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Epsilon-hypercyclic operators

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Abstract. For each fixed number ε in $(0, 1)$ we construct a bounded linear operator on the Banach space ℓ_1 having a certain orbit which intersects every cone of aperture ε, but with every orbit avoiding a certain ball of radius d, for every $d > 0$. This answers a question from [**8**]. On the other hand, if T is an operator on the Banach space X such that for every $\varepsilon > 0$ there is a point in X whose orbit under the action of T meets every cone of aperture ε, then T has a dense orbit.

1. *Introduction*

The aim of this paper is to study some (variations of) density properties of orbits of bounded linear operators acting on a real or complex separable Banach space X. Using a Functional Analysis terminology, an operator $T \in \mathcal{B}(X)$ is said to be *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\mathcal{O}rb(x, T) = \{T^n x \, ; \, n \geq 0\}$ of x under the action of T is dense in X. A vector x with dense orbit is called a *hypercyclic* vector for T.

While the first examples of Banach and Hilbert space hypercyclic operators are relatively recent ([**10**]), there is now an important literature on hypercyclicity properties and the dynamics of bounded linear operators. We refer the reader to the recent book [**1**] for more on this topic. It is natural in this context to investigate which properties of the orbit of a vector, weaker than denseness, imply either that the orbit itself is in fact dense, or that the operator is hypercyclic (i.e. some other orbit is dense in X). Let us mention here some of the results in this direction:

– if the orbit $\mathcal{O}rb(x, T)$ is somewhere dense in X, then it is dense in X [**3**]. This implies in particular that if the union of finitely many orbits $\mathcal{O}rb(x_1, T)$, $\mathcal{O}rb(x_2, T)$,

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..., $\mathcal{O}rb(x_n, T)$ is dense in X, then one of these orbits must already be dense. This result was proved directly in [**5**] and [**9**].

– suppose that for some positive number d the orbit of $x \in X$ meets every open ball $B(y, d)$ of radius d. Then $\mathcal{O}rb(x, T)$ is not necessarily dense in X, but T must be hypercyclic [**6**].

– if x is a frequently hypercyclic vector for T, then $T \oplus T$ must be hypercyclic on $X \oplus X$ [**7**].

Recall that x is a frequently hypercyclic vector for T if for every non empty open subset U of X, the set of positive integers n such that $T^n x \in U$ has positive lower density; in other words, for every non empty open subset U of X, there exists a sequence $(n_k)_{k\geq 0}$ with $n_k = O(k)$ such that $T^{n_k} x \in U$.

On the other hand, some conditions on the orbit, which may look strong enough at first sight, do not imply that the operator is hypercyclic. For instance:

– there exist operators which are weakly hypercyclic, i.e. for which there exists a vector x whose orbit is weakly dense in X, but still are not hypercyclic: examples of weighted shifts having this property are given in [**4**].

– for every $\varepsilon > 0$, there exists a bounded operator on the space $\ell_1(\mathbb{N})$ which has the following property: there exists a vector $x \in \ell_1(\mathbb{N})$ such that for every non empty open subset U of $\ell_1(\mathbb{N})$, there exists a sequence $(n_k)_{k\geq 0}$ with $n_k = O(k^{1+\varepsilon})$ such that $T^{n_k} x \in U$, but $T \oplus T$ is not hypercyclic [**2**]. This shows that the result of [**7**] that every frequently hypercyclic operator on a Banach space satisfies the Hypercyclicity Criterion is in a sense optimal.

We investigate in this paper a weaker version of Feldman's result [**6**] already mentioned above: it states that if given a positive ε there exists a vector x such that for every $y \in X$ $||T^n x - y|| \leq \varepsilon$ for some integer n, then T is hypercyclic.

Definition. Let ε be a number in $(0, 1)$. If x is a vector of X, we say that x is *ε-hypercyclic* if for every non zero vector $y \in X$ there exists an integer n such that $||T^n x - y|| \leq \varepsilon ||y||$. The operator T is *ε-hypercyclic* if it admits an ε-hypercyclic vector.

