Axiomatic theory of spectrum III. — Semiregularities

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The notion of regularity in a Banach algebra was introduced and studied in [KM] and [MM]. A non-empty subset R of a unital Banach algebra $\mathcal{A}$ is called a regularity if it satisfies the following two conditions:

(i) if $a \in \mathcal{A}$ and $n \in \mathbf{N}$, then $a \in R \Leftrightarrow a^n \in R$,

(ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac + bd = 1_{\mathcal{A}}$ then $ab \in R \Leftrightarrow a, b \in R$.

The axioms of regularities are weak enough so that there are plenty of examples that appear naturally in Banach algebras and operator theory. On the other hand they are strong enough so that they have interesting consequences, especially the spectral mapping theorem for the corresponding spectrum $\sigma_R(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin R\}$.

In fact the axioms (i) and (ii) of regularities can be divided in two halves, each of them implying a one-way spectral mapping theorem.

The aim of this paper is to study systematically semiregularities defined in this way. There are many natural examples of such classes that satisfy only one half of the axioms of regularities. The corresponding spectra include the exponential spectrum, the Weyl spectrum, T-Weyl spectrum, Kato essential spectrum, various essential spectra etc.

All Banach algebras considered in this paper are complex and unital.

Lower semiregularities

Definition 1. Let R be a non-empty subset of a Banach algebra $\mathcal{A}$. Then R is called a lower semiregularity if

(i) $a \in \mathcal{A}, n \in \mathbf{N}, a^n \in R \Rightarrow a \in R$,

(ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac + bd = 1_{\mathcal{A}}$ and $ab \in R$ then $a, b \in R$.

For a lower semiregularity R define the corresponding spectrum σ_R by $\sigma_R(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin R\}$.

Clearly the intersection $R = \bigcap_\alpha R_\alpha$ of any system of lower semiregularities is again a lower semiregularity. The corresponding spectra satisfy $\sigma_R(a) = \bigcup_\alpha \sigma_{R_\alpha}(a)$ for all $a \in \mathcal{A}$.

Denote by $\mathrm{Inv}\,(\mathcal{A})$ the set of all invertible elements of a Banach algebra $\mathcal{A}$.

Lemma 2. Let $R \subset \mathcal{A}$ be a lower semiregularity. Then:

(i) $1_{\mathcal{A}} \in R$;

(ii) $\mathrm{Inv}\,(\mathcal{A}) \subset R$;

(iii) if $a \in R$, $b \in \mathrm{Inv}\,(\mathcal{A})$ and $ab = ba$ then $ab \in R$;

(iv) $\sigma_R(a) \subset \sigma(a)$;

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(v) (translation property) $\sigma_R(a+\lambda) = \lambda + \sigma_R(a)$.

Proof. (i) Let $b \in R$. We have $1 \cdot 1 + b \cdot 0 = 1$ and $1 \cdot b = b \in R$. Thus $1 \in R$.

(ii) Let $a \in \mathrm{Inv}\,(\mathcal{A})$. Then $a \cdot a^{-1} + a^{-1} \cdot 0 = 1$ and $a \cdot a^{-1} = 1 \in R$. Hence $a \in R$.

(iii) We have $(ab) \cdot 0 + b^{-1} \cdot b = 1$ and $(ab) \cdot b^{-1} = a \in R$ so that $ab \in R$.

The remaining statements are clear.

Remark 3. Suppose that $R \subset \mathcal{A}$ is a non-empty subset satisfying

$$a, b \in \mathcal{A}, ab = ba, ab \in R \Rightarrow a, b \in R. \tag{1}$$

Then clearly R is a lower semiregularity.

Theorem 4. Let $R \subset \mathcal{A}$ be a lower semiregularity and $a \in \mathcal{A}$. Then

$$f(\sigma_R(a)) \subset \sigma_R(f(a))$$

for each locally non-constant function f analytic on a neighbourhood of $\sigma(a)$.

