Clamped elastic-ideally plastic beams and Prandtl-Ishlinskii hysteresis operators *

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Abstract

We consider a model for one-dimensional transversal oscillations of an elastic-ideally plastic beam. It is based on the von Mises model of plasticity and leads after a dimensional reduction to a fourth-order partial differential equation with a hysteresis operator of Prandtl-Ishlinskii type whose weight function is given explicitly. In this paper, we study the case of clamped beams involving a kinematic hardening in the stress-strain relation. As main result, we prove the existence and uniqueness of a weak solution. The method of proof, based on spatially semidiscrete approximations, strongly relies on energy dissipation properties of one-dimensional hysteresis operators.

1 Introduction

The use of hysteresis operators in the modeling of hysteretic stress-strain relations that are commonplace in nonlinear elastoplasticity dates back to the pioneering works of Prandtl [7] and Ishlinskii [2]. The *Prandtl-Ishlinskii hysteresis model* describes the time evolution of the relation between strain and stress in the form

$$\sigma(t) = \int_0^\infty \varphi(q)\, \mathfrak{s}_q[\varepsilon](t)\, dq\,. \tag{1.1}$$

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Here, $\varphi : (0,\infty) \to \mathbb{R}$ is a nonnegative weight function satisfying suitable integrability conditions. The symbol $\mathfrak{s}_q$ denotes the one-dimensional *stop operator* or *Prandtl's elastic-perfectly plastic element* with yield limit q. Between the thresholds $\pm q$, the behavior is linear elastic (with elasticity modulus 1), while along the upper (lower) threshold $+q$ $(-q)$ we have irreversible plastic yielding and can only move to the right (left).

The operator $\mathfrak{s}_q$ is a special case of the abstract stop operator $\mathfrak{S}_Z$ in a separable Hilbert space X associated with a closed and convex set $Z \subset X$. It is defined as the solution operator, which with each given input function $v \in W^{1,1}(0,T;X)$ and a given initial datum $\chi_0 \in Z$ associates the unique solution $\chi \in W^{1,1}(0,T;X)$ to the variational inequality

$$\begin{aligned} &\chi(t) \in Z \quad \forall\, t \in [0,T], \quad \chi(0) = \chi_0, \\ &(\dot\chi(t) - \dot v(t)\,,\, z - \chi(t)) \ge 0 \quad \forall\, z \in Z, \quad \text{for a.e. } t \in (0,T). \end{aligned} \tag{1.2}$$

Here, and throughout this paper, the superimposed dot denotes differentiation with respect to time, and $(\,\cdot\,,\,\cdot\,)$ is a scalar product in X. The theory of such variational inequalities goes back to [6], and further special properties related to the geometry of the set Z have been established in [3, 4]. In this paper, we restrict ourselves to the canonical choice of initial conditions

$$\chi_0 = \mathrm{Proj}_Z(v(0)), \tag{1.3}$$

where $\mathrm{Proj}_Z : X \to Z$ is the orthogonal projection onto Z. We then simply write $\chi = \mathfrak{S}_Z[v]$ instead of $\chi = \mathfrak{S}_Z[\chi_0, v]$. In this setting, the one-dimensional stop $\mathfrak{s}_q$ is just another notation for $\mathfrak{S}_{[-q,+q]}$.

The stop operator forms a corner stone of the mathematical theory of hysteresis operators. For a thorough treatment of its analytical and geometrical properties, we refer to the monographs [1], [3], [4], [8].

Although the Prandtl-Ishlinskii operator is easily understood and rather intuitive, its use in the physical and engineering literature is still nonstandard. The main reasons are the following: on the one hand, the operator appears to be entirely phenomenological, and its weight function φ is a priori unknown and must be identified; on the other hand, well-established three-dimensional plasticity models like those by von Mises or Tresca are available.

In the recent paper [5], the authors have demonstrated that the use of the three-dimensional single-yield von Mises plasticity model leads in the case of transversal vibrations of thin one-dimensional elastoplastic rectangular beams to a model in which a one-dimensional multi-yield Prandtl-Ishlinskii operator occurs. For the transversal component w of the displacement, it resulted a fourth-order PDE of the form

$$\varrho w_{tt} - \frac{\varrho h^2}{3} w_{xxtt} + \mathcal{P}[w_{xx}]_{xx} = g, \tag{1.4}$$

where

$$\mathcal{P}[w_{xx}](x,t) = \int_0^\infty \varphi(q)\, \mathfrak{s}_q[w_{xx}(x,\cdot)](t)\, dq, \tag{1.5}$$

with φ given by the formula

$$\varphi(q) = \begin{cases} 0\,, & \text{if} \quad 0 \le q \le \frac{R}{Eh}\,, \\ \frac{R^3}{E^2 h}\, q^{-4}\,, & \text{if} \quad q > \frac{R}{Eh}\,. \end{cases} \tag{1.6}$$