In particular, the orbit of x must intersect every cone of a fixed aperture. This is in a sense a "scaled" version of the ε-density considered in Feldman's work. It is obviously weaker than Feldman's condition, and in a sense more natural in this context. The following question was proposed in [**8**]:

QUESTION 1.1. *Suppose that T is a bounded operator on X which admits for some $\varepsilon \in (0, 1)$ an ε-hypercyclic vector. Is it true that T is hypercyclic?*

The restriction $\varepsilon \in (0, 1)$ comes from the fact the zero vector is trivially 1-hypercyclic for any operator T.

The main result of this paper gives a negative answer to Question 1.1:

THEOREM 1.2. *For every $\varepsilon \in (0, 1)$ there exists an ε-hypercyclic operator on the space $\ell_1(\mathbb{N})$ which is not hypercyclic.*

Still:

THEOREM 1.3. *If T is a bounded operator on X which is ε-hypercyclic for every $\varepsilon > 0$, then T must be hypercyclic.*

We therefore obtain, together with Feldman's result, the geometric statement stated in the abstract.

Theorems 1.2 and 1.3 are proved in the next section. Surprisingly enough, our construction for the proof of Theorem 1.2 really uses the ℓ_1-norm, and we are unable to adapt it to the Hilbertian setting. Thus the following question is still open:

QUESTION 1.4. *Let $\varepsilon \in (0,1)$ and suppose that $T \in \mathcal{B}(H)$ is an ε-hypercyclic operator acting on a Hilbert space H. Must T be hypercyclic?*

2. *Proof of Theorem 1.2*

2.1. *Outline of the proof of Theorem 1.2.* Fix $\varepsilon \in (0,1)$ and a positive integer a such that $\varepsilon > 2^{-a+1}$. Let X be the space $\ell_1(\mathbb{N})$ endowed with the canonical basis $(e_n)_{n\geq 0}$. Our operator T will act on the ℓ_1-direct sum $Y = \bigoplus_{i=0}^{\infty} X$ of countably many copies of $\ell_1(\mathbb{N})$.

Let $(y^{(k)})_{k\geq 1}$ be a sequence of vectors of Y which has the following properties:

(i) the set $\{y^{(k)}\ ;\ k \geq 1\}$ is dense in Y;

(ii) each $y^{(k)} \in Y$ can be written as a sequence $y^{(k)} = (y_1^{(k)}, \ldots, y_{k-1}^{(k)}, 0, \ldots)$, where each $y_j^{(k)}$ is a vector of $X = \ell_1(\mathbb{N})$ which is in the linear span of the vectors e_i, $i \leq k-1$;

(iii) $2^{-k} \leq \|y_j^{(k)}\|$ for every $j = 0, \ldots, k-1$, and $\|y^{(k)}\| \leq \frac{2^k}{1+2^{-a}}$.

For each $k \geq 1$ and each $j \leq k-1$, define $z_j^{(k)} = y_j^{(k)} + 2^{-a}\|y_j^{(k)}\| e_{k^2+j}$: it is a perturbation of the vector $y_j^{(k)}$ obtained by adding to it a (not too small) multiple of the basis vector e_{k^2+j}, which is far away from the support of $y_j^{(k)}$. We have $2^{-k} \leq \|z_j^{(k)}\|$ for each $j \leq k-1$. We then define $z^{(k)} \in Y$ by $z^{(k)} = (z_0^{(k)}, \ldots, z_{k-1}^{(k)}, 0, \ldots)$. Clearly $\|z^{(k)}\| \leq 2^k$.

Set $n_0 = n'_0 = 0$. Our goal is to construct by induction a sequence $(S_j)_{j\geq 1}$ of bounded operators on X and two strictly increasing sequences of positive integers $(n_k)_{k\geq 1}$ and $(n'_k)_{k\geq 1}$ such that $n'_{k-1} \leq n'_{k-1} + n_{k-1} \leq n_k < n_k + k < n'_k$ for every $k \in \mathbb{N}$ and the six following properties hold true:

(a) each operator S_j is bounded and invertible with $\|S_j^{-1}\| \leq 2$;

(b) $S_j e_0 = e_0$ for every $j \in \mathbb{N}$;

(c) $\|S_j S_{j-1} \ldots S_1\| \leq 2^{a+1}$ for every $j \in \mathbb{N}$;

(d) $S_{n'_k} \ldots S_2 S_1 = I$ (the identity operator) for every $k \in \mathbb{N}$;

(e) $S_j = I$ for every $k \in \mathbb{N}$ and every j such that $n_k - n_{k-1} < j \leq n_k + k$);

(f) $\|S_{n_k} \cdots S_j S_{j+1} z_j^{(k)}\| \leq 2^{-2k-a}$ for every $k \in \mathbb{N}$ and every $j = 0, \ldots, k-1$.