Proof. Suppose on the contrary that $\lambda \in f(\sigma_R(a)) \setminus \sigma_R(f(a))$. Since the function $f(z) - \lambda$ has only a finite number of zeros $\alpha_1, \dots, \alpha_n$ in $\sigma(a)$, we can write

$$f(z) - \lambda = (z - \alpha_1)^{k_1} \dots (z - \alpha_n)^{k_n} g(z),$$

for some $k_i \geq 1$ and a function g analytic on a neighbourhood of $\sigma(a)$ such that $g(z) \neq 0 \quad (z \in \sigma(a))$. Thus

$$f(a) - \lambda = (a - \alpha_1)^{k_1} \dots (a - \alpha_n)^{k_n} g(a),$$

where $f(a) - \lambda \in R$ and $g(a) \in \mathrm{Inv}\,(\mathcal{A})$. By Lemma 2 (iii), $(a-\alpha_1)^{k_1} \cdots (a-\alpha_n)^{k_n} \in R$.

Let $i \in \{1, \dots, n\}$. For certain polynomials p, q we have

$$(z - \alpha_i)^{k_i} \cdot p(z) + \Big(\prod_{j \neq i} (z - \alpha_j)^{k_j}\Big) \cdot q(z) = 1.$$

The corresponding identity for z replaced by a gives $(a - \alpha_i)^{k_i} \in R$. Thus $a - \alpha_i \in R$ and $\alpha_i \notin \sigma_R(a) \quad (i = 1, \dots, n)$. Hence $\lambda \notin f(\sigma_R(a))$, a contradiction.

Corollary 5. Let $R \subset \mathcal{A}$ be a lower semiregularity and $0 \notin R$. Then $p(\sigma_R(a)) \subset \sigma_R(p(a))$ for all polynomials p.

Proof. It is sufficient to verify the inclusion for the constant polynomials $p(z) \equiv \lambda$. In this case we have $p(\sigma_R(a)) \subset \{\lambda\}$ and $\sigma_R(p(a)) = \sigma_R(\lambda \cdot 1_{\mathcal{A}}) = \{\lambda\}$.

Theorem 6. Let $R \subset \mathcal{A}$ be a lower semiregularity satisfying the following condition: if $c = c^2 \in R$, $a \in \mathcal{A}$ and $ac = ca$ then $c + (1-c)a \in R$. Then $f(\sigma_R(a)) \subset \sigma_R(f(a))$ for all $a \in \mathcal{A}$ and f analytic on a neighbourhood of $\sigma(a)$.

Proof. Let U be the domain of definition of f. Suppose on the contrary that $\lambda \in f(\sigma_R(a)) \setminus \sigma_R(f(a))$. Let U_1 be the union of all components of U where f is identically equal to λ, and $U_2 = U \setminus U_1$. Let h be defined by

$$h(z) = \begin{cases} 0 & (z \in U_1), \\ 1 & (z \in U_2). \end{cases}$$

Then we can write

$$f(z) - \lambda = h(z)(z-\alpha_1)^{k_1} \cdots (z-\alpha_n)^{k_n} \cdot g(z)$$

where $\alpha_1, \dots, \alpha_n \in \sigma(a) \cap U_2$, g is analytic on U and $g(z) \neq 0 \quad (z \in \sigma(a))$. Set $q(z) = (z-\alpha_1)^{k_1} \cdots (z-\alpha_n)^{k_n}$. Thus $f(a)-\lambda = h(a)q(a)g(a) = h(a)q(a)\big(1-h(a)+g(a)h(a)\big)$, where $1 - h(a) + g(a)h(a) \in \operatorname{Inv}(\mathcal{A})$. We have $f(a) - \lambda \in R$ and so, by Lemma 2 (iii), $h(a)g(a) \in R$.

Consider the function r defined by

$$r(z) = \begin{cases} q(z)^{-1} & (z \in U_1), \\ 0 & (z \in U_2). \end{cases}$$

Then $q(z)(1-h(z)) \cdot r(z) + h(z) \cdot 1 = 1$ and $q(a)h(a) \in R$ and so $q(a) \in R, h(a) \in R$. As in Theorem 4, $q(a) \in R$ implies $a - \alpha_i \in R \quad (i = 1, \dots, n)$ and so $\alpha_i \notin \sigma_R(a)$.

