Commutant lifting theorem for n-tuples of contractions

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Abstract: We show that the commutant lifting theorem for n-tuples of commuting contractions with regular dilations fails to be true. A positive answer is given for operators which "double intertwine" given n-tuples of contractions.

The commutant lifting theorem is one of the most important results of the Sz. Nagy—Foias dilation theory. It is usually stated in the following way:

Theorem. Let T and T' be contractions in Hilbert spaces H and H'. Let $A : H \to H'$ be a contraction which intertwines T and T', i.e. $AT = T'A$. Let $V \in B(K_+)$ and $V' \in B(K'_+)$ be the minimal isometric dilations of T and T'. Then there exists a contraction $B : K_+ \to K'_+$ such that $BV = V'B$ and $AP_H = P_{H'}B$.
(We denote by P_M the orthogonal projection onto a closed subspace M).

The commutant lifting theorem has been studied intensely (see [3]) because, apart from its interesting operator-theoretic consequences, it has a number of applications concerning interpolation problems, in the control theory and even in some pure technical problems.

The aim of this paper is to study the commutant lifting theorem for n-tuples of commuting contractions. It is well-known that in general (for $n \geq 3$) n-tuples of commuting contractions have no dilations. A pair of commuting contractions has a (Ando) dilation, but this dilation is not unique and the structure of the corresponding space is rather complicated. By example VII.6.3 of [3] the commutant lifting theorem fails to be true for the Ando dilations. Therefore we restrict ourselves to the case of commuting contractions having a regular dilation, which exhibits many properties similar to the case of a single contraction.

We use the method of Timotin [7] which relates the commutant lifting theorem with a problem of finding a positive semidefinite extension of some partial operator-valued matrix.

Let I be an index set and let, for each $\alpha \in I$, a Hilbert space H_α be given. Let operators $T_{\alpha,\beta} : H_\beta \to H_\alpha$ be given for all $\alpha, \beta \in I$. We say that the matrix $(T_{\alpha,\beta})_{\alpha,\beta\in I}$ is positive semidefinite if

$$\sum_{\alpha,\beta\in I} < T_{\alpha,\beta} h_\beta, h_\alpha > \geq 0$$

for every function $h : \alpha \mapsto h_\alpha \in H_\alpha$ with a finite support.

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If $(T_{\alpha,\beta})_{\alpha,\beta\in I}$ is a positive semidefinite matrix then it is well-known that there is a (uniquely determined up to a unitary equivalence) Hilbert space K and isometries $W_\alpha : H_\alpha \to K$ such that $K = \bigvee_{\alpha\in I} W_\alpha H_\alpha$ and $T_{\alpha,\beta} = W_\alpha^* W_\beta \quad (\alpha, \beta \in I)$. The space K will be called the representation space of the matrix $(T_{\alpha,\beta})_{\alpha,\beta\in I}$ (see [7]).

Conversely, if $T_{\alpha,\beta} = W_\alpha^* W_\beta \quad (\alpha, \beta \in I)$ for some isometries $W_\alpha : H_\alpha \to K$, then the matrix $(T_{\alpha,\beta})_{\alpha,\beta\in I}$ is positive semidefinite.

Lemma 1. Let I be a finite set and let $(T_{\alpha,\beta})_{\alpha,\beta\in I}$ be a positive semidefinite operator-valued matrix. Let $h_\alpha \in H_\alpha \ (\alpha \in I)$. Then the complex matrix $\bigl(< T_{\alpha,\beta} h_\beta, h_\alpha >\bigr)_{\alpha,\beta\in I}$ is positive semidefinite.

Proof. Let c_α be complex numbers. We have

$$0 \le \sum_{\alpha,\beta\in I} < T_{\alpha,\beta} c_\beta h_\beta, c_\alpha h_\alpha >= \sum_{\alpha,\beta\in I} < T_{\alpha,\beta} h_\beta, h_\alpha > c_\beta \bar{c}_\alpha$$

so that the matrix $\bigl(< T_{\alpha,\beta} h_\beta, h_\alpha >\bigr)_{\alpha,\beta\in I}$ is positive semidefinite.

