ORBITS, WEAK ORBITS AND LOCAL CAPACITY OF OPERATORS

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Let T be an operator on a Banach space X. We give a survey of results concerning orbits $\{T^n x : n = 0, 1, \dots\}$ and weak orbits $\{\langle T^n x, x^* \rangle : n = 0, 1, \dots\}$ of T where $x \in X$ and $x^* \in X^*$. Further we study the local capacity of operators and prove that there is a residual set of points $x \in X$ with the property that the local capacity $\operatorname{cap}(T, x)$ is equal to the global capacity $\operatorname{cap} T$. This is an analogy to the corresponding result for the local spectral radius.

INTRODUCTION

Let T be a bounded linear operator acting on a (real or complex) Banach space X and let $x \in X$. The orbit of x under the operator T is the sequence $\{T^n x : n = 0, 1, \dots\}$.

The properties of orbits of different points may differ very much — the orbits of some points may be "regular" while other points may have very "irregular" orbits.

Many deep results and problems of operator theory may be formulated using the notion of orbits, For example, T has no nontrivial invariant subspace if and only if the orbit of each non-zero vector $x \in X$ spans the whole space. Similarly, T has no nontrivial closed invariant subset if and only if the orbit of each $x \neq 0$ is dense; in this case all orbits are extremely irregular.

Analogously, weak orbits under T are sequences $\{\langle T^n x, x^* \rangle : n = 0, 1, \dots\}$ where $x \in X$ and $x^* \in X^*$ are fixed. This notion is also closely related to the invariant subspace problem — the main idea of the celebrated Scott Brown technique is the construction of a weak orbit with very definite properties.

Many results for both orbits and weak orbits have their parallel for continuous one parameter semigroups of operators. In this context, orbits are closely related to stability results for semigroups of operators.

In the last section of this paper we study also polynomial orbits. By the polynomial orbit of $x \in X$ we mean the set $\{p(T)x : p \text{ polynomial}\}$. Apart from the invariant subspace problem this notion is closely connected with the notion of capacity of operators introduced by P. Halmos.

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The aim of this paper is to give a survey of results concerning orbits, weak orbits and polynomial orbits of operators. We always try to construct all types of orbits as regular as possible.

Many results concerning orbits of operators on complex Hilbert spaces may be found in [2]. For results concerning semigroups of operators see [14].

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Denote by $\mathcal{L}(X)$ the set of all bounded linear operators acting on a Banach space X.

We say that a subset $M \subset X$ is residual if its complement $X \setminus M$ is of the first category. Clearly a subset $M \subset X$ is residual if and only if it contains a dense G_δ-set.

I. ORBITS IN COMPLEX BANACH SPACES

In this section X will be a complex Banach space and $T \in \mathcal{L}(X)$.

It is known that there is a residual set of points $x \in X$ with the property that the local spectral radius $r_x(T) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$ is equal to the spectral radius

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} = \inf_n \|T^n\|^{1/n},$$

see [15], [5]. In particular, for x in this residual set, there are infinitely many powers such that $\|T^n x\|$ is "large".

Moreover, by [9], there are always points $x \in X$ such that $\|T^n x\|$ is "large" for all powers $n \geq 0$.

More precisely, we have the following results:

THEOREM 1.1. *Let $T \in \mathcal{L}(X)$, let $(a_n)_{n\geq 0}$ be a sequence of positive numbers such that $a_n \to 0$. Then:*

(i) the set of all $x \in X$ with the property that

$$\|T^n x\| \geq a_n \|T^n\| \qquad \text{for infinitely many } n\text{'s}$$

is residual.

(ii) Let $k \geq 0$. Then in each ball in X of radius greater than $\max\{a_j : j \geq k\}$ there is a vector u such that

$$\|T^n u\| \geq a_n r(T^n) \qquad (n \geq k).$$

In particular, there is a dense set of points $x \in X$ with the property that $\|T^n x\| \geq a_n r(T^n)$ for all but a finite number of n's. Further, there exist points $x \in X$ such that $\|T^n x\| \geq a_n r(T^n)$ for all $n \geq 0$.

PROOF. (i) For $k \in \mathbf{N}$ set

$$M_k = \{x \in X : \text{ there exists } n \geq k \text{ such that } \|T^n x\| > a_n \|T^n\|\}.$$

Clearly M_k is an open set. We prove that M_k is dense. Let $x \in X$ and $\varepsilon > 0$. Choose $n \geq k$ such that $a_n \varepsilon^{-1} < 1$. There exists $z \in X$ of norm one such that $\|T^n z\| > a_n \varepsilon^{-1} \|T^n\|$. Then

$$2a_n \|T^n\| < \|T^n(2\varepsilon z)\| \leq \|T^n(x+\varepsilon z)\| + \|T^n(x-\varepsilon z)\|$$

so that either $\|T^n(x+\varepsilon z)\| > a_n\|T^n\|$ or $\|T^n(x-\varepsilon z)\| > a_n\|T^n\|$. Thus either $x+\varepsilon z \in M_k$ or $x - \varepsilon z \in M_k$ so that $\operatorname{dist}\{x, M_k\} \leq \varepsilon$. Since x and ε were arbitrary, the set M_k is dense.

By the Baire category theorem the intersection $\bigcap_{k=1}^{\infty} M_k$ is a dense G_δ-set, hence it is residual. Clearly each $x \in \bigcap_{k=1}^{\infty} M_k$ satisfies $\|T^n x\| > a_n \|T^n\|$ for infinitely many n's.

In particular, for $a_n = n^{-1}$ we obtain

$$r_x(T) = \limsup_{n\to\infty} \|T^n x\|^{1/n} \geq \limsup_{n\to\infty} \Big(\frac{\|T^n\|}{n}\Big)^{1/n} = r(T)$$

for all x in a residual subset of X.

(ii) was proved in [9]; for Hilbert space operators see also [2].

COROLLARY 1.2. *The set* $\big\{x \in X : \limsup_{n\to\infty} \|T^n x\|^{1/n} = r(T)\big\}$ *is residual. The set* $\big\{x \in X : \liminf_{n\to\infty} \|T^n x\|^{1/n} = r(T)\big\}$ *is dense.*

In particular, there is a dense subset of points $x \in X$ *with the property that the limit* $\lim_{n\to\infty} \|T^n x\|^{1/n}$ *exists (and is equal to* $r(T)$*).*

The existence of the limit $\lim_{n\to\infty} \|T^n x\|^{1/n}$ was also studied in [1].

In general it is not possible to replace the word "dense" in Corollary 1.2 by "residual".

EXAMPLE 1.3. Let H be a separable Hilbert space with an orthonormal basis $\{e_j : j = 0, 1, \ldots\}$ and let S be the backward shift, $Se_0 = 0$, $Se_j = e_{j-1}$ $(j \geq 1)$. Then $r(S) = 1$ and the set $\{x \in H : \liminf_{n\to\infty} \|S^n x\|^{1/n} = 0\}$ is residual.

In particular, the set $\{x \in H :$ the limit $\lim_{n\to\infty} \|S^n x\|^{1/n}$ exists$\}$ is of the first category (but it is always dense by Corollary 1.2).

PROOF. For $k \in \mathbf{N}$ let

$$M_k = \{x \in X : \text{ there exists } n \geq k \text{ such that } \|S^n x\| < k^{-n}\}.$$

Clearly M_k is an open subset of X. Further, M_k is dense in X. To see this, let $x \in X$ and $\varepsilon > 0$. Let $x = \sum_{j=0}^{\infty} \alpha_j e_j$ and choose $n \geq k$ such that $\sum_{j=n}^{\infty} |\alpha_j|^2 < \varepsilon^2$. Set $y = \sum_{j=0}^{n-1} \alpha_j e_j$. Then $\|y - x\| < \varepsilon$ and $S^n y = 0$. Thus $y \in M_k$ and M_k is a dense open subset of X.

By the Baire category theorem the set $M = \bigcap_{k=0}^{\infty} M_k$ is a dense G_δ-subset of X, hence it is residual.

Let $x \in M$. For each $k \in \mathbf{N}$ there is $n_k \geq k$ such that $\|S^{n_k} x\| < k^{-n_k}$ so that $\liminf_{n\to\infty} \|S^n x\|^{1/n} = 0$.

Since the set $\{x \in H : \limsup_{n\to\infty} \|S^n x\|^{1/n} = r(S) = 1\}$ is also residual, we see that the set $\{x \in H :$ the limit $\lim_{n\to\infty} \|S^n x\|^{1/n}$ exists$\}$ is of the first category.

It is also possible to combine conditions (i) and (ii) of Theorem 1.1 and to obtain points $x \in X$ with $\|T^n x\| \ge a_n \cdot \|T^n\|$ for all n; in this case, however, there is a restriction on the sequence (a_n). The next lemma and its corollary essentially improve the estimates of [2], Proposition 2.B.2.

LEMMA 1.4. *Let X, Y be complex Banach spaces, let $T_n \in \mathcal{L}(X,Y)$ $(n = 1,2,\dots)$ be a sequence of operators, let a_n be positive numbers such that $\sum_{n=1}^{\infty} (\alpha_n)^{2/3} < 1/4$. Let $x \in X$. Then there exists $u \in X$ such that $\|u - x\| < 1/4$ and $\|T_n u\| \ge a_n \|T_n\|$ for all $n \ge 1$.*

PROOF. Without loss of generality we can assume that all operators T_n are non-zero.

Choose $\delta > 0$ such that $(1+\delta)\sum_{n=1}^{\infty} \alpha_n^{2/3} < 1/4$. Set $\varepsilon_n = (1+\delta)\alpha_n^{2/3}$ $(n = 1,2,\dots)$ so that $\sum_{n=1}^{\infty} \varepsilon_n < 1/4$. For each n find $z_n \in X$ of norm one such that $\|T_n z_n\| \ge (1+\delta)^{-1}\|T_n\|$.

The proof will be done in several steps.

A. For each $k \in \mathbf{N}$ there are complex numbers $\lambda_1, \dots, \lambda_n$, $|\lambda_n| \le \varepsilon_n$ $(n = 1, \dots, k)$ such that

$$\Big\|T_n\Big(x + \sum_{n=1}^{k} \lambda_n z_n\Big)\Big\| \ge a_n \|T_n\| \qquad (n = 1, \dots, k).$$

Proof. Fix $k \in \mathbf{N}$. Write

$$\Lambda = \big\{\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbf{C}^k : |\lambda_n| \le \varepsilon_n \quad (n = 1, \dots, k)\big\}.$$

For $\lambda \in \Lambda$ set $u_\lambda = x + \sum_{n=1}^{k} \lambda_n z_n$.

For $j = 1, \dots, k$ let $\Lambda_j = \big\{\lambda \in \Lambda : \|T_j u_\lambda\| < a_j \|T_j\|\big\}$. Let $1 \le j \le k$ and suppose that $\lambda, \lambda' \in \Lambda_j$ where $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\lambda' = (\lambda_1, \dots, \lambda_{j-1}, \lambda_j', \lambda_{j+1}, \dots, \lambda_k)$. Then

$$|\lambda_j - \lambda_j'|(1+\delta)^{-1}\|T_j\| \le |\lambda_j - \lambda_j'| \cdot \|T_j z_j\| = \|T_j(u_\lambda - u_{\lambda'})\| \le \|T_j u_\lambda\| + \|T_j u_\lambda'\| < 2a_j\|T_j\|$$

so that $|\lambda_j - \lambda_j'| < 2a_j(1+\delta)$. Thus, for fixed $\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_k$, the set $\big\{\nu \in \mathbf{C} : (\lambda_1, \dots, \lambda_{j-1}, \nu, \lambda_{j+1}, \dots, \lambda_k) \in \Lambda_j\big\}$ is contained in a ball of radius $2a_j(1+\delta)$.