Suppose that (n_k), (n'_k) and (S_j) have been constructed so as to satisfy properties (a) to (f). Consider on Y the operator T which is the backward shift with operatorweights S_j^{-1}: for any sequence $(v_j)_{j\geq 0}$ of Y,

$$T(v_0, v_1, \ldots) = (S_1^{-1}v_1, S_2^{-1}v_2, \ldots).$$

Clearly T is bounded on Y with $\|T\| \leq 2$ by (a). For any $n \in \mathbb{N}$ we have

$$T^n(v_0, v_1, \ldots) = (S_1^{-1} \ldots S_n^{-1}v_n, S_2^{-1} \ldots S_{n+1}^{-1}v_{n+1}, \ldots, S_{j+1}^{-1} \ldots S_{n+j}^{-1}v_{n+j}, \ldots).$$

For $k \in \mathbb{N}$ define $x^{(k)} \in Y$ by

$$x^{(k)} = (\underbrace{0, \ldots, 0}_{n_k}, S_{n_k} \ldots S_1 z_0^{(k)}, S_{n_k} \ldots S_2 z_1^{(k)}, \ldots, S_{n_k} \ldots S_k z_{k-1}^{(k)}, 0, \ldots).$$

By (f), we have $\|x^{(k)}\| \leq 2^{-2k-a}\|z^{(k)}\| \leq 2^{-k-a}$, and thus the vector

$$x = \sum_{k=1}^{\infty} x^{(k)}$$

belongs to Y.

2.2. *The vector x is ε-hypercyclic for T.* Let $k \in \mathbb{N}$. Observe that by (e) we can rewrite $x^{(k)}$ as

$$x^{(k)} = (\underbrace{0, \ldots, 0}_{n_k}, S_{n_k} \ldots S_1 z_0^{(k)}, S_{n_k+1} \ldots S_2 z_1^{(k)}, \ldots, S_{n_k+k-1} \ldots S_k z_{k-1}^{(k)}, 0, \ldots),$$

and so $T^{n_k}x^{(k)} = z^{(k)}$. Clearly $T^{n_k}x^{(m)} = 0$ for $m < k$, and for $m > k$ we have $T^{n_k}x^{(m)} = x^{(m)}$ by (e) again. Hence

$$\|T^{n_k}x - z^{(k)}\| = \left\| \left(\sum_{m=k+1}^{\infty} x^{(m)} \right) \right\| \leq \sum_{m=k+1}^{\infty} 2^{-m-a} = 2^{-k-a}.$$

Let v be any non zero vector of Y. Choose $k \in \mathbb{N}$ such that $\|v - y^{(k)}\| < \varepsilon'\|v\|$, where $\varepsilon' > 0$ satisfies $(1+\varepsilon')2^{1-a} + \varepsilon' < \varepsilon$. Then $\|y^{(k)}\| < \|v\|(1+\varepsilon')$ and

$$\begin{aligned} \|T^{n_k}x - v\| &\leq \|T^{n_k}x - z^{(k)}\| + \|z^{(k)} - y^{(k)}\| + \|y^{(k)} - v\| \\ &\leq 2^{-k-a} + 2^{-a}\|y^{(k)}\| + \varepsilon'\|v\| \leq \|y^{(k)}\|2^{-a+1} + \varepsilon'\|v\| \\ &\leq \|v\|((1+\varepsilon')2^{-a+1} + \varepsilon') \leq \varepsilon\|v\|. \end{aligned}$$

Hence x is an ε-hypercyclic vector for T.

