On the Punctured Neighbourhood Theorem

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Abstract. Let X, Y, Z be Banach spaces and $X \xrightarrow{S(z)} Y \xrightarrow{T(z)} Z$ an analytically dependant sequence of operators satisfying $T(z)S(z) = 0$. We study properties of the function $z \mapsto \dim \operatorname{Ker} T(z)/\operatorname{Im} S(z)$.

Let X, Y be complex Banach spaces. Denote by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X to Y. If $Y = X$ then we write for short $\mathcal{L}(X) = \mathcal{L}(X, X)$.

Recall the well-known punctured neighbourhood theorem:

Theorem 1. Let $T \in \mathcal{L}(X)$ be a Fredholm operator. Then there exist $\varepsilon > 0$ and constants $k_1 \le \dim \operatorname{Ker} T$, $k_2 \le \operatorname{codim} \operatorname{Im} T$ such that $\dim \operatorname{Ker}(T - z) = k_1$ and $\operatorname{codim} \operatorname{Im}(T - z) = k_2$ for all z, $0 < |z| < \varepsilon$.

In this paper we study a more general situation. Let X, Y, Z be Banach spaces, let U be an open subset of $\mathbf{C}^n$, let $S : U \to \mathcal{L}(X, Y)$ and $T : U \to \mathcal{L}(Y, Z)$ be analytic operator-valued functions satisfying $T(z)S(z) = 0$ for all $z \in U$. For $z \in U$ write $\alpha(z) = \dim \operatorname{Ker} T(z)/\operatorname{Im} S(z)$.

The aim of the paper is to study the behaviour of the function $z \mapsto \alpha(z)$.

The main result of the first section is the following generalization of Theorem 1 — if $U \subset \mathbf{C}$, $w \in U$, $\operatorname{Im} T(w)$ is closed and $\alpha(w) < \infty$ then $\alpha(z) = k$ is constant in a punctured neighbourhood of w.

Clearly the classical punctured neighbourhood theorem follows easily from this generalization for sequences $0 \to X \xrightarrow{T-z} Y$ and $X \xrightarrow{T-z} Y \to 0$, respectively.

In the second section we study the case $n \ge 2$. This situation has been studied mainly in connection with the Koszul complex of an n-tuple of commuting operators.

I.

For $T \in \mathcal{L}(X, Y)$ denote by $\gamma(T)$ the Kato reduced minimum modulus, $\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}\{x, \operatorname{Ker} T\} = 1\}$ (formally we set $\gamma(0) = \infty$). Clearly $\gamma(T) > 0$ if and only if $\operatorname{Im} T$ is closed.

If M, L are closed subspaces of X then write

$$\delta(M, L) = \sup_{\substack{x \in M \\ \|x\| \le 1}} \operatorname{dist}\{x, L\}$$

and the gap between M and L is defined by $\hat{\delta}(M, L) = \max\{\delta(M, L), \delta(L, M)\}$. For the properties of the reduced minimum modulus and the gap see [6].

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The following result is due to Markus, cf. [13], Theorem 1.4.

Theorem 2. Let U be an open subset of $\mathbf{C}^n$, let $T : U \to \mathcal{L}(X, Y)$ be a norm-continuous function, let $w \in U$ and $\operatorname{Im} T(w)$ be closed. The following conditions are equivalent:

(i) the function $z \mapsto \gamma(T(z))$ is continuous at w,

(ii) $\liminf_{z \to w} \gamma(T(z)) > 0$,

(iii) $\lim_{z \to w} \delta\big(\operatorname{Ker} T(w), \operatorname{Ker} T(z)\big) = 0$,

(iv) $\lim_{z \to w} \hat{\delta}\big(\operatorname{Ker} T(w), \operatorname{Ker} T(z)\big) = 0$,

(v) $\lim_{z \to w} \delta\big(\operatorname{Im} T(z), \operatorname{Im} T(w)\big) = 0$,

(vi) $\lim_{z \to w} \hat{\delta}\big(\operatorname{Im} T(z), \operatorname{Im} T(w)\big) = 0$.

The equivalences *(iii)* $\Leftrightarrow$ *(iv)* and *(v)* $\Leftrightarrow$ *(vi)* follow from the fact that automatically $\lim_{z \to w} \delta\big(\operatorname{Ker} T(z), \operatorname{Ker} T(w)\big) = 0$ and $\lim_{z \to w} \delta(\operatorname{Im} T(w), \operatorname{Im} T(z)) = 0$.

A continuous function $T : U \to \mathcal{L}(X, Y)$ is called regular at w if $\operatorname{Im} T(w))$ is closed and T satisfies any of equivalent conditions (i) – (vi). In particular, condition (ii) implies that the set of all regularity points of T is open. Also, T is regular at w if and only if the adjoint function $z \mapsto T(z)^*$ is regular at w.