Since $\lambda \in f(\sigma_R(a))$, there is $\beta \in U_1 \cap \sigma_R(a)$. Further $h(a)$ is an idempotent in R and, by the assumption, we have $(a-\beta)(1-h(a)) + h(a) \in R$. Further $(1-h(a)) + (a-\beta)h(a) \in \operatorname{Inv}(\mathcal{A})$ and so

$$a - \beta = \big((a-\beta)(1-h(a)) + h(a)\big) \cdot \big((1-h(a)) + (a-\beta)h(a)\big) \in R.$$

This contradicts to the fact that $\beta \in \sigma_R(a)$.

Remark 7. In particular the condition of the previous theorem is satisfied if the unit element is the unique idempotent in R.

Another typical application is when $\mathcal{A}$ is the algebra of all bounded operators on a Banach space, all idempotents in R are projections onto subspaces of finite codimension and R is invariant under finite rank perturbations (for example Fredholm operators, upper (lower) semi-Fredholm operators etc.).

Theorem 8. Let $R \subset \mathcal{A}$ be a lower semiregularity. The following conditions are equivalent:

(i) R is open;

(ii) $\sigma_R(a)$ is closed for each $a \in \mathcal{A}$ and the set-valued function $a \mapsto \sigma_R(a)$ is upper semicontinuous.

Proof. Straightforward.

Remark 9. Let $R \subset \mathcal{A}$ be a lower semiregularity. Then the spectrum σ_R can be extended to n-tuples of commuting elements of $\mathcal{A}$ in such a way that

$$p\sigma_R(a_1, \dots, a_n) \subset \sigma_R(p(a_1, \dots, a_n))$$

for all commuting n-tuples $a_1, \dots, a_n \in \mathcal{A}$ and all non-constant polynomials p in n variables, see [MW]. Indeed, define

$$\sigma_R(a_1, \dots, a_n) = \{(\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n : p(a_1 - \lambda_1, \dots, a_n - \lambda_n) \notin R \text{ for all } p\}.$$

The extension is not unique; other (trivial) extension is $\sigma_R(a_1, \dots, a_n) = \emptyset$ whenever $n \geq 2$.

The first extension is the maximal among all extensions satisfying the one-way spectral mapping property (clearly the trivial extension is minimal).

We show now some examples of lower semiregularities. Of course every regularity (for examples see [KM] and [MM]) is also a lower semiregularity. Therefore we restrict here only to examples of lower semiregularities that are not regularities.

Let $\mathcal{L}(X)$ be the algebra of all bounded linear operators acting on a Banach space X. For $T \in \mathcal{L}(X)$ and $k \geq 0$ define

$$\begin{aligned}
\alpha_k(T) &= \dim N(T^{k+1})/N(T^k), \\
\beta_k(T) &= \dim R(T^k)/R(T^{k+1}), \\
\gamma_k(T) &= \dim \ker\Big(R(T^k)/R(T^{k+1}) \xrightarrow{\hat{T}} R(T^{k+1})/R(T^{k+2})\Big) \\
&= \operatorname{codim} \operatorname{Im}\Big(N(T^{k+2})/N(T^{k+1}) \xrightarrow{\tilde{T}} N(T^{k+1}/N(T^k)\Big),
\end{aligned}$$

where the operators $\hat{T}$ and $\tilde{T}$ are induced by T, see [G].

The following properties of these numbers can be found in [MM]:

(i)
$$\begin{aligned}
\alpha_0(T) &\geq \alpha_1(T) \geq \cdots, \\
\beta_0(T) &\geq \beta_1(T) \geq \cdots.
\end{aligned}$$

(ii) $\gamma_k(T) = \alpha_k(T) - \alpha_{k+1}(T) = \beta_k(T) - \beta_{k+1}(T)$ whenever the differences have sense.

(iii)
$$\begin{aligned}
\alpha_n(T^m) &= \sum_{i=0}^{m-1} \alpha_{mn+i}(T), \\
\beta_n(T^m) &= \sum_{i=0}^{m-1} \beta_{mn+i}(T), \\
\gamma_n(T^m) &= \sum_{i=0}^{2m-2} \gamma_{mn+i}(T) \cdot \min\{i+1, 2m-1-i\} \geq \sum_{i=0}^{2m-2} \gamma_{mn+i}(T).
\end{aligned}$$

