilde{\Gamma}_4} \left[-\frac{1}{\sqrt{2}} (\bar{E} \, \bar{H}) + \frac{1}{2\sqrt{2}} \left(\bar{E}^2 + (1 - \chi_\Omega) \, \bar{B}^2 \right) \right] ds.
$$

Now the integral over $\tilde{\Gamma}_1$ vanishes as the initial data for (E^j, B^j, H^j) and (E^k, B^k, H^k) coincide while the integral over $\tilde{\Gamma}_3$ yields a nonnegative contribution. Finally, taking into account that

$$
\tilde{\Gamma}_2 \cap \Omega_T = \emptyset \qquad \tilde{\Gamma}_4 \cap \Omega_T = \emptyset,
$$

we have that the integrals over $\tilde{\Gamma}_2$ and $\tilde{\Gamma}_4$ give a nonnegative contribution, as $\bar{H} = \bar{B}$. At this point (6.2) entails in particular

$$
\int_{\Omega_T} \left(\gamma \left(\frac{\partial \bar{B}}{\partial t} \right)^2 + \bar{G} \frac{\partial \bar{B}}{\partial t} \right) dx \, dt \leq 0.
$$

This in turns gives, working as in (5.16)

$$
\frac{\gamma}{2} \int_{\Omega_T} \left| \frac{\partial \bar{B}}{\partial t} \right|^2 dx \, dt \leq \frac{1}{2\gamma} \int_0^T \int_\Omega |\bar{G}|^2 \, dx \, dt \leq \int_0^T \frac{L_{\mathcal{G}}^2}{2\gamma} t \int_0^t \int_\Omega \left| \frac{\partial \bar{B}}{\partial \tau} \right|^2 dx \, d\tau \, dt.
$$

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From the Gronwall lemma, we finally obtain $\bar{B} = 0$ and therefore, by comparison, $\bar{E} = \bar{H} = 0$. Hence, all solutions coincide on U_r, independently on the way we constructed the sequence of data with compact support.

Outside U_r the solution of (4.7) is given by the explicit formula for the solution of the linear wave equation. More precisely, outside U_r (4.7) reduces to the following

$$\begin{cases} \dfrac{\partial E}{\partial t} + J_{\text{ext}} + \dfrac{\partial B}{\partial x} = 0 \\ \dfrac{\partial B}{\partial t} + \dfrac{\partial E}{\partial x} = 0. \end{cases}$$

This means that

$$\frac{\partial^2 E}{\partial t^2} - \frac{\partial^2 E}{\partial x^2} = -\frac{\partial J_{\text{ext}}}{\partial t} =: \tilde{J}$$

and the initial conditions are

$$E(x,0) = E_0^2 \qquad \frac{\partial E}{\partial t}(x,0) = -\frac{\partial B}{\partial x}(x,0) - J_{\text{ext}}(x,0) =: \tilde{E}.$$

The solution for the linear wave equation can be represented as

$$E(x,t) = \frac{1}{2}\left[E_0^2(x+t) - E_0^2(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \tilde{E}(\xi)\, d\xi + \frac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} \tilde{J}(\tau,\xi)\, d\xi\, d\tau,$$

for $(x,t) \in [\mathbb{R} \times (0,T)] \setminus U_r$, and the assertion follows. $\square$

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