Nil, nilpotent and PI-algebras

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The notions of nil, nilpotent or PI-rings (= rings satisfying a polynomial identity) play an important role in the ring theory (see e.g. [8], [11], [20]). Banach algebras with these properties have been studied considerably less and the existing results are scattered in literature. The only exception is the work of Krupnik [13], where the Gelfand theory of Banach PI-algebras is presented. However, even this work has not get so much attention as it deserves.

The present paper is an attempt to give a survey of results concerning Banach nil, nilpotent and PI-algebras.

The author would like to thank to J. Zemánek for essential completion of the bibliography.

I. Nil and nilpotent Banach algebras

All algebras are complex; we do not assume the existence of the unit element.

The set of all positive integers will be denoted by **N**. For $n \in \mathbf{N}$ we denote by $\mathcal{P}_n$ the set of all complex polynomials in n non-commuting variables.

Definition 1. Let A be a Banach algebra. We say that

1) A is nil $\Leftrightarrow$ every element of A is nilpotent, i.e. for every $x \in A$ there is $n \in \mathbf{N}$ such that $x^n = 0$,
2) A is nilpotent $\Leftrightarrow$ there exists $m \in N$ such that $x_1 \cdots x_m = 0$ for every $x_1, \dots, x_m \in A$,
3) A is algebraic $\Leftrightarrow$ for every $x \in A$ there is a polynomial $p \in \mathcal{P}_1$, $p \neq 0$ such that $p(x) = 0$,
4) A is locally finite $\Leftrightarrow$ every finite subset of A generates a finite-dimensional subalgebra,
5) A is a PI-algebra $\Leftrightarrow$ there exist $n \in \mathbf{N}$ and a polynomial $p \in \mathcal{P}_n$, $p \neq 0$ such that $p(x_1, \dots, x_n) = 0$ for all $x_1, \dots, x_n \in A$,
6) A is locally PI $\Leftrightarrow$ for every sequence $\{x_i\}_{i=1}^{\infty} \subset A$ there exist $n \in N$ and $p \in \mathcal{P}_n$, $p \neq 0$ such that $p(x_1, \dots, x_n) = 0$.

There is a number of relations among these notions:

Theorem 2 [15]. *A Banach algebra is PI if and only if it is locally PI.*

Theorem 3 [1], cf. also [12], [5] and [10]. *Let A be Banach algebra. The following properties are equivalent*

a) *A is algebraic,*
b) *A is locally finite,*
c) Rad$A$ *is nilpotent and codim* Rad$A < \infty$.

Theorem 4 [7]. *A Banach algebra is nil if and only if it is nilpotent.*

Thus we have

$$1 \Leftrightarrow 2 \Rightarrow 3 \Leftrightarrow 4 \Rightarrow 5 \Leftrightarrow 6.$$

The algebra of all $n \times n$ matrices is an example of an algebraic Banach algebra which is not nilpotent.

Any infinite dimensional commutative semisimple Banach algebra is an example of a PI-algebra which is not algebraic (by Theorem 3).

The proof of Theorem 4 consists of two steps:

1) If every element of a Banach algebra A is nilpotent then there exists $n \in \mathbf{N}$ such that $x^n = 0$ for every $x \in A$. This (as well as many other results in this paper) is a consequence of the Baire category theorem.
2) If $x^n = 0$ for every $x \in A$ then there exists $m \in \mathbf{N}$ such that $x_1 \cdots x_m = 0$ for every $x_1, \ldots, x_m \in A$. This is the so called Nagata - Higman theorem, see [16], [9] or [11], Appendix C, which is purely algebraic (actually, it has been discovered recently, that this theorem was proved already by Dubnov and Ivanov [4]).

Denote by $d(n)$ the least integer such that $x_1 \cdots x_{d(n)} = 0$ for all $x_1, \ldots, x_{d(n)} \in A$ provided that $x^n = 0$ $(x \in A)$. An interesting open problem is to determine the exact value of $d(n)$. The best known bounds are [14], [18]

$$\frac{n(n+1)}{2} \leq d(n) \leq n^2.$$

For a survey of results about the Nagata-Higman theorem, see [6].

