Long time behaviour of a singular phase transition model

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Abstract

A phase-field system, non-local in space and non-smooth in time, with heat flux proportional to the gradient of the inverse temperature, is shown to admit a unique strong thermodynamically consistent solution on the whole time axis. The temperature remains globally bounded both from above and from below, and its space gradient as well as the time derivative of the order parameter asymptotically vanish in L^2-norm as time tends to infinity.

Key words: Phase transition, nonlocal model, integrodifferential heat equation, long-time behaviour.

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1 Introduction

We follow here the classical scheme for models of temperature-induced phase transitions in a physical body Ω as e. g. in [2, 14], and derive equations for the temperature θ (we will consider the *absolute temperature* $\theta > 0$ here) and the order parameter χ characterizing the physical state of the material. For example, in a simple melting-solidification process, χ takes values in the interval $[0,1]$, where $\chi = 0$ corresponds to the solid, $\chi = 1$ to the liquid, and $0 < \chi < 1$ is the liquid fraction in a mixture of both phases. The mathematical model we describe below may or may not contain a restriction on the domain of admissible values of χ.

We consider the free energy density F in the form

$$F[\theta,\chi] \;=\; c_V\theta(1-\log\theta) + \theta\sigma(\chi) + \lambda(\chi) + (\beta+\theta)\varphi(\chi) + B[\chi]\,, \tag{1.1}$$

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where $c_V > 0$ is the specific heat, σ and λ are smooth functions describing the local dependence on χ of entropy and of latent heat, respectively, φ is a general proper, convex, and lower semicontinuous function (in the above example of solid-liquid phase transition, φ can be chosen for instance as the indicator function of the interval $[0,1]$), $\beta > 0$ is a constant parameter, and B is a non-local operator of the form

$$B[\chi](x,t) \;=\; \int_\Omega k(x,y)\, G(\chi(x,t) - \chi(y,t))\, dy\,, \tag{1.2}$$

with a given sufficiently regular symmetric kernel $k : \Omega \times \Omega \to \mathbb{R}$, and an even smooth function G which is bounded on the domain of φ together with its first two derivatives. In comparison with [8, 13], we thus remove all restrictions on the convex potential φ. An interested reader will find a more detailed discussion on non-local phase-field models e. g. in [3, 4, 7].

The corresponding densities of internal energy E and entropy S have the form

$$E[\theta,\chi] \;=\; c_V\theta + \lambda(\chi) + \beta\,\varphi(\chi) + B[\chi]\,, \tag{1.3}$$

$$S[\theta,\chi] \;=\; c_V \log\theta - \sigma(\chi) - \varphi(\chi)\,. \tag{1.4}$$

The evolution process is driven by the *energy conservation principle*

$$\frac{d}{dt}\int_\Omega E[\theta,\chi](x,t)\, dx \;=\; 0\,, \tag{1.5}$$

and by the *order parameter evolution equation*

$$\mu(\theta)\frac{\partial\chi}{\partial t} \;\in\; -\delta_\chi \int_\Omega F[\theta,\chi](x,t)\, dx\,. \tag{1.6}$$

The free energy contains a component which is Fréchet differentiable with respect to χ, and another component which is convex, but not necessarily differentiable. The symbol δ_χ thus represents alternatively the Fréchet derivative and the subdifferential which may be multivalued. This also explains the inclusion sign in (1.6). Physically, the relation (1.6) expresses the tendency of the system to move towards local minima of the total free energy with speed proportional to $1/\mu(\theta)$.

Assuming that time differentiation and space integration can be interchanged in the energy conservation law (1.5), we obtain, using the symmetries in the operator B, that

$$\int_\Omega \left(c_V\,\theta_t + \left(\lambda(\chi) + \beta\,\varphi(\chi)\right)_t + b[\chi]\,\chi_t\right)(x,t)\, dx \;=\; 0\,, \tag{1.7}$$

where we have set

$$b[\chi](x,t) \;=\; 2\int_\Omega k(x,y)\, G'(\chi(x,t) - \chi(y,t))\, dy\,. \tag{1.8}$$

Formally, by (1.7), there exists a vector function $\mathbf{q}$ (the *heat flux*) such that $\mathbf{q}\cdot\mathbf{n} = 0$ on $\partial\Omega$ ($\mathbf{n}$ is the unit outward normal) and

$$c_V\,\theta_t + \left(\lambda(\chi) + \beta\,\varphi(\chi)\right)_t + b[\chi]\,\chi_t + \operatorname{div}\mathbf{q} \;=\; 0\,. \tag{1.9}$$

Assuming now the Penrose-Fife law $\mathbf{q} := \kappa\nabla(1/\theta)$, where $\kappa > 0$ is a constant parameter characterizing the heat conduction properties of the material, we obtain from (1.6) and (1.9) the following system of balance equations for the unknowns θ and χ:

$$c_V\,\theta_t + \kappa\Delta\left(\frac{1}{\theta}\right) + (\lambda(\chi) + \beta\,\varphi(\chi))_t + b[\chi]\,\chi_t \;=\; 0\,, \tag{1.10}$$

$$\mu(\theta)\,\chi_t + \theta\sigma'(\chi) + \lambda'(\chi) + b[\chi] \;\in\; -(\beta+\theta)\,\partial\varphi(\chi)\,. \tag{1.11}$$

It is complemented with the boundary and initial conditions

$$\frac{\partial}{\partial\mathbf{n}}\left(\frac{1}{\theta}\right) \;=\; 0 \quad \text{on} \;\; \partial\Omega\times(0,\infty)\,, \tag{1.12}$$

$$\theta(x,0) = \theta_0(x)\,, \quad \chi(x,0) = \chi_0(x) \quad \text{a.e. in } \;\Omega\,, \tag{1.13}$$

where $\partial/\partial\mathbf{n}$ denotes the outward normal derivative, and θ_0, χ_0 are given functions.

It is easy to see that the system is thermodynamically consistent. The positivity of temperature and the smoothness that will be established in Theorem 2.2 below imply, by virtue of (1.4) and (1.10)–(1.11), that

$$\frac{\partial}{\partial t}S[\theta,\chi] + \operatorname{div}\left(\frac{\mathbf{q}}{\theta}\right) \;=\; \kappa\left|\nabla\left(\frac{1}{\theta}\right)\right|^2 + \frac{\mu(\theta)}{\theta}\chi_t^2 \;\geq\; 0 \quad \text{a.e.}\;, \tag{1.14}$$

which is the Second Principle of Thermodynamics in Clausius-Duhem form.

As main results, we will prove that System (1.10)–(1.13) admits, under suitable assumptions on the data, a unique strong solution with positive temperature θ. If moreover the space dimension is at most three and the internal energy is a priori bounded from below, then the temperature remains globally bounded from above and from below by a positive constant. Moreover, as $t\to\infty$, the functions χ_t and $\nabla\theta$ tend to 0 in the norm of $L^2(\Omega)$.

The situation here differs from the problem treated e. g. in [12] in several respects. On the one hand, our heat flux does not contain the regularizing linear term in θ used there. On the other hand, the Ginzburg-Landau contribution $|\nabla\chi|^2$ in the free energy, which accounts for non-local interactions, is replaced here by the integral functional B. The regularizing effect in our setting is due to the positive constant β and to a specific growth of the coefficient μ as function of θ in Eq. (1.11).

The paper is organized as follows. The main results are stated in Section 2. Section 3 is devoted to a detailed study of a class of differential inclusions. The existence and uniqueness result is proved in Section 4, global bounds are derived in Section 5, and the asymptotic behaviour of solutions is discussed in Section 6.

2 Main results

Throughout the paper, the following assumptions on the data are supposed to hold.

Hypothesis 2.1. *We consider a bounded domain $\Omega \subset \mathbb{R}^N$ with Lipschitzian boundary, N being an arbitrary integer, and for $t \geq 0$ we denote $Q_t = \Omega \times (0,t)$. In addition to the fixed positive parameters c_V, κ, and β in (1.10)–(1.11), we assume the existence of constants $0 < \mu_0 < \mu_1$ such that*

(i) $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *is a proper, convex, and lower semicontinuous function,* $\mathcal{D}(\varphi)$ *is its domain;*

(ii) $\sigma,\, \lambda \in W^{2,\infty}(\mathcal{D}(\varphi))$*;*

(iii) $G \in W^{2,\infty}(\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$, $G(z) = G(-z)$ *for all* $z \in (\mathcal{D}(\varphi) - \mathcal{D}(\varphi))$, $k \in L^\infty(\Omega \times \Omega)$, $k(x,y) = k(y,x)$ *a. e.;*

(iv) $\mu : [0,\infty) \to [0,\infty)$ *is an absolutely continuous function,* $\mu(0) > 0$*, and for a. e.* $\theta > 0$ *we have* $\mu_0(1+\theta) \;\leq\; \mu'(\theta) \;\leq\; \mu_1(1+\theta)$.