In the following we shall use the standard multiindex notation. Denote by $\mathbb{Z}$ and $\mathbb{Z}_+$ the set of all integers and of all non-negative integers, respectively. For $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$ we write $\alpha \le \beta$ if $\alpha_i \le \beta_i \quad (i = 1, \dots, n)$ and $\alpha+\beta = (\alpha_1+\beta_1, \dots, \alpha_n+\beta_n)$. For $\alpha \in \mathbb{Z}^n$ denote by α_+ and α_- its positive (negative) part, $\alpha_+ = \bigl(\max(\alpha_1, 0), \dots, \max(\alpha_n, 0)\bigr)$, $\alpha_- = \bigl(-\min(\alpha_1, 0), \dots, -\min(\alpha_n, 0)\bigr)$. Thus $\alpha = \alpha_+ - \alpha_-$. Denote further by $|\alpha| = \sum_{i=1}^n |\alpha_i|$. If $T = (T_1, \dots, T_n)$ is an n-tuple of commuting operators and $\alpha \in \mathbb{Z}_+^n$, then we write shortly $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$.

Let $T = (T_1, \dots, T_n)$ be an n-tuple of mutually commuting contractions in a Hilbert space H. Let $U = (U_1, \dots, U_n)$ be an n-tuple of mutually commuting unitary operators in a Hilbert space $K \supset H$. Then U is called the minimal regular unitary dilation of T (see [5]) if $K = \bigvee_{\alpha\in\mathbb{Z}^n} U^\alpha H$ and $T^{*\alpha_-} T^{\alpha_+} = P_H U^\alpha | H$ for every $\alpha \in \mathbb{Z}^n$.

By [5], Theorem I.9.1, $T = (T_1, \dots, T_n)$ has a minimal regular unitary dilation if and only if

$$\sum_{F\subset\{1,\dots,n\}} (-1)^{|F|} T^{*e(F)} T^{e(F)} \ge 0 \tag{1}$$

where $e(F) \in \mathbb{Z}^n$ is defined by $e(F) = (\alpha_1, \dots, \alpha_n)$ and $\alpha_i = \begin{cases} 1 & (i \in F) \\ 0 & (i \notin F). \end{cases}$

In this case denote by $K_+ = \bigvee_{\alpha\in\mathbb{Z}_+^n} U^\alpha H$ and $V_i = U_i | K_+ \quad (i = 1, \dots, n)$. Then V_i are commuting isometries in K_+ and $V = (V_1, \dots, V_n)$ is the minimal regular isometric dilation of T.

The minimal regular unitary (isometric) dilation is determined uniquely up to a unitary equivalence.

Let $T = (T_1, \dots, T_n)$ be a commuting n-tuple of contractions. Then T has a regular unitary dilation if and only if the matrix $\bigl(T_{\alpha,\beta}\bigr)_{\alpha,\beta\in\mathbb{Z}^n}$, where $H_\alpha = H$ for each $\alpha \in \mathbb{Z}^n$ and $T_{\alpha,\beta} = T^{*(\beta-\alpha)_-} T^{(\beta-\alpha)_+} \quad (\alpha, \beta \in \mathbb{Z}^n)$ is positive semidefinite (see[5], p. 34–37). The corresponding representation space is the space K of the minimal regular unitary dilation and the embeddings $W_\alpha : H_\alpha \to K \quad (\alpha \in \mathbb{Z}^n)$ are given by $W_\alpha = U^\alpha | H$. Clearly the representation space of the positive semidefinite matrix

$\left(T_{\alpha,\beta}\right)_{\alpha,\beta\in\mathbb{Z}_+^n}$, where $T_{\alpha,\beta}=T^{*(\beta-\alpha)_-}T^{(\beta-\alpha)_+}$ is the space K_+ of the minimal regular isometric dilation.