II. Approximate properties

A natural approximate analog to the notion of a nilpotent element x of a Banach algebra A is the concept of quasinilpotent elements:

$x \in A$ is quasinilpotent $\Leftrightarrow r(x) = \lim\limits_{n \to \infty} \|x^n\|^{1/n} = 0$. There are several possibilities how to define approximative versions of nil or nilpotent algebras (cf. [2]).

Definition 5. Let A be a Banach algebra. Then

1) A is radical ($=$ topologically nil) $\Leftrightarrow$ every element of A is quasinilpotent, i.e.

$$\lim_{n \to \infty} \|x^n\|^{1/n} = 0$$

for every $x \in A$,

2) A is uniformly topologically nil $\Leftrightarrow \lim_{n \to \infty} S_A(n) = 0$ where

$$S_A(n) = \sup \left\{ \|x^n\|^{1/n}, x \in A, \|x\| = 1 \right\},$$

3) A is topologically nilpotent $\Leftrightarrow \lim\limits_{n \to \infty} \|x_1 \ldots x_n\|^{1/n} = 0$ for every sequence

$$x_1, x_2, \ldots \in A, \qquad \|x_i\| = 1 \quad (i = 1, 2, \ldots),$$

4) A is uniformly topologically nilpotent $\Leftrightarrow \lim_{n\to\infty} N_A(n) = 0$ where

$$N_A(n) = \sup\left\{\|x_1 \cdots x_n\|^{1/n}, x_1, \ldots, x_n \in A, \|x_i\| = 1 \ (i = 1, \ldots, n)\right\}.$$

These notions are related in the following manner:

Theorem 6 [3]. *A Banach algebra is topologically nilpotent if and only if it is uniformly topologically nilpotent.*

Theorem 7 [2]. *If A is a topologically nilpotent Banach algebra then A is uniformly topologically nil. If A is commutative then these two notions are equivalent.*

Thus we have

$$4 \Leftrightarrow 3 \Rightarrow 2 \Rightarrow 1$$

and for commutative Banach algebras

$$4 \Leftrightarrow 3 \Leftrightarrow 2 \Rightarrow 1.$$

An example of a uniformly topologically nil algebra which is not topologically nilpotent can be found in [3] (clearly such an example is non-commutative).

If A is a uniformly topologically nil then A is clearly radical and it is not difficult to find an example of a (even commutative) radical Banach algebra which is not uniformly topologically nil: consider the ℓ^1-algebra A generated by elements x_1, x_2, ... which satisfy $\|x_i\| = 1$, $x_i x_j = 0 \ \ (i \neq j)$, and $\|x_i^i\| = 1$, $x_i^{i+1} = 0$.

Another interesting notion is the following (see [19], [22]; for related concept see also [17]):

For a bounded subset M of a Banach algebra A denote by $\|M\| = \sup\{\|x\|, x \in M\}$ and by $r(M) = \lim_{n\to\infty} \|M^n\|^{1/n}$ (the spectral radius of M), where

$$M^n = \{x_1 \ldots x_n, x_i \in M, i = 1, \ldots, n\}.$$

Note, that in this notation $N_A(n) = \|B_A\|^{1/n}$, where B_A is the closed unit ball of A and A is topologically nilpotent if and only if $r(B_A) = 0$. An interesting open problem (see [22]) is

Problem. Let A be a radical Banach algebra. Is then A finitely quasinilpotent, i.e. $r(F) = 0$ for every finite subset $F \subset A$?

It is easy to see that this is true for commutative Banach algebras. A positive answer is known also if the algebra A consists of compact operators in a Banach space, see [21].

The example mentioned above is an example of a commutative finitely quasinilpotent algebra, which is not uniformly topologically nil.

A countable version of this problem is not true. As an example it is possible to take the set $M = \{x_1, x_2, \ldots\}$ in the ℓ^1-algebra with generators $x_1, x_2, \ldots$ which satisfy the relation $x_i x_j = 0$ for $j \neq i+1$. In other words the elements of the algebra are of the form

$$y = \sum_{i<j} \alpha_{i,j} x_i x_{i+1} \cdots x_{j-1}$$

where the complex coefficients $\alpha_{i,j}$ satisfy $\|y\| = \sum_{i<j} |\alpha_{i,j}| < \infty$.