Regular functions are closely related to the exactness:

Theorem 3. ([13], Theorem 2) Let U be an open subset of $\mathbf{C}^n$, $w \in U$ and let $T : U \to \mathcal{L}(X, Y)$ be an analytic function. The following conditions are equivalent:

(i) T is regular at w,

(ii) there exists a neighbourhood $U_0 \subset U$ of w, a Banach space E and an analytic function $S : U_0 \to \mathcal{L}(E, X)$ such that $\operatorname{Im} S(z) = \operatorname{Ker} T(z) \quad (z \in U_0)$,

(iii) there exists a neighbourhood $U_0 \subset U$ of w, a Banach space E' and an analytic function $S' : U_0 \to \mathcal{L}(Y, E')$ such that $\operatorname{Im} T(z) = \operatorname{Ker} S'(z) \quad (z \in U_0)$.

In particular, if $T : U \to \mathcal{L}(X, Y)$ is regular at w and $x \in \operatorname{Ker} T(w)$ then there exist a neighbourhood U_0 of w and an analytic function $f : U_0 \to X$ such that $f(w) = x$ and $T(z)f(z) = 0 \quad (z \in U_0)$. Indeed, let $S : U_0 \to \mathcal{L}(E, X)$ be an analytic function satisfying the properties of (ii). Choose $e \in E$ with $S(w)e = x$ and set $f(z) = S(z)e$.

Lemma 4. Let U be an open subset of $\mathbf{C}^n$, let $S : U \to \mathcal{L}(X, Y)$ and $T : U \to \mathcal{L}(Y, Z)$ be functions regular in U. Suppose that $T(z)S(z) = 0$ for all $z \in U$. Then $\alpha(z)$ is constant on each connected subset of U.

Proof. Let $w \in U$ satisfy $\alpha(w) = \dim \operatorname{Ker} T(w)/\operatorname{Im} S(w) < \infty$. By Theorem 2 (iv) and (vi), $\lim_{z \to w} \hat{\delta}\big(\operatorname{Ker} T(w), \operatorname{Ker} T(z)\big) = 0$ and $\lim_{z \to w} \hat{\delta}\big(\operatorname{Im} T(w), \operatorname{Im} T(z)\big) = 0$. Thus there exists $\varepsilon > 0$ such that $\hat{\delta}\big(\operatorname{Ker} T(z), \operatorname{Ker} T(w)\big) < 1/9$ and $\hat{\delta}\big(\operatorname{Im} S(z), \operatorname{Im} S(w)\big) < 1/9$ for $z \in U$, $\operatorname{dist}\{z, w\} < \varepsilon$. By [1] this implies that

$$\alpha(z) = \dim \operatorname{Ker} T(z)/\operatorname{Im} S(z) = \dim \operatorname{Ker} T(w)/\operatorname{Im} S(w) = \alpha(w)$$

for all $z \in U$, $\operatorname{dist}\{z, w\} < \varepsilon$.

Thus $\alpha(z)$ is locally constant and a standard argument gives that $\alpha(z)$ is constant on the component of connectivity of U containing w.

If U_0 is a component of U and there is no $w \in U_0$ with $\alpha(w) < \infty$ then clearly $\alpha(z) = \infty$ on U_0.

An operator $T \in \mathcal{L}(X)$ with the property that the function $z \mapsto T - z$ is regular at 0 is called semi-regular (sometimes Kato regular). Semi-regular operators exhibit very nice properties and have been studied intensely, see e.g. [9], [10], [12].

An essential version of semi-regular operators has been also studied. Recall that if M, L are closed subspaces of X then we write $M \overset{e}{\subset} L$ (M is essentially contained in L) if $\dim M/(M \cap L) < \infty$. We summarize some of equivalent conditions characterizing essentially semi-regular operators.

Theorem 5. ([10], Theorem 3.1) Let $T \in \mathcal{L}(X)$ be an operator with closed range. The following conditions are equivalent:

(i) (Kato decomposition) there exists a decomposition $X = X_1 \oplus X_2$ such that $TX_1 \subset X_1$, $TX_2 \subset X_2$, $\dim X_1 < \infty$, $T|X_1$ is nilpotent and $T|X_2$ is an semi-regular operator,

(ii) $\bigcap_{z \neq 0} \overline{\operatorname{Im}(T - z)} \overset{e}{\subset} \operatorname{Im} T$,

(iii) $\dim \operatorname{Ker} T/N^*(T) < \infty$, where $N^*(T)$ is the set of all $x \in X$ such that there are complex numbers z_i $(i = 1, 2, \dots)$ tending to 0 and elements $x_i \in \operatorname{Ker}(T - z_i)$ such that $x = \lim_{i \to \infty} x_i$ (clearly $N^*(T) \subset \operatorname{Ker} T$),

(iv) $\dim R^*(T)/\operatorname{Im} T < \infty$ where $R^*(T)$ is the set of all $x \in X$ such that $x = \lim_{i \to \infty} x_i$ for some $x_i \in \operatorname{Im}(T - z_i)$ and some $z_i \to 0$ (clearly $\operatorname{Im} T \subset R^*(T)$).