Here, ϱ is the mass density, $2h$ is the thickness of the beam, E is the Young elasticity modulus, and R is the yield limit in the original von Mises model. It was shown in [5] that Problem (1.4)–(1.5), complemented with the initial and boundary conditions

$$w(0,t) = \mathcal{P}[w_{xx}](0,t) = w(1,t) = \mathcal{P}[w_{xx}](1,t) = 0\,, \quad 0 \le t \le T\,, \tag{1.7}$$

$$w(x,0) = w^0(x)\,, \quad w_t(x,0) = w^1(x)\,, \quad 0 \le x \le 1\,, \tag{1.8}$$

and under suitable assumptions on the data, admits a unique weak solution

$$w \in W^{1,\infty}(0,T;L^2(0,1)) \cap L^\infty(0,T;H^2(0,1)) \cap H^1(0,T;H^1(0,1))\,.$$

In this paper, we consider the case of a clamped beam, i. e., the boundary condition (1.7) is replaced by

$$w(0,t) = w_x(0,t) = w(1,t) = w_x(1,t) = 0\,. \tag{1.9}$$

It turns out that these boundary conditions are somewhat more difficult to treat than (1.7), and the analysis performed in [5] does not apply. Also, the space discretization method presented below fails for Eq. (1.4). In order to obtain existence, we thus assume the presence of a *kinematic hardening* term $\gamma\, w_{xxxx}$ in our model; that is, we replace Eq. (1.4), normalizing all physical constants to unity, by

$$w_{tt} - w_{xxtt} + \mathcal{P}[w_{xx}]_{xx} + w_{xxxx} \;=\; g\,. \tag{1.10}$$

It will be shown in the forthcoming sections that the initial-boundary value problem (1.10), (1.5), (1.8), (1.9) admits under appropriate regularity assumptions a unique weak solution

$$w \in W^{2,\infty}(0,T;H^1(0,1)) \cap W^{1,\infty}(0,T;H^2(0,1))\,.$$

The following text is divided into three sections. In Section 2, we state the main results and recall some basic facts about hysteresis operators. In Section 3, we define the corresponding space discrete approximations and derive estimates independent of the discretization parameter. Finally, in Section 4, we pass to the limit and prove the existence and uniqueness of a solution to the original problem.

2 Statement of the mathematical results

In what follows, we use the standard notations for the spaces of continuous functions and for the usual Lebesgue and Sobolev spaces. The L^2-norm is denoted by $\|\cdot\|$.

We study Problem (1.10), (1.5), (1.8), (1.9) in $Q_T := (0,1) \times (0,T)$ under the following general assumptions on the data of the system:

(H1) $g, g_t \in L^2(Q_T)$.

(H2) $w^0 \in H^3(0,1), w^1 \in H^2(0,1)$, and the following compatibility conditions are satisfied:

$$w^i(0) = w^i_x(0) = w^i(1) = w^i_x(1) = 0\,, \quad i = 0, 1\,. \tag{2.1}$$

(H3) The weight function $\varphi : (0,\infty) \to [0,\infty)$ of the Prandtl-Ishlinskii operator

$$\mathcal{P}[u] = \int_0^\infty \varphi(q)\, \mathfrak{s}_q[u]\, dq$$

is measurable and satisfies the growth condition

$$\int_0^\infty \left(1+q^2\right) \varphi(q)\, dq < +\infty\,. \tag{2.2}$$

Putting

$$u(x,t) \;=\; w^1(x) + \int_0^t (\mathcal{I}+\mathcal{P})\,[w_{xx}]\,(x,s)\,ds\,, \quad f(x,t) \;=\; w^1(x) + \int_0^t g(x,s)\,ds\,, \tag{2.3}$$

where $\mathcal{I}$ is the identity mapping, we rewrite problem (1.10), (1.5), (1.8), (1.9) in the form

$$\begin{aligned}
u_t - (\mathcal{I}+\mathcal{P})\,[w_{xx}] \;&=\; 0 \qquad \text{in } Q_T\,, && (2.4)\\
w_t \,-\, w_{xxt} + u_{xx} \;&=\; f(x,t) \quad \text{in } Q_T\,, && (2.5)\\
w(0,t) = w(1,t) = w_x(0,t) = w_x(1,t) \;&=\; 0\,, \qquad 0 \le t \le T\,, && (2.6)\\
u(x,0) \;&=\; w^1(x)\,, \quad 0 \le x \le 1\,, && (2.7)\\
w(x,0) \;&=\; w^0(x)\,, \quad 0 \le x \le 1\,. && (2.8)
\end{aligned}$$

The aim of this paper is to establish the following result.

Theorem 2.1. *Suppose that the conditions (H1)–(H3) are satisfied. Then the system (2.4)–(2.8) has a unique solution (u,w) having the following properties:*

(i) $u \in W^{2,\infty}(0,T;L^2(0,1)) \cap W^{1,\infty}(0,T;H^1(0,1)) \cap L^\infty(0,T;H^2(0,1))$.