III. Gelfand theory of Banach PI-algebras

In this section we give a survey of the Gelfand theory for Banach PI-algebras (see [13]).

To avoid technical difficulties we formulate the results only for algebras with unit. If A is a Banach PI-algebra without unit, then its unification is also a PI-algebra (by Theorem 2), so that the results can be modified for algebras without unit in obvious way.

Among all polynomial identities play an important role the standard identities. Denote by

$$e_n(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, x_{\sigma(1)} \ldots x_{\sigma(n)},$$

where S_n is the set of all permutations of the set $\{1, \ldots, n\}$ and sign $(\sigma) = \pm 1$ for even (odd) permutation σ. Clearly $e_2(x_1, x_2) = x_1 x_2 - x_2 x_1$.

Examples of Banach PI-algebras:

1) Any commutative Banach algebra A satisfies the standard identity $e_2(x, y) = 0$ for all $x, y \in A$.
2) Let A be a finite-dimensional Banach algebra, $\dim A = n$. Then it is easy to show that
$$e_{n+1}(x_1, \ldots, x_{n+1}) = 0 \qquad (x_1, \ldots, x_{n+1} \in A).$$
3) An important particular case is the algebra M_n of all $n \times n$ (complex) matrices. By 2), the algebra M_n satisfies the polynomial identity $e_{n^2+1}(x_1, \ldots, x_{n^2+1}) = 0$. Actually, a better result is true:

Theorem 8 (Amitzur, Levitzky). *The algebra M_n satisfies the standard identity*

$$e_{2n}(x_1, \ldots, x_{2n}) = 0 \qquad (x_1, \ldots, x_{2n} \in M_n).$$

It can be shown that M_n satisfies no polynomial identity of grade $\leq 2n$.

Denote by $\mathcal{F}_n$ the class of all unital Banach algebras A which satisfy the standard identity $e_n(x_1, \ldots, x_n) = 0$ for all $x_1, \ldots, x_n \in A$.

Clearly $\mathcal{F}_2$ is the class of all commutative unital Banach algebras so that classes $\mathcal{F}_n$ can be considered as a generalization of commutative Banach algebras. Further $M_n \in \mathcal{F}_{2n}$ by Theorem 8.

Denote by $\mathcal{M}(A)$ the set of all maximal two-sided ideals in a Banach algebra A.

For $M \in \mathcal{M}(A)$ denote by π_A the canonical homomorphism $\pi_M : A \to A/M$.

It is well-known that if A is a commutative Banach algebra, then maximal ideals are of codimension 1, $\cap\{M, M \in \mathcal{M}(A)\} = \mathrm{Rad}A$, and $x \in A$ is invertible $\Leftrightarrow \pi_M(x) \neq 0$ $(M \in \mathcal{M}(A))$.

The following theorem is a generalization of these facts:

Theorem 9 ([13], Theorem 21.1). *Let A be a Banach algebra of class $\mathcal{F}_{2n}$. Then*
a) *For every $M \in \mathcal{M}(A)$ is the algebra A/M isomorphic to M_l for some $l \leq n$,*
b) *$\cap\{M, M \in \mathcal{M}(A)\} = \mathrm{Rad}A$,*
c) *$x \in A$ is invertible $\Leftrightarrow \pi_M(x)$ is invertible for every $M \in \mathcal{M}(A)$.*

If A is a Banach PI-algebra then it can be shown that A satisfies a homogeneous multilinear identity, i.e. there exist $n \in \mathbf{N}$ and complex coefficients $\alpha_\sigma (\sigma \in S_n)$ not all of them equal to 0 such that

$$\sum_{\sigma \in S_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)} = 0 \qquad (x_1, \ldots, x_n \in A).$$

However, not every Banach PI-algebra satisfies a standard identity. Consider the exterior algebra A (with unit) generated by elements e_1, e_2, ... with multiplication $e_i e_j = -e_j e_i$ $(i, j \in \mathbf{N})$. This algebra is PI as $(xy - yx)z - z(xy - yx) = 0 \quad (x, y, z \in A)$ and it can be seen easily that A does not satisfy any standard identity.