We are going to study the following problem:

Problem 1. Let $T=(T_1,\dots,T_n)$ and $T'=(T_1',\dots,T_n')$ be commuting n-tuples of contractions in Hilbert spaces H and H'. Suppose that T and T' have minimal regular isometric dilations $V=(V_1,\dots,V_n)\in B(K_+)^n$ and $V'=(V_1',\dots,V_n')\in B(K_+')^n$. Let $A:H\to H'$ be a contraction satisfying $AT_i=T_i'A\quad(i=1,\dots,n)$. Does there exists a contraction $B:K_+\to K_+'$ such that $P_{H'}B=AP_H$ and $BV_i=V_i'B\quad(i=1,\dots,n)$?

Theorem 2. The answer to the previous problem is negative for $n\ge 2$.

Proof. Suppose that $T=(T_1,T_2)\in B(H)^2$, $T'=(T_1',T_2')\in B(H')^2$ and $A:H\to H'$ satisfy all the conditions required in Problem 1 and suppose that there is a contraction $B:K_+\to K_+'$ such that $P_{H'}B=AP_H$ and $BV_i=V_i'B\ \ (i=1,2)$.

By [7] there exists a (uniquely determined) Hilbert space $\tilde K$ containing K_+ and K_+' such that $\tilde K=K_+\vee K_+'$ and $B=P_{K_+'}|K_+$.

Let I be a disjoint union of two copies of $\mathbb{Z}_+^2$. The elements of these two copies will be denoted by α and $\alpha'\quad(\alpha\in\mathbb{Z}_+^2)$. Set $H_\alpha=H, H_{\alpha'}=H'\quad(\alpha\in\mathbb{Z}_+^2)$. Define operators $W_i:H_i\to\tilde K\quad(i\in I)$ by $W_\alpha=V^\alpha|H\qquad(W_\alpha:H\to K_+\subset\tilde K)$ and $W_{\alpha'}=V'^\alpha|H'\qquad(W_{\alpha'}:H'\to K_+'\subset\tilde K)$. For $i,j\in I$ define $T_{i,j}:H_j\to H_i$ by $T_{i,j}=W_i^*W_j$. Then

$$(T_{i,j})_{i,j\in I}=\begin{pmatrix}T_{\alpha,\beta} & T_{\alpha,\beta'}\\ T_{\alpha',\beta} & T_{\alpha',\beta'}\end{pmatrix}_{\alpha,\beta\in\mathbb{Z}_+^2}$$

is a positive semidefinite matrix.

For $\alpha,\beta\in\mathbb{Z}_+^2$ we have

$$T_{\alpha,\beta}=P_HV^{*\alpha}V^\beta|H=P_HV^{*(\beta-\alpha)_-}V^{(\beta-\alpha)_+}|H=T^{*(\beta-\alpha)_-}T^{(\beta-\alpha)_+},$$

analogously $T_{\alpha',\beta'}=T'^{*(\beta-\alpha)_-}T'^{(\beta-\alpha)_+}$,

$$\begin{aligned}T_{\alpha',\beta}=&P_{H'}V'^{*\alpha}BV^\beta|H=P_{H'}V'^{*\alpha}{V'}^{\beta-(\beta-\alpha)_+}BV^{(\beta-\alpha)_+}|H\\ =&P_{H'}V'^{*(\beta-\alpha)_-}BV^{(\beta-\alpha)_+}|H={T'}^{*(\beta-\alpha)_-}AT^{(\beta-\alpha)_+},\end{aligned}$$

and $T_{\alpha,\beta'}=(T_{\alpha',\beta})^*$.

In particular, $T_{\alpha',\alpha}=P_{H'}B|H=A$.

Consider the submatrix of $(T_{i,j})_{i,j\in I}$ corresponding to the rows and columns indexed by $(1,0),(0,1),(1,0)',(0,1)'$. This matrix is positive semidefinite and it has form

$$\begin{pmatrix} I & T_1^*T_2 & A^* & Y^*\\ T_2^*T_1 & I & X^* & A^*\\ A & X & I & T_1'^*T_2'\\ Y & A & T_2'^*T_1 & I\end{pmatrix},$$

where X,Y are certain operators $H\to H'$.