(ii) $w \in W^{1,\infty}(0,T;H^2(0,1)) \cap W^{2,\infty}(0,T;H^1(0,1))$.

(iii) *Eqs. (2.4)–(2.5) hold almost everywhere in Q_T.*

(iv) *The initial and boundary conditions (2.6)–(2.8) are satisfied pointwise.*

Remark 2.2. We call (u,w) having the above properties (i)–(iv) a strong solution to (2.4)–(2.8), and w a weak solution to (1.10), (1.5), (1.8), (1.9). The meaning of the conditions (i), (ii) in Theorem 2.1 is that

$$u_{tt}, u_{xt}, u_{xx}, w_{xxt}, w_{xtt} \in L^\infty(0,T;L^2(0,1))\,. \tag{2.9}$$

By virtue of the initial/boundary conditions and of embedding theorems, we then have

$$u, u_x, u_t, w, w_x, w_t, w_{xt} \;\in\; C(\overline{Q_T})\,, \quad w_{xx} \in L^2(0,1;C[0,T])\,. \tag{2.10}$$

Hence, as we will see below, $\mathcal{P}\,[w_{xx}(x,\cdot)]$ is well defined for a. e. $x \in (0,1)$.

Before proving Theorem 2.1 in the next sections, we now collect some well-known properties of the one-dimensional stop operator that can be found in a more general form in the monographs [1] or [4].

Proposition 2.3. *Let* $v_1, v_2 \in W^{1,1}(0,T)$ *be given,* $\chi_i = \mathfrak{s}_q[v_i]$, $p_i = v_i - \chi_i$, $i = 1,2$. *Then*

(i) $(\chi_1(t) - \chi_2(t))(\dot v_1(t) - \dot v_2(t)) \ \geq\ \dfrac{1}{2}\dfrac{d}{dt}(\chi_1(t) - \chi_2(t))^2$ *a. e.;*

(ii) $|\dot p_1(t) - \dot p_2(t)| + \dfrac{d}{dt}|\chi_1(t) - \chi_2(t)| \ \leq\ |\dot v_1(t) - \dot v_2(t)|$ *a. e.;*

(iii) $|\chi_1(t) - \chi_2(t)| \ \leq\ 2 \max_{0\leq\tau\leq t} |v_1(\tau) - v_2(\tau)| \quad \forall t \in [0,T]$ *;*

(iv) $\dot\chi_i(t)\, \dot p_i(t) \ =\ 0$ *a. e.*

It follows from Proposition 2.3 (ii), (iii) that the Prandtl-Ishlinskii operator $\mathcal{P}$ from Hypothesis (H3) is Lipschitz continuous in $W^{1,1}(0,T)$ and admits a Lipschitz continuous extension to $C[0,T]$. Moreover, Proposition 2.3 (iv) implies that there exists a constant $C > 0$ such that

$$0 \ \leq\ \frac{d}{dt}\mathcal{P}[v](t)\,\dot v(t) \ \leq\ C\,\dot v^2(t) \quad \text{a.e.} \tag{2.11}$$

As a consequence of Proposition 2.3 (i), we obtain the inequality

$$(\mathcal{P}[v_1](t) - \mathcal{P}[v_2](t))(\dot v_1(t) - \dot v_2(t)) \ \geq\ \frac{1}{2}\frac{d}{dt}\int_0^\infty \varphi(q)(\mathfrak{s}_q[v_1] - \mathfrak{s}_q[v_2])^2(t)\,dq \tag{2.12}$$

for every $v_1, v_2 \in W^{1,1}(0,T)$ and a. e. $t \in (0,T)$. Formula (2.12) implies in turn the two well-known hysteresis energy inequalities: choosing $v_1 = v \in W^{1,1}(0,T)$ and $v_2 = 0$ yields

$$\mathcal{P}[v](t)\,\dot v(t) \ \geq\ \frac{1}{2}\frac{d}{dt}\int_0^\infty \varphi(q)\,\mathfrak{s}_q^2[v](t)\,dq \tag{2.13}$$

almost everywhere, while, if we consider $v_1 = v \in W^{2,1}(0,T)$ and $v_2(t) = v(t-h)$ for $h > 0$ with a suitable extension to $[-h,0]$, and let h tend to 0, then (using also Proposition 2.3 (iv)),

$$\frac{d}{dt}\mathcal{P}[v](t)\,\ddot v(t) \ \geq\ \frac{1}{2}\frac{d}{dt}\int_0^\infty \varphi(q)\left(\frac{d}{dt}\mathfrak{s}_q[v]\right)^2(t)\,dq \ =\ \frac{d}{dt}\left(\frac{1}{2}\frac{d}{dt}\mathcal{P}[v](t)\,\dot v(t)\right) \tag{2.14}$$