Let us first introduce some notation. For any $C > 0$ we denote

$$\mathcal{D}_C(\varphi) \;=\; \{\chi \in \mathcal{D}(\varphi)\,;\, \partial\varphi(\chi) \cap [-C, C] \neq \emptyset\}\,. \tag{2.1}$$

By [1, Example 2.3.4], $\partial\varphi$ is maximal monotone, hence $\mathcal{D}_C(\varphi)$ is a closed (possibly unbounded or degenerate) interval for every $C > 0$. Moreover, the function φ is Lipschitz continuous with constant C on $\mathcal{D}_C(\varphi)$.

We first state the existence and uniqueness theorem. The subscript 'loc' in (2.3)–(2.4) refers only to the time variable.

Theorem 2.2. Existence and uniqueness. *Let Hypothesis 2.1 hold, and let there exist constants $0 < \theta_1 < \theta_2$ and $C_0 > 0$ such that the initial data in (1.13) satisfy the conditions*

$$\left\{\begin{array}{l} \theta_0 \in W^{1,2}(\Omega) \cap L^\infty(\Omega)\,,\ \theta_1 \;\leq\; \theta_0(x) \;\leq\; \theta_2 \ \text{ a. e. in } \Omega\,, \\ \chi_0 \in L^\infty(\Omega)\,,\ \chi_0(x) \in \mathcal{D}_{C_0}(\varphi) \ \text{ a. e. in } \Omega\,. \end{array}\right. \tag{2.2}$$

Then there exists $C > 0$ and a unique solution (θ, χ) to (1.10)–(1.13) such that

$$\theta \in L^\infty_{\text{loc}}(\Omega \times (0,\infty))\,,\ \theta_t, \Delta(1/\theta) \in L^2_{\text{loc}}(\Omega \times (0,\infty))\,, \nabla\theta \in L^\infty_{\text{loc}}(0,\infty; L^2(\Omega))\,, \tag{2.3}$$

$$\chi \in L^\infty_{\text{loc}}(\Omega \times (0,\infty))\,,\ \chi_t \in L^\infty(\Omega \times (0,\infty))\,,\ \chi(x,t) \in \mathcal{D}_C(\varphi) \ \text{ a. e.} \tag{2.4}$$

Moreover, there exist positive constants c_1, c_2, independent of t, such that

$$\theta_1 \mathrm{e}^{-c_1 t} \;\leq\; \theta(x,t) \;\leq\; \sqrt{\theta_2^2 + 2c_2 t} \quad \text{a. e.} \tag{2.5}$$

We are able to prove the global boundedness and stabilization results only in domains Ω of dimension $N \leq 3$. We state them in the following form.

Theorem 2.3. Global boundedness. *Let the assumptions of Theorem 2.2 hold, let there exist* $\varphi_0 \in \mathbb{R}$ *such that*

$$\varphi(\chi) \;\geq\; \varphi_0 \quad \forall \chi \in \mathcal{D}(\varphi)\,, \tag{2.6}$$

and let $N \leq 3$. *Then there exist constants* $0 < \theta_* < \theta^*$ *such that for a. e.* $(x,t) \in \Omega \times (0,\infty)$ *we have*

$$\theta_* \;\leq\; \theta(x,t) \;\leq\; \theta^*\,. \tag{2.7}$$

The lower bound (2.6) for φ has a clear physical meaning. It says that the internal energy (1.3) is bounded from below.

Theorem 2.4. Asymptotic behaviour. *Let the assumptions of Theorem 2.3 hold. Then we have*

$$\lim_{t\to\infty} \int_\Omega \left(|\chi_t|^2 + |\nabla\theta|^2\right)(x,t)\,dx \;=\; 0\,. \tag{2.8}$$

It would also be interesting to describe the ω-limit set of the solution trajectory. This question seems to be still open and will be a subject of further research.

The proofs of the above results are postponed to the forthcoming sections. Theorem 2.2 is proved in Section 4, while Sections 5 and 6 are devoted to the proofs of Theorems 2.3 and 2.4, respectively. Before that, we investigate in detail a certain class of differential inclusions related to Eq. (1.11).

3 Solution operators to differential inclusions

In this section we derive some properties of solution operators to differential inclusions which slightly generalize (1.11). In addition to results established in [7], we prove the inequality (3.4) which is related to the “second-order energy inequality” for the underlying relaxation-hysteresis operator, see [5], and will play a crucial role in the proof of Theorem 2.4.

Consider a function φ as in Hypothesis 2.1 (i) and fix a final time $T > 0$. For a given initial condition χ_0, and a given function $\theta \in L^1(Q_T)$, we solve the following differential inclusion

$$\alpha(\theta)\,\chi_t + \partial\varphi(\chi) \;\ni\; f[\chi,\theta] \quad \text{a. e. in } Q_T, \quad \chi(x,0) = \chi_0(x) \quad \text{a. e. in } \Omega, \tag{3.1}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is a given function and $f : L^1(Q_T) \times L^1(Q_T) \to L^\infty(Q_T)$ is a given operator satisfying the following hypothesis.

Hypothesis 3.1. *There exist positive constants* α_0, L, C *such that*

(i) $\alpha_0 \leq \alpha(\theta)$ *for all* $\theta \in \mathbb{R}$*;*

(ii) $|\alpha(\theta_1) - \alpha(\theta_2)| \ \le\ L|\theta_1 - \theta_2|$ *for all* $\theta_1,\, \theta_2 \in \mathbb{R}$*;*

(iii) $|f[\chi,\theta](x,t)| \le C$ *a. e. in* Q_T *for all* $\chi, \theta \in L^1(Q_T)$ *such that* $\chi(x,t) \in \mathcal{D}(\varphi)$ *a. e. in* Q_T*;*

(iv) $|f[\chi_1,\theta_1] - f[\chi_2,\theta_2]|_{L^1(Q_t)} \ \le\ L(|\chi_1 - \chi_2|_{L^1(Q_t)} + |\theta_1 - \theta_2|_{L^1(Q_t)})$

for all $\chi_1, \theta_1, \chi_2, \theta_2 \in L^1(Q_T)$ *and* $t \in [0,T]$.

Note that in all formulas throughout this section, we keep fixed the value of C from Hypothesis 3.1 (iii).

Proposition 3.2. *Let Hypothesis 3.1 hold, and let* $\mathcal{D}_C(\varphi)$ *be as in (2.1). Then for every* $\theta \in L^1(Q_T)$ *and for every* $\chi_0 \in L^\infty(\Omega)$, $\chi_0(x) \in \mathcal{D}_C(\varphi)$ *a. e. in* Ω*, there exists a unique solution* $\chi \in L^\infty(Q_T)$ *to Eq. (3.1) such that* $\chi_t \in L^\infty(Q_T)$*, and we have*

$$\chi(x,t) \in \mathcal{D}_C(\varphi)\,, \quad |f[\chi,\theta](x,t) - \alpha(\theta(x,t))\, \chi_t(x,t)| \le C \quad \text{a. e. in } Q_T\,. \tag{3.2}$$

In addition, there exists a positive constant M *such that the solutions* $\chi_1, \chi_2 \in L^\infty(Q_T)$ *associated with* $\chi_{01}, \chi_{02} \in \mathcal{D}_C(\varphi)$ *and* $\theta_1, \theta_2 \in L^1(Q_T)$ *satisfy for all* $t \in [0,T]$ *the inequality*

$$|(\chi_1)_t - (\chi_2)_t|_{L^1(Q_t)} + |(\chi_1 - \chi_2)(t)|_{L^1(\Omega)} \ \le\ M\Big(|\chi_{01} - \chi_{02}|_{L^1(\Omega)} + |\theta_1 - \theta_2|_{L^1(Q_t)}\Big)\,. \tag{3.3}$$

If moreover both θ_t *and* $(f[\chi,\theta])_t$ *belong to* $L^1(Q_T)$*, then for every non-negative function* $\eta \in W^{1,\infty}(0,T)$ *with compact support in* $(0,T)$ *we have*

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \Big(((f[\chi,\theta])_t\, \chi_t)\,(x,t)\,\eta(t) + \frac{1}{2}\left(\alpha(\theta)\,\chi_t^2\right)(x,t)\,\dot\eta(t)\Big)\, dx\, dt \\ &\quad \ge\ \frac{1}{2}\int_0^T\!\!\int_\Omega \left(\alpha'(\theta)\,\theta_t\,\chi_t^2\right)(x,t)\,\eta(t)\, dx\, dt\,. \end{aligned} \tag{3.4}$$

Remark 3.3. The L^1-Lipschitz continuity estimate (3.3) in Proposition 3.2 cannot be extended to $L^p(Q_T)$ for $p > 1$, see [6, Example 3], except in the case when φ is a C^1-function with locally Lipschitz continuous derivative. Strong continuity $L^1(Q_T) \to L^p(Q_T)$ of the solution mapping for $p < \infty$ follows however from the uniform L^∞-bound (3.2). Indeed, testing (3.1) by χ_t, we obtain the identity

$$\varphi(\chi)_t \ =\ -\alpha(\theta)\,\chi_t^2 + f[\chi,\theta]\,\chi_t \quad \text{a. e. in } Q_T\,, \tag{3.5}$$

which implies in particular, by virtue of (3.2) and Hypothesis 3.1, that

$$|\varphi(\chi)_t| \ \le\ C\,|\chi_t| \ \le\ \frac{2C^2}{\alpha_0} \quad \text{a. e.} \tag{3.6}$$

Let now $\theta^{(n)}, \theta$ be such that $\theta^{(n)} \to \theta$ strongly in $L^1(Q_T)$ as $n \to \infty$, and let $\chi^{(n)}, \chi$ be the corresponding solutions to Eq. (3.1). Using Proposition 3.2 and taking into account the L^∞-bound (3.2), we see that $\chi^{(n)} \to \chi$, $\chi_t^{(n)} \to \chi_t$, $\varphi(\chi^{(n)})_t \to \varphi(\chi)_t$ strongly in any $L^p(Q_T)$ for $1 \le p < \infty$ as a consequence of the Lebesgue Dominated Convergence Theorem.