Let $h_1, h_2 \in H, h_1', h_2' \in H'$ be arbitrary vectors. By Lemma 1 the complex matrix

$$\begin{pmatrix} <h_1,h_1> & <T_1^*T_2h_2,h_1> & <A^*h_1',h_1> & \bar{y} \\ <T_2^*T_1h_1,h_2> & <h_2,h_2> & \bar{x} & <A^*h_2',h_2> \\ <Ah_1,h_1'> & x & <h_1',h_1'> & <T_1'^*T_2'h_2',h_1'> \\ y & <Ah_2,h_2'> & <T_2'^*T_1'h_1',h_2'> & <h_2',h_2'> \end{pmatrix} \qquad (2)$$

is positive semidefinite, where $x = <Xh_2, h_1'>, y = <Yh_1, h_2'>$.

Lemma 3. There exist Hilbert spaces H, H', operators $T_1, T_2 \in B(H), T_1', T_2' \in B(H')$, $A : H \to H'$ satisfying all conditions of Problem 1 and vectors $h_1, h_2 \in H$, $h_1', h_2' \in H'$ such that $\|h_1\| = \|h_2\| = \|h_1'\| = \|h_2'\| = 1$, $<T_1h_1, T_2h_2> = 0$, $<T_1'h_1', T_2'h_2'> \neq 0$, $Ah_1 = h_1'$ and $Ah_2 = h_2'$.

Proof. Let $\dim H = \dim H' = 4$, let $\{e_1, \ldots, e_4\}$ and $\{f_1, \ldots, f_4\}$ be orthonormal bases of H and H', respectively. With respect to these basis set

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{pmatrix},$$

$$T_1' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad T_2' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $T_1T_2 = T_2T_1 = 0$, $T_1'T_2' = T_2'T_1' = 0$, $T_1^*T_1 + T_2^*T_2 \leq I$ and $T'^*_1T_1' + T'^*_2T_2' \leq I$ so that the pairs (T_1, T_2), (T_1', T_2') have regular isometric dilations,

$$AT_1 = T_1'A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad AT_2 = T_2'A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\|A\| \leq 1$. Set $h_1 = e_1$, $h_2 = e_2$, $h_1' = f_1$ and $h_2' = f_2$. Then $\|h_1\| = \|h_2\| = \|h_1'\| = \|h_2'\| = 1$, $<T_1h_1, T_2h_2> = <\frac{1}{2}e_3, \frac{1}{2}e_4> = 0$, $<T_1'h_1', T_2'h_2'> = <\frac{1}{4}f_3, \frac{1}{4}f_3> \neq 0$, $Ah_1 = h_1'$ and $Ah_2 = h_2'$.

Continuation of the proof of Theorem 2. Let $(V_1, V_2) \in B(K_+)^2$ and $(V_1', V_2') \in B(K_+')^2$ be the minimal regular isometric dilations of the pairs (T_1, T_2) and (T_1', T_2') which were constructed in the previous lemma. Suppose on the contrary that there is a contraction $B : K_+ \to K_+'$ such that $P_{H'}B = AP_H$ and $BV_i = V_i'B \quad (i = 1, 2)$. Let h_1, h_2, h_1', h_2' be the vectors constructed in the previous lemma.

By substituting into (2) we have

$$\begin{pmatrix} 1 & 0 & 1 & \bar{y} \\ 0 & 1 & \bar{x} & 1 \\ 1 & x & 1 & \bar{a} \\ y & 1 & a & 1 \end{pmatrix} \geq 0,$$

where $a =< T_1'h_1', T_2'h_2' >\neq 0$ and x, y are certain complex numbers. Then

$$0 \leq \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \bar{x} \\ 1 & x & 1 \end{pmatrix} = -|x|^2$$

so that $x = 0$ and

$$0 \leq \det \begin{pmatrix} 1 & \bar{x} & 1 \\ x & 1 & \bar{a} \\ 1 & a & 1 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \bar{a} \\ 1 & a & 1 \end{pmatrix} = -|a|^2 < 0,$$

a contradiction.