Note that condition (i) implies that the function $z \mapsto T - z$ is regular in a punctured neighbourhood $\{z : 0 < |z| < \varepsilon\}$ for some $\varepsilon > 0$.

General analytic operator-valued functions of one variable can be reduced to the linear case by the method of linearization, see [2], Theorem 2.6.

Theorem 6. Let $U \subset \mathbf{C}$ be an open set, $T : U \to \mathcal{L}(X, Y)$ an analytic function and $w \in U$. Then there exist a neighbourhood U_0 of w, Banach spaces Z and M, an operator $V \in \mathcal{L}(M)$ and analytic functions $A : U_0 \to \mathcal{L}(M, X \oplus Z)$, $B : U_0 \to \mathcal{L}(Y \oplus Z, M)$ such that $A(z)$ and $B(z)$ are invertible operators and

$$B(z)\big(T(z) \oplus I_Z\big)A(z) = V - zI_M \qquad (z \in U_0).$$

Let $U \subset \mathbf{C}$ be an open set and let $T : U \to \mathcal{L}(X, Y)$ be an analytic operator-valued function. Let $w \in U$. Write

$$R^*(T(w)) = \big\{y \in Y : \text{ there exist } z_k \in U, z_k \to w \text{ and } y_k \in \operatorname{Im} T(z_k) \text{ with } y_k \to y\big\},$$
$$R^{**}(T(w)) = \big\{y \in Y : \lim_{z \to w} \operatorname{dist}\{y, \operatorname{Im} T(z)\} = 0\big\}.$$

Clearly $\operatorname{Im} T(w) \subset R^{**}(T(w)) \subset R^*(T(w))$ and $R^*(T(w))$, $R^{**}(T(w))$ are closed subspaces of Y.

Similarly write

$$N^*(T(w)) = \big\{x \in X : \text{ there are } z_k \in U, x_k \in \operatorname{Ker} T(z_k) \text{ with } z_k \to w \text{ and } x_k \to x\big\},$$
$$N^{**}(T(w)) = \big\{x \in X : \lim_{z \to w} \operatorname{dist}\{x, \operatorname{Ker} T(z)\} = 0\big\}.$$

Clearly $N^{**}(T(w)) \subset N^*(T(w)) \subset \operatorname{Ker} T(w)$ and $N^*(T(w)), N^{**}(T(w))$ are closed subspaces of X.

Theorem 7. Let $U \subset \mathbf{C}$ be an open set, $T : U \to \mathcal{L}(X, Y)$ an analytic function and $w \in U$. The following statements are equivalent:
(i) $\dim R^*(T(w))/\operatorname{Im} T(w) < \infty$,
(ii) $\dim R^{**}(T(w))/\operatorname{Im} T(w) < \infty$,
(iii) $\dim \operatorname{Ker} T(w)/N^*(T(w)) < \infty$ and $\operatorname{Im} T(w)$ is closed,
(iv) $\dim \operatorname{Ker} T(w)/N^{**}(T(w)) < \infty$ and $\operatorname{Im} T(w)$ is closed.

Any of these conditions implies that there exists $\varepsilon > 0$ such that the function T is regular in the punctured neighbourhood $\{z \in U : 0 < |z - w| < \varepsilon\}$. Further $N^*(T(w)) = N^{**}(T(w))$, $R^*(T(w)) = R^{**}(T(w))$ and $\dim \operatorname{Ker} T(w)/N^*(T(w)) = \dim R^*(T(w))/\operatorname{Im} T(w)$.

Proof.
A. Suppose first that $Y = X$ and $T(z) = V - zI_X$ for some operator $V \in \mathcal{L}(X)$. We show that in this case conditions (i) – (iv) are equivalent to
(v) $V - w$ is essentially semi-regular.

Clearly *(i)* $\Rightarrow$ *(ii)* and *(iv)* $\Rightarrow$ *(iii)*.

By Theorem 5, *(i)* $\Leftrightarrow$ *(iii)* $\Leftrightarrow$ *(v)*.

(ii) $\Rightarrow$ *(v)*: Clearly (ii) implies that $\operatorname{Im} T(w)$ is closed. Further

$$\bigcap_{z \neq w} \overline{\operatorname{Im}(V - z)} \subset R^{**}(V - w)$$

so that, by Theorem 5, $V - w$ is essentially semi-regular.

Suppose now that $V - w$ is essentially semi-regular. Let $X = X_1 \oplus X_2$ be the Kato decomposition of $V - w$, i.e., $VX_1 \subset X_1$, $VX_2 \subset X_2$, $\dim X_1 < \infty$, $(V - w)|X_1$ is nilpotent and $(V - w)|X_2$ is semi-regular. It is easy to see that, for $z \neq w$, $\operatorname{Ker}(V - z) = \operatorname{Ker}((V - z)|X_2)$ and $\operatorname{Im}(V - z) = X_1 + \operatorname{Im}((V - z)|X_2)$. Thus

$$N^*(V - w) = N^{**}(V - w) = \operatorname{Ker}\big((V - w)|X_2\big)$$

and

$$R^*(V - w) = R^{**}(V - w) = X_1 + \operatorname{Im}\big((V - w)|X_2\big).$$

Hence (v) implies (iv). Further

$$\begin{aligned} &\dim \operatorname{Ker}(V - w)/N^*(V - w) = \dim \operatorname{Ker}\big((V - w)|X_1\big) \\ &= \dim X_1/(V - w)X_1 = \dim R^*(V - w)/\operatorname{Im}(V - w). \end{aligned}$$

Also the Kato decomposition implies that the function $z \mapsto V - z$ is regular in a certain punctured neighbourhood of w.