in the sense of distributions, that is, the function

$$\mathcal{D}[v](t) \ =\ \frac{1}{2}\frac{d}{dt}\mathcal{P}[v](t)\,\dot v(t) - \int_0^t \frac{d}{dt}\mathcal{P}[v](\tau)\,\ddot v(\tau)\,d\tau \tag{2.15}$$

is nonincreasing in $[0,T]$. This means, in particular, that the function

$$\mathcal{U}[v](t) \ =\ \frac{d}{dt}\mathcal{P}[v](t)\,\dot v(t) \tag{2.16}$$

has bounded variation on $[0,T]$ whenever $v \in W^{2,1}(0,T)$. A detailed discussion of these issues can be found in [4, Chapter II]. Note only that (2.13) is the usual energy inequality, where

$$\mathcal{Q}[v] = \frac{1}{2}\int_0^\infty \varphi(q)\, \mathfrak{s}_q^2[v]\, dq \tag{2.17}$$

is the so-called *clockwise admissible potential* of $\mathcal{P}$. The dissipation

$$\int_{t_1}^{t_2} \mathcal{P}[v](t)\, \dot v(t)\, dt - (\mathcal{Q}[v](t_2) - \mathcal{Q}[v](t_1)) \ \geq\ 0\,, \tag{2.18}$$

defined as the difference between the work done during a time interval $[t_1, t_2]$ and the potential increment, corresponds to the *area of hysteresis loops*. In (2.15) instead, the "dissipation" $\mathcal{D}[v]$ is related to the *curvature* of the hysteresis branches.

3 Space discretization

For a generic vector $\boldsymbol{v} = (v_0, v_1, \dots, v_n)$ we introduce the notation

$$D_k \boldsymbol{v} \ = \ n^2(v_{k+1} - 2v_k + v_{k-1})\,, \quad k = 1, \dots, n-1\,, \tag{3.1}$$

$$d_k \boldsymbol{v} \ = \ n(v_k - v_{k-1})\,, \quad k = 1, \dots, n\,. \tag{3.2}$$

We start with an easy, but useful auxiliary result.

Lemma 3.1. *Let* $\boldsymbol{v} = (v_0, v_1, \dots, v_n)$ *be such that*

$$\frac{1}{n}\sum_{k=0}^{n} v_k^2 \leq 1\,, \quad \frac{1}{n}\sum_{k=1}^{n-1} (D_k \boldsymbol{v})^2 \leq 1\,.$$

Then $\max_{k=1,\dots,n} |d_k \boldsymbol{v}| \leq 7$.

Proof. We define an auxiliary sequence $\hat v_{2j} = v_j$ for $j = 0, \dots, n$, $\hat v_{2j-1} = (v_j + v_{j-1})/2$ for $j = 1, \dots, n$. Then

$$\frac{1}{2n}\sum_{k=0}^{2n} \hat v_k^2 \leq 1\,, \quad (2n)^3 \sum_{k=1}^{2n-1} (\hat v_{k+1} - 2\hat v_k + \hat v_{k-1})^2 \leq 2\,.$$

Let $\hat v_k \geq 0$ for at least $n+1$ elements; otherwise we pass from v_k to $-v_k$. We further proceed by contradiction. Assume that for some j we have $2n(\hat v_j - \hat v_{j-1}) > 7$ (the case "< -7" is similar). Then for all k we have

$$|2n(\hat v_k - \hat v_{k-1}) - 2n(\hat v_j - \hat v_{j-1})| \ \leq\ 2n \sum_{k=1}^{2n-1} |\hat v_{k+1} - 2\hat v_k + \hat v_{k-1}| \ \leq\ \sqrt{2}\,,$$

hence $2n(\hat v_k - \hat v_{k-1}) > 7-\sqrt{2}$ for all k. In particular, $\hat v_n \ge 0$, and $\hat v_{n+j} \; > \; j\,(7-\sqrt{2})/(2n)$ for all $j = 1,\dots,n$. This yields that

$$1 \;\ge\; \frac{1}{2n}\sum_{j=1}^{n} \hat v_{n+j}^2 \;>\; \frac{(7-\sqrt{2})^2}{8n^3}\sum_{j=1}^{n} j^2 \;>\; \frac{(7-\sqrt{2})^2}{24}\,,$$

which is the desired contradiction. Hence, $2n|\hat v_k - \hat v_{k-1}| \le 7$ for all k, and the assertion follows. □

We now fix an integer $n \in \mathbb{N}$, and consider space-discrete approximations of (2.4)–(2.8) in the form

$$\dot u_k(t) - (\mathcal{I}+\mathcal{P})\,[D_k\boldsymbol{w}]\,(t) \;=\; 0\,, \qquad k = 0,1,\dots,n\,, \tag{3.3}$$

$$\dot w_k(t) - D_k\dot{\boldsymbol{w}}(t) + D_k\boldsymbol{u}(t) \;=\; f_k(t)\,, \qquad k = 1,\dots,n-1\,. \tag{3.4}$$