Thus the commutant lifting theorem is not true for n-tuples of commuting contractions with the regular dilation.

Remark 1. In the construction we have used instead of the condition $AP_H = P_{H'}B$ only the weaker condition $A = P_{H'}B|H$. Thus there is no lifting of A satisfying only this weaker condition.

Remark 2. Clearly it is possible to consider regular unitary dilations instead of isometric dilations. The reasoning given in the example remains true so that the commutant lifting theorem is not true for regular unitary dilations.

Remark 3. It is easy to modify the example of Theorem 3 for any $n \geq 2$. It is sufficient to consider the n-tuples $(T_1, T_2, 0, \dots, 0)$ and $(T_1', T_2', 0, \dots, 0)$.

Remark 4. Regular dilations are connected with the polydisc

$$D^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n, |z_i| \leq 1, i = 1, \dots, n\}$$

in the following way. An n-tuple $T = (T_1, \dots, T_n)$ of commuting contractions has a regular dilation if and only if T is unitarily equivalent to the restriction of $M_z^* \oplus W$ to an invariant subspace, where W is an n-tuple of commuting unitary operators and $M_z = (M_{z_1}, \dots, M_{z_n})$ where M_{z_i} is the operator of multiplication by z_i in the space $H^2(D^n)$ (with infinite multiplicity), see [2].

There is a parallel theory for the ball

$$B_n = \big\{(z_1, \dots, z_n) \in \mathbb{C}^n, \sum_{i=1}^n |z_i|^2 \leq 1\big\},$$

see [1], [4], or [8]. Condition (1) is then replaced by

$$I - \sum_{i=1}^n T_i^* T_i \geq 0 \tag{3}$$

and

$$\sum_{|\alpha| \leq n} (-1)^{|\alpha|} \frac{n!}{\alpha!(n - |\alpha|)!} T^{*\alpha} T^\alpha \geq 0. \tag{4}$$

An n-tuple $T=(T_1,\dots,T_n)$ of commuting contractions satisfies (3) and (4) if and only if T is unitarily equivalent to the restriction of $M_z^*\oplus W$ where $W=(W_1,\dots,W_n)$ is an n-tuple of commuting normal operators with $\sum_{i=1}^n W_i^*W_i=I$ and $M_z=(M_{z_1},\dots,M_{z_n})$, where M_{z_i} is the multiplication by z_i in $H^2(B_n)$ (for details see [4]).

The n-tuple T has then a spherical dilation, i.e. there are commuting normal operators $N=(N_1,\dots,N_n)$ such that $\sum N_i^*N_i=I$ and $T^\alpha=P_HN^\alpha|H$ for each $\alpha\in\mathbb{Z}_+^n$.

It is easy to check that the pairs $T=(T_1,T_2)$ and $T'=(T_1',T_2')$ constructed in Theorem 3 satisfy also (3) and (4). It is possible to show that the operator A from Theorem 3 can not be lifted to an operator B satisfying $N_i'B=BN_i \quad (i=1,2)$ where $N=(N_1,N_2)$, $N'=(N_1',N_2')$ are the spherical dilations of T and T'. We omit the proof since it requires more detailed information about the spherical dilations N and N'.

In the following we shall suppose that the operator A "double intertwines" $T=(T_1,\dots,T_n)$ and $T'=(T_1',\dots,T_n')$, i.e. both $AT_i=T_i'A$ and $AT_i^*=T_i'^*A$ for all i. We show that then A can be lifted to the space of the minimal regular unitary dilation.