The situation changes when we consider semisimple Banach algebras or, equivalently, Banach PI-algebras modulo radical.

Theorem 10. *Let A be a unital Banach algebra. The following statements are equivalent:*
1) *$A|\mathrm{Rad}A$ is a PI-algebra.*
2) *$A|\mathrm{Rad}A \in \mathcal{F}_{2n}$ for some $n \in N$.*
3) *There exists $m \in \mathbf{N}$ such that $x \in A$ is invertible $\Leftrightarrow \pi(x)$ is invertible for all representations (i.e. unit-preserving homomorphisms) $\pi : A \to M_m$.*
4) *There exists $m \in \mathbf{N}$ such that $x \in A$ is invertible $\Leftrightarrow \pi(x)$ is invertible for all representations $\pi : A \to M_m$ with the property*

$$|(\pi(y))_{i,j}| \leq \|y\| \qquad (y \in A, i, j = 1, \ldots, m).$$

Proof. The equivalence $1 \Leftrightarrow 2 \Leftrightarrow$ ($x \in A$ is invertible $\Leftrightarrow \pi(x)$ is invertible for all representations $\pi : A \to M_l$ $(l \leq n)$) can be found in [13], Theorems 22.1, 22.2.

To show the equivalence with 3, take $m = n!$ (in fact $m = n^2$ is sufficient), so that for any representation $\pi : A \to M_l$ with $l \leq n$ there is a representation $\tilde{\pi} : A \to M_m$, $\tilde{\pi} = \bigoplus_{i=1}^{m/l} \pi$.

For the equivalence with 4, see [13], Lemma 25.1.

For commutative Banach algebras there is a 1-1 correspondence between maximal ideals and multiplicative functionals. For PI-algebras the role of multiplicative functionals play finite-dimensional representations, but a finite-dimensional representation is not uniquely determined by its kernel. Property 4 of the previous theorem shows that it is sufficient to consider a compact set of representations.

Let A be a unital Banach algebra such that $A|\mathrm{Rad}A$ is a PI-algebra. Let $m \in \mathbf{N}$ be the integer from the previous theorem. Denote by $\mathcal{K}$ the set of all representations $\pi : A \to M_m$ with the property

$$\left|\big(\pi(y)\big)_{i,j}\right| \le \|y\| \qquad (y \in A, i, j = 1, \dots, m).$$

We define a topology on $\mathcal{K}$ in the following way: the basis of neighborhoods of an element $\pi \in \mathcal{K}$ is formed by the sets

$$U_{x_1,\dots,x_k,\varepsilon} = \{\pi' \in \mathcal{K}, \|\pi'(x_i) - \pi(x_i)\| < \varepsilon, i = 1, \dots, k\}$$

where $x_1, \dots, x_k \in A$, $\varepsilon > 0$.

Clearly $\mathcal{K}$ is a compact set. Denote by $C(\mathcal{K}, M_m)$ the algebra of all continuous functions from $\mathcal{K}$ to M_m and define a mapping $G : A \to C(\mathcal{K}, M_m)$ by

$$G(a)(\pi) = \pi(a) \qquad (a \in A, \pi \in \mathcal{K}).$$

Theorem 11. *Let A be a unital Banach algebra such that $A|\mathrm{Rad}A$ is a PI-algebra. Then the mapping $G : A \to C(\mathcal{K}, M_m)$ has the following properties:*

1) *G is a continuous homomorphism,*
2) *$a \in A$ is invertible $\Leftrightarrow$ $G(a)$ is invertible in $C(\mathcal{K}, M_m)$,*
3) *$G(a) = 0 \Leftrightarrow a \in \mathrm{Rad}A$,*
4) *$\sigma(a) = \bigcup_{\pi \in \mathcal{K}} \sigma\big(G(a)(\pi)\big)$ so that $r(a) = \max_{\pi \in \mathcal{K}} r\big(G(a)(\pi)\big)$.*

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