On the regular spectrum

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Introduction

Let T be a bounded linear operator acting in a Banach space X. Denote by $R(T) = TX$ and $N(T) = \{x \in X, Tx = 0\}$ its range and kernel, respectively.

Continuity properties of the functions $z \mapsto R(T - z)$ and $z \mapsto N(T - z)$ were studied by a number of authors. The investigation was started by Kato [9], [10], who introduced also useful concepts of the reduced minimum modulus and the gap between two closed subspaces.

The spectrum $\sigma_\gamma(T)$ was defined for Hilbert space operators by Apostol [3] as the set of all complex λ such that either $R(T - \lambda)$ is not closed or λ is a discontinuity point of the function $z \mapsto R(T - z)$. Properties of this spectrum are analogous to the properties of the ordinary spectrum. It is always a non-empty compact subset of the complex plain, contains the topological boundary of $\sigma(T)$ and satisfies the spectral mapping property.

The results of Apostol were generalized by Mbekhta [14], [15], Mbekhta and Ouahab [16], [17] and Harte [7] for operators in Banach spaces.

In this paper we continue the investigation of σ_γ. We define an essential version $\sigma_{\gamma e}$ which exhibits similar properties as σ_γ and is closely related to the theory of semi-Fredholm operators. Further we study generalized inverses for $T - \lambda$ and show that it is not possible to extend reasonably σ_γ for n-tuples of commuting operators.

The author would like to thank to the referee for drawing his attention to the paper of Rakočevič [18] which is closely related to the present paper. Some of the results are already proved in [18], especially Theorem 3.1, equivalence $1 \Leftrightarrow 2$ (see Theorem 2.1 of [18]) or the spectral mapping theorem for $\sigma_{\gamma e}$. We leave the proofs here for the sake of completeness and because they seem to us sometimes more direct. On the other hand the present paper solves some questions posed in [18]. Thus Example 2.2 gives a negative answer to both parts of Question 4 and Theorem 3.5 gives a positive answer to Question 2 of [18].

I. Semi-regular operators and spectrum σ_γ

Throughout the paper we shall denote by X a fixed complex Banach space X. Denote by $B(X)$ the algebra of all bounded linear operators in X. For $T \in B(X)$ the reduced minimum modulus of T is defined by

$$\gamma(T) = \inf\big\{\|Tx\|, x \in X, \operatorname{dist}\{x, N(T)\} = 1\big\}$$

(if $T = 0$ then we set $\gamma(T) = \infty$).

Let M_1 and M_2 be two closed subspaces of X. Then we denote by

$$\delta(M_1, M_2) = \sup\big\{\operatorname{dist}\{x, M_2\}, x \in M_1, \|x\| = 1\big\}$$

(if $M_1 = \{0\}$ then $\delta(M_1, M_2) = 0$) and the gap between M_1 and M_2 by

$$\hat{\delta}(M_1, M_2) = \max\bigl\{\delta(M_1, M_2), \delta(M_2, M_1)\bigr\}.$$

We list the most important properties of the reduced minimum modulus and the gap between two subspaces (see [10], Chapter IV):

Theorem 1.1.
1) $\gamma(T) > 0$ if and only if $R(T)$ is closed,
2) $\gamma(T) > r > 0$ if and only if for every $y \in R(T)$ there exists $x \in X$ such that $Tx = y$ and
$$\|x\| \le r^{-1}\|y\|,$$
3) $\gamma(T^*) = \gamma(T)$,
4) the set $\{T \in B(X), \gamma(T) \ge \varepsilon\}$ is norm-closed in $B(X)$ for every ε (see [2]),
5) $\delta(M_1, M_2) = \delta(M_2^\perp, M_1^\perp)$,
6) if $\hat{\delta}(M_1, M_2) < 1$ then $\dim M_1 = \dim M_2$.

For $T \in B(X)$ we have $R(T) \supset R(T^2) \supset R(T^3) \supset \dots$ and $N(T) \subset N(T^2) \subset \dots$. Denote shortly $R^\infty(T) = \bigcap_{n=0}^\infty R(T^n)$ and $N^\infty(T) = \bigvee_{n=0}^\infty N(T^n)$.

Consider the function $z \mapsto \gamma(T - z)$ defined for complex z. Although this function is not continuous in general, it has good continuity properties. From a great number of equivalent conditions characterizing the continuity points of the function $z \mapsto \gamma(T - z)$ we choose the most important:

Theorem 1.2. Let $T \in B(X)$ be an operator with closed range. The following conditions are equivalent:
1) the function $z \mapsto \gamma(T - z)$ is continuous at $z = 0$,
2) the function $z \mapsto \gamma(T - z)$ is bounded from below in a neighbourhood of 0, i.e. there exists $\varepsilon > 0$ such that $\inf_{|z|<\varepsilon} \gamma(T - z) > 0$,
3) the function $z \mapsto R(T - z)$ is continuous at 0 in the gap topology, i.e.
$$\lim_{z \to 0} \hat{\delta}\bigl(R(T), R(T - z)\bigr) = 0,$$
4) the function $z \mapsto N(T - z)$ is continuous at 0 in the gap topology, i.e.
$$\lim_{z \to 0} \hat{\delta}\bigl(N(T), N(T - z)\bigr) = 0,$$
5) $N(T) \subset R^\infty(T)$,
6) $N^\infty(T) \subset R(T)$,
7) $N^\infty(T) \subset R^\infty(T)$.

The previous theorem was proved in [16]. The equivalence of the first four conditions is true for any continuous operator-valued function $z \mapsto T(z)$; in [19] this result was attributed to Markus, see [13].

Definition 1.3. (see [16]) An operator $T \in B(X)$ is called s-regular (semi-regular) if T has closed range and satisfies any of the equivalent conditions of Theorem 1.2.

For s-regular operators the subspaces $R^\infty(T)$ and $N^\infty(T)$ can be described in another way. We start with two simple lemmas:

Lemma 1.4. Let $T \in B(X)$ be s-regular, $x \in X$ and $Tx \in R^\infty(T)$. Then $x \in R^\infty(T)$.

Proof. Let $n \geq 1$. Then there exists $y \in X$ such that $T^{n+1}y = Tx$, i.e. $x - T^n y \in N(T) \subset R^\infty(T) \subset R(T^n)$. So $x \in R(T^n)$ and as n was arbitrary, $x \in R^\infty(T)$.