Theorem 4. Let $T=(T_1,\dots,T_n)$ and $T'=(T_1',\dots,T_n')$ be commuting n-tuples of contractions in Hilbert spaces H and H', respectively. Suppose that T and T' have the minimal regular unitary dilations $U=(U_1,\dots,U_n)\in B(K)^n$ and $U'=(U_1',\dots,U_n')\in B(K')^n$. Let $A:H\to H'$ be a contraction satisfying $AT_i=T_i'A$ and $AT_i^*=T_i'^*A \quad (i=1,\dots,n)$. Then there exists a contraction $B:K\to K'$ such that $P_{H'}B=AP_H$ and $BU_i=U_i'B \quad (i=1,\dots,n)$.

Proof. For $\alpha,\beta\in\mathbb{Z}^n$ set

$$
\begin{aligned}
T_{\alpha,\beta}&=T^{*(\beta-\alpha)_-}T^{(\beta-\alpha)_+}\\
T_{\alpha',\beta'}&=T'^{*(\beta-\alpha)_-}T'^{(\beta-\alpha)_+}\\
T_{\alpha',\beta}&=AT^{*(\beta-\alpha)_-}T^{(\beta-\alpha)_+}\\
T_{\alpha,\beta}&=(T_{\alpha',\beta})^*.
\end{aligned}
$$

We show that the matrix

$$
\begin{pmatrix} T_{\alpha,\beta} & T_{\alpha,\beta'}\\ T_{\alpha',\beta} & T_{\alpha',\beta'}\end{pmatrix}_{\alpha,\beta\in\mathbb{Z}^2} \tag{5}
$$

is positive semidefinite. It is sufficient to show that, for each positive integer k, the finite matrix

$$
\begin{pmatrix} T_{\alpha,\beta} & T_{\alpha,\beta'}\\ T_{\alpha',\beta} & T_{\alpha',\beta'}\end{pmatrix}_{\substack{\alpha,\beta\in\mathbb{Z}^2\\ |\alpha|,|\beta|\le k}}
$$

is positive semidefinite.

By [5] the matrices $S=(T_{\alpha,\beta})_{\substack{\alpha,\beta\in\mathbb{Z}^2\\ |\alpha|,|\beta|\le k}}$ and $S'=(T_{\alpha',\beta'})_{\substack{\alpha,\beta\in\mathbb{Z}^2\\ |\alpha|,|\beta|\le k}}$ are positive semidefinite.

Denote by $\operatorname{diag}(A)$ the diagonal matrix

$$\operatorname{diag}(A) = \begin{pmatrix} A & 0 & \cdots \\ 0 & A & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}_{\substack{\alpha,\beta\in\mathbb{Z}^2 \\ |\alpha,\beta\leq k}}.$$

Then $\operatorname{diag}(A)\cdot S = S'\cdot\operatorname{diag}(A)$ and, by expressing the square root as a limit of polynomials, also $\operatorname{diag}(A)\cdot S^{1/2} = S'^{1/2}\cdot\operatorname{diag}(A)$. Thus

$$\begin{pmatrix} T_{\alpha,\beta} & T_{\alpha,\beta'} \\ T_{\alpha',\beta} & T_{\alpha',\beta'} \end{pmatrix}_{\substack{\alpha,\beta\in\mathbb{Z}^2 \\ |\alpha|,|\beta|\leq k}} = \begin{pmatrix} S & S\cdot\operatorname{diag}(A^*) \\ \operatorname{diag}(A)\cdot S & S' \end{pmatrix}$$

$$= \begin{pmatrix} S^{1/2} & 0 \\ 0 & S'^{1/2} \end{pmatrix}\begin{pmatrix} I & \operatorname{diag}(A^*) \\ \operatorname{diag}(A) & I \end{pmatrix}\begin{pmatrix} S^{1/2} & 0 \\ 0 & S'^{1/2} \end{pmatrix},$$

Since $\|A\|\leq 1$ we have $\begin{pmatrix} I & A^* \\ A & I \end{pmatrix}\geq 0$ and $\begin{pmatrix} I & \operatorname{diag}(A^*) \\ \operatorname{diag}(A) & I \end{pmatrix}\geq 0$. Hence the matrix (5) is also positive semidefinite.