Before proving Proposition (3.2), let us start with a space-independent problem. For a given initial condition $\chi_0 \in \mathcal{D}(\varphi)$ and a given function $\theta \in L^1(0,T)$, we consider the differential inclusion

$$\alpha(\theta(t))\,\dot{\chi}(t) + \partial\varphi(\chi(t)) \ \ni\ g(t) \quad \text{a.e. in } (0,T), \quad \chi(0) = \chi_0, \tag{3.7}$$

where $\alpha : \mathbb{R} \to \mathbb{R}$ is as in Hypothesis 3.1 and $g \in L^\infty(0,T)$ is such that

$$|g(t)| \le C \quad \text{a.e. in } (0,T). \tag{3.8}$$

We prove the following result.

Proposition 3.4. *Let Hypotheses 3.1 (i–ii) and (3.8) hold. Then for every $\theta \in L^1(0,T)$ and every $\chi_0 \in \mathcal{D}_C(\varphi)$, there exists a unique solution $\chi \in W^{1,\infty}(0,T)$ to Eq. (3.7), and we have*

$$\chi(t) \in \mathcal{D}_C(\varphi) \ \ \forall t \in [0,T]\,, \quad \left|g(t) - \alpha(\theta(t))\,\dot{\chi}(t)\right| \ \le\ C \quad \text{a.e. in } (0,T)\,. \tag{3.9}$$

In addition, there exists a positive constant R depending only on C, α_0, and L, such that the solutions $\chi_1,\, \chi_2 \in W^{1,\infty}(0,T)$ associated with $\chi_{01}, \chi_{02} \in \mathcal{D}_C(\varphi)$, $\theta_1, \theta_2 \in L^1(0,T)$, and $g_1, g_2 \in L^\infty(0,T)$ with the constraint (3.8) satisfy the inequality

$$|\dot{\chi}_1 - \dot{\chi}_2|(t) + \frac{d}{dt}|\chi_1 - \chi_2|(t) \le R\Big(|\theta_1 - \theta_2|(t) + |g_1 - g_2|(t)\Big) \quad \text{a.e. in } (0,T). \tag{3.10}$$

If moreover both θ and g belong to $W^{1,1}(0,T)$, then for every non-negative function $\eta \in W^{1,\infty}(0,T)$ with compact support in $(0,T)$ we have

$$\int_0^T \left(\dot{g}(t)\,\dot{\chi}(t)\ \eta(t) + \frac{1}{2}\,\alpha(\theta(t))\,\dot{\chi}^2(t)\,\dot{\eta}(t)\right) dt \ \ge\ \frac{1}{2}\int_0^T \alpha'(\theta(t))\,\dot{\theta}(t)\,\dot{\chi}^2(t)\,\eta(t)\,dt\,. \tag{3.11}$$

Proof of Proposition 3.4. We first prove the existence of solutions. We fix $\theta \in L^1(0,T)$, $\chi_0 \in \mathcal{D}_C(\varphi)$ and, for $n \in \mathbb{N}$ and $k = 1, \dots, n$, define the sequences

$$\alpha_k = \frac{n}{T}\int_{t_{k-1}}^{t_k} \alpha(\theta(t))\,dt\,, \ \ g_k = \frac{n}{T}\int_{t_{k-1}}^{t_k} g(t)\,dt\,, \tag{3.12}$$

$$\chi_k = \Big(\frac{n\alpha_k}{T} I + \partial\varphi\Big)^{-1}\Big(g_k + \frac{n\alpha_k}{T}\,\chi_{k-1}\Big) \tag{3.13}$$

corresponding to the partition $t_0 = 0$, $t_k = Tk/n$, where $I(u) = u$ is the identity mapping. Assume that for some $k \ge 1$ we have

$$\partial\varphi(\chi_k) \ \ni\ g_k - \frac{n\alpha_k}{T}(\chi_k - \chi_{k-1}) \ >\ C\,. \tag{3.14}$$

By hypothesis, we have $|g_k| \le C$, hence $\max \mathcal{D}_C(\varphi) \le \chi_k < \chi_{k-1}$ by the monotonicity of $\partial\varphi$. This yields, if $k - 1 > 0$, that

$$g_{k-1} - \frac{n\alpha_k}{T}(\chi_{k-1} - \chi_{k-2}) \ \ge\ g_k - \frac{n\alpha_k}{T}(\chi_k - \chi_{k-1}) \ >\ C\,, \tag{3.15}$$

and by induction we obtain $\max \mathcal{D}_C(\varphi) \leq \chi_k < \chi_{k-1} < \cdots < \chi_0$ which is a contradiction. We obtain a similar contradiction by assuming that $g_k - (n\alpha_k/T)(\chi_k - \chi_{k-1}) < -C$. Using the fact that $\alpha_k \geq \alpha_0$, we thus have for all $k = 1, \ldots, n$ that

$$\left| g_k - \frac{n\alpha_k}{T}(\chi_k - \chi_{k-1}) \right| \leq C, \quad \chi_k \in \mathcal{D}_C(\varphi), \quad |\chi_k - \chi_{k-1}| \leq \frac{2CT}{n\alpha_0}. \tag{3.16}$$

We now define the interpolates

$$\alpha^{(n)}(t) = \alpha_k, \quad g^{(n)}(t) = g_k, \quad \bar{\chi}^{(n)}(t) = \chi_k, \quad \underline{\chi}^{(n)}(t) = \chi_{k-1}, \tag{3.17}$$

$$\chi^{(n)}(t) = \chi_{k-1} + \frac{n}{T}(t - t_{k-1})(\chi_k - \chi_{k-1}), \tag{3.18}$$

for $t \in [t_{k-1}, t_k)$, continuously extended to $t = T$. The functions $\chi^{(n)}$ are bounded in $W^{1,\infty}(0,T)$ uniformly with respect to $n \in \mathbb{N}$. Passing to a subsequence, if necessary, we find $\chi \in W^{1,\infty}(0,T)$ such that $\chi(0) = \chi_0$, $\dot{\chi}^{(n)} \to \dot{\chi}$ in $L^\infty(0,T)$ weakly-star, and $\chi^{(n)} \to \chi$ uniformly in $[0,T]$. Using the inequalities

$$|\chi^{(n)}(t) - \bar{\chi}^{(n)}(t)| \leq \frac{2CT}{n\alpha_0}, \quad |\chi^{(n)}(t) - \underline{\chi}^{(n)}(t)| \leq \frac{2CT}{n\alpha_0}, \tag{3.19}$$

we also see that $\bar{\chi}^{(n)} \to \chi$, $\underline{\chi}^{(n)} \to \chi$ uniformly. Using the Mean Continuity Theorem for functions in $L^1(0,T)$, we conclude that $\alpha^{(n)}$ converge to $\alpha(\theta(\cdot))$ strongly in $L^1(0,T)$, and $g^{(n)}$ converge to g strongly in any $L^p(0,T)$ for $1 \leq p < \infty$ and weakly-star in $L^\infty(0,T)$. Let now $z \in L^\infty(0,T)$ be a test function, $z(t) \geq 0$ a. e. in $(0,T)$, and let $w \in \mathcal{D}(\varphi)$, $\xi \in \partial\varphi(w)$ be arbitrary. By construction, we have

$$(g^{(n)}(t) - \alpha^{(n)}(t)\,\dot{\chi}^{(n)}(t) - \xi)(\bar{\chi}^{(n)}(t) - w) \geq 0 \quad \text{a. e. in } (0,T), \tag{3.20}$$

hence

$$\int_0^T (g^{(n)}(t) - \alpha^{(n)}(t)\dot{\chi}^{(n)}(t) - \xi)(\bar{\chi}^{(n)}(t) - w)\, z(t)\, dt \geq 0. \tag{3.21}$$

Passing to the limit as $n \nearrow \infty$ in (3.21) we obtain

$$(g(t) - \alpha(\theta(t))\,\dot{\chi}(t) - \xi)(\chi(t) - w) \geq 0 \quad \text{a. e.} \tag{3.22}$$

Since $\partial\varphi$ is maximal monotone, the function χ satisfies Eq. (3.7). Estimate (3.9) follows from (3.16).