Lemma 1.5. Let $T \in B(X)$ be an s-regular operator. Denote by $U = \{z \in \mathbf{C}, |z| < \gamma(T)\}$. Then for every $\lambda \in U$ and $x \in N(T-\lambda)$ there exists an analytic function $f : U \to X$ such that $(T-z)f(z) = 0 \quad (z \in U)$ and $f(\lambda) = x$.

Proof. By [16], Theorem 2.10., $T - z$ is s-regular for $z \in U$. By [19], Theorem 2, there exists a Banach space Y and an analytic operator-valued function $S : U \to B(Y, X)$ such that $R(S(z)) = N(T(z)) \quad (z \in U)$. Choose $y \in Y$ such that $S(\lambda)y = x$ and set $f(z) = S(z)y$. Clearly f satisfies all conditions of Lemma 1.5.

Theorem 1.6. Let $T \in B(X)$ be s-regular and let r be a positive number, $r \leq \gamma(T)$. Then
1) $N^\infty(T) = \bigvee_{|\lambda|<r} N(T-\lambda)$,
2) $R^\infty(T) = \bigcap_{|\lambda|<r} R(T-\lambda)$.

Proof. 1) Denote by $U = \{z \in \mathbf{C}, |z| < \gamma(T)\}$. Let $\lambda \in U$ and $x \in N(T-\lambda)$. Then there exists an analytic function $f : U \to X$ such that $(T-z)f(z) = 0 \quad (z \in U)$ and $f(\lambda) = x$. Let $f(z) = \sum_{i=0}^\infty a_i z^i \quad (z \in U)$, where $a_i \in X$. The equality $(T-z)f(z) = 0$ implies $Ta_0 = 0$ and $Ta_i = a_{i-1} \quad (i = 1, 2, \ldots)$. Thus $T^n a_n = 0$ and $a_n \in N(T^n) \subset N^\infty(T)$, so that

$$x = f(\lambda) = \sum_{i=0}^\infty a_i \lambda^i \in N^\infty(T).$$

Hence $\bigvee_{|\lambda|<\gamma(T)} N(T-\lambda) \subset N^\infty(T)$.

Conversely, let $0 < r \leq \gamma(T)$ and $x \in N(T^n)$, i.e. $T^n x = 0$. Set $a_0 = T^{n-1}x$, $a_1 = T^{n-2}x, \ldots, a_{n-1} = x$. As $x \in N(T^n) \subset R^\infty(T)$, we can find $a_n \in X$ such that $Ta_n = x = a_{n-1}$ and $\|a_n\| \leq 2r^{-1}\|a_{n-1}\|$. By Lemma 1.4, $a_n \in R^\infty(T)$, so that we can inductively construct elements $a_i \quad (i = n+1, n+2, \ldots)$, such that $Ta_i = a_{i-1}$ and $\|a_i\| \leq 2r^{-1}\|a_{i-1}\| \quad (i = n, n+1, \ldots)$. Set $f(z) = \sum_{i=0}^\infty a_i z^i$. Clearly this series converges for $|z| < r/2$ and $(T-z)f(z) = 0$, i.e. $f(z) \in N(T-z) \quad (|z| < r/2)$. Further

$$x = a_{n-1} = \frac{1}{2\pi i} \int_{|z|=r/4} \frac{f(z)}{z^n} dz \in \bigvee_{|z|<r} N(T-z).$$

2) Let $0 < r \leq \gamma(T)$ and $x \in \bigcap_{|z|<r} R(T-z)$.

By [19] there exists an analytic function $f(z) = \sum_{i=0}^\infty a_i z^i$ such that $(T-z)f(z) = x$ $(|z| < r)$. Hence $Ta_0 = x$ and $Ta_i = a_{i-1} \quad (i = 1, 2, \ldots)$, so that $x \in R^\infty(T)$.

Conversely, let $x \in R^\infty(T)$ and $|\lambda| < \gamma(T)$. Choose r, $|\lambda| < r < \gamma(T)$. Similarly as in 1) we can find points $a_i \in X$ such that $a_0 = x$, $Ta_i = a_{i-1}$ and $\|a_i\| \le r^{-1}\|a_{i-1}\|$ for $i = 1, 2, \dots$. Set $f(z) = \sum_{i=1}^\infty a_i z^{i-1}$. Then $f(z)$ is defined and

$$(T - z)f(z) = x \qquad \text{for} \qquad |z| < r.$$

Thus $x \in R(T - \lambda)$ and

$$R^\infty(T) \subset \bigcap_{|z|<\gamma(T)} R(T - z)$$

We shall need the following lemma (for better use we state it in a little bit more general form):

Lemma 1.7. Let $T \in B(X)$ be an operator with a closed range. Suppose that, for $k = 1, 2, \dots$, there exist a finite dimensional subspace $F_k \subset N(T)$ such that $N(T) \subset \overline{R(T^k)} + F_k$. Then $R(T^k)$ is closed for each k.

In particular, if $R(T)$ is closed and $N(T) \subset \bigcap_{k=0}^\infty \overline{R(T^k)}$ then T is s-regular.

Proof. We prove by induction on k that $R(T^k)$ is closed.

Suppose that $k \ge 1$ and $\overline{R(T^k)} = R(T^k)$. Let $u \in \overline{R(T^{k+1})}$. By the induction assumption $u \in R(T^k)$, i.e. $u = T^k v$ for some $v \in X$. Further there are elements $v_j \in X \quad (j = 1, 2, \dots)$ such that $T^{k+1} v_j \to u \qquad (j \to \infty)$. Thus $T(T^k v_j - T^{k-1}v) \to 0$. Consider the operator $\tilde{T} : X/N(T) \to R(T)$ induced by T. Clearly $\tilde{T}$ is bounded below and $\tilde{T}(T^k v_j - T^{k-1}v + N(T)) \to 0$, so that $T^k v_j - T^{k-1}v + N(T) \to 0 \qquad (j \to \infty)$ in the quotient space $X/N(T)$. Thus there exist vectors $k_j \in N(T)$ such that $T^k v_j + k_j \to T^{k-1}v$. Since $k_j \in N(T) \subset R(T^k) + F_k$ and $R(T^k) + F_k$ is closed, we have $T^{k-1}v = T^k a + f$ for some $a \in X$ and $f \in F_k \subset N(T)$. Hence $u = T^k v = T^{k+1}a \in R(T^{k+1})$ and $R(T^{k+1})$ is closed.

The following theorem gives another characterization of s-regular operators (cf. [3], Lemma 1.4 and [15], Theorem 2.1).