2.3. *The operator T is not hypercyclic on Y.* Suppose on the contrary that there is a vector $v = (v_0, v_1, \ldots) \in Y$ hypercyclic for T. Then there exists an increasing sequence $(m_j)_{j\geq 0}$ of integers such that the quantity $\|T^{m_j}v - (e_0, 0, \ldots)\|$ tends to zero as j tends to infinity. In particular, reading this on the first coordinate yields that $\|S_1^{-1}S_2^{-1} \ldots S_{m_j}^{-1}v_{m_j} - e_0\|$ tends to zero. Here assumptions (b) and (c) come into play:

$$\begin{aligned} \|v_{m_j} - e_0\| &= \|v_{m_j} - S_{m_j} \cdots S_1 e_0\| \leq \|S_{m_j} \cdots S_1\| \cdot \|S_1^{-1} \cdots S_{m_j}^{-1}v_{m_j} - e_0\| \\ &\leq 2^{a+1}\|S_1^{-1} \cdots S_{m_j}^{-1}v_{m_j} - e_0\| \end{aligned}$$

Hence $\|v_{m_j} - e_0\|$ tends to zero, thus $\|v_{m_j}\|$ tends to 1, which contradicts the assumption that v belongs to Y.

2.4. *Construction of the sequences* $(n_k)_{k\geq 0}$, $(n'_k)_{k\geq 0}$ *and* $(S_j)_{j\geq 1}$. Recall that we set formally $n_0 = n'_0 = 0$. Define the numbers n_k, n'_k inductively by setting

$$n_k = n'_{k-1} + 4k + 2a + 1 + n_{k-1}$$

and

$$n'_k = n_k + 5k + 2a + 1.$$

We define the operators S_j by induction: at step k the operators S_j are constructed for $n'_{k-1} < j \leq n'_k$. So let $k \geq 1$ and suppose that $S_j \in B(X)$ are already defined and invertible for $j \leq n'_{k-1}$. For $0 \leq i \leq k-1$ write

$$w_i^{(k)} = S_1^{-1} \cdots S_i^{-1} y_i^{(k)}$$

and

$$\alpha_i^{(k)} = 2^{-a} \|y_i^{(k)}\| \cdot \|S_1^{-1} \cdots S_i^{-1} e_{k^2+i}\|.$$

Note that for $k = 1$ we have $w_0^{(1)} = y_0^{(1)}$ and $\alpha_0^{(1)} = 2^{-a}\|y_0^{(1)}\|$.

At step $k \geq 2$ we have already defined in particular the invertible operators $S_1, \ldots, S_{k-1}$, since $k - 1 \leq n'_{k-1}$.

We define the operators S_j, $n'_{k-1} < j \leq n'_k$, by defining $S_j e_i$, depending on the values of i and j:

- For $i < k^2$, define

(1) $S_j e_i = e_i$ for $n'_{k-1} < j \leq n'_k$.

- For $k^2 \leq i \leq k^2 + k - 1$, define

(2a) $S_j e_i = 2e_i$ $(n'_{k-1} < j \leq n'_{k-1} + a)$;
(2b) $S_j e_i = -\frac{w_{i-k^2}^{(k)}}{2^a \alpha_{i-k^2}^{(k)}} + e_i$ $(j = n'_{k-1} + a + 1)$;
(2c) $S_j e_i = \frac{1}{2} e_i$ $(n'_{k-1} + a + 1 < j < n'_{k-1} + 2a + 4k + 1 = n_k - n_{k-1})$;
(2d) $S_j e_i = e_i$ $(n_k - n_{k-1} < j \leq n_k + k)$;
(2e) $S_j e_i = 2e_i$ $(n_k + k < j \leq n_k + 5k + a)$;
(2f) $S_j e_i = \frac{w_{i-k^2}^{(k)}}{2^a \alpha_{i-k^2}^{(k)}} + e_i$ $(j = n_k + 5k + a + 1)$;
(2g) $S_j e_i = \frac{1}{2} e_i$ $(n_k + 5k + a + 1 < j \leq n_k + 5k + 2a + 1 = n'_k)$.

- For $i > k^2 + k - 1$, define

(3a) $S_j e_i = \frac{1}{2} e_i$ $(n'_{k-1} < j \leq n_k - n_{k-1})$;
(3b) $S_j e_i = e_i$ $(n_k - n_{k-1} < j \leq n_k + k)$;
(3c) $S_j e_i = \frac{1}{2} e_i$ $(n_k + k < j \leq n'_k - 1)$;
(3d) $S_j e_i = 2^{n'_k - n'_{k-1} - n_{k-1} - k - 1} e_i$ $(j = n'_k)$.

For $k \in \mathbb{N}$ let $M_k = \overline{\mathrm{sp}}[e_i\ ;\ i = 0 \ldots k^2 + k - 1]$ and $L_k = \overline{\mathrm{sp}}[e_i\ ;\ i > k^2 + k - 1]$.