We now prove inequality (3.10) which also implies uniqueness of solutions to Eq. (3.7). Let $\chi_{01}, \chi_{02} \in \mathcal{D}_C(\varphi)$, $\theta_1, \theta_2 \in L^1(0,T)$, and g_1, $g_2 \in L^\infty(0,T)$ be functions satisfying (3.8), and let $\chi_1, \chi_2 \in W^{1,\infty}(0,T)$ be corresponding solutions of Eq. (3.7). For $i = 1, 2$ and $t \in (0,T)$ put

$$\xi_i(t) = g_i(t) - \alpha(\theta_i(t))\,\dot{\chi}_i(t). \tag{3.23}$$

As $\partial\varphi$ is monotone and $\xi_i(t) \in \partial\varphi(\chi_i(t))$ for $i = 1, 2$ a. e. in $(0, T)$, we have

$$\big(\xi_1(t) - \xi_2(t)\big)\,\big(\chi_1(t) - \chi_2(t)\big) \;\geq\; 0 \quad \text{a. e.} \tag{3.24}$$

We test the identity

$$\begin{aligned} \xi_1(t) - \xi_2(t) \;+\; \alpha(\theta_1(t))\,\big(\dot\chi_1(t) - \dot\chi_2(t)\big) = \;& \dot\chi_2(t)\,\big(\alpha(\theta_2(t)) - \alpha(\theta_1(t))\big) \\ & + g_1(t) - g_2(t) \quad \text{a. e.} \end{aligned} \tag{3.25}$$

by the sign of $\xi_1(t) - \xi_2(t)$ if $\xi_1(t) \neq \xi_2(t)$, or otherwise by the sign of $\chi_1(t) - \chi_2(t)$. By virtue of (3.24), this yields

$$\begin{aligned} & \big|\xi_1 - \xi_2\big|(t) \;+\; \alpha(\theta_1(t))\,\frac{d}{dt}\big|\chi_1 - \chi_2\big|(t) \\ & \quad \leq\; \big|\dot\chi_2(t)\big|\,\big|\alpha(\theta_1(t)) - \alpha(\theta_2(t))\big| + \big|g_1 - g_2\big|(t) \quad \text{a. e. ,} \end{aligned} \tag{3.26}$$

hence

$$\begin{aligned} & \alpha(\theta_1(t))\,\left(\big|\dot\chi_1 - \dot\chi_2\big|(t) + \frac{d}{dt}\big|\chi_1 - \chi_2\big|(t)\right) \\ & \quad \leq\; 2\big|g_1 - g_2\big|(t) + 2\,\big|\dot\chi_2(t)\big|\,\big|\alpha(\theta_1(t)) - \alpha(\theta_2(t))\big| \quad \text{a. e.} \end{aligned} \tag{3.27}$$

Using the estimates

$$\alpha(\theta_i(t)) \;\geq\; \alpha_0 \;\text{ for }\; i = 1, 2, \quad \big|\dot\chi_2(t)\big| \;\leq\; \frac{2C}{\alpha_0} \quad \text{a. e. ,} \tag{3.28}$$

and Hypothesis 3.1, we obtain from (3.27) that

$$\begin{aligned} & \big|\dot\chi_1 - \dot\chi_2\big|(t) + \frac{d}{dt}\big|\chi_1 - \chi_2\big|(t) \\ & \quad \leq\; \frac{2}{\alpha_0}\big|g_1 - g_2\big|(t) + \left(\frac{4CL}{\alpha_0^2}\right)\big|\theta_1 - \theta_2\big|(t) \quad \text{a. e. ,} \end{aligned} \tag{3.29}$$

that is exactly (3.10).

To prove inequality (3.11), we fix η with the required properties and $h_0 > 0$ such that $\operatorname{supp}\eta \subset (0, T - h_0)$. For $0 < h < h_0$ we use the monotonicity of $\partial\varphi$ which yields that

$$\begin{aligned} & (g(t+h) - g(t))\,(\chi(t+h) - \chi(t)) \\ & \quad \geq\; \big(\alpha(\theta(t+h))\,\dot\chi(t+h) - \alpha(\theta(t))\,\dot\chi(t)\big)\,(\chi(t+h) - \chi(t)) \end{aligned} \tag{3.30}$$

for a. e. $t \in (0, T - h)$. Testing the above inequality by $\eta(t)$, dividing by h^2, and integrating by parts, we obtain

$$
\begin{aligned}
(3.31)\quad & \int_0^{T-h} \left(\left(\frac{g(t+h)-g(t)}{h}\right)\left(\frac{\chi(t+h)-\chi(t)}{h}\right)\eta(t) \right.\\
&\qquad \left. +\frac{1}{2}\alpha(\theta(t))\left(\frac{\chi(t+h)-\chi(t)}{h}\right)^2 \dot\eta(t)\right) dt\\
&\geq \int_0^{T-h} \left(\left(\frac{\alpha(\theta(t+h))-\alpha(\theta(t))}{h}\right)\left(\frac{\chi(t+h)-\chi(t)}{h}\right)\dot\chi(t+h) \right.\\
&\qquad \left. -\frac{1}{2}\alpha'(\theta(t))\,\dot\theta(t)\left(\frac{\chi(t+h)-\chi(t)}{h}\right)^2\right)\eta(t)\, dt\,.
\end{aligned}
$$

By the Mean Continuity Theorem for the Lebesgue integral, the difference quotients converge as $h \searrow 0+$ to the derivatives strongly in $L^1(0,T)$, hence pointwise almost everywhere. Thanks to the L^∞ bound for $\dot\chi$, we can pass to the limit in (3.31) as $h \searrow 0+$ and obtain (3.11). The proof of Proposition 3.4 is complete. ■

We now use this result to prove Proposition 3.2.

Proof of Proposition 3.2. For given $\theta \in L^1(Q_T)$ and $\chi_0 \in L^\infty(\Omega)$, $\chi_0(x) \in \mathcal{D}_C(\varphi)$ a. e., we prove the existence of a unique solution to (3.1) by the Banach contraction argument. We define the set

$$
(3.32)\qquad S \ := \left\{ v \in L^\infty(Q_T)\ :\ \begin{array}{l} v_t \in L^\infty(Q_T)\,,\ |v_t|_{L^\infty(Q_T)} \leq 2C/\alpha_0\,,\\ v(x,0)\ =\ \chi_0(x)\ \text{ a. e. in } \Omega \end{array} \right\}
$$

as a closed subset of $L^1(Q_T)$ endowed with the weighted norm

$$
(3.33)\qquad |v|_{RL} := \int_0^T \mathrm{e}^{-2RLt} \int_\Omega |v(x,t)|\, dx\, dt\,,
$$

where $R = R(C,\alpha_0,L)$ and L are as in Hypothesis 3.1 and Proposition 3.4. For an arbitrary $\tilde\chi \in S$, we put $\tilde g(x,t) = f[\tilde\chi,\theta](x,t)$, and define $\chi(x,t)$ as the solution of the differential inclusion

$$
(3.34)\qquad \begin{aligned} \alpha(\theta(x,t))\,\chi_t(x,t) + \partial\varphi(\chi(x,t)) &\ \ni\ \tilde g(x,t) && \text{a. e. in } Q_T\,,\\ \chi(x,0) &\ =\ \chi_0(x) && \text{a. e. in } \Omega\,. \end{aligned}
$$

For almost every $x \in \Omega$, this inclusion is of the form (3.7) with right-hand side satisfying (3.8). By Proposition 3.4, the function χ belongs to S, and we may define the solution mapping $\mathcal{T} : S \to S : \tilde\chi \mapsto \chi$. We check that $\mathcal{T}$ is a contraction with respect to the norm (3.33). Indeed, we integrate the estimate (3.10) with $\theta_1 = \theta_2$, $g_i(t) = f[\tilde\chi_i,\theta_i](t)$ for $i = 1,2$ from 0 to t. This, thanks to Hypothesis 3.1 (iv), leads to

$$
(3.35)\qquad \int_\Omega |\chi_1 - \chi_2|(x,t)\, dx \ \leq\ RL \int_0^t \int_\Omega |\tilde\chi_1 - \tilde\chi_2|(x,\tau)\, dx\, d\tau\,.
$$

Now, multiplying both sides of this inequality by e^{-2RLt}, and integrating over $[0,T]$, we infer that

$$|\chi_1 - \chi_2|_{RL} \ \le\ \frac{1}{2}|\tilde\chi_1 - \tilde\chi_2|_{RL}\,. \tag{3.36}$$