Theorem 1.8. Let $T \in B(X)$ be an operator with closed range. The following conditions are equivalent:
1) T is s-regular,
2) $N(T) \subset \bigvee_{z \ne 0} N(T - z)$,
3) $R(T) \supset \bigcap_{z \ne 0} \overline{R(T - z)}$.

Proof. Implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ follow from the previous theorem (note that $R(T - z)$ is closed for $|z| < \gamma(T)$ by [16], Theorem 2.10).

$2 \Rightarrow 1$. Let $\lambda \ne 0$ and $x \in N(T - \lambda)$. Then $Tx = \lambda x$ and $x = \frac{T^n x}{\lambda^n} \in R(T^n)$, so that $x \in R^\infty(T)$. Thus $\bigvee_{\lambda \ne 0} N(T - \lambda) \subset \overline{R^\infty(T)}$, so that $N(T) \subset \overline{R^\infty(T)} \subset \bigcap_{n=0}^\infty \overline{R(T^n)}$ and T is s-regular by the previous lemma.

$3 \Rightarrow 1$. Let $x \in N(T^n)$ and $\lambda \ne 0$. Then

$$(T - \lambda)(T^{n-1} + \lambda T^{n-2} + \ldots + \lambda^{n-1})x = T^n x - \lambda^n x = -\lambda^n x,$$

so that $x \in R(T-\lambda)$. Thus $N(T^n) \subset R(T-\lambda)$. Hence $N^\infty(T) \subset \bigcap_{z\neq 0} \overline{R(T-z)} \subset R(T)$ and T is s-regular.

Definition 1.9. Let $T \in B(X)$. Denote by $\sigma_\gamma(T) = \{\lambda \in \mathbf{C}, T-\lambda \text{ is not s} - \text{regular}\}$.

For properties of $\sigma_\gamma(T)$ see [3] and [15]. The spectrum $\sigma_\gamma(T)$ is always a non-empty compact subset of $\mathbf{C}$ and

$$\partial\sigma(T) \subset \sigma_\gamma(T) \subset \sigma(T).$$

More precisely, $\sigma_\gamma(T) \subset \sigma_\pi(T) \cap \sigma_\delta(T)$, where $\sigma_\pi(T)$ is the approximate point spectrum of T,

$$\sigma_\pi(T) = \big\{\lambda, \inf\{\|(T-\lambda)x\|, x \in X, \|x\| = 1\} = 0\big\}$$

and $\sigma_\delta(T) = \{\lambda, (T-\lambda)X \neq X\}$ is the defect spectrum of T.

The set $\{\lambda \in \sigma_\gamma(T), R(T-\lambda) \text{ is closed}\}$ is at most countable and

$$\sigma_\gamma(T) = \{\lambda, \lim_{z\to\lambda} \gamma(T-z) = 0\}$$

(this limit always exists).

Further $\sigma_\gamma(f(T)) = f(\sigma_\gamma(T))$ for every function f analytic in a neighbourhood of $\sigma(T)$ (in particular for every polynomial).

II. Generalized spectra

The axiomatic theory of spectrum was introduced by Żelazko [20]. A generalized spectrum in a Banach algebra A is a set-valued function $\tilde\sigma$ which assigns to every n-tuple $a_1, \dots, a_n$ of commuting elements of A a non-empty compact subset of $\mathbf{C}^n$ such that

1) $\tilde\sigma(a_1, \dots, a_n) \subset \prod_{i=1}^n \sigma(a_i)$,
2) $\tilde\sigma(p(a_1, \dots, a_n)) = p(\tilde\sigma(a_1, \dots, a_n))$ for every m-tuple $p = (p_1, \dots, p_m)$ of polynomials in n variables.

Sometimes, a generalized spectrum is defined first only for single elements and one is looking for its extension for n-tuples of commuting elements, see e.g. [6]. We show that σ_γ can not be extended to a generalized spectrum. We start with the following simple criterion:

Theorem 2.1. Let $\tilde\sigma$ be a generalized spectrum defined in a Banach algebra A, let $a, b \in A$ and $ab = ba$. Then $0 \in \tilde\sigma(ab)$ if and only if either $0 \in \tilde\sigma(a)$ or $0 \in \tilde\sigma(b)$.

Proof. If $0 \in \tilde\sigma(a)$ then there exists $\lambda \in \mathbf{C}$ such that $(0, \lambda) \in \tilde\sigma(a, b)$. Then $0 = 0 \cdot \lambda \in \tilde\sigma(ab)$. Similarly $0 \in \tilde\sigma(b)$ implies $0 \in \tilde\sigma(ab)$.

Conversely, let $0 \notin \tilde\sigma(a)$ and $0 \notin \tilde\sigma(b)$. Then

$$\begin{aligned}\tilde\sigma(ab) &= \{\lambda\mu, (\lambda, \mu) \in \tilde\sigma(a, b)\} \subset \{\lambda\mu, \lambda \in \tilde\sigma(a), \mu \in \tilde\sigma(b)\} \\ &\subset \{\lambda\mu, \lambda \neq 0, \mu \neq 0\} = \mathbf{C} - \{0\},\end{aligned}$$

i.e. $0 \notin \tilde\sigma(ab)$.

Example 2.2. We construct two commuting s-regular opertors such that their product is not s-regular.

Let H be the Hilbert space with an orthonormal basis $\{e_{i,j}\}$ where i and j are integers such that $ij \le 0$. Define opertors T and $S \in B(H)$ by

$$Te_{i,j} = \begin{cases} 0 & \text{if } i = 0, j > 0, \\ e_{i+1,j} & \text{otherwise} \end{cases}$$

and

$$Se_{i,j} = \begin{cases} 0 & \text{if } j = 0, i > 0, \\ e_{i,j+1} & \text{otherwise.} \end{cases}$$

Then

$$TSe_{ij} = STe_{ij} = \begin{cases} 0 & \text{if } i = 0, j \ge 0 \quad \text{or} \quad j = 0, i \ge 0, \\ e_{i+1,j+1} & \text{otherwise,} \end{cases}$$

so that T and S commute.

Further $N(T) = \bigvee\{e_{0,j}, j > 0\} \subset R^\infty(T)$, $N(S) = \bigvee\{e_{i,0}, i > 0\} \subset R^\infty(S)$ and both $R(T)$ and $R(S)$ are closed. Thus T and S are s-regular.

On the other hand $TSe_{0,0} = 0$, i.e. $e_{0,0} \in N(TS)$ and $e_{0,0} \notin R(TS)$, so that TS is not s-regular.