We prescribe “boundary conditions”

$$w_0(t) = w_n(t) = 0\,,\quad w_{-1}(t) = w_1(t)\,,\quad w_{n+1}(t) = w_{n-1}(t)\,, \tag{3.5}$$

and initial conditions

$$w_k(0) \;=\; w_k^0 := w^0(k/n)\,, \qquad k = 1,\dots,n-1\,, \tag{3.6}$$

$$u_k(0) \;=\; w_k^1 := w^1(k/n)\,, \qquad k = 0,1,\dots,n\,. \tag{3.7}$$

The right-hand side $f_k(t)$ is defined as

$$f_k(t) \;=\; w_k^1 + n\int_{(k-1)/n}^{k/n}\int_0^t g(x,\tau)\,d\tau\,dx\,,\quad k = 1,\dots,n-1\,. \tag{3.8}$$

Note that formula (3.1) under the additional hypothesis $v_0 = v_n = 0$ generates on $\mathbb{R}^{n-1}$ a symmetric negative semidefinite matrix $\tilde{\boldsymbol{D}}$ with entries $\tilde D_{kk} = -2n^2$, $\tilde D_{k(k-1)} = \tilde D_{(k-1)k} = n^2$, $\tilde D_{ij} = 0$ otherwise. With this notation, Eq. (3.4) for $t = 0$ reads $(\boldsymbol{I}-\tilde{\boldsymbol{D}})\,\dot{\boldsymbol{w}}(0) = (\boldsymbol{I}-\tilde{\boldsymbol{D}})\,\boldsymbol{w}^1$, hence

$$\dot w_k(0) \;=\; w_k^1\,,\quad k = 1,\dots,n-1\,. \tag{3.9}$$

The ODE system (3.3)–(3.8) has a unique global solution by virtue of the Lipschitz continuity of the operator $\mathcal{I}+\mathcal{P}$ and the invertibility of the matrix $\boldsymbol{I}-\tilde{\boldsymbol{D}}$ in (3.4). We now derive some estimates that will enable us to pass to the limit as $n\to\infty$. We denote by $C_1, C_2, \dots$ any constant that depends possibly on the data, but not on the discretization parameter n.

The estimates are based on the “second-order energy inequality” (2.14). We differentiate (3.4) twice and (3.3) once, test (3.4) by $\ddot w_k(t)$, use summation by parts and the conditions (3.5), and obtain for a. e. $t \in (0,t)$ the identity (omitting the argument t for simplicity)

$$\frac{1}{n}\sum_{k=1}^{n-1} \dddot w_k\,\ddot w_k + \frac{1}{n}\sum_{k=1}^{n} d_k\,\dddot{\boldsymbol{w}}\;d_k\ddot{\boldsymbol{w}} + \frac{1}{n}\sum_{k=0}^{n} \delta_k\,\frac{d}{dt}(\mathcal{I}+\mathcal{P})[D_k\boldsymbol{w}]\,D_k\ddot{\boldsymbol{w}} \;=\; \frac{1}{n}\sum_{k=1}^{n-1} \ddot f_k\,\ddot w_k\,, \tag{3.10}$$

with $\delta_k = 1$ for $k = 1,\dots,n-1$, $\delta_0 = \delta_n = 1/2$. Now, by virtue of (2.14)–(2.15), the function

$$V(t) := \frac{1}{n}\sum_{k=1}^{n-1} \ddot{w}_k^2(t) + \frac{1}{n}\sum_{k=1}^{n}(d_k\ddot{\boldsymbol{w}})^2(t) + \frac{1}{n}\sum_{k=0}^{n}\delta_k \frac{d}{dt}(\mathcal{I}+\mathcal{P})[D_k\boldsymbol{w}](t)\, D_k\dot{\boldsymbol{w}}(t)$$
$$-2\int_0^t \frac{1}{n}\sum_{k=1}^{n-1}\ddot{f}_k(\tau)\,\ddot{w}_k(\tau)\,d\tau$$

is nonincreasing in $(0,T)$. By (2.11), we have $V_*(t) \le V(t) \le V^*(t)$ a. e., where both the functions

$$V_*(t) := \frac{1}{n}\sum_{k=1}^{n-1} \ddot{w}_k^2(t) + \frac{1}{n}\sum_{k=1}^{n}(d_k\ddot{\boldsymbol{w}})^2(t) + \frac{1}{n}\sum_{k=0}^{n}\frac{1}{2}(D_k\dot{\boldsymbol{w}})^2(t)$$
$$-\int_0^t \left(\|g_t(\tau)\|^2 + \frac{1}{n}\sum_{k=1}^{n-1}(\ddot{w}_k)^2(\tau)\right)d\tau\,,$$

$$V^*(t) := \frac{1}{n}\sum_{k=1}^{n-1} \ddot{w}_k^2(t) + \frac{1}{n}\sum_{k=1}^{n}(d_k\ddot{\boldsymbol{w}})^2(t) + \frac{1}{n}\sum_{k=0}^{n}C_1\,(D_k\dot{\boldsymbol{w}})^2(t)$$
$$+\int_0^t \left(\|g_t(\tau)\|^2 + \frac{1}{n}\sum_{k=1}^{n-1}(\ddot{w}_k)^2(\tau)\right)d\tau\,,$$