B. Let now $T(z)$ be a general analytic operator-valued function. By Theorem 6 there exist a neighbourhood U_0 of w, Banach spaces Z, M, an operator $V \in \mathcal{L}(M)$ and analytic functions $A : U_0 \to \mathcal{L}(M, X \oplus Z)$, $B : U_0 \to \mathcal{L}(Y \oplus Z, M)$ whose values are invertible operators, such that

$$B(z)\big(T(z) \oplus I_Z\big)A(z) = V - zI_Z \qquad (z \in U_0).$$

For $z \in U_0$ we have

$$\operatorname{Ker}(V-zI)=\operatorname{Ker}\big((T(z)\oplus I_Z)A(z)\big)=A(z)^{-1}\operatorname{Ker}\big(T(z)\oplus I_Z\big)=A(z)^{-1}\operatorname{Ker}T(z)$$

and

$$\operatorname{Im}(V-zI)=\operatorname{Im}\big(B(z)(T(z)\oplus I_Z)\big)=B(z)(\operatorname{Im}T(z)+Z).$$

Thus

$$\begin{aligned}N^*(V-wI)&=A(w)^{-1}N^*(T(w)),\\ N^{**}(V-wI)&=A(w)^{-1}N^{**}(T(w)),\\ R^*(V-wI)&=B(w)\big(R^*(T(w))+Z\big)\qquad\text{and}\\ R^{**}(V-wI)&=B(w)\big(R^{**}(T(w))+Z\big).\end{aligned}$$

Hence all the statements for the function $T(z)$ are equivalent to the corresponding statements for $V-zI$ and the general case reduces to the previous case.

Remark 8. Let $U\subset\mathbf{C}$, $w\in U$ and let $T:U\to\mathcal{L}(X,Y)$ be an analytic function. Then $\dim\operatorname{Ker}T(w)/N^*(T(w))$ can be interpreted as the "jump" in the kernel of $T(z)$; similarly $\dim R^*(T(w))/\operatorname{Im}T(w)$ signifies the jump in the range of $T(z)$. It is interesting to note that these two numbers are always equal.

Theorem 9. Let U be an open subset of $\mathbf{C}$ and $w\in U$. Suppose that $S:U\to\mathcal{L}(X,Y)$, $T:U\to\mathcal{L}(Y,Z)$ are analytic functions satisfying $T(z)S(z)=0\quad(z\in U)$, $\alpha(w)<\infty$ and $\operatorname{Im}T(w)$ is closed. Then there exist $\varepsilon>0$ and a constant $c\le\alpha(w)$ such that $\alpha(z)=c$ for all z, $0<|z-w|<\varepsilon$.

Proof. By [14], Lemma 2.1, $\alpha(z)\le\alpha(w)$ for all z in a neighbourhood of w. Using the previous theorem, both $z\mapsto S(z)$ and $z\mapsto T(z)$ are regular in a certain punctured neighbourhood of w so that, by Lemma 4, $\alpha(z)$ is constant in this punctured neighbourhood.

II.

In this section we study analytic operator-valued functions of n-variables.

It is not possible to expect the punctured neighbourhood theorem for $n\ge2$; the proper generalization seems to be

Conjecture 10. Let $U\subset\mathbf{C}^n$ be open, let $S:U\to\mathcal{L}(X,Y)$ and $T:U\to\mathcal{L}(X,Y)$ be analytic on U. Suppose that $T(z)S(z)=0$, $\operatorname{Im}T(z)$ is closed and $\alpha(z)=\dim\operatorname{Ker}T(z)/\operatorname{Im}S(z)<\infty\quad(z\in U)$. Let $k\in\mathbf{N}$. Then the set $\{z\in U:\alpha(z)\ge k\}$ is analytic in U.

Recall that a set $M\subset U$ is called analytic in U if for each $w\in U$ there exist a neighbourhood U_0 of w and analytic (scalar-valued) functions $f_1,\dots,f_r$ such that $M\cap U_0=\{z\in U_0:f_1(z)=\cdots=f_r(z)=0\}$.