Corollary 2.3. There exists no generalized spectrum $\tilde{\sigma}$ such that $\tilde{\sigma}(T) = \sigma_\gamma(T)$ for every $T \in B(X)$.

Remark 2.4. Note that one implication in Theorem 2.1 is true for σ_γ:
if $TS = ST$ and either $0 \in \sigma_\gamma(T)$ or $0 \in \sigma_\gamma(S)$ then $0 \in \sigma_\gamma(TS)$ (see [15], Lemma 4.15).

Another drawback of the spectrum σ_γ is that it is not upper semicontinuous. For this it is sufficient to show that the set of all s-regular operators is not open.

Example 2.5. Let H be the Hilbert space with an orthonormal basis

$$\{e_{i,j}, i, j \text{ integers}, i \ge 1\}.$$

Let $T \in B(H)$ be defined by

$$Te_{i,j} = \begin{cases} e_{i,j+1} & \text{if } j \ne 0, \\ 0 & \text{if } j = 0. \end{cases}$$

Clearly $N(T) = \vee\{e_{i,0}, i \ge 1\} \subset R^\infty(T)$ and $R(T)$ is closed, so that T is s-regular.

Let $\varepsilon > 0$. Define $S \in B(H)$ by

$$Se_{i,j} = \begin{cases} \frac{\varepsilon}{i} e_{i,1} & \text{if } j = 0, \\ 0 & \text{if } j \ne 0. \end{cases}$$

Clearly $\|S\| = \varepsilon$ and S is an infinite dimensional compact operator so that $R(S)$ is not closed. Denote $M = \vee\{e_{i,1}, i \ge 1\}$. We have $R(T) \perp M$ and $R(S) \subset M$, so that

$(T+S)x \in M$ implies $x \in N(T)$ and $(T+S)x = Sx$. Thus $R(T+S) \cap M = SN(T) = R(S)$, so that $R(T+S)$ is not closed. Therefore $T+S$ is not s-regular.

III. Essential case

In this section we admit finite dimensional jumps in $N(T-z)$ or $R(T-z)$.

If M_1 and M_2 are subspaces of X then we shall write shorty $M_1 \subset_e M_2$ if there exists a finite dimensional subspace $F \subset X$ such that $M_1 \subset M_2 + F$. In this case we may assume that $F \subset M_1$. Clearly $M_1 \subset_e M_2$ if and only if $\dim\bigl(M_1|(M_1 \cap M_2)\bigr) < \infty$.

Theorem 3.1. Let $T \in B(X)$ be an operator with closed range. Then the following conditions are equivalent:

1) $N(T) \subset_e R^\infty(T)$,
2) $N^\infty(T) \subset_e R(T)$,
3) $N^\infty(T) \subset_e R^\infty(T)$,
4) there exists a decomposition $X = X_1 \oplus X_2$ such that $\dim X_1 < \infty$, $TX_1 \subset X_1$, $TX_2 \subset X_2$, $T|X_1$ is nilpotent and $T|X_2$ is an s-regular operator,
5) $N(T) \subset_e \bigvee_{z \neq 0} N(T-z)$,
6) $R(T) \supset_e \bigcap_{z \neq 0} \overline{R(T-z)}$,
7) $\dim\bigl(N(T)|\tilde{N}(T)\bigr) < \infty$, where $\tilde{N}(T)$ is the set of all $x \in X$ such that there are complex numbers $\lambda_i \quad (i = 1, 2, \dots)$ tending to 0 and elements $x_i \in N(T - \lambda_i)$ such that $x = \lim_{i\to\infty} x_i \quad$ (clearly $\tilde{N}(T) \subset N(T)$),
8) $\dim(\tilde{R}(T)|R(T)) < \infty$ where $\tilde{R}(T)$ is the set of all $x \in X$ such that $x = \lim_{i\to\infty} x_i$ for some $x_i \in R(T-\lambda_i)$ and some $\lambda_i \to 0$. (Clearly $R(T) \subset \tilde{R}(T)$).

Proof. Implications $4 \Rightarrow 3$, $3 \Rightarrow 1$ and $3 \Rightarrow 2$ are clear.

$1 \Rightarrow 4$ and $2 \Rightarrow 4$. We prove these two implications simultanously. The proof will be done in several steps.

a) Either 1) or 2) implies $N(T^n) \subset_e R(T^k)$ for every n, k, i.e. there are finite dimensional subspaces $F_{n,k} \subset N(T^n)$ such that

$$N(T^n) \subset R(T^k) + F_{n,k}. \tag{$*$}$$

Suppose first $N(T) \subset_e R^\infty(T)$. We prove $(*)$ by induction on n.

The statement is clear for $n = 1$.

Suppose that we have found subspaces $F_{m,k} \subset N(T^m)$ for every $m \le n-1$ and every k such that $(*)$ holds. Choose a subspace $F'_{n,k} \subset X$ such that $TF'_{n,k} = F_{n-1,k+1} \cap R(T)$ and $\dim F'_{n,k} = \dim(F_{n-1,k+1} \cap R(T)) \le \dim F_{n-1,k+1} < \infty$.

Then

$$\begin{aligned} N(T^n) &= T^{-1}N(T^{n-1}) \subset T^{-1}\bigl(R(T^{k+1}) + F_{n-1,k+1}\bigr) \\ &\subset \bigl(R(T^k) + N(T)\bigr) + \bigl(F'_{n,k} + N(T)\bigr) \subset R(T^k) + F'_{n,k} + R(T^k) + F_{1,k} = R(T^k) + F_{n,k}, \end{aligned}$$

where $F_{n,k} = F'_{n,k} + F_{1,k} \subset N(T^n)$.

We prove that 2) implies $(*)$. Suppose $N^\infty(T) \subset_e R(T)$. We prove $(*)$ by induction on k. The statement is clear for $k=1$. Suppose $(*)$ is true for every n and every $l \le k-1$. Then $N(T^{n+1}) \subset R(T^{k-1}) + F_{n+1,k-1}$, so that

$$TN(T^{n+1}) \subset R(T^k) + TF_{n+1,k-1}.$$

Further $TN(T^{n+1}) = N(T^n) \cap R(T)$ and $N(T^n) \subset R(T) + F_{n,1}$ where $F_{n,1} \subset N(T^n)$, so that

$$\begin{aligned} N(T^n) &\subset \big(R(T) \cap N(T^n)\big) + F_{n,1} = TN(T^{n+1}) + F_{n,1} \\ &\subset R(T^k) + TF_{n+1,k-1} + F_{n,1} = R(T^k) + F_{n,k}, \end{aligned}$$

where $F_{n,k} = TF_{n+1,k} + F_{n,1} \subset N(T^n)$.

b) Condition $(*)$ implies by Lemma 1.7 that $R(T^k)$ is closed for each k.

c) We construct now the decomposition $X = X_1 \oplus X_2$. Suppose that T satisfies $(*)$.