with a suitable constant $C_1 > 1$, are continuous. For a. e. $0 < s < t < T$, we have $V_*(t) \le V(t) \le V(s) \le V^*(s)$. Hence, in particular,

$$V_*(t) \;\le\; V^*(0) \quad \forall t \ge 0\,. \tag{3.11}$$

The proof that $V^*(0)$ is bounded from above by a constant is more delicate. As a consequence of Hypothesis (H2), we have indeed, with some $\bar{C} \ge 1$,

$$\frac{1}{n}\sum_{k=0}^{n}(D_k\dot{\boldsymbol{w}})^2(0) \;\le\; \bar{C}\|w^1_{xx}\|^2\,. \tag{3.12}$$

For the remainder, we differentiate (3.4) and test by $\ddot{w}_k(t)$, to obtain for every $t \in [0,T]$ that

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k^2(t) + \frac{1}{n}\sum_{k=1}^{n}(d_k\ddot{\boldsymbol{w}})^2(t) \;=\; \frac{1}{n}\sum_{k=1}^{n} d_k\dot{\boldsymbol{u}}(t)\, d_k\ddot{\boldsymbol{w}}(t) + \frac{1}{n}\sum_{k=1}^{n-1}\dot{f}_k(t)\,\ddot{w}_k(t)\,, \tag{3.13}$$

hence, for $t = 0$,

$$\frac{1}{n}\sum_{k=1}^{n-1}\ddot{w}_k^2(0) + \frac{1}{n}\sum_{k=1}^{n}(d_k\ddot{\boldsymbol{w}})^2(0) \;\le\; C_2\left(\frac{1}{n}\sum_{k=1}^{n}(d_k\dot{\boldsymbol{u}})^2(0) + \|g(0)\|^2\right). \tag{3.14}$$

Eq. (3.3) and the Lipschitz continuity of the operator $\mathcal{I}+\mathcal{P}$ entail that for $2 \le k \le n-1$ we have

$$
\begin{aligned}
|d_k \dot{\boldsymbol{u}}(0)| &\le C_3\, n |D_k \boldsymbol{w}^0 - D_{k-1}\boldsymbol{w}^0| \\
&= C_3\, n^3 \left| \int_{(k-1)/n}^{k/n} (w^0_x(x+1/n) - 2w^0_x(x) + w^0_x(x-1/n))\, dx \right| \\
&= C_3\, n^3 \left| \int_{(k-1)/n}^{k/n} \int_{x-(1/n)}^{x} \int_{y}^{y+(1/n)} w^0_{xxx}(z)\, dz\, dy\, dx \right| ,
\end{aligned}
$$

hence

$$
\begin{aligned}
\frac{1}{n}\sum_{k=2}^{n-1} (d_k \dot{\boldsymbol{u}})^2(0) &\le C_3^2 n^2 \int_{1/n}^{1-(1/n)} \int_{x-(1/n)}^{x} \int_{y}^{y+(1/n)} |w^0_{xxx}(z)|^2\, dz\, dy\, dx \\
&\le C_3^2 \int_0^1 |w^0_{xxx}(z)|^2\, dz\,. \qquad (3.15)
\end{aligned}
$$

It remains to estimate the terms $(1/n)(d_1\dot{\boldsymbol{u}})^2(0)$ and $(1/n)(d_n\dot{\boldsymbol{u}})^2(0)$ in (3.14). We have

$$
(1/n)(d_1\dot{\boldsymbol{u}})^2(0) \le C_3^2 n^5 |w^0(2/n) - 4w^0(1/n)|^2\,. \qquad (3.16)
$$

By virtue of (2.6), we have for all $x \in [0,1]$ the identity

$$
w^0(x) = \frac{1}{2} x^2 w^0_{xx}(0) + \frac{1}{2}\int_0^x w^0_{xxx}(y)(x-y)^2\, dy\,,
$$

hence

$$
w^0(2x) - 4w^0(x) = 2\int_0^x (2w^0_{xxx}(2y) - w^0_{xxx}(y))(x-y)^2\, dy\,.
$$

For every $x \in [0,1/2]$ we thus have by Hölder's inequality

$$
|w^0(2x) - 4w^0(x)| \le 4\,\|w^0_{xxx}\| \left(\int_0^x (x-y)^4\, dy\right)^{1/2} \le \frac{4}{\sqrt{5}} x^{5/2}\,\|w^0_{xxx}\|\,, \qquad (3.17)
$$

and combining (3.17) with (3.16) we conclude that

$$
(1/n)(d_1\dot{\boldsymbol{u}})^2(0) \le \frac{16}{5}\, C_3^2\, \|w^0_{xxx}\|^2\,. \qquad (3.18)
$$