2.5. *Boundedness and invertibility of the operators* S_j. We show first by induction on k that the operators S_j, $n'_{k-1} < j \leq n'_k$, defined above are bounded, invertible and upper triangular, and that their inverses S_j^{-1} are also bounded and upper triangular.

As mentioned above, for $k = 1$ we have $w_0^{(1)} = y_0^{(1)} \in \mathbb{C} \cdot e_0$, so the operators S_j, $j \leq n'_1$ are upper triangular. Moreover, for each $j \leq n'_1$ we have $S_j(M_1) \subseteq M_1$, $S_j(L_1) \subseteq L_1$. The operator $S_j|M_1$ is upper triangular with a positive main diagonal and $S_j|L_1$ is a nonzero scalar multiple of the identity operator. So S_j is bounded and invertible and its inverse S_j^{-1} is also bounded and upper triangular.

Suppose that $k \geq 2$ and that the operators S_j, S_j^{-1}, $j \leq n'_{k-1}$, are bounded, invertible and upper triangular. For $0 \leq i \leq k-1$, $y_i^{(k)}$ belongs to the linear span of the vectors e_l, $l = 0 \dots k-1$, and so this is also the case for the vector $w_i^{(k)} = S_1^{-1} \cdots S_{k-1}^{-1} y_i^{(k)}$. Hence the operators S_j, $n'_{k-1} < j \leq n'_k$ defined by (1) – (3) are upper triangular. As above, we conclude that they are also bounded and invertible, and that their inverses S_j^{-1} are also bounded and upper triangular.

We now have to show that the operators S_j satisfy conditions (a)–(f).

2.6. *Proof of properties (b), (e) and (d).* By definition, $S_j e_0 = e_0$ for all j and S_j is equal to the identity operator for $n_k - n_{k-1} < j \leq n_k + k$. Hence conditions (b) and (e) are satisfied trivially. Then we have to prove by induction on k that $S_{n'_k} \cdots S_1 = I$, i.e., that $S_{n'_k} \cdots S_{n'_{k-1}+1} = I$:

- for $i < k^2$, clearly $S_{n'_k} \cdots S_{n'_{k-1}+1} e_i = e_i$ since all the operators S_j, $n'_{k-1}+1 \leq j \leq n'_k$, act on e_i as the identity operator by (1);
- for $i > k^2 + k - 1$ it is also easy to check using property (3) that $S_{n'_k} \cdots S_{n'_{k-1}+1} e_i = e_i$ (just multiply all coefficients together);
- for $k^2 \leq i \leq k^2 + k - 1$ we have that $S_{n'_k} \cdots S_{n'_{k-1}+1} e_i$ is equal to

$$S_{n'_k} \cdots S_{n'_{k-1}+a+1}(2^a e_i) = S_{n'_k} \cdots S_{n'_{k-1}+a+2}\Big(-\frac{w_{i-k^2}^{(k)}}{\alpha_{i-k^2}^{(k)}} + 2^a e_i\Big)$$

by (2b). Then since $w_{i-k^2}^{(k)}/\alpha_{i-k^2}^{(k)}$ is supported by the first k vectors e_l, $l = 0, \dots, k-1$, by (2c),(2d) and (2e) applied successively this quantity is equal to

$$S_{n'_k} \cdots S_{n'_{k-1}+2a+4k+2}\Big(-\frac{w_{i-k^2}^{(k)}}{\alpha_{i-k^2}^{(k)}} + 2^{-4k} e_i\Big)$$

and so equal to

$$S_{n'_k} \cdots S_{n_k+k+1}\Big(-\frac{w_{i-k^2}^{(k)}}{\alpha_{i-k^2}^{(k)}} + 2^{-4k} e_i\Big) = S_{n'_k} \cdots S_{n_k+5k+a+1}\Big(-\frac{w_{i-k^2}^{(k)}}{\alpha_{i-k^2}^{(k)}} + 2^a e_i\Big).$$

Then the expression in (2f) destroys the quantity $w_{i-k^2}^{(k)}/\alpha_{i-k^2}^{(k)}$ in this expression, and we eventually get that

$$S_{n'_k} \cdots S_{n'_{k-1}+2a+4k+2}\Big(-\frac{w_{i-k^2}^{(k)}}{\alpha_{i-k^2}^{(k)}} + 2^{-4k} e_i\Big) = S_{n'_k} \cdots S_{n_k+5k+a+2}(2^a e_i) = e_i.$$

Hence $S_{n'_k} \cdots S_{n'_{k-1}+1} = I$ and property (d) is proved.