The conjecture is true in the following particular cases:

A. if the ranges and kernels of $S(z)$ and $T(z)$ are complemented subspaces, see Theorem 14 below. In particular, the conjecture is true for operators in Hilbert spaces.
B. if either $S(z) \equiv 0$ or $T(z) \equiv 0$; this means that the other function is upper (lower) semi-Fredholm-valued and the conjecture reduces to the statement about defect indices of semi-Freholm-valued functions, see [5].
C. if the sequence $X \xrightarrow{S(z)} Y \xrightarrow{T(z)} Z$ is a part of a Fredholm complex vanishing at the ends, see [7], [8], [11] or Theorem 18 below.

We start with the following lemma:

Lemma 11. Let $U \subset \mathbf{C}^n$ be an open subset, let $T : U \to \mathcal{L}(X, Y)$ be an analytic function, let $k \in \mathbf{N}$. Then the set $\{z \in U : \dim \operatorname{Im} T(z) < k\}$ is analytic.

Proof. If $x_1, \dots, x_k \in X$, $y_1^*, \dots, y_k^* \in Y^*, z \in U$ and $\dim \operatorname{Im} T(z) < k$ then the vectors $T(z)x_1, \dots, T(z)x_k$ are linearly independent and $\det\big(\langle T(z)x_i, y_j^*\rangle\big) = 0$.

On the other hand, if $\dim \operatorname{Im} T(z) \geq k$ then there are vectors $x_1, \dots, x_k \in X$, $y_1^*, \dots, y_k^* \in Y^*$ such that $\det\big(\langle T(z)x_i, y_j^*\rangle\big) \neq 0$. Thus

$$\begin{aligned}&\big\{z \in U : \dim \operatorname{Im} T(z) < k\big\}\\ =&\big\{z \in U : \det\big(\langle T(z)x_i, y_j^*\rangle\big) = 0 \text{ for all } x_1, \dots, x_k \in X, y_1^*, \dots, y_k^* \in Y^*\big\}\end{aligned}$$

which is an analytic set, see [3], p. 86.

Corollary 12. Let $S : U \to \mathcal{L}(X, Y)$ and $T : U \to \mathcal{L}(Y, Z)$ be analytic functions and let $k \in \mathbf{N}$. Then the set $\big\{z \in U : \dim \operatorname{Im} S(z)/\big(\operatorname{Im} S(z) \cap \operatorname{Ker} T(z)\big) < k\big\}$ is analytic.

Proof. Clearly $\dim \operatorname{Im} S(z)/\big(\operatorname{Im} S(z) \cap \operatorname{Ker} T(z)\big) = \dim \operatorname{Im}\big(T(z)S(z)\big)$ so that the corollary follows from the previous lemma.

Lemma 13. Let U be an open subset of $\mathbf{C}^n$, let $S : U \to \mathcal{L}(X, Y)$ and $T : U \to \mathcal{L}(Y, Z)$ be analytic functions satisfying $T(z)S(z) = 0 \quad (z \in U)$. Suppose that there are Banach spaces X_1 and Z_1 and regular analytic functions $S_1 : U \to \mathcal{L}(X_1, Y)$, $T_1 : U \to \mathcal{L}(Y, Z_1)$ satisfying

$$\operatorname{Ker} T_1(z) \subset \operatorname{Im} S(z) \subset \operatorname{Ker} T(z) \subset \operatorname{Im} S_1(z)$$

and $\dim \operatorname{Im} S_1(z)/\operatorname{Ker} T_1(z) < \infty \quad (z \in U)$. Then the set

$$\big\{z \in U : \dim \operatorname{Ker} T(z)/\operatorname{Im} S(z) \geq k\big\}$$

is analytic in U.

Proof. The situation is illustrated by the following diagram:

$$X \xrightarrow{S(z)} Y \xrightarrow{T(z)} Z$$

Fig. 1

We can assume that U is connected. For each j set

$$A_j = \{z \in U : \dim \operatorname{Im} S(z)/\operatorname{Ker} T_1(z) \le j\}$$

and

$$B_j = \{z \in U : \dim \operatorname{Im} S_1(z)/\operatorname{Ker} T(z) \le j\}.$$

By Corollary 12, A_j and B_j are analytic sets. As in the proof of Lemma 4 (or using Theorem 3) it is easy to that there is a constant c such that $\dim \operatorname{Im} S_1(z)/\operatorname{Ker} T_1(z) = c$ in U. Thus

$$\begin{aligned} &\bigl\{z \in U : \dim \operatorname{Ker} T(z)/\operatorname{Im} S(z) \ge k\bigr\} \\ =&\bigl\{z \in U : \dim \operatorname{Im} S_1(z)/\operatorname{Ker} T(z) + \dim \operatorname{Im} S(z)/\operatorname{Ker} T_1(z) \le c-k\bigr\} \\ =&\bigcup_{i=0}^{c-k} A_i \cap B_{c-k-i}. \end{aligned}$$

The last set is clearly analytic.

Let $T \in \mathcal{L}(X, Y)$. An operator $S \in \mathcal{L}(Y, X)$ is called a generalized inverse of T if $TST = T$ and $STS = S$. If S is a generalized inverse of T then TS and ST are projections satisfying $\operatorname{Im}(TS) = \operatorname{Im} T$ and $\operatorname{Ker}(ST) = \operatorname{Ker} T$. Thus T has a generalized inverse if and only if both $\operatorname{Ker} T$ and $\operatorname{Im} T$ are complemented subspaces of X and Y, respectively.