In a similar way, we estimate $(1/n)(d_n\dot{\boldsymbol{u}})^2(0)$ to obtain that

$$
\frac{1}{n}\sum_{k=1}^{n} (d_k \dot{\boldsymbol{u}})^2(0) \le C_4\, \|w^0_{xxx}\|^2\,. \qquad (3.19)
$$

From (3.11)–(3.19) and Gronwall's inequality, we deduce the estimate

$$
\frac{1}{n}\sum_{k=1}^{n-1} \ddot{w}_k^2(t) + \frac{1}{n}\sum_{k=1}^{n} (d_k \ddot{\boldsymbol{w}})^2(t) + \frac{1}{n}\sum_{k=0}^{n} (D_k \dot{\boldsymbol{w}})^2(t) \le C_5 \quad \forall t \in [0,T]\,, \qquad (3.20)
$$

and, by comparison,

$$\frac{1}{n}\sum_{k=0}^{n}\ddot{u}_k^2(t) + \frac{1}{n}\sum_{k=1}^{n-1}(D_k\boldsymbol{u})^2(t) \;\le\; C_6 \quad \text{a. e.} \tag{3.21}$$

We now use Lemma 3.1 to derive the final estimate. For $k = 0,\dots,n$, set $p_k(t) = \dot{w}_k(t) - u_k(t)$. We have $\dot{p}_k(t) = \ddot{w}_k(t) - \dot{u}_k(t)$ a. e. and, as a consequence of (3.4), $D_k\dot{\boldsymbol{p}}(t) = \ddot{w}_k(t) - \dot{f}_k(t)$. By Lemma 3.1, we have $\max |d_k\dot{\boldsymbol{p}}(t)| \le C_7$ and, by (3.20),

$$\frac{1}{n}\sum_{k=1}^{n}(d_k\dot{\boldsymbol{u}})^2(t) \;\le\; C_8 \quad \text{a. e.} \tag{3.22}$$

4 Passage to the limit

With the intention to let n tend to ∞, we define the interpolates

$$\begin{aligned}
w^{(n)}(x,t) &= \frac{1}{2}(w_k + w_{k-1})(t) + \Big(x - \frac{k-1}{n}\Big)\, d_k\boldsymbol{w}(t) + \frac{1}{2}\Big(x - \frac{k-1}{n}\Big)^2 D_k\boldsymbol{w}(t)\,,\\
u^{(n)}(x,t) &= \frac{1}{2}(u_k + u_{k-1})(t) + \Big(x - \frac{k-1}{n}\Big)\, d_k\boldsymbol{u}(t) + \frac{1}{2}\Big(x - \frac{k-1}{n}\Big)^2 D_k\boldsymbol{u}(t)\,,\\
\bar{w}^{(n)}(x,t) &= w_k(t)\,,\\
\bar{u}^{(n)}(x,t) &= u_k(t)\,,\\
\bar{f}^{(n)}(x,t) &= f_k(t)\,,
\end{aligned}$$

for $x \in [(k-1)/n, k/n)$, $k = 1,\dots,n$, and $t \in [0,T]$, continuously extended to $x = 1$, with the convention $D_n\boldsymbol{u}(t) = f_n(t) + D_n\dot{\boldsymbol{w}}(t)$ or, equivalently, $u_{n+1}(t) = 2u_n(t) - u_{n-1}(t) + 2\dot{w}_{n-1}(t) + (1/n^2)f_n(t)$. In (3.20)–(3.22), we derived the estimates

$$\max_{0\le t\le T}\Big(\|\bar{w}_{tt}^{(n)}(t)\|^2 \,+\, \|w_{xtt}^{(n)}(t)\|^2 \,+\, \|w_{xxt}^{(n)}(t)\|^2\Big) \;\le\; C_5\,, \tag{4.1}$$

$$\max_{0\le t\le T}\Big(\|\bar{u}_{tt}^{(n)}(t)\|^2 \,+\, \|u_{xt}^{(n)}(t)\|^2 \,+\, \|u_{xx}^{(n)}(t)\|^2\Big) \;\le\; C_9\,. \tag{4.2}$$

Furthermore, for $x \in [(k-1)/n, k/n)$ and $t \in [0,T]$ we have

$$|\bar{u}^{(n)}(x,t) - u^{(n)}(x,t)| \;\le\; 2|u_k(t) - u_{k-1}(t)| + \frac{1}{2}|u_{k+1}(t) - u_k(t)|\,, \tag{4.3}$$

and similarly for $\bar{u}_t^{(n)} - u_t^{(n)}$, $\bar{w}^{(n)} - w^{(n)}$, etc. This, together with (3.22), yields in particular that

$$\int_0^1 |\bar{u}_t^{(n)}(x,t) - u_t^{(n)}(x,t)|^2\,dx \;\le\; \frac{25}{4n}\sum_{k=1}^{n+1}|\dot{u}_k(t) - \dot{u}_{k-1}(t)|^2 \;\le\; \frac{C_{10}}{n^2}\,. \tag{4.4}$$