Hence $\mathcal{T}$ is a contraction on S, and the Banach fixed point theorem yields the existence and uniqueness of a solution $\chi \in S$ of the differential inclusion (3.1). Estimate (3.2) follows directly from (3.9). Finally, in order to prove (3.3), take $\chi_{01}, \chi_{02} \in \mathcal{D}_C(\varphi)$, $\theta_1, \theta_2 \in L^1(Q_T)$, and let χ_1, χ_2 be the corresponding solutions of Eq. (3.1). For almost all x we use (3.10) with $g_i(t) = f[\chi_i, \theta_i](x,t)$, $i = 1,2$. Integrating over Ω and over $(0,t)$ for $t \in (0,T]$, and using Hypothesis 3.1, we obtain that

$$\begin{aligned}\int_0^t\int_\Omega |(\chi_1)_t - (\chi_2)_t|(x,s)\,dx\,ds + \int_\Omega |\chi_1 - \chi_2|(x,t)\,dx - |\chi_{01} - \chi_{02}|_{L^1(\Omega)}\\ \le\ \int_0^t\int_\Omega (RL|\chi_1 - \chi_2|(x,s) + R(L+1)|\theta_1 - \theta_2|(x,s))\,dx\,ds\,.\end{aligned} \tag{3.37}$$

Gronwall's argument then yields

$$\begin{aligned}\int_0^t\int_\Omega |(\chi_1)_t - (\chi_2)_t|(x,s)\,dx\,ds + \int_\Omega |\chi_1 - \chi_2|(x,t)\,dx\\ \le \mathrm{e}^{RLt}\left(|\chi_{01} - \chi_{02}|_{L^1(\Omega)} + R(L+1)\int_0^t\int_\Omega |\theta_1 - \theta_2|(x,s)\,dx\,ds\right)\,,\end{aligned} \tag{3.38}$$

and (3.3) follows. Inequality (3.4) is a direct consequence of (3.11). ■

4 Existence and uniqueness

This section is devoted to the proof of Theorem 2.2. We rewrite Eq. (1.11) as

$$\frac{\mu(\theta)}{\beta+\theta}\,\chi_t + \partial\varphi(\chi) \ \ni\ -\frac{\theta}{\beta+\theta}\,\sigma'(\chi) - \frac{1}{\beta+\theta}\,(\lambda'(\chi) + b[\chi])\,. \tag{4.1}$$

We see that it is of the form (3.1), and we may use Proposition 3.2 for any $T > 0$ with some suitable $C > 0$ independent of T as an immediate consequence of Hypothesis 2.1. This enables us to define a mapping $A : L^1_{\mathrm{loc}}(\Omega \times (0,\infty)) \to L^\infty_{\mathrm{loc}}(\Omega \times (0,\infty))$ which with each $\theta \in L^1_{\mathrm{loc}}(\Omega \times (0,\infty))$ associates the solution χ of (4.1) satisfying the initial condition $\chi(x,0) = \chi_0(x)$ a. e. in Ω. Problem (1.10)–(1.13) is thus of the form

$$c_V\,\theta_t + \kappa\Delta\left(\frac{1}{\theta}\right) + (\lambda(\chi) + \beta\,\varphi(\chi))_t + b[\chi]\,\chi_t \ =\ 0\,, \tag{4.2}$$

$$\chi \ =\ A[\theta]\,, \tag{4.3}$$

$$\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{\theta}\right) \ =\ 0 \quad \text{on } \partial\Omega \times (0,\infty)\,, \tag{4.4}$$

$$\theta(x,0) = \theta_0(x) \quad \text{a. e. in } \Omega\,. \tag{4.5}$$

4.1 Uniqueness

Let $\theta, \tilde{\theta}$ be two positive solutions of (4.2)–(4.5) with the prescribed regularity, and set $\chi = A[\theta]$, $\tilde{\chi} = A[\tilde{\theta}]$. We test the difference of equations (4.2) written for θ and $\tilde{\theta}$ by the sign of $\theta - \tilde{\theta}$ (which is equal to the sign of $(1/\tilde{\theta}) - (1/\theta)$) and obtain

$$(4.6) \qquad c_V \frac{d}{dt} \int_\Omega |\theta - \tilde{\theta}|(x,t)\, dx \\ \leq \int_\Omega \left| (\lambda(\chi) + \beta\,\varphi(\chi))_t + b[\chi]\,\chi_t - \big(\lambda(\tilde{\chi}) + \beta\,\varphi(\tilde{\chi})\big)_t - b[\tilde{\chi}]\,\tilde{\chi}_t \right|\, dx\,.$$

Integrating this relation from 0 to t and using (3.3) and (3.5) leads to the inequality

$$(4.7) \qquad \int_\Omega |\theta - \tilde{\theta}|(x,t)\, dx \;\leq\; C_1 \left(1 + \sup_{(x,\tau)\in Q_t} \theta(x,\tau) \right) \int_0^t \int_\Omega |\theta - \tilde{\theta}|(x,\tau)\, dx\, d\tau$$

with some constant C_1 independent of t. As θ is locally bounded, we may use Gronwall's argument and conclude that $\theta = \tilde{\theta}$ a. e.

4.2 A cut-off system

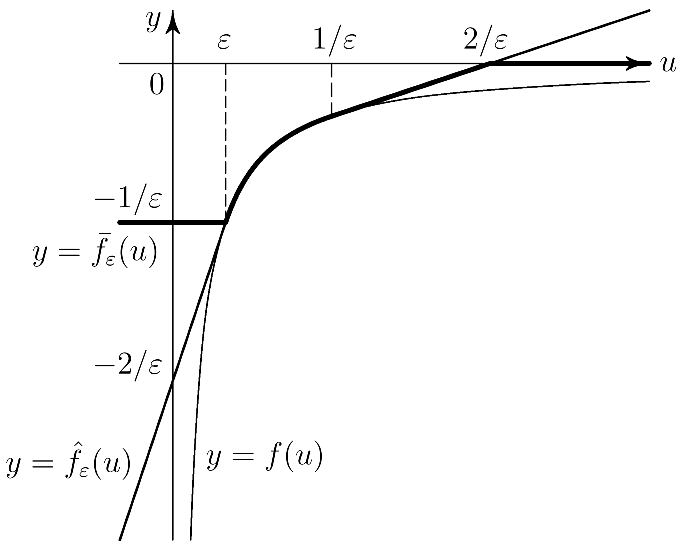

Figure 1: The cut-off diagram.

We fix some $T > 0$ and choose a parameter $\varepsilon \in (0,1)$ which will be specified later on.

Let $\bar f_\varepsilon, \hat f_\varepsilon, f$ be auxiliary functions defined as

$$f(u) \;=\; \begin{cases} -\infty & \text{for } u \le 0\,, \\ -\frac{1}{u} & \text{for } u > 0\,, \end{cases} \tag{4.8}$$

$$\hat f_\varepsilon(u) \;=\; \begin{cases} \frac{u}{\varepsilon^2} - \frac{2}{\varepsilon} & \text{for } u < \varepsilon\,, \\ -\frac{1}{u} & \text{for } \varepsilon \le u \le \frac{1}{\varepsilon}\,, \\ \varepsilon^2 u - 2\varepsilon & \text{for } u > \frac{1}{\varepsilon}\,, \end{cases} \tag{4.9}$$

$$\bar f_\varepsilon(u) \;=\; \begin{cases} -\frac{1}{\varepsilon} & \text{for } u < \varepsilon\,, \\ \hat f_\varepsilon(u) & \text{for } \varepsilon \le u \le \frac{2}{\varepsilon}\,, \\ 0 & \text{for } u > \frac{2}{\varepsilon}\,, \end{cases} \tag{4.10}$$

see Figure 1. We replace (4.2)–(4.5) by the cut-off system in Q_T

$$c_V\, \hat f_\varepsilon(u)_t - \kappa \Delta u \;=\; \left(\lambda(\chi^\varepsilon) + \beta\, \varphi(\chi^\varepsilon)\right)_t + b[\chi^\varepsilon]\, \chi^\varepsilon_t\,, \tag{4.11}$$

$$\chi^\varepsilon \;=\; A[-\bar f_\varepsilon(u)]\,, \tag{4.12}$$

$$\frac{\partial u}{\partial \mathbf{n}} \;=\; 0 \quad \text{on } \partial\Omega \times (0,T)\,, \tag{4.13}$$

$$u(x,0) = \frac{1}{\theta_0(x)} \quad \text{a.e. in } \Omega\,. \tag{4.14}$$

For each fixed $\varepsilon \in (0,1)$, this is a regular system with Lipschitz continuous nonlinearities which admits a unique solution (u, χ^ε) with $u \in L^2(0,T; W^{2,2}(\Omega))$, $u_t \in L^2(Q_T)$, $\nabla u \in L^\infty(0,T; L^2(\Omega))$, and $\chi^\varepsilon, \chi^\varepsilon_t \in L^\infty(Q_T)$.