Let μ be the Lebesgue measure on Λ. Then $\mu(\Lambda) = \prod_{n=1}^{k} (\pi\varepsilon_n^2)$ and, by the Fubini theorem,

$$\mu(\Lambda_j) \le 4\pi(1+\delta)^2 a_j^2 \prod_{\substack{1 \le n \le k \\ n \ne j}} (\pi\varepsilon_n^2) = \frac{4a_j^2(1+\delta)^2}{\varepsilon_j^2}\mu(\Lambda) \le 4a_j^{2/3}\mu(\Lambda).$$

Thus

$$\mu\Big(\Lambda \setminus \bigcup_{j=1}^{k} \Lambda_j\Big) \ge \mu(\Lambda)\Big(1 - 4\sum_{j=1}^{k} a_j^{2/3}\Big) > 0.$$

Hence there exists $\lambda \in \Lambda \setminus \bigcup_{j=1}^k \Lambda_j$. In other words, $u = u_\lambda$ satisfies $\|T_j u\| \geq a_j\|T_j\| \quad (j = 1, \dots, k)$ and $\|u - x\| \leq \sum_{n=1}^k |\lambda_n| \leq \sum_{n=1}^k \varepsilon_n < 1/4$.

B. The set $M = \bigl\{x + \sum_{n=1}^\infty \lambda_n z_n \,:\, |\lambda_n| \leq \varepsilon_n \quad (n = 1, 2, \dots)\bigr\}$ is totally bounded.

Proof. We must show that for each $\eta > 0$ there is a finite η-net in M.

Find $k \in \mathbf{N}$ such that $\sum_{n=k+1}^\infty \varepsilon_n < \eta/2$. Set

$$M_k = \Bigl\{x + \sum_{n=1}^k \lambda_n z_n : |\lambda_n| \leq \varepsilon_n \quad (n = 1, 2, \dots, k)\Bigr\}.$$

Clearly M_k is compact so that there exists a finite set $F \subset M_k$ such that $\operatorname{dist}\{u, F\} \leq \eta/2$ for all $u \in M_k$. Clearly F is the required η-net for M.

Proof of Lemma 1.4. By A, for each $k \in \mathbf{N}$ there is $u_k \in M$ with

$$\|T_n u_k\| \geq a_n \|T_n\| \qquad (n = 1, \dots, k).$$

By B, there is a convergent subsequence (u_{k_j}) of (u_k). Denote by $u \in X$ its limit. Clearly $\|u - x\| \leq \limsup_{j\to\infty} \|u_{k_j} - x\| \leq \sum_{n=1}^\infty \varepsilon_n < 1/4$ and

$$\|T_n u\| \geq a_n \|T_n\| \qquad (n = 1, 2, \dots).$$

COROLLARY 1.5. *Let $T \in \mathcal{L}(X)$. Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers satisfying $\sum_{n=0}^\infty a_n^{2/3} < \infty$. Then there is a dense subset $L \subset X$ such that, for each $x \in L$, there is $k \in \mathbf{N}$ with*

$$\|T^n x\| \geq a_n \|T^n\| \qquad (n \geq k).$$

Further, there are points $x \in X$ such that $\|T^n x\| \geq a_n \|T^n\|$ for all $n \geq 0$.

PROOF. Let $x \in X$ and $\varepsilon > 0$. Find $k \in \mathbf{N}$ and s such that $\Bigl(4 \sum_{n=k}^\infty a_n^{2/3}\Bigr)^{3/2} < s < \varepsilon$. Set $a'_n = \frac{a_n}{s}$. Then $\sum_{n=k}^\infty {a'}_n^{2/3} < 1/4$ so that, by Lemma 1.4, there exists $u \in X$ with $\|u - \frac{x}{s}\| < 1/4$ and $\|T^n u\| \geq a'_n \|T^n\| \quad (n \geq k)$. Thus $\|su - x\| < \varepsilon$ and $\|T^n(su)\| \geq a_n \|T^n\| \quad (n \geq k)$.

A better estimate can be obtained using the essential norm.

For $T \in \mathcal{L}(X)$ and a closed subspace $M \subset X$ denote by $T|M$ the restriction $T|M : M \to X$. For $T \in \mathcal{L}(X)$ let $\|T\|_\mu = \inf\bigl\{\|T|M\| : M \subset X, \operatorname{codim} M < \infty\bigr\}$. This quantity belongs to "measures of non-compactness" since $\|T\|_\mu = 0$ if and only if T is compact (for more details see [8]).

For Hilbert space operators $\|T\|_\mu$ is equal to the essential norm $\|T\|_e = \inf\{\|T + K\| : K \in \mathcal{K}(X)\}$ where $\mathcal{K}(X)$ denotes the ideal of all compact operators acting on X.

The following lemma (see [10], Lemma 1) is a useful technical tool in many constructions. It plays the role of the "orthogonal complement" in general Banach spaces.

LEMMA 1.6. *Let F be a finite dimensional subspace of a Banach space X, let $\varepsilon > 0$. Then there exists a closed subspace $M \subset X$ of finite codimension such that*

$$\|f + m\| \geq (1-\varepsilon)\max\{\|f\|, \|m\|/2\}$$

for all $f \in F$ and $m \in M$.

THEOREM 1.7. *Let $T \in \mathcal{L}(X)$, let $(a_n)_{n\geq 0}$ be a sequence of positive numbers satisfying $\sum_{n=0}^{\infty} a_n < \infty$. Then each ball in X of radius greater than $2\sum_{n=0}^{\infty} a_n$ contains a point u such that*

$$\|T^n u\| \geq a_n \|T^n\|_\mu \qquad (n = 0, 1, \ldots). \tag{1}$$

PROOF. The statement is trivial if $\dim X < \infty$. Suppose that X is infinite dimensional.

Let $x \in X$ and $\varepsilon > 2\sum_{n=0}^{\infty} a_n$. We show that there is $u \in X$ such that $\|u - x\| < \varepsilon$ and (1).

Let $\delta > 0$ satisfy $(1+\delta)a_0 + 2(1+\delta)^3 \sum_{n=1}^{\infty} a_n < \varepsilon$. We construct inductively a convergent sequence $(u_k)_{k\geq 0}$ whose limit will satisfy the required conditions.

Let $u_0 \in X$ be any vector satisfying $\|u_0 - x\| = a_0(1+\delta)$ and $\|u_0\| \geq a_0(1+\delta)$ (for example, $u_0 = x + a_0(1+\delta)x/\|x\|$ will do).

If $u_k \in X$ has already been constructed then set $E_k = \bigvee\{T^n u_k : 0 \leq n \leq k+1\}$. By Lemma 1.6, there is a closed subspace $Y_k \subset X$ of finite codimension such that $\|e + y\| \geq (1+\delta)^{-1}\max\{\|e\|, \|y\|/2\}$ for all $e \in E_k, y \in Y_k$. Let $Z_k = \bigcap_{s=0}^{k+1}\bigcap_{j=0}^{k} T^{-s}Y_j$. Clearly $\operatorname{codim} Z_k < \infty$ so that there is $z_{k+1} \in Z_k$ of norm one such that $\|T^{k+1}z_{k+1}\| \geq (1+\delta)^{-1}\|T^{k+1}\|_\mu$. Clearly $T^s z_{k+1} \in Y_j$ for all $s \leq k+1$ and $j \leq k$. Set $u_{k+1} = u_k + 2(1+\delta)^3 a_{k+1} z_{k+1}$. Then $\|u_{k+1} - u_k\| = 2(1+\delta)^3 a_{k+1}$ so that the sequence (u_k) constructed in this way is Cauchy. Denote by u its limit, $u = u_0 + \sum_{k=1}^{\infty} 2(1+\delta)^3 a_k z_k$. Clearly $\|u - x\| \leq \|u_0 - x\| + 2(1+\delta)^3\sum_{k=1}^{\infty} a_k < \varepsilon$ and $\|u\| \geq (1+\delta)^{-1}\|u_0\| \geq a_0$. For each $n \geq 1$ we have

$$\begin{aligned}
\|T^n u\| &= \Big\|T^n\Big(u_n + \sum_{k=n+1}^{\infty} 2(1+\delta)^3 a_k z_k\Big)\Big\| \geq (1+\delta)^{-1}\|T^n u_n\| \\
&= (1+\delta)^{-1}\Big\|T^n\Big(u_{n-1} + 2(1+\delta)^3 a_n z_n\Big)\Big\| \geq \frac{(1+\delta)^{-2}}{2}\big\|T^n\big(2(1+\delta)^3 a_n z_n\big)\big\| \\
&= (1+\delta)a_n\|T^n z_n\| \geq a_n\|T^n\|_\mu.
\end{aligned}$$

If X is a Hilbert space then it is possible to take in the previous proof $Y_k = E_k^\perp$. The sequence (z_k) is then orthonormal and it is possible to obtain a better result, which improves [2], Theorem 2.A.7 (compare also Remark 2.A.8 of [2] with the just proved Theorem 1.7).

COROLLARY 1.8. *Let T be an operator on a Hilbert space H. Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers satisfying $\sum_{n=0}^{\infty} a_n^2 < \infty$. Then in each ball of radius greater than $\left(\sum_{n=0}^{\infty} a_n^2\right)^{1/2}$ there exists a point x such that $\|T^n x\| \geq a_n\|T^n\|_e \quad (n \geq 0)$.*

Further, there exists a dense subset $L \subset H$ such that for each $x \in L$ there is $k \in \mathbf{N}$ with

$$\|T^n x\| \geq a_n \|T^n\|_e \qquad (n \geq k).$$

Another result which is true for Hilbert space operators is the following theorem, see [2], Corollary 3.6. We give an alternative proof which can be adapted to Banach space operators.

LEMMA 1.9. *Let H be a Hilbert space, let $T \in \mathcal{L}(H)$ be a non-nilpotent operator. Then there exists $x \in H$ such that*

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} = \infty.$$

PROOF. We distinguish two cases.

A. There exists a subspace $M \subset H$ of finite codimension such that $\|T^n|M\| \leq \frac{1}{2}\|T^n\|$ for infinitely many n's.