The next result shows that Conjecture 10 is true for operators with generalized inverses. We adopt the method of [4].

Theorem 14. Let U be an open subset of $\mathbf{C}^n$, let $S : U \to \mathcal{L}(X, Y)$ and $T : U \to \mathcal{L}(Y, Z)$ be analytic functions. Suppose that $T(z)S(z) = 0$, $\dim \operatorname{Ker} T(z)/\operatorname{Im} S(z) < \infty$ and the operators $S(z)$ and $T(z)$ have generalized inverses for $z \in U$. Let $k \in \mathbf{N}$. Then the set $\{z \in U : \alpha(z) \ge k\}$ is analytic in U.

Proof. Let $w \in U$. Let V be a generalized inverse of $S(w)$, i.e., $VS(w)V = V$ and $S(w)VS(w) = S(w)$. Set $P = I - S(w)V$. Then P is a projection, $\operatorname{Ker} P = \operatorname{Im} S(w)$.

For z close to w, the operator $I + (S(z) - S(w))V$ is invertible. Define $P(z) \in \mathcal{L}(Y)$ by $P(z) = P\bigl(I + (S(z) - S(w))V\bigr)^{-1} \in \mathcal{L}(Y)$. Clearly the function $z \mapsto P(z)$ is regular at w since $\operatorname{Im} P(z) = \operatorname{Im} P$ is constant. We prove $\operatorname{Ker} P(z) \subset \operatorname{Im} S(z)$. Let $y \in \operatorname{Ker} P(z)$, i.e., $0 = P(z)y = P\bigl(I + (S(z) - S(w))V\bigr)^{-1} y$. Then

$$\bigl(I + (S(z) - S(w))V\bigr)^{-1} y \in \operatorname{Ker} P = \operatorname{Im} S(w)$$

For some $x \in X$ we have

$$y = \bigl(I + (S(z) - S(w))V\bigr) S(w)x = S(z)VS(w)x \in \operatorname{Im} S(z).$$

Similarly, let W be a generalized inverse of $T(w)$. Set $Q = I - WT(w)$. Then Q is a projection with $\operatorname{Im} Q = \operatorname{Ker} T(w)$. For z close to w define $Q(z) \in \mathcal{L}(Y)$ by

$Q(z) = \big(I + W(S(z) - S(w))\big)^{-1}Q$. Clearly the function $z \mapsto Q(z)$ is regular since $\operatorname{Ker} Q(z) = \operatorname{Ker} Q$ is constant. We have

$$WT(z) = WT(w) + W(T(z) - T(w)) = I - Q + W(T(z) - T(w))$$

so that

$$\big(I + W(T(z) - T(w))\big)^{-1} WT(z) = I - \big(I + W(T(z) - T(w))\big)^{-1} Q = I - Q(z).$$

Consequently, $\operatorname{Ker} T(z) \subset \operatorname{Im} Q(z)$.

Thus we have $\operatorname{Ker} P(z) \subset \operatorname{Im} S(z) \subset \operatorname{Ker} T(z) \subset \operatorname{Im} Q(z)$ and

$$\dim \operatorname{Im} Q(w)/\operatorname{Ker} P(w) = \dim \operatorname{Im} Q/\operatorname{Ker} P = \dim \operatorname{Ker} T(w)/\operatorname{Im} S(w) < \infty.$$

As in Lemma 4 we have that $\dim \operatorname{Im} Q(z)/\operatorname{Ker} P(z) < \infty$ in a neighbourhood of w. The rest follows from Lemma 13.

Corollary 15. Conjecture 10 is true for operators in Hilbert spaces.

In the following we consider a complex

$$0 \longrightarrow X_0 \xrightarrow{\delta_0(z)} X_1 \xrightarrow{\delta_1(z)} \cdots \xrightarrow{\delta_{n-1}(z)} X_n \longrightarrow 0, \tag{1}$$

where $X_0, \dots, X_n$ are Banach spaces, operators $\delta_j(z)$ satisfy $\delta_j(z)\delta_{j-1}(z) = 0$ and depend analytically on a parameter $z \in U$, where U is an open subset of $\mathbf{C}^n$.

Suppose that complex (1) is Fredholm, i.e., $\dim \operatorname{Ker} \delta_j(z)/\operatorname{Im} \delta_{j-1}(z) < \infty$ for all $j = 0, \dots, n$ and $z \in U$ (formally we set $\delta_{-1}(z) = 0$ and $\delta_n(z) = 0$).

Let $k \in \mathbf{N}$. It is a folklore among specialists in the sheaf theory that the set $\big\{z \in U : \dim \operatorname{Ker} \delta_j(z)/\operatorname{Im} \delta_{j-1}(z) \geq k\big\}$ is analytic. This result is stated without proof (for the Koszul complex of a commuting n-tuple of operators) in [7] and [8]; cf also [11]. Since apparently there is no elementary proof of this result, we include the proof here.