In the same way, we obtain from (3.20)–(3.21) that

$$\sup_{Q_T}|\bar{u}^{(n)} - u^{(n)}| + \sup_{Q_T}|\bar{w}^{(n)} - w^{(n)}| + \sup_{Q_T}|\bar{w}_t^{(n)} - w_t^{(n)}| \;\le\; \frac{C_{11}}{n}\,. \tag{4.5}$$

Combining the above estimates, and possibly selecting a suitable subsequence of $n \to \infty$, we find that there exist functions u, w in the appropriate Sobolev spaces such that the following convergences take place:

$$\left.\begin{array}{r} w_{xxt}^{(n)} \to w_{xxt},\ w_{xtt}^{(n)} \to w_{xtt},\ \bar{w}_{tt}^{(n)} \to w_{tt}, \\ \bar{u}_{tt}^{(n)} \to u_{tt},\ u_{xx}^{(n)} \to u_{xx},\ u_{xx}^{(n)} \to u_{xx}, \end{array}\right\} \text{ weakly-* in } L^\infty(0,T;L^2(0,1))\,. \tag{4.6}$$

Then, by compact embedding,

$$\left.\begin{array}{r} u^{(n)} \to u,\ u_x^{(n)} \to u_x,\ u_t^{(n)} \to u_t, \\ w^{(n)} \to w,\ w_x^{(n)} \to w_x,\ w_t^{(n)} \to w_t,\ w_{xt}^{(n)} \to w_{xt}, \end{array}\right\} \text{ strongly in } C(\overline{Q_T})\,. \tag{4.7}$$

By virtue of (3.3)–(3.3), the identities

$$\bar{u}_t^{(n)} - (\mathcal{I}+\mathcal{P})\left[w_{xx}^{(n)}\right] \;=\; 0\,, \tag{4.8}$$

$$\bar{w}_t^{(n)} \;-\; w_{xxt}^{(n)} + u_{xx}^{(n)} \;=\; \bar{f}^{(n)}(x,t)\,. \tag{4.9}$$

hold a. e. in Q_T. The operator $\mathcal{I}+\mathcal{P}$ is invertible on $C[0,T]$, and its inverse is Lipschitz continuous (see [4, Corollary II.3.4]). By (4.4) and (4.7), $\bar{u}_t^{(n)} \to u_t$ strongly in $L^2(0,1;C[0,T])$, hence $w_{xx}^{(n)} \to (\mathcal{I}+\mathcal{P})^{-1}[u_t]$ strongly in $L^2(0,1;C[0,T])$. Clearly, $\bar{f}^{(n)} \to f$ strongly in $L^2(0,1;C[0,T])$, so that we can pass to the limit in (4.8)–(4.9) to see that (2.4)–(2.5) are satisfied a. e. in Q_T.

The convergence of the initial conditions easily follows from the inequalities

$$|\bar{w}^{(n)}(x,0) - w^0(x)| \le \int_{\frac{k-1}{n}}^{\frac{k}{n}} |w_x^0(x)|\,dx\,, \quad |\bar{w}_t^{(n)}(x,0) - w^1(x)| \le \int_{\frac{k-1}{n}}^{\frac{k}{n}} |w_x^1(x)|\,dx\,, \tag{4.10}$$

for $x \in [(k-1)/n, k/n)$. To check the boundary conditions, just notice that

$$\begin{aligned} |w_x^{(n)}(0,t)|^2 &= n^2\,|w_1(t)|^2 = \frac{1}{4n^2}|D_0\boldsymbol{w}(t)|^2 \le \frac{1}{4n^2}\sum_{k=0}^{n}(D_k\boldsymbol{w})^2(t) \le \frac{C_{12}}{n}\,, \\ |w^{(n)}(0,t)|^2 &= \frac{1}{4}|w_1(t)|^2 \le \frac{C_{12}}{4n^3}\,, \end{aligned}$$

and similarly for $x = 1$. The uniqueness is easy as well: consider two solutions (u^*, w^*), (u_*, w_*), and set $\hat{w} = w^* - w_*$, $\hat{u} = u^* - u_*$, $\hat{p} = \hat{w}_t - \hat{u}$. We obtain from (2.5) that $\hat{w}_t = \hat{p}_{xx}$. Differentiating this identity with respect to t, and testing by $\hat{w}_t$, we obtain that

$$\int_0^1 \hat{w}_{tt}\,\hat{w}_t + \hat{w}_{xtt}\,\hat{w}_{xt} + ((\mathcal{I}+\mathcal{P})[w_{xx}^*] - (\mathcal{I}+\mathcal{P})[(w_*)_{xx}])\;\hat{w}_{xxt}\,dx = 0\,,$$

whence, by (2.12), $w^* = w_*$. The proof of Theorem 2.1 is thus complete.

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