Let $\{f_1, \dots, f_r\}$ be an orthonormal basis in $M^\perp$. Let $A = \{n \in \mathbf{N} : \|T^n|M\| \leq \frac{1}{2}\|T^n\|\}$, so that A is an infinite set. For $j = 1, \dots, r$ set

$$A_j = \Big\{n \in A : \|T^n f_j\| \geq \frac{1}{3r}\|T^n\|\Big\}.$$

We show that $\bigcup_{j=1}^r A_j = A$. Suppose on the contrary that there is $n \in A \setminus \bigcup_{j=1}^r A_j$. Let $x \in H$, $\|x\| = 1$ and $\|T^n x\| > \frac{5}{6}\|T^n\|$. Write x as $x = \sum_{j=1}^r \alpha_j f_j + u$ where $\alpha_j \in \mathbf{C}$ and $u \in M$. Then $|\alpha_j| \leq 1$, $\|u\| \leq 1$ and

$$\|T^n x\| \leq \sum_{j=1}^r |\alpha_j| \cdot \|T^n f_j\| + \|T^n u\| \leq r \frac{1}{3r}\|T^n\| + \frac{1}{2}\|T^n\| = \frac{5}{6}\|T^n\|,$$

a contradiction. Thus $A = \bigcup_{j=1}^r A_j$ and there exists $j \in \{1, \dots, r\}$ such that A_j is infinite. Hence

$$\sum_{n=0}^{\infty} \frac{\|T^n f_j\|}{\|T^n\|} \geq \sum_{n \in A_j} \frac{\|T^n f_j\|}{\|T^n\|} \geq \sum_{n \in A_j} \frac{1}{3r} = \infty.$$

B. For each subspace $M \subset H$ of finite codimension, $\|T^n|M\| > \frac{1}{2}\|T^n\|$ for all but a finite number of n's.

We construct inductively a convergent sequence $(x_k) \subset H$ and an increasing sequence (n_k) of positive integers such that $\|T^{n_j} x_k\| \geq \frac{1}{2j} \cdot \|T^{n_j}\| \quad (j \leq k)$. Then the limit $x = \lim_{k \to \infty} x_k$ will satisfy $\|T^{n_j} x\| \geq \frac{1}{2j} \cdot \|T^{n_j}\|$ for all j, so that

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} \geq \sum_{j=1}^{\infty} \frac{\|T^{n_j} x\|}{\|T^{n_j}\|} \geq \sum_{j=1}^{\infty} \frac{1}{2j} = \infty.$$

Let $n_1 = 1$, $x_1 \in H$, $\|x_1\| = 1$ and $\|Tx_1\| > \|T\|/2$. Let $k \in \mathbf{N}$ and suppose that we have found $x_k \in H$ and $n_1 < n_2 < \cdots < n_k$ such that $\|T^{n_j} x_k\| \geq \frac{1}{2j}\|T^{n_j}\| \quad (j \leq k)$. Let

$$M = \bigcap_{s \leq n_k} T^{-s}\Big(\vee\{T^j x_i : 0 \leq j \leq n_k, 1 \leq i \leq k\}^{\perp}\Big).$$

Clearly $\operatorname{codim} M < \infty$ so that there are $n_{k+1} > n_k$ and a vector $u_{k+1} \in M$ of norm one such that $\|T^{n_{k+1}} u_{k+1}\| > \frac{1}{2}\|T^{n_{k+1}}\|$. Then

$$\Big\|T^{n_{k+1}}\Big(x_k + \frac{u_{k+1}}{k+1}\Big)\Big\| + \Big\|T^{n_{k+1}}\Big(x_k - \frac{u_{k+1}}{k+1}\Big)\Big\| \geq \frac{2\|T^{n_{k+1}} u_{k+1}\|}{k+1} \geq \frac{\|T^{n_{k+1}}\|}{k+1}$$

so that either $x_{k+1} = x_k + \frac{u_{k+1}}{k+1}$ or $x_{k+1} = x_k - \frac{u_{k+1}}{k+1}$ will satisfy $\|T^{n_{k+1}} x_{k+1}\| \geq \frac{\|T^{n_{k+1}}\|}{2(k+1)}$. Further $T^{n_j} x_k \perp T^{n_j} u_{k+1} \quad (j \leq k)$ so that

$$\|T^{n_j} x_{k+1}\| \geq \|T^{n_j} x_k\| \geq \frac{1}{2j}\|T^{n_j}\| \qquad (j \leq k).$$

Let (x_k) be the sequence constructed in the above described way. For $m < k$ we have $\|x_k - x_m\|^2 = \sum_{i=m+1}^{k} \frac{1}{i^2}$. Thus the sequence (x_k) is convergent and its limit x satisfies the required condition.

THEOREM 1.10. *Let T be a non-nilpotent operator on a Hilbert space H. Then the set $\Big\{x \in H : \sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} = \infty\Big\}$ is residual.*

PROOF. For $k \in \mathbf{N}$ let $M_k = \{x \in H : \sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} > k\}$. Clearly M_k is open. To show that M_k is dense, let $x \in H$ and $\varepsilon > 0$. By the previous lemma there is $u \in H$ such that $\sum_{n=0}^{\infty} \frac{\|T^n u\|}{\|T^n\|} = \infty$. Clearly we can assume that $\|u\| = \varepsilon$. Then

$$\sum_{n=0}^{\infty} \frac{\|T^n(x+u)\|}{\|T^n\|} + \sum_{n=0}^{\infty} \frac{\|T^n(x-u)\|}{\|T^n\|} \geq 2\sum_{n=0}^{\infty} \frac{\|T^n u\|}{\|T^n\|} = \infty$$

so that either $x + u$ or $x - u$ belongs to M_k. Thus M_k is an open dense subset of X and $M = \bigcap_{k=0}^{\infty} M_k$ is residual. The points of M satisfy the required property.

For Banach space operators the previous statements are not true:

EXAMPLE 1.11. There are a Banach space X and a non-nilpotent operator $T \in \mathcal{L}(X)$ such that $\sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} < \infty$ for all $x \in X$.

PROOF. Let X be the ℓ_1 space with the standard basis $\{e_0, e_1, \ldots\}$. Let $T \in \mathcal{L}(X)$ be the weighted backward shift defined by $Te_0 = 0$ and $Te_k = (\frac{k+1}{k})^2 e_{k-1} \quad (k \geq 1)$. For $n \in \mathbf{N}$ we have

$$T^n e_k = \begin{cases} 0 & (n > k), \\ \frac{(k+1)^2}{(k-n+1)^2} e_{k-n} & (n \leq k) \end{cases}$$

and $\|T^n\| = (n+1)^2$. Thus $r(T) = 1$.

Let $x \in X$, $x = \sum_{k=0}^{\infty} \alpha_k e_k$ where $\sum_{k=0}^{\infty} |\alpha_k| < \infty$. Then

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} = \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \frac{|\alpha_k|(k+1)^2}{(n+1)^2(k-n+1)^2} = \sum_{k=0}^{\infty} |\alpha_k| \sum_{n=0}^{k} \frac{(k+1)^2}{(n+1)^2(k-n+1)^2}.$$

We have

$$\sum_{n=0}^{k} \frac{(k+1)^2}{(n+1)^2(k-n+1)^2} = \sum_{n=0}^{[k/2]} \frac{(k+1)^2}{(n+1)^2(k-n+1)^2} + \sum_{n=[k/2]+1}^{k} \frac{(k+1)^2}{(n+1)^2(k-n+1)^2}$$
$$\leq \sum_{n=0}^{[k/2]} \frac{4}{(n+1)^2} + \sum_{n=[k/2]+1}^{k} \frac{4}{(k-n+1)^2} \leq 8 \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{4\pi^2}{3}.$$

Thus

$$\sum_{n=0}^{\infty} \frac{\|T^n x\|}{\|T^n\|} \leq \sum_{k=0}^{\infty} |\alpha_k| \cdot \frac{4\pi^2}{3} = \frac{4\|x\|\pi^2}{3} < \infty.$$

REMARK 1.12. Let H be a Hilbert space, let $T \in \mathcal{L}(H)$ be an non-nilpotent operator and let $c < 2$. Using the method of proof of Theorem 1.9 it is easy to check that the set

$$\left\{ x \in H : \sum_{n=0}^{\infty} \Big(\frac{\|T^n x\|}{\|T^n\|} \Big)^c = \infty \right\}$$

is residual. For $c = 2$ the statement is not true; an example will be given later.

For Banach space X and $T \in \mathcal{L}(X)$ it is possible to show that

$$\left\{ x \in X : \sum_{n=0}^{\infty} \Big(\frac{\|T^n x\|}{\|T^n\|} \Big)^c = \infty \right\}$$

is residual for all $c < 1$. By the previous example, this is not true for $c = 1$.

II. ORBITS IN REAL BANACH SPACES

The main technical difficulty in generalizing the results of the previous section to the real case is the lack of approximate eigenvalues. Most of the results that do not use approximate eigenvalues remain unchanged in the real case.

This is true for Theorem 1.1 (i), Lemma 1.6, Theorems 1.7, 1.8, 1.10 and Remark 1.12. Because of different geometry of the real line Theorem 1.5 is modified in the following way:

THEOREM 2.1. *Let T be an operator in a real Banach space X. Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers satisfying $\sum_{n=0}^{\infty} a_n^{1/2} < \infty$. Then there is a dense subset $L \subset X$ such that, for each $x \in L$, there is $k \in \mathbf{N}$ with*

$$\|T^n x\| \geq a_n \|T^n\| \qquad (n \geq k).$$

Other results can be proved in the real case by using the complexification of a real Banach space.

Let X be a real Banach space. Set $X_c = \{x + iy : x, y \in X\}$. Define a norm in X_c by

$$\|x + iy\| = \inf \sum_{j=1}^{n} |\alpha_j + i\beta_j| \cdot \|x_j\| \qquad (x, y \in X)$$

where the infimum is taken over all $n \in \mathbf{N}$, $\alpha_j, \beta_j \in \mathbf{R}$ and $x_j \in X$ such that $\sum_{j=1}^{n}(\alpha_j + i\beta_j)x_j = x + iy$. With naturally defined algebraic operations, X_c is a complex Banach space called the complexification of X.

It is easy to see that

$$\max\{\|x\|, \|y\|\} \le \|x + iy\| \le \|x\| + \|y\| \qquad (x, y \in X).$$

Let T be an operator on X. The complexification of T is the operator $T_c \in \mathcal{L}(X_c)$ defined by $T_c(x + iy) = Tx + iTy \quad (x, y \in X)$. Clearly $\|T\| \le \|T_c\| \le 2\|T\|$. By the spectrum of T we understand the spectrum of its complexification T_c. Similarly we define the spectral radius $r(T) = \max\{|\lambda| : \lambda \in \sigma(T_c)\} = \lim_{n\to\infty} \|T^n\|^{1/n}$. In the same way we use for operators in real Banach spaces the essential spectrum $\sigma_e(T) = \sigma_e(T_c) = \{\lambda \in \mathbf{C} : T_c - \lambda \text{ is not Fredholm}\}$, the essential spectral radius $r_e(T) = r_e(T_c) = \max\{|\lambda| : \lambda \in \sigma_e(T)\}$ and the upper semi-Fredholm spectrum $\sigma_{\pi e}(T) = \sigma_{\pi e}(T_c) = \{\lambda \in \mathbf{C} : T_c - \lambda \text{ is not upper semi-Fredholm}\}$. Equivalently, $\lambda \in \sigma_{\pi e}(T)$ if and only if, for each subspace $M \subset X_c$ of finite codimension, $(T_c - \lambda)|M$ is not bounded below. Recall that $\partial\sigma_e(T) \subset \sigma_{\pi e}(T)$, see [7].