We now use the maximum principle to show that $\theta := 1/u$ solves (4.2)–(4.5) in Q_T provided $\varepsilon = \varepsilon(T)$ is sufficiently small.

Maximum principle 1.

Put $\hat\theta^\varepsilon = -\hat f_\varepsilon(u)$, $\bar\theta^\varepsilon = -\bar f_\varepsilon(u)$. By (3.5) and (4.1) we have

$$\mu(\bar\theta^\varepsilon)(\chi^\varepsilon_t)^2 + \bar\theta^\varepsilon(\sigma(\chi^\varepsilon) + \varphi(\chi^\varepsilon))_t + \left(\lambda(\chi^\varepsilon) + \beta\, \varphi(\chi^\varepsilon)\right)_t + b[\chi^\varepsilon]\, \chi^\varepsilon_t \;=\; 0\,, \tag{4.15}$$

hence we may rewrite Eq. (4.11) in the form

$$c_V\, \hat\theta^\varepsilon_t + \kappa \Delta u \;=\; \mu(\bar\theta^\varepsilon)(\chi^\varepsilon_t)^2 + \bar\theta^\varepsilon(\sigma(\chi^\varepsilon) + \varphi(\chi^\varepsilon))_t\,. \tag{4.16}$$

We may find a constant $c_1 > 0$ independent of ε such that

$$\hat\theta^\varepsilon_t + \frac{\kappa}{c_V}\Delta u \;\ge\; -c_1 \bar\theta^\varepsilon \quad \text{a.e.} \tag{4.17}$$

We now test the inequality (4.17) by the non-positive function $p(x,t) = -(\theta_1\, \mathrm{e}^{-c_1 t} - \hat\theta^\varepsilon(x,t))^+$, where $z^+ = \max\{0, z\}$ denotes the positive part of a real number z and θ_1 is defined in (2.2), and obtain

$$\int_\Omega \left(\hat\theta^\varepsilon_t p - \frac{\kappa}{c_V} \langle \nabla u, \nabla p \rangle \right) dx \;\le\; c_1 \int_\Omega |\bar\theta^\varepsilon||p|\, dx \;\le\; c_1 \int_\Omega |\hat\theta^\varepsilon||p|\, dx \quad \text{a.e.} \tag{4.18}$$

We have $\langle \nabla p, \nabla u\rangle \le 0$ a. e., and (4.18) yields that

$$\int_\Omega p\,\left(p+\theta_1\,\mathrm{e}^{-c_1 t}\right)_t\,dx \;\le\; c_1\int_\Omega |p|\,\left(|p|+\theta_1\,\mathrm{e}^{-c_1 t}\right)\,dx\,, \tag{4.19}$$

hence

$$\frac{d}{dt}\int_\Omega p^2(x,t)\,dx \;\le\; 2c_1\int_\Omega p^2(x,t)\,dx \quad \text{a. e.} \tag{4.20}$$

Let us fix some $\varepsilon > 0$ such that

$$\varepsilon \;\le\; \min\left\{\frac{1}{\theta_2}\,,\,\theta_1\,\mathrm{e}^{-c_1 T}\right\}\,, \tag{4.21}$$

with θ_2 specified in (2.2). Then $u(x,0)\in(\varepsilon, 1/\varepsilon)$, hence $\bar\theta^\varepsilon(x,0)=\theta_0(x)$ and $p(x,0)=0$ a. e. From (4.20) combined with Gronwall's Lemma it follows that $p\equiv 0$, hence $\hat\theta^\varepsilon(x,t)\ge\theta_1\,\mathrm{e}^{-c_1 t}$ a. e. We have $\varepsilon\le\theta_1\,\mathrm{e}^{-c_1 t}\le 1/\varepsilon$, hence

$$u(x,t) \;\le\; \hat f_\varepsilon^{-1}\left(\theta_1\,\mathrm{e}^{-c_1 t}\right) \;=\; \frac{1}{\theta_1}e^{c_1 t} \;\le\; \frac{1}{\varepsilon} \quad \text{a. e.} \tag{4.22}$$

Maximum principle 2.

By virtue of (4.15) and Hypothesis 2.1 (iv), there exists $C_2 > 0$ such that for a. e. $(x,t)\in Q_T$ we have

$$|\chi_t^\varepsilon(x,t)| \;\le\; \frac{C_2}{\bar\theta^\varepsilon(x,t)}\,. \tag{4.23}$$

Taking (4.22) into account, we may find $c_2 > 0$ such that

$$\frac{1}{c_V}\left|\left(\lambda(\chi^\varepsilon)+\beta\,\varphi(\chi^\varepsilon)\right)_t + b[\chi^\varepsilon]\,\chi_t^\varepsilon\right|(x,t) \;\le\; \frac{c_2}{\bar\theta^\varepsilon(x,t)} \;=\; c_2\,\max\{\varepsilon, u(x,t)\}\,. \tag{4.24}$$

For $w\in\mathbb{R}$ put $M_\varepsilon(w)=c_2\,\max\{\varepsilon,w\}$. From (4.11) and (4.24) it follows that

$$\hat f_\varepsilon(u)_t \;-\; \frac{\kappa}{c_V}\Delta u + M_\varepsilon(u) \;\ge\; 0 \quad \text{a. e.} \tag{4.25}$$

Let us consider

$$\varepsilon \;\le\; \min\left\{\theta_1\,e^{-c_1 T}\,,\,\frac{1}{\sqrt{\theta_2^2+2c_2 T}}\right\}\,, \tag{4.26}$$

and define $v\in C^1([0,T])$ as the unique solution of the ODE

$$\frac{d}{dt}\hat f_\varepsilon(v) \,+\, M_\varepsilon(v) \;=\; 0\,, \quad v(0)=\frac{1}{\theta_2}\in\left(\varepsilon,\frac{1}{\varepsilon}\right)\,. \tag{4.27}$$

As long as $v(t)$ stays in the interval $[\varepsilon, 1/\varepsilon]$, Eq. (4.27) has the form $\dot v(t) + c_2 v^3(t) = 0$, hence

$$v(t) \;=\; \frac{1}{\sqrt{\theta_2^2 + 2c_2 t}} \qquad \text{for } t \in [0,T]\,. \tag{4.28}$$

Subtracting (4.25) from the relation

$$\hat f_\varepsilon(v)_t \;-\; \frac{\kappa}{c_V}\Delta v + M_\varepsilon(v) \;=\; 0\,, \tag{4.29}$$

we obtain

$$\Big(\hat f_\varepsilon(v) - \hat f_\varepsilon(u)\Big)_t \;-\; \frac{\kappa}{c_V}\Delta(v-u) + M_\varepsilon(v) - M_\varepsilon(u) \;\le\; 0\,. \tag{4.30}$$

Let H_δ for $\delta > 0$ be a regularization of the Heaviside function, say,

$$H_\delta(w) \;=\; \max\{0, \min\{1, w/\delta\}\} \qquad \text{for } w \in \mathbb{R}\,.$$

Testing (4.30) by $H_\delta(v-u)$ and letting $\delta \searrow 0+$ (note that both $\hat f_\varepsilon$ and M_ε are non-decreasing functions) we obtain

$$\frac{d}{dt}\int_\Omega \left(\Big(\hat f_\varepsilon(v) - \hat f_\varepsilon(u)\Big)^+\right) dx + \int_\Omega (M_\varepsilon(v) - M_\varepsilon(u))^+\,dx \;\le\; 0\,. \tag{4.31}$$

We have $u(x,0) \ge v(0)$, hence $u(x,t) \ge v(t) \ge \varepsilon$ a. e. in $\Omega \times (0,T)$. Using also (4.22), we thus obtain that $\hat f_\varepsilon(x,t) = -1/u(x,t)$, hence $\theta = 1/u$ satisfies (4.2)–(4.3) a. e. in $\Omega \times (0,T)$ whenever (4.26) holds. Since $T > 0$ has been arbitrarily chosen, we conclude from the above computations that the unique global solution of (4.2)–(4.5) satisfies (2.5), and Theorem 2.2 is proved.

5 Global bounds

We use here a variant of the Alikakos-Moser iteration scheme, see e. g. [10, 14], to derive the bounds for $\theta(x,t)$ stated in Theorem 2.3. To simplify the formulas, we denote by $|\cdot|_p$ the usual norm in $L^p(\Omega)$ for $1 \le p \le \infty$. We make repeated use of the well-known interpolation inequality

$$|v|_q \;\le\; K\,\left(\delta^\alpha\,|\nabla v|_2 \,+\, \delta^{-\gamma}\,|v|_1\right) \qquad \text{for } 1 \ge \frac{1}{q} > \frac{1}{2} - \frac{1}{N}\,, \tag{5.1}$$

which holds for every $v \in W^{1,2}(\Omega)$ and every $\delta \in (0,1)$ with $\alpha = 1 - N((1/2) - (1/q))$, $\gamma = N(1 - (1/q))$, and with a constant $K > 0$ independent of v and δ. In fact, we deal only with values of q between 2 and 4, so that within this range, the constant K in (5.1) can be taken independent of q. On the other hand, the hypothesis $N \le 3$ is motivated by the fact that inequality (5.1) has to hold for all $q \in [2,4]$.