We need the following modification of Lemma 13:

Lemma 16. Let U be an open subset of $\mathbf{C}^n$, let $S : U \to \mathcal{L}(X, Y)$ and $T : U \to \mathcal{L}(Y, Z)$ be analytic functions satisfying $T(z)S(z) = 0 \quad (z \in U)$. Suppose that there are Banach spaces X_1, Z_1, finite dimensional Banach spaces F, G and regular analytic functions $S_1 : U \to \mathcal{L}(X_1, Y \oplus F)$ and $T_1 : U \to \mathcal{L}(Y \oplus G, Z_1)$ such that $\operatorname{Im} S_1(z) \supset \operatorname{Ker} T(z) \supset \operatorname{Im} S(z)$, $\operatorname{Im} S(z) + G \supset \operatorname{Ker} T_1(z)$ and $\dim\big(\operatorname{Im} S_1(z) + G\big)/\operatorname{Ker} T_1(z) < \infty \quad (z \in U)$, see Fig. 2. Let $k \in \mathbf{N}$. Then the set $\{z \in U : \alpha(z) \geq k\}$ is analytic in U.

$$X \xrightarrow{S(z)} Y \xrightarrow{T(z)} Z$$

Fig. 2

Proof. Set $Y' = Y \oplus F \oplus G$. For $z \in U$ define operators $S'(z) : X \oplus G \to Y'$, $T'(z) : Y' \to Z \oplus F$, $S_1'(z) : X_1 \oplus G \to Y'$ and $T_1'(z) : Y' \to Z_1 \oplus F$ by

$$\begin{aligned}
S'(z)(x \oplus g) &= S(z)x + g,\\
T'(z)(y \oplus f \oplus g) &= T(z)y + f,\\
S_1'(z)(x_1 \oplus g) &= S_1(z)x_1 + g,\\
T_1'(z)(y \oplus f \oplus g) &= T_1(z)(y \oplus g) + f
\end{aligned}$$

for all $x \in X$, $f \in F$, $g \in G$ and $x_1 \in X_1$. Thus $\operatorname{Im} S'(z) = \operatorname{Im} S(z) + G$, $\operatorname{Ker} T'(z) = \operatorname{Ker} T(z) + G$, $\operatorname{Im} S_1'(z) = \operatorname{Im} S_1(z) + G$ and $\operatorname{Ker} T_1'(z) = \operatorname{Ker} T_1(z)$. We have

$$\operatorname{Im} S_1'(z) \supset \operatorname{Ker} T'(z) \supset \operatorname{Im} S'(z) \supset \operatorname{Ker} T_1'(z)$$

and

$$\dim \operatorname{Im} S_1'(z) / \operatorname{Ker} T_1'(z) = \dim\bigl(\operatorname{Im} S_1(z) + G\bigr) / \operatorname{Ker} T_1(z) < \infty.$$

By Lemma 13, the set $\bigl\{z \in U : \dim \operatorname{Ker} T'(z) / \operatorname{Im} S'(z) \geq k\bigr\}$ is analytic in U. This set, however, is equal to the set $\{z \in U : \alpha(z) \geq k\}$.

Lemma 17. Let U be an open subset of $\mathbf{C}^n$, let $S : U \to \mathcal{L}(X, Y)$ and $T : U \to \mathcal{L}(Y, Z)$ be analytic functions satisfying $T(z)S(z) = 0$ and $\alpha(z) < \infty \quad (z \in U)$. Let $w \in U$. Suppose that there are finite dimensional spaces G, H, a neighbourhood U_1 of w and a regular analytic function $T_1 : U_1 \to \mathcal{L}(Y \oplus G, Z \oplus H)$ such that $T_1(z)|Y = T(z)$. Then there exist a finite dimensional space F, a neighbourhood U_2 of w and a regular analytic function $S_1 : U_2 \to \mathcal{L}(X \oplus F, Y \oplus G)$ such that $S_1(z)|X = S(z)$ and $\operatorname{Im} S_1(z) = \operatorname{Ker} T_1(z) \supset \operatorname{Ker} T(z)$, see Fig. 3.

$$X \xrightarrow{S(z)} Y \xrightarrow{T(z)} Z$$

Fig. 3

Proof. For $z \in U_1$ we have

$$\dim \operatorname{Ker} T_1(z) / \operatorname{Im} S(z) = \dim \operatorname{Ker} T_1(z) / \operatorname{Ker} T(z) + \dim \operatorname{Ker} T(z) / \operatorname{Im} S(z) < \infty.$$

Let $y_1, \ldots, y_r$ be linearly independent vectors in $\operatorname{Ker} T_1(w)$ such that

$$\operatorname{Im} S(w) \vee \{y_1, \ldots, y_r\} = \operatorname{Ker} T_1(w).$$