The proof of Theorem 1 (ii) (the existence of vectors x with $\|T^n x\|$ large for all n) is based on the existence of approximate eigenvalues and thus it can not be used in the real case. In [12] it was proved for real Banach space operators under an additional assumption that $r(T) = 1$ and T is power bounded ($\sup_n \|T^n\| < \infty$). We prove a variant of this result in general.

LEMMA 2.2. *Let X be a real Banach space, $T \in \mathcal{L}(X)$, let $r(T) = 1$, $\alpha \in \sigma_e(T)$, $|\alpha| = 1$. Then there is a positive constant C (depending only on α) with the following property: for each $n \in \mathbf{N}$ and each subspace $Y \subset X$ of finite codimension there exists a vector $y \in Y$ of norm one with $\|T^j y\| \ge C \quad (j = 0, 1, \ldots, n)$.*

PROOF. There is $k \in \mathbf{N}$ such that $\min\bigl\{|\alpha^j - 1|, |\alpha^{j+1} - 1|, \ldots, |\alpha^{j+k} - 1|\bigr\} \le 1/6$ for all $j \in \mathbf{N}$. This is clear if $\alpha = e^{2\pi i t}$ with t rational; if t is irrational then the set $\{\alpha^j : j = 0, 1, \ldots\}$ is dense in the unit circle so that there is $k \in \mathbf{N}$ such that $\{1, \alpha, \alpha^2, \ldots, \alpha^k\}$ is a 1/6-net in the unit circle so that the same is true also for the set $\{\alpha^j, \alpha^{j+1}, \ldots, \alpha^{j+k}\}$.

Set $C = \bigl(6 \max\{1, \|T\|, \|T^2\|, \ldots, \|T^k\|\}\bigr)^{-1}$.

We have $\alpha \in \partial\sigma_e(T) \subset \sigma_{\pi e}(T)$. Let $n \in \mathbf{N}$ and let Y be a subspace of X of finite codimension. Let $X_c = X + iX$ be the complexification of X and $Y_c = Y + iY$. Clearly Y_c is a subspace of finite codimension in X_c. Then there exists a vector $z \in Y_c$ of norm one such that $\|T^j z - \alpha^j z\| \le 1/6 \quad (j = 0, 1, \ldots, n + k)$. Express $z = u + iv$ for some $u, v \in Y$. Then either $\|u\| \ge 1/2$ or $\|v\| \ge 1/2$. Without loss of generality we can assume that $\|u\| \ge 1/2$.

Let $j \le n$. Find $j' \in \{j, j+1, \dots, j+k\}$ such that $|\alpha^{j'} - 1| \le 1/6$. Then

$$\|T^{j'}u - u\| \le \|T^{j'}z - z\| \le \|T^{j'}z - \alpha^{j'}z\| + \|\alpha^{j'}z - z\| \le 1/6 + 1/6 = 1/3$$

so that

$$\|T^{j'}u\| \ge \|u\| - \|T^{j'}u - u\| \ge 1/2 - 1/3 = 1/6.$$

Further $\|T^{j'}u\| \le \|T^{j'-j}\| \cdot \|T^j u\|$ so that $\|T^j u\| \ge \frac{1}{6\|T^{j'-j}\|} \ge C$ for all $j \le n$.

The next result is a weaker form of Theorem 1.1 (ii) for real Banach spaces.

THEOREM 2.3. *Let X be a real Banach space, $T \in \mathcal{L}(X)$, let $(a_n)_{n\ge 0}$ be a sequence of positive numbers, $a_n \to 0$. Then there exists a dense subset $L \subset X$ with the property that for each $x \in L$ there is a constant $c > 0$ with*

$$\|T^n x\| \ge c \cdot a_n r(T^n) \qquad (n = 0, 1, \dots).$$

PROOF. By replacing a_n by $\sup\{a_j : j \ge n\}$ we can assume that $a_n \searrow 0$. We can also assume that $r(T) = 1$.

We distinguish two cases:

A. $r_e(T) < 1$.

Find $\alpha \in \sigma(T)$ with $|\alpha| = 1$. Then α is an isolated eigenvalue of T_c. Let $M \subset X_c$ be the corresponding spectral subspace and let P be the Riesz projection onto M. Then $\dim M < \infty$ and $(T_c - \alpha)|M$ is a nilpotent operator.

Suppose that w be a non-zero vector in M. Let $k \in \mathbf{N}$ satisfy $(T_c - \alpha)^k w = 0$ and $(T_c - \alpha)^{k-1} w \ne 0$. Let $Q \in \mathcal{L}(M)$ be a projection satisfying $Qw = w$ and $Q\big(\ker(T_c - \alpha)^{k-1}|M\big) = 0$. Then $Q(T_c - \alpha)T_c^{j-1}w = 0 \quad (j = 1, 2, \dots)$ so that $QT_c^j w = \alpha Q T_c^{j-1} w$. By induction we get $QT_c^j w = \alpha^j Q w = \alpha^j w$ for all $j \ge 0$. Thus

$$\|T_c^j w\| \ge \|Q\|^{-1} |\alpha^j| \cdot \|w\| = \|Q\|^{-1}\|w\| \qquad (j \ge 0).$$

Set $Z = \{z \in X_c : Pz \ne 0\}$. For $z \in Z$ we have $\|T_c^j P z\| = \|P T_c^j z\| \le \|P\| \cdot \|T_c^j z\|$ so that

$$\|T_c^j z\| \ge \|P\|^{-1} \|T_c^j P z\| \ge \frac{\|Pz\|}{\|P\| \cdot \|Q\|} \qquad (j \ge 0).$$

Clearly Z is an open dense subset of X_c. It is sufficient to show that $Z \cap X$ is dense in X since all vectors $y \in Z \cap X$ satisfy

$$\inf\{\|T^j y\| : j = 0, 1, \dots\} \ge \frac{\|Py\|}{\|P\| \cdot \|Q\|} > 0.$$

Let $x \in X$ and $\varepsilon > 0$. Let $u, v \in X$, $u + iv \ne 0$ and $T_c(u + iv) = \alpha(u + iv)$. Then $u + iv \in M$ so that $P(u + iv) = u + iv \ne 0$. Thus either $Pu \ne 0$ or $Pv \ne 0$. Consequently at least one of the vectors $x, x + \varepsilon u, x + \varepsilon v$ is in $X \cap Z$ and $X \cap Z$ is dense in X (in fact $X \cap Z$ is also open so that it is even a residual subset of X).

B. $r_e(T)=1$.

Find $\alpha\in\sigma_e(T)$ with $|\alpha|=1$. Let C be the constant from the previous lemma. Find an increasing sequence $m_1<m_2<\cdots$ such that $a_{m_j}<2^{-(j+1)}$.

Let $x\in X$ and $\varepsilon>0$. We construct inductively vectors $x_j \quad (j=0,1,\dots)$ in the following way:

Set $x_0=x$. If $j\ge 1$ and $x_{j-1}\in X$ has already been constructed then let E_j be the finite dimensional subspaces defined by $E_j=\vee\{T^n x_s : n=0,\dots,m_j, s=0,\dots,j-1\}$. By Lemma 1.6 find a subspace $Y_j\subset X$ of finite codimension satisfying $\|e+y\|\ge\max\{\|e\|/2,\|y\|/4\} \quad (e\in E_j, y\in Y_j)$ and a vector $u_j\in Y_j$ of norm one such that $\|T^n u_j\|\ge C \quad (n=0,\dots,m_j)$. Let $x_j=x_{j-1}+\frac{\varepsilon u_j}{2^j}$. Clearly the sequence (x_j) constructed in this way is convergent; denote its limit by $u=x+\sum_{j=1}^\infty \frac{\varepsilon u_j}{2^j}$. We have $\|u-x\|\le\sum_{j=1}^\infty\left\|\frac{\varepsilon u_j}{2^j}\right\|=\varepsilon$.

Let $j\ge 1$, $n\in\mathbf{N}$ and $m_j<n\le m_{j+1}$. Then

$$\|T^n u\|=\lim_{k\to\infty}\|T^n x_k\|=\lim_{k\to\infty}\Big\|T^n\Big(x_{j+1}+\varepsilon\sum_{s=j+2}^{k}\frac{u_s}{2^s}\Big)\Big\|\ge\lim_{k\to\infty}\frac{1}{2}\|T^n x_{j+1}\|$$

$$=\lim_{k\to\infty}\frac{1}{2}\Big\|T^n\Big(x+\varepsilon\sum_{s=1}^{j+1}\frac{u_s}{2^s}\Big)\Big\|\ge\frac{1}{8}\frac{\varepsilon\|T^n u_{j+1}\|}{2^{j+1}}\ge\frac{C\varepsilon}{8\cdot 2^{j+1}}\ge\frac{C\varepsilon}{8}a_n.$$

Thus $\|T^n u\|\ge\frac{C\varepsilon}{8}a_n$ for all $n\ge m_1$ so that there there is a positive constant c with $\|T^n u\|\ge c\cdot a_n$ for all $n\ge 0$.

Consequently, Corollary 1.2 remains true for operators in real Banach spaces.

COROLLARY 2.4. *Let T be an operator acting on a real Banach space X. Then the set $\{x\in X : \limsup_n\|T^n x\|^{1/n}=r(T)\}$ is residual and the set $\{x\in X : \liminf_n\|T^n x\|^{1/n}=r(T)\}$ is dense in X.*

COROLLARY 2.5. *Let T be an operator acting on a real Banach space X, let $(a_n)_{n\ge 0}$ be a sequence of positive numbers such that $\sup_n a_n<1$ and $\lim_n a_n=0$. Then there exists a norm-one vector $x\in X$ such that $\|T^n x\|\ge a_n r(T^n)$ for all $n\ge 0$.*

PROOF. Without loss of generality we can assume that $r(T)=1$ and $1>a_0\ge a_1\ge\cdots$. Let ε be a positive number satisfying $1>(1+\varepsilon)a_0$. For $n\ge 0$ set

$$b_n=\max\{a_{i_1}\cdots a_{i_m}(1+\varepsilon)^m : m\in\mathbf{N}, i_1+\cdots i_m=n\}.$$

Clearly $b_{n+k}\ge b_n a_k(1+\varepsilon) \quad (n,k\ge 0)$. Further $\lim_n b_n=0$. Indeed, let $\delta>0$ and choose j such that $a_j(1+\varepsilon)<\delta$ and $(a_0(1+\varepsilon))^j<\delta$. Let $n\ge j^2$ and $b_n=a_{i_1}\cdots a_{i_m}(1+\varepsilon)^m$. Then either $i_s<j$ for all s or there is s with $i_s\ge j$; in both cases it is easy to verify that $b_n<\delta$. Thus $b_n\to 0$ and by Theorem 2.3 there exists a positive constant C and a vector $u\in X$ such that $\|T^j u\|\ge C\cdot b_j \quad (j=0,1,\dots)$. Set $C'=\inf_j\frac{\|T^j\|}{b_j}>0$. Fix k such that $\frac{\|T^k u\|}{b_k}<C'(1+\varepsilon)$. Set $x=\frac{T^k u}{\|T^k u\|}$. Then $\|x\|=1$ and

$$\|T^n x\|=\frac{\|T^{k+n}u\|}{\|T^k u\|}\ge\frac{C'\cdot b_{k+n}}{C'(1+\varepsilon)b_k}\ge a_n$$

for all $n\ge 0$.