(iv) If A, B, C, D are mutually commuting operators satisfying $AC + BD = I$ then

$$\begin{aligned}
\max\{\alpha_n(A), \alpha_n(B)\} &\leq \alpha_n(AB) \leq \alpha_n(A) + \alpha_n(B), \\
\max\{\beta_n(A), \beta_n(B)\} &\leq \beta_n(AB) \leq \beta_n(A) + \beta_n(B), \\
\max\{\gamma_n(A), \gamma_n(B)\} &\leq \gamma_n(AB) \leq \gamma_n(A) + \gamma_n(B).
\end{aligned}$$

(v) if $AC + BD = I$ then $R(A^n B^n)$ is closed $\Leftrightarrow$ $R(A^n)$ and $R(B^n)$ are closed;

(vi) if $R(T^n)$ is closed and $\gamma_i(T) < \infty \quad (i \geq n-1)$ then $R(T^j)$ is closed for every $j \geq n-1$.

Write $N^\infty(T) = \bigcup_{i\geq 0} N(T^i)$ and $R^\infty(T) = \bigcap_{i\geq 0} R(T^i)$.

Fix $m \geq 0$. The previous properties imply that the following subsets of $\mathcal{L}(X)$ are lower semiregularities:

(1) $\{T \in \mathcal{L}(X) : \dim N(T) \leq m\} = \{T : \sup \alpha_i(T) \leq m\}$,
(2) $\{T \in \mathcal{L}(X) : \dim N^\infty(T) \leq m\} = \{T : \sum \alpha_i(T) \leq m\}$,
(3) $\{T \in \mathcal{L}(X) : \lim \alpha_i(T) \leq m\}$,
(4) $\{T \in \mathcal{L}(X) : \operatorname{codim} R(T) \leq m\} = \{T : \sup \beta_i(T) \leq m\}$,
(5) $\{T \in \mathcal{L}(X) : \operatorname{codim} R^\infty(T) \leq m\} = \{T : \sum \beta_i(T) \leq m\}$,
(6) $\{T \in \mathcal{L}(X) : \lim \beta_i(T) \leq m\}$,
(7) $\{T \in \mathcal{L}(X) : \sup \gamma_i(T) \leq m\}$,
(8) $\{T \in \mathcal{L}(X) : \sum \gamma_i(T) \leq m\}$,
(9) $\{T \in \mathcal{L}(X) : \limsup \gamma_i(T) \leq m\}$.
(10) $\{T \in \mathcal{L}(X) : \dim N(T) \leq m$ and R(T) is closed$\}$,
(11) $\{T \in \mathcal{L}(X) : \dim N^\infty(T) \leq m$ and R(T) is closed$\}$,
(12) $\{T \in \mathcal{L}(X) :$ there exists n such that $\alpha_n(T) < \infty$ and $R(T^{n+1})$ is closed$\}$,
(13) $\{T \in \mathcal{L}(X) : \sup \gamma_i(T) \leq m$ and $R(T)$ is closed$\}$,
(14) $\{T \in \mathcal{L}(X) : \sum \gamma_i(T) \leq m$ and $R(T)$ is closed$\}$,
(15) $\{T \in \mathcal{L}(X) :$ there is n_0 such that $\gamma_n(T) \leq m$ and $R(T^n)$ is closed $(n \geq n_0)\}$.

Note that the range in classes (4) – (6) is closed automatically.

Denote by $\Phi(X), \Phi_+(X)$ and $\Phi_-(X)$ the set of all Fredholm, upper semi-Fredholm and lower semi-Fredholm operators acting on a Banach space X. It is easy to see that these classes are even regularities. The union

(16) $\Phi_+(X) \cup \Phi_-(X)$

is a lower semiregularity. Indeed, by [O2], it satisfies condition (1) of Remark 3. The corresponding spectrum was studied by Kato [K], Oberai [O2], Gramsch and Lay [GL], and others. It is sometimes called the Kato essential spectrum (σ_K). By Remark 9 the one-way spectral mapping $\sigma_K(f(T)) \subset f(\sigma_K(T))$ is satisfied for all functions analytic on a neighbourhood of $\sigma(T)$.

Upper semiregularities

Definition 10. A subset $R \subset \mathcal{A}$ is called an upper semiregularity if

(i) $a \in R, n \in \mathbf{N} \Rightarrow a^n \in R$,
(ii) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac + bd = 1_{\mathcal{A}}$ and $a, b \in R$ then $ab \in R$,
(iii) R contains a neighbourhood of the unit element $1_{\mathcal{A}}$.