2.7. *Proof of property (a).* We now have prove by induction on k that $\|S_j^{-1}\| \leq 2$ for every j with $n'_{k-1} < j \leq n'_k$. Let $k \geq 1$ and suppose that $\|S_j^{-1}\| \leq 2$ for every $j \leq n'_{k-1}$. For $0 \leq i \leq k-1$ we have

$$S_1^{-1} \cdots S_i^{-1} e_{k^2+i} = \|S_1^{-1} \cdots S_i^{-1} e_{k^2+i}\| \cdot e_{k^2+i}$$

since the operators $S_1^{-1}, \dots, S_i^{-1}$ just multiply the vector e_{k^2+i} by some coefficient. Thus

$$S_1^{-1} \cdots S_i^{-1} z_i^{(k)} = S_1^{-1} \cdots S_i^{-1} y_i^{(k)} + 2^{-a}\|y_i^{(k)}\| S_1^{-1} \cdots S_i^{-1} e_{k^2+i} = w_i^{(k)} + \alpha_i^{(k)} e_{k^2+i}.$$

Let $r = \text{card}\,\{s\ ;\ 1 \leq s \leq i \text{ and } S_s \neq I\}$. Then $\|S_i^{-1} \cdots S_i^{-1} y_i^{(k)}\| \leq 2^r \cdot \|y_i^{(k)}\|$ by the induction assumption and $\|S_1^{-1} \cdots S_i^{-1} e_{k^2+i}\| = 2^r$ by (3a). Hence

$$\alpha_i^{(k)} = 2^{-a}\|y_i^{(k)}\| 2^r \geq 2^{-a}\|S_i^{-1} \cdots S_i^{-1} y_i^{(k)}\| \geq 2^{-a}\|w_i^{(k)}\|.$$

Clearly $\|S_j^{-1}\| \leq 2$ for all j with $n'_{k-1} < j \leq n'_k$, $j \neq n'_{k-1} + a + 1$ and $j \neq n_k + 5k + a + 1$. In order to prove that $\|S_j^{-1}\| \leq 2$ in these two cases, we only have to check that $\|S_j^{-1} e_i\| \leq 2$ for every $i \geq 0$: observe that at this point we use the ℓ_1-norm in a crucial way.

- If $i < k^2$ then , $\|S^{-1}_{n'_{k-1}+a+1} e_i\| = \|e_i\| \leq 2$ and $\|S^{-1}_{n_k+5k+a+1} e_i\| \leq 2$ by (1).
- Similarly, if $i > k^2 + k - 1$ then $\|S^{-1}_{n'_{k-1}+a+1} e_i\| \leq 2$ and $\|S^{-1}_{n_k+5k+a+1} e_i\| \leq 2$ by (3).
- Let $k^2 \leq i \leq k^2 + k - 1$. Then

$$S_{n'_{k-1}+a+1} S_{n_k+5k+a+1} e_i S_{n_k+5k+a+1} S_{n'_{k-1}+a+1} e_i = e_i.$$

So $\|S^{-1}_{n_k+5k+a+1} e_i\| = \|S_{n'_{k-1}+a+1} e_i\| \leq 1$ and $\|S^{-1}_{n'_{k-1}+a+1} e_i\| = \|S_{n_k+5k+a+1} e_i\| \leq 2$.

This proves (a).

2.8. *Proof of property (f).* Let $k \in \mathbb{N}$ and $0 \leq i \leq k-1$. Then

$$\begin{aligned}
\|S_{n_k} \cdots S_{i+1} z_i^{(k)}\| &= \|S_{n_k} \cdots S_1 (S_1^{-1} \cdots S_i^{-1}) z_i^{(k)}\| \\
&= \|S_{n_k} \cdots S_{n_1} (w_i^{(k)} + \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|S_{n_k} \cdots S_{n'_{k-1}+1} (w_i^{(k)} + \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|S_{n_k} \cdots S_{n'_{k-1}+a+1} (w_i^{(k)} + 2^a \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|S_{n_k} \cdots S_{n'_{k-1}+a+2} (2^a \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|S_{n_k} \cdots S_{n'_{k-1}+2a+4k+2} (2^{-4k} \alpha_i^{(k)} e_{k^2+i})\| \\
&= \|2^{-4k} \alpha_i^{(k)} e_{k^2+i}\| \\
&= 2^{-4k} \alpha_i^{(k)}.
\end{aligned}$$