III. WEAK ORBITS

Some results concerning orbits remain true also for weak orbits. An example is the statement of Theorem 1.1 (i):

THEOREM 3.1. *Let T be an operator in a (real or complex) Banach space X, let $(a_n)_{n\geq 0}$ be a sequence of positive numbers such that $a_n \to 0$. Then the set of all pairs $(x, x^*) \in X \times X^*$ such that*

$$|\langle T^n x, x^*\rangle| \geq a_n \|T^n\| \qquad \text{for infinitely many } n\text{'s}$$

is residual in $X \times X^*$.

In particular, the set $\{(x,x^*) \in X \times X^* : \limsup_{n\to\infty} |\langle T^n x, x^*\rangle|^{1/n} = r(T)\}$ is residual in $X \times X^*$.

PROOF. (i) For $k \in \mathbf{N}$ set

$$M_k = \{(x,x^*) \in X \times X^* : \text{ there exists } n \geq k \text{ such that } |\langle T^n x, x^*\rangle| > a_n \|T^n\|\}.$$

Clearly M_k is an open subset of $X \times X^*$. We prove that M_k is dense. Let $x \in X, x^* \in X^*$ and $\varepsilon > 0$. Choose $n \geq k$ such that $a_n < \varepsilon^2$. There is a vector $u \in X$ of norm one such that $\|T^n u\| > \frac{a_n}{\varepsilon^2}\|T^n\|$. Let $u^* \in X^*$ satisfy $\|u^*\| = 1$ and $\langle T^n u, u^*\rangle = \|T^n u\|$. We have

$$\begin{aligned}
&|\langle T^n(x+\varepsilon u), x^* + \varepsilon u^*\rangle| + |\langle T^n(x+\varepsilon u), x^* - \varepsilon u^*\rangle| \\
&\qquad + |\langle T^n(x-\varepsilon u), x^* + \varepsilon u^*\rangle| + |\langle T^n(x-\varepsilon u), x^* - \varepsilon u^*\rangle| \\
&\geq \Big| \langle T^n(\varepsilon u + x), \varepsilon u^* + x^*\rangle + \langle T^n(\varepsilon u + x), \varepsilon u^* - x^*\rangle \\
&\qquad + \langle T^n(\varepsilon u - x), \varepsilon u^* + x^*\rangle + \langle T^n(\varepsilon u - x), \varepsilon u^* - x^*\rangle \Big| \\
&= |4\langle T^n \varepsilon u, \varepsilon u^*\rangle| = 4\varepsilon^2 \|T^n u\| > 4a_n \|T^n\|.
\end{aligned}$$

Thus there is a pair

$$(y,y^*) \in \{(x+\varepsilon u, x^* + \varepsilon u^*), (x+\varepsilon u, x^* - \varepsilon u^*), (x-\varepsilon u, x^* + \varepsilon u^*), (x - \varepsilon u, x^* - \varepsilon u^*)\}$$

such that $|\langle T^n y, y^*\rangle| > a_n \|T^n\|$. Hence $(y,y^*) \in M_k$ and M_k is dense in $X \times X^*$.

By the Baire category theorem the intersection $M = \bigcap_{k=1}^{\infty} M_k$ is a residual subset of $X \times X^*$ and all pairs $(y,y^*) \in M$ satisfy $|\langle T^n y, y^*\rangle| > a_n \|T^n\|$ for infinitely many n's.

In particular, for $a_n = n^{-1}$ we obtain that

$$\limsup_{n\to\infty} |\langle T^n y, y^*\rangle|^{1/n} \geq \limsup_{n\to\infty} \Big(\frac{\|T^n\|}{n}\Big)^{1/n} = r(T)$$

for all pairs (y,y^*) in a residual subset of $X \times X^*$.

The weak version of Theorem 1.1 (ii) is an open problem. It may be stated as follows:

PROBLEM 3.2. Let T be an operator on a complex Banach space X, let (a_n) be a sequence of positive numbers such that $a_n \to 0$. Do there exist vectors $x \in X$ and $x^* \in X^*$ such that

$$|\langle T^n x, x^* \rangle| \geq a_n \cdot r(T^n)$$

for all $n = 0, 1, \ldots$?

For real Banach spaces this is not true, see [12].

A positive answer was shown in [12] for positive operators on Banach lattices and for non-unitary isometries on Hilbert spaces. We show a positive answer for C_{00} operators (i.e., $T^n \to 0$ and $T^{*n} \to 0$ strongly) on Hilbert spaces. Operators of this class play an important role in the results concerning the existence of invariant subspaces for contractions with rich spectrum, see e.g. [3], [4]. It is an interesting question whether it is possible to obtain these results using the weak orbits instead of the Scott Brown technique.

LEMMA 3.3. *Let T be an operator on a complex Hilbert space H such that $1 \in \sigma(T)$, $\|T^n x\| \to 0$ and $\|T^{*n} x\| \to 0$ for all $x \in H$. Then, for all $\varepsilon > 0$, $\delta > 0$, $n \in \mathbf{N}$ and each subspace $M \subset H$ of finite codimension, there exists a vector $z \in M$ of norm one with*

$$\begin{aligned} Re\,\langle T^j z, z \rangle &\geq 1 - \delta \qquad (j \leq n), \\ Re\,\langle T^j z, z \rangle &\geq -\varepsilon \qquad (j > n). \end{aligned} \tag{2}$$

PROOF. By the uniform boundedness theorem we have $\sup_n \|T^n\| < \infty$ so that $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \leq 1$. Since $1 \in \sigma(T)$ we have $r(T) = 1$. Further 1 is not an eigenvalue of T since $T^n \to 0$ strongly. This implies in particular that H is infinite dimensional.

Since $\sigma(T) \setminus \sigma_e(T)$ contains only isolated eigenvalues in the unbounded component of $\mathbf{C} \setminus \sigma_e(T)$, we have $1 \in \sigma_e(T)$. Clearly $1 \in \partial\sigma_e(T) \subset \sigma_{\pi e}(T)$. This means that for all $\delta > 0$, $k \in \mathbf{N}$ and $M \subset H$, $\operatorname{codim} M < \infty$ there is a vector $z \in M$ of norm one such that $\|T^j z - z\| \leq \delta \quad (0 \leq j \leq k)$. Hence $\operatorname{Re}\langle T^j z, z\rangle = \operatorname{Re}\langle z, z\rangle + \operatorname{Re}\langle T^j z - z, z\rangle \geq 1 - \delta$.

Denote by A the set of all $\varepsilon > 0$ for which (2) is true for all $\delta > 0, n \in \mathbf{N}$ and $M \subset H$, $\operatorname{codim} M < \infty$.

Clearly $\varepsilon \in A$ implies $(\varepsilon, \infty) \subset A$. Further A is non-empty since T is power bounded (clearly $\sup_n \|T^n\| \in A$).

We show that $\varepsilon \in A$ implies $\frac{3\varepsilon}{4} \in A$. Hence $\inf A = 0$ and $A = (0, \infty)$.

Suppose that $\varepsilon \in A$. Let $n \in \mathbf{N}$, $\delta > 0$ and $M \subset H$, $\operatorname{codim} M < \infty$. We may assume that $\delta < 1$. By the assumption there is $z \in M$ of norm one such that

$$\begin{aligned} \operatorname{Re}\langle T^j z, z \rangle &\geq 1 - \delta \qquad (j \leq n), \\ \operatorname{Re}\langle T^j z, z \rangle &\geq -\varepsilon \qquad (j > n). \end{aligned}$$

Since $\lim_{j\to\infty} \|T^j z\| = 0$ and $\lim_{j\to\infty} \|T^{*j} z\| = 0$ there exists $m \geq n$ such that $\|T^j z\| \leq \varepsilon/6$ and $\|T^{*j} z\| \leq \varepsilon/6$ for all $j \geq m$. Consider the subspace $Y = \big(\vee\{z, Tz, \ldots, T^m z\}\big)^\perp$. Clearly $\operatorname{codim} Y < \infty$. Let $Y' = M \cap \bigcap_{j=0}^m T^{-j} Y$. Since $\operatorname{codim} T^{-j} Y < \infty$ for all j, we have $\operatorname{codim} Y' < \infty$. By the assumption there is $u \in Y'$ of norm one such that

$$\begin{aligned} \operatorname{Re}\langle T^j u, u \rangle &\geq 1 - \delta \qquad (j \leq m), \\ \operatorname{Re}\langle T^j u, u \rangle &\geq -\varepsilon \qquad (j > m). \end{aligned}$$

Since $T^j u \in Y$ for $j = 0, 1, \ldots, m$, we have $T^j u \perp T^i z$ for all $i, j \leq m$. In particular, $u \perp z$ and $\|z + u\| = \sqrt{2}$. Set $v = \frac{z+u}{\sqrt{2}}$. Then $v \in M$ and $\|v\| = 1$.

For $0 \leq j \leq n$ we have

$$\mathrm{Re}\,\langle T^j v, v\rangle = \frac{1}{2}\big(\mathrm{Re}\,\langle T^j z, z\rangle + \mathrm{Re}\,\langle T^j u, u\rangle\big) \geq \frac{1}{2}(1 - \delta + 1 - \delta) = 1 - \delta.$$

For $n < j \leq m$ we have

$$\mathrm{Re}\,\langle T^j v, v\rangle = \frac{1}{2}\big(\mathrm{Re}\,\langle T^j z, z\rangle + \mathrm{Re}\,\langle T^j u, u\rangle\big) \geq \frac{1}{2}(-\varepsilon + 1 - \delta) \geq \frac{-\varepsilon}{2} > \frac{-3\varepsilon}{4}.$$

Finally, for $j > m$ we have

$$\begin{aligned}\mathrm{Re}\,\langle T^j v, v\rangle &= \frac{1}{2}\big(\mathrm{Re}\,\langle T^j z, z\rangle + \mathrm{Re}\,\langle T^j u, u\rangle + \mathrm{Re}\,\langle T^j z, u\rangle + \mathrm{Re}\,\langle T^j u, z\rangle\big)\\ &\geq \frac{1}{2}\big(-\|T^j z\| - \varepsilon - \|T^j z\| - \|T^{*j} z\|\big) \geq \frac{1}{2}(-\varepsilon - \frac{3}{6}\varepsilon) = -\frac{3\varepsilon}{4}.\end{aligned}$$

Since δ, n and M were arbitrary, we have $\frac{3\varepsilon}{4} \in A$. Hence (2) is true for all $\varepsilon > 0$.

COROLLARY 3.4. *Let T be an operator acting on a complex Hilbert space H such that $1 \in \sigma(T)$, $T^n x \to 0$ and $T^{*n} x \to 0$ for all $x \in H$. Let $(a_n)_{n\geq 0}$ be a sequence of positive numbers such that $1 > a_0 \geq a_1 \geq \cdots$ and $a_n \to 0$. Then there exists $x \in X$ of norm one such that*

$$Re\,\langle T^n x, x\rangle \geq a_n \qquad (n = 0, 1, \ldots).$$

PROOF. Let $\sqrt{a_0} < d < 1$ and $\varepsilon = \sqrt{1 - d^2}$.