The definitions of upper and lower semiregularities are only seemingly asymmetric. In fact condition (iii) was for lower semiregularities satisfied automatically.

Clearly R is a regularity if and only if it is both lower and upper semiregularity.

Again define $\sigma_R(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin R\}$. Clearly the intersection of any family of upper semiregularities is again upper semiregularity. Also the mapping $a \mapsto \sigma_R(a)$ is upper semicontinuous if and only if R is open.

Remark 11. If $R \subset \mathcal{A}$ is a semigroup then conditions (i) and (ii) of Definition 10 are satisfied. Thus a semigroup containing a neighbourhood of the unit element is an upper semiregularity.

Lemma 12. Let $R \subset \mathcal{A}$ be an upper semiregularity, let $a \in R \cap \mathrm{Inv}\,(\mathcal{A})$. Then there exists $\varepsilon > 0$ such that $\{b \in \mathcal{A} : ab = ba, \|b - a\| < \varepsilon\} \subset R$.

Proof. Let $\delta > 0$ satisfy $\{c \in \mathcal{A} : \|c - 1_{\mathcal{A}}\| < \delta\} \subset R$. Let $a \in R \cap \mathrm{Inv}\,(\mathcal{A})$. Set $\varepsilon = \frac{\delta}{\|a^{-1}\|}$. Suppose that $b \in \mathcal{A}$, $ab = ba$ and $\|b - a\| < \varepsilon$. Then $\|a^{-1}b - 1\| = \|a^{-1}(b - a)\| \le \|a^{-1}\| \cdot \|b - a\| < \delta$ and so $a^{-1}b \in R$. Further $a \cdot a^{-1} + (a^{-1}b) \cdot 0 = 1$, hence $b = a \cdot (a^{-1}b) \in R$.

Lemma 13. Let $R \subset \mathcal{A}$ be an upper semiregularity, $a_n \in R \cap \mathrm{Inv}\,(\mathcal{A}) \quad (n = 1, 2, \dots)$, $a \in \mathrm{Inv}\,(\mathcal{A})$, $a_n \to a$ and $a_n a = a a_n$. Then $a \in R$.

Proof. For each n we have $a_n \cdot a_n^{-1} + (a_n^{-1}a) \cdot 0 = 1$. Further $a_n^{-1}a \to 1$ so that $a_n^{-1}a \in R$ for n large enough. Thus $a = a_n \cdot (a_n^{-1}a) \in R$.

Theorem 14. Let $R \subset \mathcal{A}$ be an upper semiregularity, let $a \in \mathcal{A}$. Let M be a component of $\mathbf{C} \setminus \sigma(a)$. Then either $M \subset \sigma_R(a)$ or $M \cap \sigma_R(a) = \emptyset$.

Proof. Let $L = \{a - \lambda : \lambda \in M, a - \lambda \in R\}$. By Lemma 12, L is open and, by Lemma 13, it is relatively closed in M. Thus either $L = \emptyset$ or $L = M$.

Corollary 15. Let $R \subset \mathcal{A}$ be an upper semiregularity. Then $\lambda \cdot 1_{\mathcal{A}} \in R$ for each nonzero complex number λ.

Proof. Consider the element $a = 0$. The set $M = \{\lambda \in \mathbf{C} : \lambda \neq 0\}$ is a component of $\mathbf{C} \setminus \sigma(0)$. Further $1 \in R$, so $\lambda \in R$ for all $\lambda \in M$.

Lemma 16. Let $R \subset \mathcal{A}$ be an upper semiregularity, let $a \in R$, $b \in R \cap \mathrm{Inv}\,(\mathcal{A})$ and $ab = ba$. Then $ab \in R$.

Proof. We have $a \cdot 0 + b \cdot b^{-1} = 1$, so $ab \in R$.

Denote by $\hat{\sigma}(a)$ the polynomially convex hull of $\sigma(a)$.

Theorem 17. Let $R \subset \mathcal{A}$ be an upper semiregularity, let $a \in \mathcal{A}$. Then $\sigma_R(a) \subset \hat{\sigma}(a)$. Further $\sigma_R(a) \setminus \sigma(a)$ is a union of some bounded components of $\mathbf{C} \setminus \sigma(a)$.