If $N(T) \subset R^\infty(T)$ then T is s-regular and we can take $X_1 = \{0\}, X_2 = X$.

Therefore we may assume that $N(T) \not\subset R(T^k)$ for some k and we take the smallest k with this property, i.e. $N(T) \subset R(T^{k-1})$. Find a subspace L_1 such that

$$N(T) = L_1 \oplus \big(N(T) \cap R(T^k)\big).$$

Clearly $1 \le \dim L_1 = r < \infty$.

As $L_1 \subset N(T) \subset R(T^{k-1})$, we can find a subspace L_k such that $\dim L_k = r$ and $T^{k-1}L_k = L_1$. Set $L_i = T^{k-i}L_k \quad (i = 1, \dots, k)$. Clearly $L_i \subset R(T^{k-i})$ and $L_i \cap R(T^{k-i+1}) = \{0\}$ for every i. Therefore subspaces L_k, L_{k-1}, …, L_1 and $R(T^k)$ are linearly independent in the following sense: if $l_i \in L_i$ $(1 \le i \le k)$, $x \in R(T^k)$ and $x + l_1 + \cdots + l_k = 0$, then $x = l_1 = \ldots = l_k = 0$.

Let $x_1, \dots, x_r$ be a basis in L_1. As $x_1, \dots, x_r$ are linearly independent modulo $R(T^k) + L_2 + \ldots + L_k$, we can find linear functionals $f_1, \dots, f_r \in (R(T^k) + L_2 + \ldots L_k)^\perp$ such that $\langle x_i, f_j \rangle = \delta_{ij} \quad (1 \le i, j \le r)$. Set

$$Y_1 = \bigvee_{i=1}^{k} L_i \qquad \text{and} \qquad Y_2 = \bigcap_{j=0}^{k-1} \bigcap_{i=1}^{r} \ker(T^{*j} f_i).$$

Clearly $\dim Y_1 < \infty$, $TY_1 \subset Y_1$ and $(T|Y_1)^k = 0$. Further $TY_2 \subset Y_2$. Indeed, if $x \in Y_2$ then

$$\langle Tx, T^{*j} f_i \rangle = \langle x, T^{*(j+1)} f_i \rangle = 0 \qquad \text{for} \qquad 0 \le j \le k-2$$

and $\langle Tx, T^{*(k-1)} f_i \rangle = \langle T^k x, f_i \rangle = 0$.

Find $y_1, \dots, y_r \in L_k$ such that $x_i = T^{k-1} y_i \ \ (1 \le i \le r)$. Then

$$\{T^j y_i, 0 \le j \le k-1, 1 \le i \le r\}$$

form a basis of Y_1 and

$$\{T^j y_i, 0 \le j \le k-1, 1 \le i \le r\} \qquad \text{and} \qquad \{T^{*j} f_i, 0 \le j \le k-1, 1 \le i \le r\}$$

form a biorthogonal system. Thus it is easy to show that $X = Y_1 \oplus Y_2$.

Denote by $T_1 = T|Y_1$ and $T_2 = T|Y_2$. We have $N(T) = N(T_1) \oplus N(T_2) = L_1 \oplus N(T_2)$ and $R^\infty(T) = R^\infty(T_1) \oplus R^\infty(T_2) = R^\infty(T_2)$.

If T satisfies 1), i.e. $\dim\bigl(N(T)|(N(T) \cap R^\infty(T))\bigr) < \infty$ then

$$\begin{aligned}\dim\bigl(N(T_2)|(N(T_2) \cap R^\infty(T_2))\bigr) &= \dim\bigl(N(T)|(N(T) \cap R^\infty(T))\bigr) - r \\ &< \dim\bigl(N(T)|(N(T) \cap R^\infty(T))\bigr) < \infty.\end{aligned}$$

and we can repeat the same construction for T_2. After a finite number of steps we obtain a decomposition $X = X_1 \oplus X_2$ such that $\dim X_1 < \infty$, $TX_1 \subset X_1$, $TX_2 \subset X_2$, $T|X_1$ is nilpotent and $N(T|X_2) \subset R^\infty(T)$, i.e. $T|X_2$ is s-regular.

Similarly, if T satisfies 2), i.e.

$$\dim\bigl(N^\infty(T)|(N^\infty(T) \cap R(T))\bigr) = a < \infty,$$

then

$$\begin{aligned}\dim(N^\infty(T_2)|(N^\infty(T_2) \cap R(T_2))) &= a - \dim(N^\infty(T_1)|(N^\infty(T_1) \cap R(T_1))) \\ &= a - \dim(Y_1| \bigvee_{i=1}^{k-1} L_i) = a - r < a,\end{aligned}$$

so that after a finite number of steps we obtain the required decomposition $X = X_1 \oplus X_2$.

$1 \Rightarrow 7$: Since $\tilde{N}(T|X_2) = N(T|X_2)$ by Lemma 1.5, we have $\dim(N(T)|\tilde{N}(T)) = \dim(N(T|X_1)|\tilde{N}(T|X_1) = \dim N(T|X_1) < \infty$.

$7 \Rightarrow 5$: Clearly $\tilde{N}(T) \subset \bigvee_{z\neq 0} N(T-z)$.

$5 \Rightarrow 1$: It is easy to see that $N(T-z) \subset R^\infty(T)$ for $z \neq 0$. Thus

$$N(T) \subset_e \bigvee_{z\neq 0} N(T-z) \subset \overline{R^\infty(T)}.$$

By Lemma 1.7 we have $\overline{R(T^k)} = R(T^k)$ for each k, so that $N(T) \subset_e R^\infty(T)$.

$4 \Rightarrow 8$: By condition 2 of Theorem 1.2 $\tilde{R}(T|X_2) = R(T|X_2)$, so that

$$\dim(\tilde{R}(T)|R(T)) \le \dim X_1 < \infty.$$

$8 \Rightarrow 6$: Clearly $\bigcap_{z\neq 0} \overline{R(T-z)} \subset \tilde{R}(T)$.

$6 \Rightarrow 2$: This follows from the inclusion $N^\infty(T) \subset \bigcap_{z\neq 0} \overline{R(T-z)}$ (see the proof of Theorem 1.7).