Denote its representation space by $\tilde{K}$.

Since the representation spaces of matrices $(T_{\alpha,\beta})_{\alpha,\beta\in\mathbb{Z}^2}$ and $(T_{\alpha',\beta'})_{\alpha,\beta\in\mathbb{Z}^2}$ are unitarily equivalent to K and K', respectively, we may consider K and K' as subspaces of $\tilde{K}$ and $\tilde{K} = K\vee K'$.

Set $B = P_{K'}|K$. Clearly $\|B\|\leq 1$.

For each $\beta\in\mathbb{Z}^n$ we have

$$AP_H U^\beta|H = AT^{*\beta_-}T^{\beta_+} = T_{0',\beta} = P_{H'}P_{K'}U^\beta|H = P_{H'}BU^\beta|H.$$

Since $\bigvee_{\beta\in\mathbb{Z}^n} U^\beta H = K$ we conclude that $AP_H = P_{H'}B$.

For $i = 1,\ldots,n$ denote by $e_i = (\underbrace{0,\ldots,0}_{i-1},1,0,\ldots,0)\in\mathbb{Z}^n$. We have

$$T_{\alpha',\beta} = T_{(\alpha+e_i)',\beta+e_i}\quad (\alpha,\beta\in\mathbb{Z}^n, i = 1,\ldots,n).$$

Then, for each $\alpha,\beta\in\mathbb{Z}^n, i = 1,\ldots,n, h\in H$ and $h'\in H'$, we have

$$< BU_iU^\beta h, U'^\alpha h' > = < P_{H'}U'^{*\alpha}BU^{\beta+e_i}h, h' > = < T_{\alpha',\beta+e_i}h, h' >$$
$$= < T_{(\alpha-e_i)',\beta}h, h' > = < P_{H'}U'^{*\alpha}U'_iBU^\beta h, h' > = < U'_iBU^\beta h, U'^\alpha h' > .$$

Since $K = \bigvee_{\beta\in\mathbb{Z}^n} U^\beta H$ and $K' = \bigvee_{\beta\in\mathbb{Z}^n} U'^\beta H'$ we conclude that $BU_i = U'_iB \quad (i = 1,\ldots,n)$.

Remark 5. Consider the minimal regular isometric dilations $V\in B(K_+)^n$ and $V'\in B(K'_+)^n$ of T and T' and let $A: H\to H'$ double intertwine T and T'. As in the previous theorem we can get a contractive lifting $C: K_+\to K'_+$ such that $P_{H'}C = AP_H$, but instead of the intertwining property $V'_iC = CV_i$ it is possible to prove only the Toeplitz type property $V_i'^* C V_i = C \quad (i = 1, \ldots, n)$. The lifting to the space of the minimal regular unitary dilation is the "symbol" of C, cf. [6].

References

[1] A. Athavale, On the intertwining of joint isometries, J. Operator Theory 23 (1990), 339–350.

[2] R.E. Curto, F.H. Vasilescu, Standard operator models in the polydisc, to appear.

[3] C. Foias, A.E. Frazho, The commutant lifting approach to interpolation problems, Birkhäuser Verlag, Basel, 1990.

[4] V. Müller, F.H. Vasilescu, Standard models for some commuting multioperators, Proc. Amer. Math. Soc. 117 (1993), 979–989.

[5] B. Sz.-Nagy, C. Foias, Harmonic analysis of operators on Hilbert spaces, North-Holland, Amsterdam, 1970.

[6] B. Sz.-Nagy, C. Foias, Toeplitz type operators and hyponormality, in Dilation theory, Toeplitz operators and other topics, Operator Theory: Advances and Applications, Vol. 11, Birkhäuser Verlag (Basel, 1983), 371–388.

[7] D. Timotin, Completions of matrices and the commutant lifting theorem, J. Funct. Anal. 104 (1992), 291–298.

[8] F.H. Vasilescu, An operator-valued Poisson kernel, J. Funct. Anal. 110 (1990), 47–72.

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