Proof. For $|\lambda|$ big enough we have $1 - \frac{a}{\lambda} \in R$, so $a - \lambda = -\lambda(1 - \frac{a}{\lambda}) \in R$. By Theorem 14 the unbounded component of $\mathbf{C} \setminus \sigma(a)$ is disjoint with $\sigma_R(a)$ and thus $\sigma_R(a) \subset \hat{\sigma}(a)$.

Corollary 18. $\sigma_R(a) \cup \sigma(a)$ is a compact subset of $\mathbf{C}$ for all $a \in \mathcal{A}$.

Theorem 19. Let $R \subset \mathcal{A}$ be an upper semiregularity, let $a \in \mathcal{A}$. Then $\sigma_R(p(a)) \subset p(\sigma_R(a))$ for all nonconstant polynomials p.

Moreover, if $\sigma_R(a) \neq \emptyset$ for all $b \in \mathcal{A}$ then $\sigma_R(p(a)) \subset p(\sigma_R(a))$ for all polynomials p.

Proof. Let p be a nonconstant polynomial. Let $\lambda \notin p(\sigma_R(a))$. Write $p(z) - \lambda = \beta \cdot (z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n}$ where $n \geq 1$ and $\beta \in \mathbf{C}$, $\beta \neq 0$. Thus

$$p(a) - \lambda = \beta \cdot (a - \alpha_1)^{k_1} \cdots (a - \alpha_n)^{k_n}.$$

By the assumption $\alpha_i \notin \sigma_R(a) \quad (i = 1, \ldots, n)$. Thus $a - \alpha_i \in R$ and $(a - \alpha_i)^{k_i} \in R$. As in Theorem 4 we have $(z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n} \in R$ and $p(a) - \lambda \in R$, i.e., $\lambda \notin \sigma_R(p(a))$. Thus $\sigma_R(p(a)) \subset p(\sigma_R(a))$ for all non-constant polynomials.

Suppose that $\sigma_R(b) \neq \emptyset$ for all $b \in \mathcal{A}$. Let $p(z) \equiv \lambda$ be a constant polynomial. Then

$$\sigma_R(p(a)) = \sigma_R(\lambda \cdot 1_{\mathcal{A}}) \subset \{\lambda\} = p(\sigma_R(a)).$$

Theorem 20. Let $R \subset \mathcal{A}$ be an upper semiregularity. Suppose that R satisfies the condition

$$b \in R \cap \operatorname{Inv}(\mathcal{A}) \Rightarrow b^{-1} \in R. \tag{2}$$

Then $\sigma_R(f(a)) \subset f(\sigma_R(a))$ for all $a \in \mathcal{A}$ and all locally non-constant functions f analytic on a neighbourhood of $\sigma(a) \cup \sigma_R(a)$.

Further $\sigma_R(f(a)) \subset f(\sigma_R(a) \cup \sigma(a))$ for all functions f analytic on a neighbourhood of $\sigma_R(a) \cup \sigma(a)$.

Proof. Suppose first that f is locally non-constant and suppose on the contrary that there is $\lambda \in \sigma_R(f(a)) \setminus f(\sigma_R(a))$. Then $f(a) - \lambda = q(a)g(a)$ where $q(a) = (z - \alpha_1)^{k_1} \cdots (a - \alpha_n)^{k_n}$ and g is a function analytic and non-zero on a neighbourhood of $\sigma(a) \cup \sigma_R(a)$. By the assumption $f(a) - \lambda \notin R$ and $\alpha_i \notin \sigma_R(a)$, i.e., $a - \alpha_i \in R \quad (i = 1, \ldots, n)$. As in Theorem 4 we obtain that $q(a) \in R$. Further there are a compact neighbourhood V of $\sigma(a) \cup \sigma_R(a)$ and rational functions $\frac{p_n(z)}{q_n(z)}$ with poles outside V such that $\frac{p_n(z)}{q_n(z)} \to g(z)$ uniformly on V. We can assume that the polynomials p_n, q_n are non-constant and $p_n(z) \neq 0$ on $\sigma(a) \cup \sigma_R(a)$.