Then (f) is proved by observing that

$$2^{-4k}\alpha_i^{(k)} \quad = \quad 2^{-4k} 2^{-a} \|y_i^{(k)}\| \,.\, \|S_1^{-1} \cdots S_i^{-1} e_{k^2+i}\| \leq 2^{-4k} 2^{-a} 2^k \cdot 2^k = 2^{-2k-a}.$$

2.9. *Proof of property (c).* It remains to show that $\|S_j \cdots S_1\| \leq 2^{a+1}$ for all $j \in \mathbb{N}$. By (d), it is sufficient to show that $\|S_j \cdots S_{n'_{k-1}}\| \leq 2^{a+1}$ for all $k \in \mathbb{N}$ and $n'_{k-1} < j \leq n'_k$. Equivalently, using again the ℓ_1-norm, it must be proved that $\|S_j \cdots S_{n'_{k-1}+1} e_i\| \leq 2^{a+1}$ for every $i \geq 0$ and $n'_{k-1} < j \leq n'_k$.

• For $i < k^2$ this is clear since the operators S_j, $n'_{k-1} < j \leq n'_k$, act on e_i as the identity operator.

• For $i > k^2 + k - 1$ this is also clear: $\|S_j \cdots S_{n'_{k-1}+1} e_i\| \leq 1$ for all j, $n'_{k-1} + 1 \leq j \leq n'_k$ (just multiply the coefficients, the worst case being when $j = n'_k$).

• For $k^2 \leq i \leq k^2 + k - 1$, the sequence

$$S_{n'_{k-1}+1} e_i, S_{n'_{k-1}+2} S_{n'_{k-1}+1} e_i, \ldots, S_{n'_k} \cdots S_{n'_{k-1}+1} e_i$$

is equal to $2e_i, \ldots, 2^a e_i, -\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^a e_i, \ldots, -\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^{-4k} e_i, \ldots, -\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^{-4k} e_i, -\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^a e_i, 2^a e_i, \ldots, e_i$. Hence

$$\max_{n'_{k-1}+1 \leq j \leq n'_k} \|S_j \cdots S_{n'_{k-1}+1} e_i\| = \left\| -\frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} + 2^a e_i \right\| = \left| \frac{w^{(k)}_{i-k^2}}{\alpha^{(k)}_{i-k^2}} \right| + 2^a \leq 2^{a+1}.$$

This proves (c).

Thus the operators S_j, $j \in \mathbb{N}$, satisfy all the properties (a) to (f), and consequently the operator T defined here is ε-hypercyclic but not hypercyclic on $Y = \bigoplus_{\ell_1} \ell_1(\mathbb{N})$. This finishes the proof of Theorem 1.2.

2.10. *Possible extensions of the method.* In the same way it is possible to construct a non-hypercyclic operator T on $\ell_1(\mathbb{N})$ on an infinite-dimensional separable Hilbert space such that $T \oplus T$ is ε-hypercyclic. Indeed, consider the space Y as in Theorem 1.2 and a sequence of pairs of vectors $(y^{(k)}, \widetilde{y}^{(k)})$ which is dense in $Y \oplus Y$. In the same way one can construct a vector $x \oplus \widetilde{x} \in Y \oplus Y$ which is ε-hypercyclic for $T \oplus T$. One can even have that $T_n = \underbrace{T \oplus \cdots \oplus T}_{n}$ is ε-hypercyclic for each $n \in \mathbb{N}$. Details are left to the reader.

3. *Proof of Theorem 1.3*

3.1. *Point spectrum of the adjoint of an ε-hypercyclic operator.* We need the following auxiliary result:

LEMMA 3.1. *Let $0 < \varepsilon < 1$ and let $T \in B(X)$ be an ε-hypercyclic operator. Then the point spectrum $\sigma_p(T^*)$ of the adjoint of T is empty.*

Proof. Suppose on the contrary that α belongs to $\sigma_p(T^*)$. Let $y^* \in X^*$ satisfy $\|y^*\| = 1$ and $T^* y^* = \alpha y^*$, and let $x \in X$ be an ε-hypercyclic vector for T. We distinguish two cases.