The following elementary estimate will also be of interest.

Lemma 5.1. *Let* $w, h : [0,\infty) \to [0,\infty)$ *be absolutely continuous functions such that* h *is non-decreasing, and for some* $a, \varrho > 0$ *we have*

$$\dot{w}(t) + a\, w^{\varrho}(t) \ \le\ h(t) \quad a.\,e. \tag{5.2}$$

Then for all $t \ge 0$ *we have*

$$a\, w^{\varrho}(t) \ \le\ h(t) + (a\, w^{\varrho}(0) - h(0))^{+}\,. \tag{5.3}$$

Proof. We show that $V(t) := (a\, w^{\varrho}(t) - h(t))^{+}$ is a Lyapunov function in $[0,\infty)$. Indeed, V is absolutely continuous, and if $\dot{V}(t) \ne 0$ for some t, then $a\, w^{\varrho}(t) - h(t) > 0$, hence $\dot{w}(t) < 0$, and $\dot{V}(t) = a\varrho\, w^{\varrho-1}\, \dot{w}(t) - \dot{h}(t) \le 0$. Consequently, V is non-increasing, hence $V(t) \le V(0)$ for all t and (5.3) follows. ∎

5.1 An upper bound

Integrating Eq. (1.10) over Ω yields the energy conservation principle

$$\frac{d}{dt}\int_{\Omega} (c_V\theta(x,t) + \lambda(X) + \beta\varphi(X) + B[X])\; dx \ =\ 0\,, \tag{5.4}$$

hence there exists a constant $C_3 > 0$ independent of t such that

$$\int_{\Omega} \theta(x,t)\, dx \ \le\ C_3\,. \tag{5.5}$$

We further test Eq. (1.10) by θ^{r} for $r \ge 3$ and use the inequalities (4.23)–(4.24) for $\varepsilon = 0$ to obtain

$$\frac{1}{r+1}\frac{d}{dt}|\theta(t)|_{r+1}^{r+1} + r\frac{\kappa}{c_V}\int_{\Omega} |\nabla\theta(t)|^2\, \theta(t)^{r-3}\, dx \ \le\ c_2\, |\theta(t)|_{r-1}^{r-1}\,. \tag{5.6}$$

Put $\psi_r = \theta^{(r-1)/2}$. Then (5.6) reads

$$\frac{1}{r+1}\frac{d}{dt}\, |\psi_r(t)|_{2+4/(r-1)}^{2+4/(r-1)} + \frac{4r\kappa}{c_V(r-1)^2}\; |\nabla\psi_r(t)|_2^2 \ \le\ c_2\; |\psi_r(t)|_2^2\ , \tag{5.7}$$

hence

$$\frac{d}{dt}\, |\psi_r(t)|_{2+4/(r-1)}^{2+4/(r-1)} + \frac{4\kappa}{c_V}\, |\nabla\psi_r(t)|_2^2 \ \le\ c_2\, (r+1)\, |\psi_r(t)|_2^2\ . \tag{5.8}$$

In what follows, we denote by $C_4, C_5, \ldots$ positive constants independent of r and t. Using (5.1) for $v = \psi_r$, $q = 2$, and $\delta = \tilde{\delta}/\sqrt{r+1}$ with suitably chosen $\tilde{\delta}$ independent of r ($0 < \tilde{\delta} < \min\{1, (2\kappa/(c_V c_2))^{1/2}/K\}$, say), we have

$$\frac{d}{dt}\, |\psi_r(t)|_{2+4/(r-1)}^{2+4/(r-1)} + C_4\; |\nabla\psi_r(t)|_2^2 \ \le\ C_5\, (r+1)^{1+N/2}\, |\psi_r(t)|_1^2\ . \tag{5.9}$$

The next step consists in a repeated use of (5.1) with $q = 2 + 4/(r-1)$ in order to estimate the term $|\nabla\psi_r(t)|_2$ from below by $|\psi_r(t)|_{2+4/(r-1)}$. All related values of

q are between 2 and 4, hence the exponents α, γ in (5.1) satisfy the inequalities $1 - N/4 \le \alpha \le 1$, $N/2 \le \gamma \le 3N/4$. This enables us to choose an appropriate δ independent of r and obtain

$$\frac{d}{dt}\,|\psi_r(t)|_{2+4/(r-1)}^{2+4/(r-1)} + C_6\,|\psi_r(t)|_{2+4/(r-1)}^{2} \;\le\; C_7\,(r+1)^{1+N/2}\,|\psi_r(t)|_1^2 \,. \tag{5.10}$$

By (5.5) we have that $|\psi_3(t)|_1 \le C_3$, hence (5.13) for $r = 3$ can be written in the form

$$\frac{d}{dt}\,|\psi_3(t)|_4^4 + C_6\,|\psi_3(t)|_4^2 \;\le\; C_8 \,. \tag{5.11}$$

For $r \ge 5$ we have

$$|\psi_r(t)|_1^2 \;=\; \left|\psi_{(r+1)/2}(t)\right|_2^4 , \tag{5.12}$$

hence

$$\frac{d}{dt}\,|\psi_r(t)|_{2+4/(r-1)}^{2+4/(r-1)} + C_6\,|\psi_r(t)|_{2+4/(r-1)}^{2} \;\le\; C_9\,(r+1)^{1+N/2}\left|\psi_{(r+1)/2}(t)\right|_2^4 . \tag{5.13}$$

For $k \in \mathbb{N}$ put $r_k = 2^k + 1$, $q_k = 2 + 4/(r_k - 1)$ and

$$\phi_k(t) \;=\; \max_{0\le s\le t}\,|\psi_{r_k}(s)|_2^2 \,. \tag{5.14}$$

Set $b = \theta_2 \max\{1, |\Omega|\}$. For all $r \ge 3$ and $q \ge 2$ we have $|\psi_r(0)|_q^2 \le b^{r-1}$. From the inequality

$$|\psi_{r_k}(t)|_2^2 \;\le\; |\Omega|^{1/(2^{k-1}+1)}\,|\psi_{r_k}(t)|_{q_k}^2 ,$$

as well as from Lemma 5.1, (5.11), and (5.13) it follows that

$$\phi_1(t) \;\le\; C_{10} , \tag{5.15}$$

$$\phi_k(t) \;\le\; C_{10}\left(b^{2^k} + (2^k+2)^{1+N/2}\phi_{k-1}^2(t)\right) \tag{5.16}$$

for all $t \ge 0$. This inequality fits with the framework of [14, Lemma 4.1.5] which is essentially due to Laurençot in [11]. We can however proceed in a straightforward way and introduce a new variable $z_k(t) = \phi_k^{2^{-k}}(t) = |\theta(t)|_{2^k}$. Inequalities (5.15)–(5.16) can be rewritten as

$$z_1(t) \;\le\; C_{11} , \tag{5.17}$$

$$z_k(t) \;\le\; \left(C_{11}\,(2^k+2)^{1+N/2}\right)^{2^{-k}}\,\max\{b, z_{k-1}(t)\} \,. \tag{5.18}$$

For $y_k(t) := \log(\max\{b, z_k(t)\})$ we thus obtain

$$y_k(t) \;\le\; y_1(t) + \sum_{j=1}^{\infty} 2^{-j}\,\log(C_{11}\,(2^j+2)^{1+N/2}) \;\le\; C_{12} , \tag{5.19}$$

independently of k and t. In particular, we have $|\theta(t)|_{2^k} \le C_{13}$ independently of k and t, hence, taking a bigger C_{13}, if necessary,

$$\theta(x,t) \;<\; C_{13} \quad \text{for a.e. } (x,t) \in \Omega \times (0,\infty) \,. \tag{5.20}$$

5.2 A lower bound

Using (1.11), we rewrite Eq. (1.10) in the form

$$c_V\,\theta_t + \kappa\Delta\left(\frac{1}{\theta}\right) \;=\; \mu(\theta)\,\chi_t^2 + \theta\left(\sigma(\chi)+\varphi(\chi)\right)_t \tag{5.21}$$

which yields (cf. the Clausius-Duhem inequality (1.14))

$$\begin{aligned} c_V\,\frac{\theta_t}{\theta} + \kappa\,\mathrm{div}\left(\frac{1}{\theta}\nabla\frac{1}{\theta}\right) \;&=\; \kappa\left|\nabla\frac{1}{\theta}\right|^2 + \frac{\mu(\theta)}{\theta}\,\chi_t^2 + \left(\sigma(\chi)+\varphi(\chi)\right)_t \\ &\geq\; \left(\sigma(\chi)+\varphi(\chi)\right)_t \quad \text{a. e.} \end{aligned} \tag{5.22}$$