By Theorem 19 this means that $p_n(a) \in R$, $q_n(a) \in R$. By the assumption $q_n(a)^{-1} \in R$. Thus $p_n(a)q_n(a)^{-1} \in R$ and, by Lemma 13, $g(a) = \lim p_n(a)q_n(a)^{-1} \in R$. Since $q(a) \in R$ and $g(a) \in R \cap \operatorname{Inv}(\mathcal{A})$, we have $f(a) - \lambda \in R$, a contradiction.

Suppose now that f is analytic on a neighbourhood of $\sigma(a) \cup \sigma_R(a)$ and $\lambda \in \sigma_R(f(a)) \setminus f(\sigma(a) \cup \sigma_R(a))$. Let U be the domain of definition of f, $U = U_1 \cup U_2$ where U_1, U_2 are disjoint open sets, $f|U_1 \equiv \lambda$ and f is not identically equal to λ on any nonempty open subset of U_2. By the assumption $(\sigma_R(a) \cup \sigma(a)) \cap U_1 = \emptyset$, so U_2 is an open neighbourhood of $\sigma(a) \cup \sigma_R(a)$. The proof proceeds as in the first part.

In many cases the inclusion $\sigma_R(f(a)) \subset f(\sigma_R(a))$ is true for all analytic functions. By Theorem 20 this is true if R satisfies (2) and $R \subset \operatorname{Inv}(\mathcal{A})$, i.e., $\sigma_R(a) \supset \sigma(a)$ for all a.

More generally, one can show that $\sigma_R(f(a)) \subset f(\sigma_R(a))$ for all f if R satisfies (2) and each component of $\sigma(a)$ meats $\sigma_R(a)$.

Another typical situation is described in the following theorem.

Theorem 21. Let $R \subset \mathcal{L}(X)$ be an upper semiregularity satisfying (2) such that
(i) if $T \in R$ and $F \in \mathcal{L}(X)$ is a finite rank operator commuting with T, then $T + F \in R$,

(ii) if $T \in \mathcal{L}(X)$, U_1, U_2 are disjoint open sets, $\sigma(T) \subset U_1 \cup U_2$ and $\sigma_R(T) \subset U_2$, then the spectral projection of T corresponding to U_1 is of finite rank.

Then $\sigma_R(f(T) \subset f(\sigma_R(T))$ for all $T \in \mathcal{L}(X)$ and f analytic on a neighbourhood of $\sigma(T) \cup \sigma_R(T)$.

Proof. Let f is analytic on a neighbourhood of $\sigma(a) \cup \sigma_R(a)$ and suppose that there is $\lambda \in \sigma_R(f(a)) \setminus f(\sigma_R(a))$. Let U_1, U_2 be disjoint open sets, $f|U_1 \equiv \lambda$ and f is not identically equal to λ on any nonempty open subset of U_2. By the assumption $\sigma_R(a) \cap U_1 = \emptyset$, so $\sigma_R(T) \subset U_2$. Let h be defined by

$$h(z) = \begin{cases} 0 & (z \in U_1), \\ 1 & (z \in U_2). \end{cases}$$

By (ii), $I - h(a)$ is a finite rank projection. We can write

$$f(z) - \lambda = h(z)(z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n} g(z)$$

for some $\alpha_1, \dots, \alpha_n \in \sigma(a) \cap U_2$, g analytic on $U_1 \cup U_2$ and $g(z) \neq 0 \quad (z \in \sigma(a))$. Set $q(z) = (z - \alpha_1)^{k_1} \cdots (z - \alpha_n)^{k_n}$.

We have $\alpha_i \notin \sigma_R(T)$, so $T - \alpha_i \in R$ and, as in Theorem 4, $q(a) \in R$.

Further $f(T) - \lambda = h(T)q(T)g(T) \in R$ and by assumption (i), $q(T)g(T) \in R$. As in Theorem 20 we can get $g(T) \in R \cap \operatorname{Inv}(\mathcal{L}(X))$ and so $q(T) \in R$, a contradiction.

Examples.

(1) Let R be the principal component of $\operatorname{Inv}(\mathcal{A})$. Then R is an open semigroup and so an upper semiregularity. The corresponding spectrum is the exponential spectrum σ_{exp} of Harte [H1]. By Theorems 17 and 20, $\sigma(a) \subset \sigma_{exp}(a) \subset \hat{\sigma}(a)$ and $f\sigma_{exp}(a) \subset \sigma_{exp}(f(a))$ for each function f analytic on a neighbourhood of $\sigma_{exp}(a)$.