Definition 3.2. We say that an operator $T \in B(X)$ is essentially s-regular if $R(T)$ is closed and T satisfies any of the equivalent conditions of Theorem 3.1.

Remark 3.3. Condition 4 of Theorem 3.1 is the Kato decomposition which was proved in [9] for semi-Fredholm operators. Clearly, essentially s-regular operators are a generalization of semi-Fredholm operators.

This notion is closely related to quasi-Fredholm operators, see [11], [12].

Corollary 3.4. (cf. [18]). Let $T \in B(X)$.

1) If T is essentially s-regular, then T^n is essentially s-regular for every n.

2) T is essentially s-regular if and only if $T^* \in B(X^*)$ is essentially s-regular.

Proof. 1) Let $X = X_1 \oplus X_2$ be the Kato decomposition for T (see condition 4 of Theorem 3.1). Clearly the same decomposition satisfies all conditions for T^n.

2) We have $X^* = X_2^\perp \oplus X_1^\perp$ where $\dim X_2^\perp = \operatorname{codim} X_2 = \dim X_1 < \infty$, $T^* X_2^\perp \subset X_2^\perp$, $T^* X_1^\perp \subset X_1^\perp$, $T^*|X_2^\perp$ is a nilpotent operator and $T^*|X_1^\perp$ is isometrically isomorphic to $(T|X_2)^*$, so that $T^*|X_1^\perp$ is s-regular and T^* is essentially s-regular.

Conversely, if T^* is essentially s-regular, then $R(T)$ and $R(T^n)$ are closed for every n and $T^{**} \in B(X^{**})$ is essentially s-regular, so that $N(T^{**}) \subset_e R^\infty(T^{**})$. Further $N(T) = N(T^{**}) \cap X$ and $R(T^n) = R(T^{**n}) \cap X$ for every n, so that $R^\infty(T) = R^\infty(T^{**}) \cap X$ and $N(T) \subset_e R^\infty(T)$.

Theorem 3.5. Let $A, B \in B(X)$, $AB = BA$. If AB is essentially s-regular then A and B are essentially s-regular.

Proof. We have $N(A) \subset N(AB) \subset_e R^\infty(AB) \subset R^\infty(A)$, so that it is sufficient to prove that $R(A)$ is closed.

There exists a finite-dimensional subspace $F \subset X$ such that $N(AB) \subset R(AB) + F$. We prove that $R(A) + F$ is closed. Let $v_j \in X$, $f_j \in F$ and $Av_j + f_j \to u$. Then $BAv_j + Bf_j \to Bu$ and $Bu \in R(AB) + BF$ since $R(AB) + BF$ is closed. Thus $Bu = ABv + Bf$ for some $v \in X$ and $f \in F$, i.e.

$$Av + f - u \in N(B) \subset N(AB) \subset R(AB) + F \subset R(A) + F.$$

Hence $u \in R(A) + F$ and $R(A) + F$ is closed.

The closeness of $R(A)$ follows from the following lemma, which is a particular case of lemma of Neubauer, see [11], Proposition 2.1.1.

Lemma 3.6. Let $T \in B(X)$, let $F \subset X$ be a finite-dimensional subspace. Suppose that $R(T) + F$ is closed. Then $R(T)$ is closed.

Proof. Without loss of generality we can assume $R(T) \cap F = \{0\}$. Let $S : X|\operatorname{Ker} T \oplus F \to X$ be defined by $S\big((x + \operatorname{Ker} T) \oplus f\big) = Tx + f \in R(T) + F$. Then S is a bounded injective operator onto $R(T) + F$. Hence S is bounded below and $R(T) = S(X|\operatorname{Ker} T \oplus \{0\})$ is closed.

Definition 3.7. Let $T \in B(X)$. Denote by

$$\sigma_{\gamma e}(T) = \{\lambda \in \mathbf{C}, T - \lambda \text{ is not essentially s} - \text{regular}\}.$$

Theorem 3.8. (cf. [18]). Let $\dim X = \infty$ and $T \in B(X)$. Then

1) $\sigma_{\gamma e}(T) \subset \sigma_\gamma(T)$ and $\sigma_\gamma(T) - \sigma_{\gamma e}(T)$ consists of at most countably many isolated points,
2) $\sigma_{\gamma e}(T)$ is a non-empty compact set,
3) $\partial\sigma_e(T) \subset \sigma_{\gamma e}(T) \subset \sigma_e(T)$, where $\sigma_e(T)$ denotes the essential spectrum of T. More precisely, $\sigma_{\gamma e}(T) \subset \sigma_{\pi e}(T) \cap \sigma_{\delta e}(T)$, where $\sigma_{\pi e}(T)$ is the essential approximate point spectrum of T,

$$\begin{aligned}\sigma_{\pi e}(T) &= \{\lambda, T - \lambda \text{ is not upper semi} - \text{Fredholm}\} \\ &= \{\lambda, R(T-\lambda) \text{ is not closed}\} \cup \{\lambda, \dim N(T-\lambda) = \infty\}\end{aligned}$$

and

$$\sigma_{\delta e}(T) = \{\lambda, T - \lambda \text{ is not lower semi} - \text{Fredholm}\} = \{\lambda, \operatorname{codim} R(T-\lambda) = \infty\}.$$

Proof. 1) Let $\lambda \in \sigma_\gamma(T) - \sigma_{\gamma e}(T)$. Then $T - \lambda$ is essentially s-regular, so that there exists a decomposition $X = X_1 \oplus X_2$ with $TX_1 \subset X_1$, $TX_2 \subset X_2$, $\dim X_1 < \infty$, $(T-\lambda)|X_1$ nilpotent and $(T-\lambda)|X_2$ s-regular. Then $(T-z)|X_2$ is s-regular in a certain neighbourhood U of λ and $(T-z)|X_1$ is s-regular (even invertible) for every $z \neq \lambda$. It is easy to see that $T - z$ is s-regular for $z \in U - \{\lambda\}$, i.e. $U \cap \sigma_\gamma(T) = \{\lambda\}$. Clearly $\sigma_\gamma(T) - \sigma_{\gamma e}(T)$ is at most countable.

2) If $\lambda \notin \sigma_{\gamma e}(T)$ then either $\lambda \notin \sigma_\gamma(T)$ or $\lambda \in \sigma_\gamma(T) - \sigma_{\gamma e}(T)$. In both cases $U \cap \sigma_{\gamma e}(T) = \emptyset$ for some neighbourhood U od λ. Hence $\sigma_{\gamma e}(T)$ is closed.