Find $m_0 \in \mathbf{N}$ such that $a_{m_0} < \frac{\varepsilon^2}{4}$. Find a vector $u_0 \in H$ of norm one such that $\mathrm{Re}\,\langle T^n u_0, u_0\rangle \geq \frac{a_0}{d^2}$ for all $n \leq m_0$ and $\mathrm{Re}\,\langle T^n u_0, u_0\rangle \geq \frac{-\varepsilon^2}{4}$ for all $n > m_0$. Set $x_0 = d u_0$.

We construct inductively sequences $(u_k), (x_k) \subset H$ and an increasing sequence (m_k) such that the limit $x = \lim_{k\to\infty} x_k$ will satisfy the required conditions.

Suppose that $k \geq 1$, $m_{k-1} \in \mathbf{N}$ and vectors $u_0, \ldots, u_{k-1}, x_0, \ldots, x_{k-1} \in H$ have already been constructed. Choose $m_k > m_{k-1}$ such that

$$\begin{aligned} &a_{m_k} \leq \frac{\varepsilon^2}{2^{k+3}},\\ &\|T^n x_{k-1}\| \leq \frac{\varepsilon^2}{2^{k+3}} \quad \text{and}\\ &\|T^{*n} x_{k-1}\| \leq \frac{\varepsilon^2}{2^{k+3}}\end{aligned}$$

for all $n > m_k$. Let $M_k = \bigcap_{j=0}^{m_k} T^{-j}\big(\vee\{T^n u_i : 0 \leq i \leq k-1, 0 \leq n \leq m_k\}^{\perp}\big)$. Clearly $\operatorname{codim} M_k < \infty$. Find $u_k \in M_k$ of norm one such that $\mathrm{Re}\,\langle T^n u_k, u_k\rangle \geq \frac{1}{2} \quad (n \leq m_k)$

and $\operatorname{Re}\langle T^n u_k, u_k\rangle \ge -\frac{1}{8} \quad (n > m_k)$. Then $T^n u_k \perp u_i$ and $u_k \perp T^n u_i \quad (n \le m_k, i < k)$. In particular, $u_k \perp u_i$.

Set $x_k = x_{k-1} + \frac{\varepsilon u_k}{2^{k/2}}$.

Clearly the sequence (x_k) constructed in this way is convergent. Denote by x its limit, $x = du_0 + \sum_{i=1}^{\infty} \frac{\varepsilon u_i}{2^{i/2}}$. Then $\|x\|^2 = d^2 + \sum_{i=1}^{\infty} \frac{\varepsilon^2}{2^i} = d^2 + \varepsilon^2 = 1$. For $n \le m_0$ we have

$$\operatorname{Re}\langle T^n x, x\rangle = \operatorname{Re}\langle T^n du_0, du_0\rangle + \sum_{i=1}^{\infty} \operatorname{Re}\left\langle \frac{\varepsilon T^n u_i}{2^{i/2}}, \frac{\varepsilon u_i}{2^{i/2}}\right\rangle \ge d^2 \operatorname{Re}\langle T^n u_0, u_0\rangle \ge a_0 \ge a_n.$$

For $m_0 < n \le m_1$ we have

$$\begin{aligned}\operatorname{Re}\langle T^n x, x\rangle &= \operatorname{Re}\langle T^n x_0, x_0\rangle + \sum_{i=1}^{\infty} \operatorname{Re}\left\langle \frac{\varepsilon T^n u_i}{2^{i/2}}, \frac{\varepsilon u_i}{2^{i/2}}\right\rangle \\ &\ge \frac{-d^2\varepsilon^2}{4} + \frac{1}{2}\sum_{i=1}^{\infty} \frac{\varepsilon^2}{2^i} \ge \frac{\varepsilon^2}{4} \ge a_{m_0} \ge a_n.\end{aligned}$$

Let $k \ge 1$ and $m_k < n \le m_{k+1}$. Then

$$\begin{aligned}\operatorname{Re}\langle T^n x, x\rangle &= \operatorname{Re}\langle T^n x_{k-1}, x\rangle + \operatorname{Re}\langle T^n (x - x_{k-1}), x_{k-1}\rangle + \sum_{i,j=k}^{\infty} \operatorname{Re}\left\langle \frac{\varepsilon T^n u_i}{2^{i/2}}, \frac{\varepsilon u_j}{2^{j/2}}\right\rangle \\ &\ge -\|T^n x_{k-1}\| - \|T^{*n} x_{k-1}\| + \sum_{i=k}^{\infty} \frac{\varepsilon^2}{2^i} \operatorname{Re}\langle T^n u_i, u_i\rangle \\ &\ge -\frac{2\varepsilon^2}{2^{k+3}} + \frac{\varepsilon^2}{2^k} \operatorname{Re}\langle T^n u_k, u_k\rangle + \sum_{i=k+1}^{\infty} \frac{\varepsilon^2}{2^i} \operatorname{Re}\langle T^n u_i, u_i\rangle \\ &\ge -\frac{3\varepsilon^2}{2^{k+3}} + \frac{1}{2}\sum_{i=k+1}^{\infty} \frac{\varepsilon^2}{2^i} = \frac{\varepsilon^2}{2^{k+3}} \ge a_n.\end{aligned}$$

The next result, which is something between the statement of Theorem 3.1 and Problem 3.2, is a generalization of [12]. We need the following lemma:

LEMMA 3.5. *Let X be a real or complex Banach space, $T \in \mathcal{L}(X)$, $r_e(T) = 1$, $n_0 \in \mathbf{N}$, $\varepsilon > 0$, $m \in \mathbf{N}$. Then there are numbers $n_0 < n_1 < \cdots < n_m$ such that, in each subspace $M \subset X$ of finite codimension, there exists a vector $x \in M$ of norm one with $\|T^{n_j} x - x\| \le \varepsilon \quad (j = 1, 2, \ldots, m)$.*

PROOF. Suppose first that X is a complex Banach space and $T \in \mathcal{L}(X)$. Let $\lambda \in \sigma_e(T)$, $|\lambda| = 1$. Then $\lambda \in \sigma_{\pi e}(T)$. Find $s \in \mathbf{N}$ such that $s > n_0$ and $|\lambda^s - 1| \le \varepsilon/2m$. Then

$$|\lambda^{sj} - 1| = |\lambda^s - 1| \cdot |\lambda^{s(j-1)} + \lambda^{s(j-2)} + \cdots + 1| \le \varepsilon/2$$

for $j = 1, 2, \ldots, m$. Let $x \in M$ be a vector of norm one satisfying $\|T^{sj} x - \lambda^{sj} x\| \le \varepsilon/2 \quad (j = 1, \ldots, m)$ so that $\|T^{sj} x - x\| \le \|T^{sj} x - \lambda^{sj} x\| + \|\lambda^{sj} x - x\| \le \varepsilon$.

This finishes the proof in the complex case.

In the real case consider the complexification X_c of X and $T_c \in \mathcal{L}(X_c)$. As in the complex case find a vector $x \in M_c$ of norm one and $s > n_0$ such that $\|T_c^{sj}x - x\| \le \varepsilon/2 \quad (j = 1, \dots, m)$. Let $x = u + iv$ where $u, v \in X$. Then $\|u\| + \|v\| \ge \|x\| = 1$ so that $\max\{\|u\|, \|v\|\} \ge 1/2$; without loss of generality we can assume that $\|u\| \ge 1/2$. Set $y = \frac{u}{\|u\|}$. Then $y \in M$, $\|y\| = 1$ and $\|T^{sj}y - y\| = \frac{1}{\|u\|}\|T^{sj}u - u\| \le 2\|T^{sj}x - x\| \le \varepsilon \quad (j = 1, \dots, m)$.

THEOREM 3.6. *Let X be a real or complex Banach space, $T \in \mathcal{L}(X)$ and let $(a_j)_{j\ge1}$ be a sequence of positive numbers with $a_j \to 0$. Then there exist $x \in X$, $x^* \in X^*$ and an increasing sequence (n_j) of positive integers such that*

$$Re\,\langle T^{n_j}x, x^*\rangle \ge a_j \cdot r(T)^{n_j}$$

for all $j \ge 1$.

PROOF. Without loss of generality we can assume that $r(T) = 1$ and that $1/16 \ge a_0 \ge a_1 \ge \cdots$. We distinguish two cases:

A. Suppose that T^n does not tend to 0 weakly, so that there are $x \in X$, $x^* \in X^*$ and $c > 0$ such that $|\langle T^n x, x^*\rangle| \ge c$ for infinitely many n's.

In the real case $|\langle T^n x, x^*\rangle| \le \max\{\mathrm{Re}\,\langle T^n x, x^*\rangle, \mathrm{Re}\,\langle T^n x, -x^*\rangle\}$; in the complex case

$$|\langle T^n x, x^*\rangle| \le \sqrt{2} \cdot \max\{\mathrm{Re}\,\langle T^n x, x^*\rangle, \mathrm{Re}\,\langle T^n x, ix^*\rangle, \mathrm{Re}\,\langle T^n x, -x^*\rangle, \mathrm{Re}\,\langle T^n x, -ix^*\rangle\}.$$

In both cases there are $c_1 > 0$ and $x_1^* \in X^*$ such that $\mathrm{Re}\,\langle T^n x, x_1^*\rangle \ge c_1$ for infinitely many powers n. By considering a suitable multiple of x_1^* we get the statement of the Theorem.

B. Suppose that $T^n \to 0$ weakly.

Using the uniform boundedness theorem twice it is easy to show that $\sup\{\|T^n\| : n = 0, 1, \dots\} = M < \infty$. Further $T^n \to 0$ weakly implies that there are no eigenvalues of modulus 1, i.e., $r_e(T) = 1$. Let $s = 8M$. Find numbers $m_k \in \mathbf{N}$ such that $0 = m_0 < m_1 < m_2 < \cdots$ and

$$a_j \le \frac{1}{16s^{2k}} \qquad (k \ge 0, j > m_k).$$

We construct inductively sequences $(u_k)_{k\ge0} \subset X$, $(u_k^*)_{k\ge0} \subset X^*$ and an increasing sequence of positive integers (n_j) in the following way:

Set $u_0 = 0$ and $u_0^* = 0$. Let $k \ge 0$ and suppose that $u_0, \dots, u_k \in X$, $u_0^*, \dots, u_k^* \in X^*$ and numbers $n_1, \dots, n_{m_k}$ have already been constructed. Write $x_k = \sum_{i=1}^k \frac{u_i}{s^{i-1}}$ and $x_k^* = \sum_{i=1}^k \frac{u_i^*}{s^{i-1}}$. Find q_k such that $|\langle T^j x_k, x_k^*\rangle| \le \frac{1}{16s^{2k}} \quad (j \ge q_k)$. Find numbers $n_{m_k+1}, \dots, n_{m_{k+1}}$ satisfying the properties of Lemma 3.5 such that

$$\max\{n_{m_k}, q_k\} < n_{m_k+1} < n_{m_k+2} < \cdots < n_{m_{k+1}}.$$

Let $E_k = \vee\{T^{n_j}u_i : 0 \le j \le m_{k+1}, 0 \le i \le k\}$. By Lemma 1.6 there exists a subspace Y_k of finite codimension such that

$$\|e + y\| \ge \max\{\|e\|/2, \|y\|/4\} \quad (e \in E_k, y \in Y_k).$$

Let $u_{k+1} \in Y_k \cap \big(\vee\{T^{*n_j} u_i^* : 0 \le j \le m_{k+1}, 0 \le i \le k\}\big)^\perp$ be a vector of norm one such that

$$\|T^{n_j} u_{k+1} - u_{k+1}\| < 1/16 \qquad (m_k < j \le m_{k+1}).$$

Find $u_{k+1}^* \in E_k^\perp$ such that $\|u_{k+1}^*\| = 1$ and

$$\langle u_{k+1}, u_{k+1}^* \rangle = \operatorname{dist}\{u_{k+1}, E_k\} \ge 1/4.$$

Note that $\langle T^{n_j} u_i, u_{k+1}^* \rangle = 0$ and $\langle T^{n_j} u_{k+1}, u_i^* \rangle = 0$ for all $i \le k$ and $j \le m_{k+1}$.