• *First case.* We have either $\langle x, y^*\rangle = 0$, or $|\alpha| \leq 1$. Choose $t > (\|x\|+1)/(1-\varepsilon)$ and $y \in X$ with $\|y\| = 1$ and $\langle y, y^*\rangle > 1-\varepsilon/t$. Since x is an ε-hypercyclic vector for T, there exists an $n \geq 0$ such that $\|T^n x - ty\| \leq \varepsilon\|ty\| = t\varepsilon$. So $|\langle T^n x - y, y^*\rangle| \leq t\varepsilon$. On the other hand,

$$|\langle T^n x - ty, y^*\rangle| \geq |\langle ty, y^*\rangle| - |\langle T^n x, y^*\rangle| \geq t - \varepsilon - |\alpha|^n |\langle x, y^*\rangle| \geq t - 1 - \|x\|.$$

Thus $t - 1 - \|x\| \leq t\varepsilon$ and so $t \leq (1 + \|x\|)/(1-\varepsilon)$, a contradiction.

• *Second case.* We have $\langle x, y^*\rangle \neq 0$ and $|\alpha| > 1$. Choose $y \in X$ such that $0 \neq \|y\| < |\langle x, y^*\rangle|/(1+\varepsilon)$. There exists $n \geq 0$ such that $\|T^n x - y\| \leq \varepsilon\|y\|$, and thus $|\langle T^n x - y, y^*\rangle| \leq \varepsilon\|y\|$. On the other hand,

$$|\langle T^n x - y, y^*\rangle| \geq |\langle T^n x, y^*\rangle| - |\langle y, y^*\rangle| \geq |\alpha^n| \cdot |\langle x, y^*\rangle| - \|y\| > |\langle x, y^*\rangle| - \|y\|.$$

Thus $|\langle x, y^*\rangle| - \|y\| < \varepsilon\|y\|$, and so $\|y\| > (|\langle x, y^*\rangle|)(1+\varepsilon)$, a contradiction again. □

3.2. *Proof of Theorem 1.3.* Lemma 3.1 shows that we can assume that X is infinite dimensional. We are going to prove that T is topologically transitive, i.e. that for every nonempty open subsets U and V of X there exists an integer $n \in \mathbb{N}$ such that $T^n(U) \cap V$ is nonempty. Let $u \in U$ and $v \in V$ be two nonzero vectors of U and V respectively, and let $\delta > 0$ be so small that $B(u,\delta) \subseteq U$, $B(v,\delta) \subseteq V$ and $\delta < \min(\|u\|, \|v\|)$. Let $x \in X$ be an ε-hypercyclic vector for T, where $\varepsilon < \delta/(6\max(\|u\|, \|v\|))$. There exists $n_0 \geq 0$ such that $\|T^{n_0}x - u\| \leq \varepsilon\|u\| < \delta$, and so $T^{n_0}x$ belongs to U. Let us now show that there exist infinitely many n's such that $T^n x$ belongs to V. Suppose on the contrary that there are only finitely many such integers $n_1, \ldots, n_k$. As above, for each $v' \in X$ with $\|v' - v\| < \frac{2\delta}{3}$ there exists an integer $n(v')$ which satisfies $\|T^{n(v')}x - v'\| \leq \varepsilon\|v'\| \leq 2\varepsilon\|v\| < \delta/3$. Since $\|T^{n(v')}x - v\| \leq \|T^{n(v')}x - v'\| + \|v' - v\| < \delta$, we have $n(v') \in \{n_1, \ldots, n_k\}$ and the ball $B(v, (2\delta)/3)$ is covered by a finite number of balls $B(T^{n_1}x, \delta/3), \ldots, B(T^{n_k}x, \delta/3)$. However, in an infinite dimensional space this is not possible. Hence there are infinitely many n's with $\|T^n x - v\| < \delta$, and in particular, there exists $n_1 > n_0$ such that $T^{n_1}x$ is in V. So $T^{n_1-n_0}T^{n_0}x = T^{n_1}x \in V \cap T^{n_1-n_0}(U)$, and consequently T is hypercyclic.

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