The non-emptiness of $\sigma_{\gamma e}(T)$ follows from the inclusion $\partial\sigma_e(T) \subset \sigma_{\gamma e}(T)$ which will be proved next.

3) Suppose $\lambda \in \partial\sigma_e(T)$ and $\lambda \notin \sigma_{\gamma e}(T)$. Then $T - \lambda$ is essentially s-regular so that $R(T-\lambda)$ is closed and there exists a decomposition $X = X_1 \oplus X_2$ such that $\dim X_1 < \infty$, $TX_1 \subset X_1$, $TX_2 \subset X_2$, $(T-\lambda)|X_1$ is nilpotent and $(T-\lambda)|X_2$ is s-regular. Choose a sequence $\lambda_n \to \lambda$ such that $\lambda_n \notin \sigma_e(T)$, i.e. $T - \lambda_n$ is Fredholm. We have

$$\dim N((T-\lambda_n)|X_2) \le \dim N(T-\lambda_n) < \infty$$

and, from the regularity of $T|X_2$ and property 6 of Theorem 1.1 we conclude that

$$\dim N((T-\lambda)|X_2) < \infty$$

and also $\dim N(T-\lambda) < \infty$.

Similarly we can prove $\operatorname{codim} R(T-\lambda) < \infty$, so that $T - \lambda$ is a Fredholm operator and $\lambda \notin \sigma_e(T)$, a contradiction.

Thus $\partial\sigma_e(T) \subset \sigma_{\gamma e}(T)$.

If $\lambda \in \sigma_{\gamma e}(T)$, then $T - \lambda$ is not semi-Fredholm by Remark 3.3, so that $\lambda \in \sigma_{\pi e}(T) \cap \sigma_{\delta e}(T)$.

Remark 3.9. In fact we have proved $\partial\sigma_e(T) \subset \sigma_{\pi e}(T)$ and $\partial\sigma_e(T) \subset \sigma_{\delta e}(T)$, which is not so trivial as in the non-essential case (see [8], cf. also [1]).

Theorem 3.10. Let $T \in B(X)$. Then $\sigma_{\gamma e}(f(T)) = f(\sigma_{\gamma e}(T))$ for every function f analytic in a neighbourhood of $\sigma(T)$.

Proof. It is sufficient to prove that $0 \notin \sigma_{\gamma e}(f(T))$ if and only if $T-\lambda$ is essentially s-regular whenever $f(\lambda)=0$.

Since f has only a finite number of zeros $\lambda_1,\dots,\lambda_n$ in $\sigma(T)$ we can write $f(z) = (z-\lambda_1)^{m_1}\cdots(z-\lambda_n)^{m_n}h(z)$ where h is analytic in a neighbourhood of $\sigma(T)$ and $f(z)\neq 0$ for $z\in\sigma(T)$.

We have $f(T) = (T-\lambda_1)^{m_1}\cdots(T-\lambda_n)^{m_n}h(T)$. If $f(T)$ is essentially s-regular, then $T-\lambda_1,\dots,T-\lambda_n$ are essentially s-regular by Theorem 3.5.

Conversely, suppose that $T-\lambda_1,\dots,T-\lambda_n$ are essentially s-regular. Denote by $q(z) = (z-\lambda_1)\cdots(z-\lambda_n)$ and $p(z) = (z-\lambda_1)^{m_1}\cdots(z-\lambda_m)^{m_n}$. Then

$$N(q(T)) = \bigvee_{i=1}^{n} N(T-\lambda_i)$$

and

$$R(q(T)^m) = \bigcap_{i=1}^{n} R((T-\lambda_i)^m)$$

for every m (see [15], Lemmas 5.2 and 5.3). Thus $R(q(T))$ is closed. Further $N(T-\lambda_i)\subset R^\infty(T-\lambda_j)$ for $j\neq i$ and $N(T-\lambda_i))\subset R^\infty(T-\lambda_i)+F_i$ for some finite-dimensional subspace $F_i\subset X$. Thus

$$N(T-\lambda_i)\subset \bigcap_{j=1}^{n} R^\infty(T-\lambda_j)+F_i$$

and

$$N(q(T))\subset \bigcap_{i=1}^{n} R^\infty(T-\lambda_i)+F_1+\cdots+F_n = R^\infty(q(T))+F_1+\cdots+F_n.$$

Hence $q(T)$ is essentially s-regular. If $m=\max\{m_i, 1\le i\le n\}$, then $q(T)^m$ is essentially s-regular by Corollary 3.4 and $p(T)$ is essentially s-regular by Theorem 3.5. Further $h(T)$ is an invertible operator commuting with $p(T)$. Thus $N(f(T)) = N(p(T))$ and $R(f(T)^n) = R(p(T)^n)$ for every n, so that $R(f(T))$ is closed and

$$N(f(T)) = N(p(T)) \subset_e R^\infty(p(T)) = R^\infty(f(T)).$$

Hence $f(T)$ is essentially s-regular.

Problem 3.11. Example 2.5 shows that $\sigma_{\gamma e}(T)$ is not stable under compact perturbations. We do not know if it is stable under finite-dimensional perturbations. Equivalently, taking into account the Kato decomposition, we can reformulate this question as follows:

Let T be s-regular and A a finite-dimensional operator. Is then $T+A$ essentially s-regular?

IV. Generalized inverses

Let $T \in B(X)$. We say that $S \in B(X)$ is a generalized inverse of T if $TST = T$ and $STS = S$. In this case TS is a bounded projection onto $R(T)$ and ST is a bounded projection with $N(ST) = N(T)$. Thus it is easy to see that T has a generalized inverse if and only if $R(T)$ is closed and both $N(T)$ and $R(T)$ are ranges of bounded projections.

An operator T is called regular if T is s-regular and has a generalized inverse.

Let T be an operator in a Hilbert space H. Then there is an analytic generalized inverse of $T - z$ defined on the open set $G = \mathbf{C} - \sigma_\gamma(T)$ (see [3], Theorem 2.5). More precisely, there exists an analytic operator-valued function $S : G \to B(X)$ such that $(T - z)S(z)(T - z) = T - z$ and $S(z)(T - z)S(z) = S(z)$ for all $z \in G$. One can see easily that $\mathbf{C} - \sigma_\gamma(T)$ is the largest open set with this property.

If T is an operator in a Banach space X then another necessary condition for existence of an analytic generalized inverse of $T - z$ is that $R(T - z)$ and $N(T - z)$ are ranges of bounded projections. We show that this is already a sufficient condition.