Continue the inductive construction and set $x = \sum_{i=1}^\infty \frac{u_i}{s^{i-1}}$ and $x^* = \sum_{i=1}^\infty \frac{u_i^*}{s^{i-1}}$.

To show that x, x^* and the sequence (n_j) satisfy the required properties, let $k \ge 0$ and $m_k < j \le m_{k+1}$. We have

$$\begin{aligned}
\operatorname{Re}\langle T^{n_j} x, x^* \rangle &= \operatorname{Re}\Big\langle T^{n_j}\big(x_k + \sum_{i=k}^\infty \frac{u_{i+1}}{s^i}\big), x_k^* + \sum_{i=k}^\infty \frac{u_{i+1}^*}{s^i}\Big\rangle \\
&= \operatorname{Re}\langle T^{n_j} x_k, x_k^* \rangle + \frac{1}{s^{2k}} \operatorname{Re}\langle T^{n_j} u_{k+1}, u_{k+1}^* \rangle + \sum_{i=k+1}^\infty \frac{1}{s^{2i}} \operatorname{Re}\langle T^{n_j} u_{i+1}, u_{i+1}^* \rangle \\
&\ge -\frac{1}{16 s^{2k}} + \frac{1}{s^{2k}}\big(\operatorname{Re}\langle u_{k+1}, u_{k+1}^* \rangle - \operatorname{Re}\langle u_{k+1} - T^{n_j} u_{k+1}, u_{k+1}^* \rangle\big) - \sum_{i=k+1}^\infty \frac{M}{s^{2i}} \\
&\ge \frac{1}{s^{2k}}\big(-\frac{1}{16} + \frac{1}{4} - \frac{1}{16} - \frac{2M}{s^2}\big) \ge \frac{1}{16 s^{2k}} \ge a_j.
\end{aligned}$$

COROLLARY 3.7. (cf. [16]) *Let X be a real or complex Banach space, let $T \in \mathcal{L}(X)$, $1 \le p < \infty$, $r(T) \ne 0$. Then the set*

$$\Big\{(x, x^*) \in X \times X^* : \sum_{n=0}^\infty \Big(\frac{|\langle T^n x, x^* \rangle|}{r(T^n)}\Big)^p = \infty\Big\}$$

is residual in $X \times X^*$. Consequently (see [17]), the set

$$\Big\{x \in X : \sum_{n=0}^\infty \Big(\frac{\|T^n x\|}{r(T^n)}\Big)^p = \infty\Big\}$$

is residual in X.

PROOF. For $k \in \mathbf{N}$ set

$$M_k = \Big\{(x, x^*) \in X \times X^* : \sum_{n=0}^\infty \Big(\frac{|\langle T^n x, x^* \rangle|}{r(T^n)}\Big)^p > k\Big\}.$$

Clearly M_k is open in $X \times X^*$. To show that M_k is dense, let $x \in X$, $x^* \in X^*$ and $\varepsilon > 0$. By the previous lemma for a suitable sequence (a_n) there are $u \in X$ and $u^* \in X^*$

such that $\sum_{n=0}^\infty \Big(\frac{|\langle T^n u,u^*\rangle|}{r(T^n)}\Big)^p = \infty$. Clearly we can assume that $\|u\| < \varepsilon$ and $\|u^*\| < \varepsilon$. Since

$$\begin{aligned}\Big(\frac{|\langle T^n u,u^*\rangle|}{r(T^n)}\Big)^p &= \Big|\frac{\langle T^n(u+x),u^*+x^*\rangle}{4r(T^n)} + \frac{\langle T^n(u+x),u^*-x^*\rangle}{4r(T^n)} \\ &\quad + \frac{\langle T^n(u-x),u^*+x^*\rangle}{4r(T^n)} + \frac{\langle T^n(u-x),u^*-x^*\rangle}{4r(T^n)}\Big|^p \\ &\le \max\Big\{\frac{|\langle T^n(u+x),u^*+x^*\rangle|}{r(T^n)}, \frac{|\langle T^n(u+x),u^*-x^*\rangle|}{r(T^n)}, \\ &\quad \frac{|\langle T^n(u-x),u^*+x^*\rangle|}{r(T^n)}, \frac{|\langle T^n(u-x),u^*-x^*\rangle|}{r(T^n)}\Big\}^p,\end{aligned}$$

we have that $\sum_{n=0}^\infty \Big(\frac{|\langle T^n y,y^*\rangle|}{r(T^n)}\Big)^p = \infty$ for at least one pair

$$(y,y^*) \in \big\{(x+u,x^*+u^*),(x+u,x^*-u^*),(x-u,x^*+u^*),(x-u,x^*-u^*)\big\}.$$

Thus M_k is dense in $X \times X^*$ and

$$M = \bigcap_k M_k = \Big\{(x,x^*) \in X \times X^* : \sum_{n=0}^\infty \Big(\frac{|\langle T^n x,x^*\rangle|}{r(T)^n}\Big)^p = \infty\Big\}$$

is residual in $X \times X^*$.

The second statement can be proved similarly (or we can use the fact that the canonical projection $X \times X^* \to X$ maps residual subsets of $X \times X^*$ onto residual subsets of X).

THEOREM 3.8. *Let X,Y be complex Banach spaces, $T_n \in \mathcal{L}(X,Y)$ $(n = 1,2,\dots)$. Let (a_n) be a sequence of positive numbers satisfying $\sum_{n=1}^\infty a_n^{1/3} < 1/4$. Let $B \subset X$, $B^* \subset Y^*$ be balls of radii equal to $1/4$. Then there exist $x \in B$ and $y^* \in B^*$ such that*

$$|\langle T_n x,y^*\rangle| \ge a_n\|T_n\|$$

for all $n = 1,2,\dots$.

PROOF. By Lemma 1.4 there exists $x \in B$ such that $\|T_n x\| \ge a_n^{1/2}\|T_n\|$ for all n.

Consider operators $S_n : Y^* \to \mathbf{C}$ defined by $S_n y^* = \langle T_n x,y^*\rangle$ $(y^* \in Y^*)$. Clearly $\|S_n\| = \|T_n x\| \ge a_n^{1/2}\|T_n\|$ for all n. Using Lemma 1.4 again there exists $y^* \in B^*$ such that

$$|\langle T_n x,y^*\rangle| = \|S_n y^*\| \ge a_n^{1/2}\|S_n\| \ge a_n\|T_n\|$$

for all $n = 1,2,\dots$.

COROLLARY 3.9. *Let X be a complex Banach space, $T \in \mathcal{L}(X)$, and let $(a_n)_{n\ge0}$ be a sequence of positive numbers satisfying $\sum_{n=0}^\infty {a_n}^{1/3} < \infty$. Then there is* *a dense subset $L \subset X \times X^*$ with the following property: for all pairs $(x, x^*) \in L$ there is $k \in \mathbf{N}$ such that*

$$|\langle T^n x, x^* \rangle| \geq a_n \|T^n\| \qquad (n \geq k).$$

If X is a real Banach space and $T \in \mathcal{L}(X)$ then the condition $\sum {a_n}^{1/3} < \infty$ must be replaced by $\sum {a_n}^{1/4} < \infty$ (cf. Theorem 2.1).

COROLLARY 3.10. *Let X be a real or complex Banach space, let $T \in \mathcal{L}(X)$. Then the set $\{(x, x^*) \in X \times X^* : \liminf_{n\to\infty} |\langle T^n x, x^* \rangle|^{1/n} = r(T)\}$ is dense in $X \times X^*$.*

EXAMPLE 3.11. There is a Hilbert space H and a non-nilpotent operator $T \in \mathcal{L}(H)$ such that

$$\sum_{n=0}^{\infty} \frac{|\langle T^n x, y \rangle|}{\|T^n\|} < \infty$$

for all $x, y \in H$.

PROOF. For $k \geq 1$ let H_k be the $(k+1)$-dimensional Hilbert space with an orthonormal basis $e_{k0}, e_{k1}, \ldots, e_{kk}$. Let $S_k \in \mathcal{L}(H_k)$ be the shift operator defined by $S_k e_{k0} = 0$, $S_k e_{kj} = e_{k,j-1}$ $(j \geq 1)$. Set $H = \bigoplus_{k=1}^{\infty} H_k$ and $T = \bigoplus_{k=1}^{\infty} \frac{1}{2^k} S_k$. Then $\|T^n\| = 2^{-n^2}$ for all n.

Let $k \in \mathbf{N}$ and $x_k, y_k \in H_k$, $x_k = \sum_{j=0}^{k} \alpha_j e_{kj}$, $y_k = \sum_{j=0}^{k} \beta_j e_{kj}$. Then

$$\sum_{n=0}^{\infty} \frac{|\langle T^n x_k, y_k \rangle|}{\|T^n\|} = \sum_{n=0}^{k} \frac{2^{n^2}}{2^{kn}} \Big| \sum_{j=0}^{k-n} \alpha_{j+n} \beta_j \Big| \leq \sum_{n=0}^{k} \frac{1}{2^{n(k-n)}} \Big(\sum_{j=0}^{k-n} |\alpha_{j+n}|^2 \Big)^{1/2} \Big(\sum_{j=0}^{k-n} |\beta_j|^2 \Big)^{1/2}$$

$$\leq \|x_k\| \cdot \|y_k\| \sum_{n=0}^{k} \frac{1}{2^{n(k-n)}} \leq \|x_k\| \cdot \|y_k\| \Big(1 + \sum_{n=0}^{k-1} \frac{1}{2^n} \Big) \leq 3 \|x_k\| \cdot \|y_k\|.$$

Let $x, y \in H$, $x = \sum_{k=1}^{\infty} x_k$, $y = \sum_{k=1}^{\infty} y_k$ where $x_k, y_k \in H_k$. Then

$$\sum_{n=0}^{\infty} \frac{|\langle T^n x, y \rangle|}{\|T^n\|} \leq \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{|\langle T^n x_k, y_k \rangle|}{\|T^n\|}$$

$$\leq \sum_{k=1}^{\infty} 3 \|x_k\| \cdot \|y_k\| \leq 3 \Big(\sum_{k=1}^{\infty} \|x_k\|^2 \Big)^{1/2} \cdot \Big(\sum_{k=1}^{\infty} \|y_k\|^2 \Big)^{1/2} = 3 \|x\| \cdot \|y\| < \infty.$$

REMARK 3.12. In the previous example we have $\sum_{n=0}^{\infty} \Big(\frac{\|T^n x\|}{\|T^n\|} \Big)^2 < \infty$ for all $x \in H$.