Since T_1 is regular, for $i = 1, \ldots, r$, there exists a $(Y \oplus G)$-valued analytic function ϕ_i defined in a neighbourhood of w such that $T_1(z)\phi(z) = 0$ and $\phi_i(w) = y_i$. Let F be an r-dimensional space with a basis $f_1 \ldots, f_r$ and define $S_1(z) : X \oplus F \to Y \oplus G$ by

$$S_1(z)\Bigl(x \oplus \sum_{i=1}^{r} \beta_i f_i\Bigr) = S(z)x + \sum_{i=1}^{r} \beta_i \phi(z) y_i \qquad (x \in X, \beta_i \in \mathbf{C}).$$

Clearly $T_1(z)S_1(z) = 0$ and $\operatorname{Im} S_1(w) = \operatorname{Ker} T_1(w)$ so that there is a neighbourhood of w where $\operatorname{Ker} T_1(z) = \operatorname{Im} S_1(z)$, see [14]. Thus S_1 is regular in a neighbourhood of w and satisfies all the required conditions.

Theorem 18. Let $X_0, X_1, \dots, X_n$ be Banach spaces, U an open subset of $\mathbf{C}^n$. Let

$$0 \longrightarrow X_0 \xrightarrow{\delta_0(z)} X_1 \xrightarrow{\delta_1(z)} \cdots \xrightarrow{\delta_{n-1}(z)} X_n \longrightarrow 0$$

be a Fredholm complex analytically dependent on $z \in U$ (i.e., $\delta_j(z)\delta_{j-1}(z) = 0$ and $\dim \operatorname{Ker} \delta_j(z)/\operatorname{Im} \delta_{j-1}(z) < \infty$ for all $\in U$ and $j = 0, \dots, n$).

Let $0 \le j \le n$ and $k \in \mathbf{N}$. Then the set $\{z \in U : \dim \operatorname{Ker} \delta_j(z)/\operatorname{Im} \delta_{j-1}(z) \ge k\}$ is analytic in U.

Proof. Let $w \in U$. Using Lemma 17 repeatedly it is easy to see by the downward induction that there are finite dimensional spaces F_{j-1}, F_j and a regular analytic function $S(z) : X_{j-1} \oplus F_{j-1} \to X_j \oplus F_j$ defined in a neighbourhood of w such that $S(z)|X_{j-1} = \delta_{j-1}(z)$ and $\operatorname{Im} S(z) \supset \operatorname{Ker} \delta_j(z)$. In particular, $\dim \operatorname{Im} S(z)/\operatorname{Ker} \delta_{j-1}(z) < \infty$.

Consider the "adjoint" complex

$$0 \longleftarrow X_0^* \xleftarrow{\delta_0^*(z)} X_1^* \xleftarrow{\delta_1^*(z)} \cdots \xleftarrow{\delta_{n-1}^*(z)} X_n^* \longleftarrow 0$$

where we write for short $\delta_j^*(z)$ instead of $(\delta_j(z))^*$. Since this complex is also Fredholm, similarly as above there exist finite dimensional spaces G_j and G_{j+1} and a regular analytic function $T(z) : X_{j+1}^* \oplus G_{j+1} \to X_j^* \oplus G_j$ defined in a neighbourhood of w such that $\operatorname{Im} T(z) \supset \operatorname{Ker}(\delta_{j-1}^*(z))$ and $\dim \operatorname{Im} T(z)/\operatorname{Ker} \delta_{j-1}^*(z) < \infty$. Further the operator $S^*(z) : X_j^* \oplus F^* \to X_{j-1}^* \oplus F_{j-1}^*$ satisfies

$$\operatorname{Ker} S^*(z) = (\operatorname{Im} S(z))^\perp \subset (\operatorname{Ker} \delta_j(z))^\perp + F_j^* = \operatorname{Im} \delta_j^*(z) + F_j^*.$$

By Lemma 16, the set $\{z : \dim \operatorname{Ker} \delta_{j-1}^*(z)/\operatorname{Im} \delta_j^*(z) \ge k\}$ is analytic. Since

$$\dim \operatorname{Ker} \delta_{j-1}^*(z)/\operatorname{Im} \delta_j^*(z) = \dim \operatorname{Ker} \delta_j(z)/\operatorname{Im} \delta_{j-1}(z),$$

this finishes the proof.

Let $A = (A_1, \dots, A_n)$ be an n-tuple of commuting operators on a Banach space X. Denote by $\sigma_T(A)$ the Taylor spectrum of A. The essential spectrum $\sigma_{Te}(A)$ of A is defined as the set of all $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n$ such that the Koszul complex of the n-tuple $(A_1 - \lambda_1, \dots, A_n - \lambda_n)$ is not Fredholm.

Corollary 19. ([7], [8]) Let $A = (A_1, \dots, A_n)$ be an n-tuple of commuting operators on a Banach space X. Then the set $\sigma_T(A) \setminus \sigma_{Te}(A)$ is analytic in $\mathbf{C}^n \setminus \sigma_{Te}(A)$.

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