The function

$$w(x,t) \;:=\; \log C_{13} - \log\theta$$

is positive and satisfies the inequality

$$c_V\,w_t - \kappa\,\mathrm{div}\left(\frac{1}{\theta^2}\nabla w\right) \;\leq\; -\left(\sigma(\chi)+\varphi(\chi)\right)_t \quad \text{a. e.} \tag{5.23}$$

Integrating this inequality from 0 to t and using the assumption (2.6) we obtain that

$$\int_\Omega w(x,t)\,dx \;\leq\; C_{14}\,. \tag{5.24}$$

We now argue as in the previous subsection and test Eq. (5.23) by w^r for $r \geq 1$. From the uniform upper bounds for $|\varphi(\chi)_t|$, $|\chi_t|$, and θ it follows that

$$\frac{1}{r+1}\frac{d}{dt}|w(t)|_{r+1}^{r+1} + r\frac{\kappa}{c_V C_{13}^2}\int_\Omega |\nabla w(t)|^2\, w(t)^{r-1}\,dx \;\leq\; C_{15}\,|w(t)|_r^r\,. \tag{5.25}$$

Put $\Psi_r(x,t) = w^{(r+1)/2}(x,t)$. Then

$$\frac{d}{dt}|\Psi_r(t)|_2^2 + C_{16}\,|\nabla\Psi_r(t)|_2^2 \;\leq\; C_{17}(r+1)\,\left(1+|\Psi_r(t)|_2^2\right)\,. \tag{5.26}$$

Using again the interpolation inequality, we obtain similarly as in (5.8)–(5.10) for all $r \geq 1$ that

$$\frac{d}{dt}|\Psi_r(t)|_2^2 + C_{18}\,|\Psi_r(t)|_2^2 \;\leq\; C_{19}(r+1)^{1+N/2}\,\left(1+|\Psi_r(t)|_1^2\right)\,. \tag{5.27}$$

By (5.24), we have $|\Psi_1(t)|_1 \leq C_{20}$, hence

$$\frac{d}{dt}\,|\Psi_1(t)|_2^2 + C_{18}\,|\Psi_1(t)|_2^2 \;\leq\; C_{21}\,. \tag{5.28}$$

For $r \geq 3$ we have

$$|\Psi_r(t)|_1^2 \;=\; \left|\Psi_{(r-1)/2}(t)\right|_2^4\,, \tag{5.29}$$

hence

$$\frac{d}{dt}\,|\Psi_r(t)|_2^2 + C_{18}\,|\Psi_r(t)|_2^2 \;\le\; C_{22}\,(r+1)^{1+N/2}\left(1+\left|\Psi_{(r-1)/2}(t)\right|_2^4\right). \tag{5.30}$$

For $k \in \mathbb{N}$ put $r_k = 2^k - 1$, and

$$\Phi_k(t) = 1 + \max_{0\le s\le t} |\Psi_{r_k}(s)|_2^2\,. \tag{5.31}$$

Set $B = (\log C_{13} - \log\theta_1)\,\max\{1, |\Omega|\}$. For all $r \ge 1$ we have $|\Psi_r(0)|_2^2 \le B^{r+1}$, and from Lemma 5.1, (5.28), and (5.30) it follows that

$$\Phi_1(t) \;\le\; C_{23}\,, \tag{5.32}$$

$$\Phi_k(t) \;\le\; C_{24}\left(B^{2^k} + 2^{k(1+N/2)}\Phi_{k-1}^2(t)\right) \tag{5.33}$$

for all $t \ge 0$. Similarly as in the previous subsection we derive the estimate

$$w(x,t) \;<\; C_{25} \quad \text{for a.e. } (x,t) \in \Omega\times(0,\infty)\,, \tag{5.34}$$

and the proof of Theorem 2.3 is complete.

6 Asymptotic behaviour

The statement of Theorem 2.4 fits with the general framework of Proposition 5.4 in [9]. We however give here a direct and elementary proof which takes into account the fact that the function

$$\mathcal{E}(t) \;:=\; \int_\Omega \left(\kappa\left|\nabla\frac{1}{\theta}\right|^2 + \mu(\theta)\,\chi_t^2\right)(x,t)\,dx \tag{6.1}$$

may be discontinuous.

It follows from (5.22) for all $T > 0$ that

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \left(\kappa\left|\nabla\frac{1}{\theta}\right|^2 + \frac{\mu(\theta)}{\theta}\,\chi_t^2\right)(x,t)\,dx\,dt + \int_\Omega (\sigma(\chi)+\varphi(\chi))\,(x,T)\,dx \\ &\quad= c_V \int_\Omega (\log\theta(x,T) - \log\theta_0(x))\,dx + \int_\Omega (\sigma(\chi_0)+\varphi(\chi_0))\,(x)\,dx \\ &\quad\le c_V \int_\Omega (\theta(x,T) - \log\theta_1)\,dx + \int_\Omega (\sigma(\chi_0)+\varphi(\chi_0))\,(x)\,dx\,, \end{aligned} \tag{6.2}$$

hence $\mathcal{E} \in L^1(0,\infty)$ as a consequence of (5.5). Testing Eq. (1.10) by θ_t/θ^2 we further obtain for a.e. $t > 0$ that

$$C_{26}\int_\Omega \theta_t^2(x,t)\,dx + \frac{d}{dt}\int_\Omega \kappa\left|\nabla\frac{1}{\theta}\right|^2(x,t)\,dx \;\le\; C_{27}\,. \tag{6.3}$$

Using (3.4) we find a constant C_{28} such that for every $T > 0$ and every non-negative function $\eta \in W^{1,\infty}(0,T)$ with compact support in $(0,T)$ we have

$$(6.4) \qquad \int_0^T \int_\Omega \mu(\theta)\, \chi_t^2(x,t)\, \dot\eta(t)\, dx\, dt \ \geq\ -\int_0^T \eta(t) \left(C_{28} + C_{26} \int_\Omega \theta_t^2(x,t)\, dx \right) dt\,.$$

Put $C^* = C_{27} + C_{28}$. We now claim that

$$(6.5) \qquad \textit{the function } \ q(t) \ :=\ C^*\, t - \mathcal{E}(t) \ \ \textit{is non-decreasing.}$$

Indeed, for every $T > 0$ and every non-negative function $\eta \in W^{1,\infty}(0,T)$ with compact support in $(0,T)$ we have

$$\begin{aligned}(6.6) \qquad \int_0^T q(t)\, \dot\eta(t)\, dt \ &=\ \int_0^T \left(-C^* + \frac{d}{dt} \int_\Omega \kappa \left|\nabla \frac{1}{\theta}\right|^2 (x,t)\, dx \right) \eta(t)\, dt \\ &\quad - \int_0^T \int_\Omega \mu(\theta)\, \chi_t^2(x,t)\, \dot\eta(t)\, dx\, dt \\ &\leq\ \int_0^T (-C^* + C_{27} + C_{28})\, \eta(t)\, dt \ \leq\ 0\,,\end{aligned}$$

and (6.5) follows. This implies, in particular, that $\mathcal{E}$ has bounded variation on every bounded time interval, and $\mathcal{E}(t+) \leq \mathcal{E}(t-)$ for every $t > 0$.

For $n \in \mathbb{N}$ we define the sequences

$$(6.7) \qquad \varepsilon_n \ =\ \operatorname{ess\,sup}\{\mathcal{E}(t)\,;\, t \in [n, n+1]\}\,, \quad \delta_n \ =\ \min\left\{1, \frac{\varepsilon_n}{C^*}\right\}\,,$$

and find $t_n \in [n, n+1]$ such that $\mathcal{E}(t_n-) = \varepsilon_n$. For a. e. $t \in [t_n - \delta_n, t_n]$ we have

$$\mathcal{E}(t) \ \geq \varepsilon_n - C^*(t_n - t)\,,$$

hence

$$\begin{aligned}(6.8) \ \int_{t_n-\delta_n}^{t_n} \mathcal{E}(t)\, dt \ &\geq\ \int_{t_n-\delta_n}^{t_n} (\varepsilon_n - C^*(t_n - t))\, dt \ =\ \frac{1}{2C^*}\left(\varepsilon_n^2 - (\varepsilon_n - C^*\delta_n)^2\right) \\ &\geq\ \frac{1}{2} C^* \delta_n^2\,.\end{aligned}$$

We thus have

$$(6.9) \qquad \frac{1}{2} C^* \sum_{n=1}^\infty \delta_n^2 \ \leq\ \sum_{n=1}^\infty \int_{n-1}^{n+1} \mathcal{E}(t)\, dt \ \leq\ 2 \int_0^\infty \mathcal{E}(t)\, dt\,.$$

Since $\mathcal{E}$ belongs to $L^1(0,\infty)$, we obtain that $\lim_{n\to\infty} \delta_n = 0$, and the proof of Theorem 2.4 is complete.

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