Indeed, using the notation of the previous example, for $x_k \in H_k$ we have

$$\sum_{n=0}^{\infty} \Big(\frac{\|T^n x_k\|}{\|T^n\|} \Big)^2 = \sum_{n=0}^{k} \Big(\frac{2^{n^2}}{2^{nk}} \Big(\sum_{j=0}^{k-n} |\alpha_{j+n}|^2 \Big)^{1/2} \Big)^2 \leq \sum_{n=0}^{k} \frac{1}{2^{2n(k-n)}} \|x_k\|^2 \leq 3 \|x_k\|^2$$

and for $x = \bigoplus_{k=1}^{\infty} x_k \in H$ we have

$$\sum_{n=0}^{\infty}\Big(\frac{\|T^n x\|}{\|T^n\|}\Big)^2 = \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\Big(\frac{\|T^n x_k\|}{\|T^n\|}\Big)^2 \le \sum_{k=1}^{\infty} 3\|x_k\|^2 = 3\|x\|^2 < \infty.$$

REMARK 3.13. Using the method of Lemma 1.9 it is possible to show that for each Hilbert space operator $T \in \mathcal{L}(H)$ and all $c < 1$ there are vectors $x, y \in H$ with

$$\sum_{n=0}^{\infty}\Big(\frac{|\langle T^n x, y\rangle|}{\|T^n\|}\Big)^c = \infty$$

(one can even get $y = x$).

For Banach space operators this is true for all $c < 1/2$.

IV. LOCAL CAPACITY

In this section we replace the powers of an operator by the set of all polynomials. All Banach spaces are supposed to be complex.

Denote by $\mathcal{P}_n^1$ the set of all monic polynomials of degree n (by monic we mean that the leading coefficient is equal to 1). For $T \in \mathcal{L}(X)$ we write $\operatorname{cap}_n T = \inf\{\|p(T)\| : p \in \mathcal{P}_n^1\}$ and the capacity of T is defined by

$$\operatorname{cap} T = \lim_{n\to\infty} (\operatorname{cap}_n T)^{1/n} = \inf_n (\operatorname{cap}_n T)^{1/n},$$

see [6]. Clearly the capacity is related to the spectral radius, $\operatorname{cap}_n T \le \|T^n\|$ for all n and $\operatorname{cap} T \le r(T)$.

For $T \in \mathcal{L}(X)$, $x \in X$ and $n \in \mathbf{N}$ write $\operatorname{cap}_n(T, x) = \inf\{\|p(T)x\| : p \in \mathcal{P}_n^1\}$. In general the limit $\lim_{n\to\infty}\big(\operatorname{cap}_n(T, x)\big)^{1/n}$ does not exist; the local capacity $\operatorname{cap}(T, x)$ is defined by

$$\operatorname{cap}(T, x) = \limsup_{n\to\infty}\big(\operatorname{cap}_n(T, x)\big)^{1/n}.$$

By [6], $\operatorname{cap} T = \operatorname{cap}\sigma(T)$; recall that the classical capacity of a nonempty compact subset $K \subset \mathbf{C}$ is defined by $\operatorname{cap} K = \lim_{n\to\infty}(\operatorname{cap}_n K)^{1/n}$, where $\operatorname{cap}_n K = \inf\{\|p\|_K : p \in \mathcal{P}_n^1\}$ and $\|p\|_K = \sup\{|p(z)| : z \in K\}$.

Since $\sigma(T) \setminus \sigma_e(T)$ consists of some bounded components of $\mathbf{C} \setminus \sigma_e(T)$ and at most countably many isolated points, it is easy to see that $\operatorname{cap}\sigma_e(T) = \operatorname{cap}\sigma(T)$.

A nonempty compact subset $K \subset \mathbf{C}$ is called algebraic if $p(K) = \{0\}$ for some nonzero polynomial p.

The basic results for local capacities are similar to those for the spectral radius.

THEOREM 4.1. *Let X be a complex Banach space, $T \in \mathcal{L}(X)$ and let $\varepsilon > 0$. Then:*

(i) the set of all $x \in X$ with the property that

$$\operatorname{cap}_n(T, x) \ge (n+1)^{-(2+\varepsilon)}(\operatorname{cap} T)^n \qquad \textit{for infinitely many } n\text{'s}$$

is residual.

In particular, the set $\big\{x \in X : \operatorname{cap}(T,x) = \limsup_{n\to\infty}\big(\operatorname{cap}_n(T,x)\big)^{1/n} = \operatorname{cap} T\big\}$ *is residual.*

(ii) there is a dense subset $R \subset X$ *with the property that, for each* $x \in R$*, there exists* $k \in \mathbf{N}$ *with*

$$\operatorname{cap}_n(T,x) \geq (n+1)^{-(2+\varepsilon)}(\operatorname{cap} T)^n \qquad (n \geq k).$$

In particular, the set $\big\{x \in X : \lim_{n\to\infty}\big(\operatorname{cap}_n(T,x)\big)^{1/n} = \operatorname{cap} T\big\}$ *is dense.*

PROOF. (ii) is a reformulation of [10].

To show (i), we need the following lemma:

LEMMA 4.2. *Let* $T \in \mathcal{L}(X)$*,* $x \in X$*,* $\eta > 0$ *and* $k \in \mathbf{N}$*. Suppose that the set* $K = \sigma_e(T)$ *is not algebraic. Then there exists* $n \geq k$ *such that the set*

$$L = \big\{y \in X : \|y-x\| < \eta, \|p(T)y\| > (n+1)^{-(2+\eta)}\|p\|_K \text{ for all } p \text{ with } \deg p \leq n\big\}$$

is a non-empty open subset of X*.*

PROOF. Choose $n \geq k$ such that $(n+1)^{-\eta} < \frac{\eta(1-\eta)}{4}$.

For a polynomial $p(z) = \sum_{i=0}^n \alpha_i z^i$ set $|p| = \sum_{i=0}^n |\alpha_i|$. Since K is not an algebraic set, the norms $|\cdot|$ and $\|\cdot\|_K$ are equivalent on the set of polynomials of degree $\leq n$ so that there exists a constant $c > 0$ such that $|p| \leq c \cdot \|p\|_K$ for all polynomials p with $\deg p \leq n$.

We prove that the set L is open. Let $y \in L$. By a compactness argument, there exists $\delta > 0$ such that $\|p(T)y\| > \delta + (n+1)^{-(2+\eta)}$ for all polynomials p with $\deg p \leq n$ and $\|p\|_K = 1$.

Let $y' \in X$, $\|y'-x\| < \eta$ and $\|y'-y\| < \frac{\delta}{c\cdot\max\{1,\|T\|^n\}}$. Let $p(z) = \sum_{i=0}^n \alpha_i z^i$ be a polynomial with $\|p\|_K = 1$. Then

$$\|p(T)(y'-y)\| \leq \sum_{i=0}^n |\alpha_i| \cdot \|T\|^i \|y'-y\| \leq |p| \max\{1, \|T\|, \dots, \|T\|^n\} \cdot \|y-y'\| < \delta$$

and

$$\|p(T)y'\| \geq \|p(T)y\| - \|p(T)(y'-y)\| > \delta + \frac{1}{(n+1)^{2+\eta}} - \delta = \frac{1}{(n+1)^{2+\eta}}.$$

Thus $y' \in L$.

It is sufficient to show that L is non-empty. Set $E = \bigvee\{x, Tx, \dots, T^n x\}$. By [10], Lemma 3 there exists $u \in X$ of norm one such that

$$\|p(T)(e+u)\| \geq \frac{1-\eta}{2(n+1)^2} r_e(p(T))$$

for all $e \in E$ and all polynomials p, $\deg p \leq n$. Note that

$$r_e(p(T)) = \max\big\{|\mu| : \mu \in \sigma_e(p(T))\big\} = \max\big\{|p(\lambda)| : \lambda \in \sigma_e(T)\big\} = \|p\|_K.$$

Set $y = x + \frac{\eta u}{2}$. Then $\|y - x\| = \eta/2 < \eta$ and, for all polynomials p of degree $\leq n$,

$$\|p(T)y\| = \frac{\eta}{2}\Big\|p(T)\Big(\frac{2x}{\eta} + u\Big)\Big\| \geq \frac{\eta}{2} \cdot \frac{1-\eta}{2(n+1)^2}\|p\|_K > \frac{1}{(n+1)^{2+\eta}}\|p\|_K.$$

Hence $y \in L$.

Proof of Theorem 4.1 (i). If $\sigma_e(T)$ is an algebraic set then $\operatorname{cap} T = \operatorname{cap} \sigma_e(T) = 0$, so that the statement is trivial.

Suppose that the set $K = \sigma_e(T)$ is not algebraic. Fix $\varepsilon > 0$. For each $k \in \mathbf{N}$ let

$$M_k = \big\{y \in X : \text{ there exists } n \geq k \text{ such that } \operatorname{cap}_n(T, y) > (n+1)^{-(2+\varepsilon)} \operatorname{cap}_n K\big\}.$$

For each $x \in X$ and $\eta \in (0, \varepsilon)$ let $n = n(x, \eta, k) \geq k$ and $L = L(x, \eta, k) \subset X$ be as constructed in Lemma 4.2. For $y \in L(x, \eta, k)$ we have

$$\begin{aligned}\operatorname{cap}_n(T, y) &= \inf\{\|p(T)y\| : p \in \mathcal{P}_n^1\} \geq \inf\big\{(n+1)^{-(2+\eta)}\|p\|_K : p \in \mathcal{P}_n^1\big\}\\ &\geq (n+1)^{-(2+\eta)} \operatorname{cap}_n K > (n+1)^{-(2+\varepsilon)} \operatorname{cap}_n K.\end{aligned}$$

Thus $\bigcup_{x,\eta} L(x, \eta, k) \subset M_k$ and $\bigcup_{x,\eta} L(x, \eta, k)$ is an open dense subset of X. Hence M_k is residual for all k and so is the intersection $\bigcap_k M_k$.

For $k \in \mathbf{N}$ and $y \in M$ there is $n \geq k$ with

$$\operatorname{cap}_n(T, y) > (n+1)^{-(2+\varepsilon)} \operatorname{cap}_n K.$$

In particular, for $y \in \bigcap_{k=1}^{\infty} M_k$ we have

$$\operatorname{cap}(T, y) = \limsup_{n\to\infty} (\operatorname{cap}_n(T, y))^{1/n} \geq \limsup_{n\to\infty} \big(\operatorname{cap}_n K\big)^{1/n} = \operatorname{cap} K = \operatorname{cap} \sigma_e(T) = \operatorname{cap} T.$$

The opposite inequality $\operatorname{cap}(T, y) \leq \operatorname{cap} T$ is always true.

Example 1.3 shows that in general we can not expect to have

$$\liminf_{n\to\infty} (\operatorname{cap}(T, y))^{1/n} = \operatorname{cap} T$$

for all y in a residual set. Thus (ii) cannot be improved.

REMARK 4.3. The statement of Theorem 4.1 can be generalized to commuting n-tuples of operators. The analogue of the second statement was proved in [11]; the analogue of part (i) can be proved as above.

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