We start with a local version of this result, which was essentially proved in [14], Theorem 2.6, see also [7], Theorem 9.

Theorem 4.1. Let $T \in B(X)$ be a regular operator. Then there exists an open neighbourhood U of 0 and an analytic function $S : U \to B(X)$ such that $(T-z)S(z)(T-z) = T - z$ and $S(z)(T - z)S(z) = S(z)$ for all $z \in U$.

Proof. Let $S \in B(X)$ be a generalized inverse of T, i.e. $TST = T$ and $STS = S$. Set $U = \{z \in \mathbf{C}, |z| < \|S\|^{-1}\}$. For $z \in U$ define $P(z) = \sum_{i=0}^{\infty} S^i(I - ST)z^i$. Clearly the sum converges for $z \in U$. We have $(I - ST)S = 0 = T(I - ST)$, therefore $P(z)^2 = \sum_{i=0}^{\infty} z^i S^i(I - ST)^2 = P(z)$ and

$$(T - z)P(z) = (T - z)\sum_{i=0}^{\infty} S^i(I - ST)z^i$$

$$= T(I - ST) + \sum_{i=1}^{\infty} z^i\big[TS^i(I - ST) - S^{i-1}(I - ST)\big] = \sum_{i=1}^{\infty}(TS - I)S^{i-1}(I - ST)z^i.$$

Let $x \in X$. Then $T(I - ST)x = 0$, so that $(I - ST)x \in N(T) \subset R^\infty(T)$.

Further, if $y \in R^\infty(T)$, $y = Tz$ then $y = Tz = TSTz = TSy$ and, by Lemma 1.4, $Sy \in R^\infty(T)$. Thus $S(R^\infty(T)) \subset R^\infty(T)$.

Finally, for $u \in R(T)$, $u = Tv$ we have $(TS - I)u = (TS - I)Tv = 0$. Thus

$$(TS - I)S^{i-1}(I - ST) = 0 \qquad (i \geq 1)$$

and $(T - z)P(z) = 0$.

For $z \in G$ set $S(z) = \sum_{i=0}^{\infty} S^{i+1}z^i$. Then

$$S(z)(T - z) + P(z) = \sum_{i=0}^{\infty} S^{i+1}z^i(T - z) + \sum_{i=0}^{\infty} S^i(I - ST)z^i$$

$$= ST + I - ST + \sum_{i=1}^{\infty}\big[S^{i+1}T - S^i + S^i(I - ST)\big]z^i = I,$$

hence $S(z)(T-z)=I-P(z)$. We have $(T-z)S(z)(T-z)=(T-z)(I-P(z))=T-z$ and $S(z)(T-z)S(z)=(I-P(z))S(z)=S(z)$ since

$$P(z)S(z)=\left(\sum_{i=0}^{\infty} z^i S^i (I-ST)\right)\left(\sum_{i=0}^{\infty} S^{i+1} z^i\right)=0.$$

Clearly $P(z)$ is a bounded projection onto $N(T-z)$.

Remark 4.2. Let $S(z)$ be the function constructed in the previous theorem and let $\lambda,\mu\in U$. Then

$$\begin{aligned} S(\lambda)-S(\mu) &= \sum_{i=0}^{\infty}(\lambda^i-\mu^i)S^{i+1}=(\lambda-\mu)\sum_{i=1}^{\infty}(\lambda^{i-1}+\lambda^{i-2}\mu+\cdots+\mu^{i-1})S^{i+1} \\ &= (\lambda-\mu)\left(\sum_{j=0}^{\infty}\lambda^j S^{j+1}\right)\left(\sum_{k=0}^{\infty}\lambda^k S^{k+1}\right)=(\lambda-\mu)S(\lambda)S(\mu). \end{aligned}$$

Thus $S(z)$ satisfies the resolvent identity and so it is not only a generalized inverse of $T-z$ but also a generalized resolvent in the sense of [4] or [5].

The next theorem shows that it is possible to find a global analytic general inverse of $T-z$. It is an open question if there always exists a global analytic general resolvent.

Theorem 4.3. Let $T\in B(X)$. Denote by $G=\{z\in\mathbf{C}, T-z \text{ is regular}\}$. Then G is an open set and there exists an analytic function $S:G\to B(X)$ such that

$$(T-z)S(z)(T-z)=T-z$$

and

$$S(z)(T-z)S(z)=S(z)\qquad (z\in G).$$

Proof. For $z\in G$ define the operator $M(z):B(X)\to B(X)$ by

$$M(z)A=(T-z)A(T-z)\qquad (A\in B(X)).$$

Clearly $M:G\to B(B(X))$ is an analytic function. Let $\lambda\in G$. By the previous theorem there exists a neighbourhood U of λ and an analytic function $S_1:U\to B(X)$ such that $(T-z)S_1(z)(T-z)=T-z$ and $S_1(z)(T-z)S_1(z)=S_1(z)\quad (z\in U)$.

Let $\mu\in U$ and $A\in B(X)$. Set $A_1=S_1(\mu)(T-\mu)A(T-\mu)S_1(\mu)$. Then

$$M(\mu)A_1=(T-\mu)S_1(\mu)(T-\mu)A(T-\mu)S_1(\mu)(T-\mu)=(T-\mu)A(T-\mu)=M(\mu)A$$

and

$$\|A_1\|\le\|S_1(\mu)\|^2\|(T-\mu)A(T-\mu)\|=\|S_1(\mu)\|^2\|M(\mu)A\|.$$

Thus $\gamma(M(\mu))\ge\|S_1(\mu)\|^{-2}$ so that $\gamma(M(z))$ is bounded from below in a certain neighbourhood of λ. Further function $z\mapsto T-z\in B(X)$ is an analytic vector-valued function and, by the definition of G, $T-z\in R(M(z))$ for every $z\in G$.

By [19], Theorem 2, there exists an analytic function $S_2 : G \to B(X)$ such that $M(z)S_2(z) = T - z$, i.e. $(T-z)S_2(z)(T-z) = T-z$ for $z \in G$. Set

$$S(z) = S_2(z)(T-z)S_2(z) \qquad (z \in G).$$

Then

$$(T-z)S(z)(T-z) = (T-z)S_2(z)(T-z)S_2(z)(T-z) = T-z$$

and

$$\begin{aligned} S(z)(T-z)S(z) &= S_2(z)(T-z)S_2(z)(T-z)S_2(z)(T-z)S_2(z) \\ &= S_2(z)(T-z)S_2(z) = S(z) \end{aligned}$$

for every $z \in G$.

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