(2) Let $R = \{T \in \Phi(X) : \operatorname{ind} T = 0\}$. Again R is an open semigroup and thus an upper semiregularity. The corresponding spectrum is the Weyl spectrum (sometimes also called Schechter spectrum) $\sigma_W(T) = \{\lambda \in \mathbf{C} : T - \lambda \notin \Phi(X) \text{ or } \operatorname{ind} T \neq 0\}$, see [S], [O1]. It is well-known that $\sigma_W(T) = \bigcap \sigma(T + K)$ where the intersection is taken over the set of all compact operators K. By Theorem 21 we have $\sigma_W(f(T)) \subset f(\sigma_W(T))$ for each function f analytic on a neighbourhood of $\sigma(T)$.

(3) More generally, let J be a closed two-sided ideal in a Banach algebra $\mathcal{A}$ and $R = \{a \in \mathcal{A} : (a+J) \cap \operatorname{Inv}(\mathcal{A}) \neq \emptyset\}$. It is easy to check that R is a semigroup containing $\operatorname{Inv}(\mathcal{A})$ and so an upper semiregularity. The corresponding spectrum was studied in [H2]

(4) Let $\mathcal{A} = \mathcal{L}(X)$ and $m \geq 0$. Clearly the following sets are upper semiregularities:
$\{T \in \Phi(X) : \operatorname{ind} T \geq m\}$,
$\{T \in \Phi(X) : \operatorname{ind} T \leq -m\}$,
$\{T \in \Phi(X) : \operatorname{ind} T \in m\mathbf{Z}\}$,
$\{T \in \Phi_+(X) : \operatorname{ind} T \geq m\}$,
$\{T \in \Phi_-(X) : \operatorname{ind} T \leq -m\}$,
$\{T \in \Phi_+(X) : \operatorname{ind} T \leq -m\}$,
$\{T \in \Phi_-(X) : \operatorname{ind} T \geq m\}$.
In particular, for $m = 0$, the last two classes are

$\Phi_+^-(X) = \{T \in \Phi_+(X) : \operatorname{ind} T \le 0\}$ and
$\Phi_-^+(X) = \{T \in \Phi_-(X) : \operatorname{ind} T \ge 0\}$.

These classes and the corresponding spectra were studied by Rakočević [R1], [R2] and Zemánek [Z2].

The corresponding spectra were called the essential approximate point spectrum and the essential defect spectrum and denoted by σ_{ea} and σ_{ed}, respectively.

They satisfy

$$\sigma_{ea}(T) = \bigcap\{\sigma_\pi(T+K) : K \text{ compact}\},$$
$$\sigma_{ed}(T) = \bigcap\{\sigma_\delta(T+K) : K \text{ compact}\}$$

(where σ_π and σ_δ denote the approximate point spectrum and defect spectrum, respectively), and the one-way spectral mapping theorem for all analytic functions, cf. Theorem 21.

Note that similar classes
$\mathcal{B}_+(X) = \{T \in \Phi_+(X) :$ ascent $(T) < \infty\}$ and
$\mathcal{B}_-(X) = \{T \in \Phi_-(X) :$ descent $(T) < \infty\}$,

and the corresponding spectra $\sigma_{\mathcal{B}_+}$ and $\sigma_{\mathcal{B}_-}$ (called Browder essential approximate point spectrum and Browder essential defect spectrum e.g. in [R1] and [Z2], and upper (lower) semi-Browder spectra in [H3], [KMR]) exhibit much nicer properties. Not only are these classes regularities but it is possible to extend the spectra $\sigma_{\mathcal{B}_+}$ and $\sigma_{\mathcal{B}_-}$ to all commuting n-tuples of operators such that the multivariable spectral mapping property is satisfied, see [KMR].

Further, for single operators,

$$\sigma_{\mathcal{B}_+}(T) = \bigcap\{\sigma_\pi(T+K) : K \text{ compact}, TK = KT\},$$
$$\sigma_{\mathcal{B}_-}(T) = \bigcap\{\sigma_\delta(T+K) : K \text{ compact}, TK = KT\}.$$

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