A continuous semicharacter

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Abstract: *We exhibit an example of a continuous proper semicharacter on a Banach algebra. This gives an answer to the problem posed by Z. Słodkowski and W. Żelazko.*

A semicharacter on a Banach algebra A is a complex-valued function f defined on A such that, for every commutative subalgebra $A_0 \subset A$, the restriction $f|A_0$ is a multiplicative linear functional (=character) on A_0 (we do not assume continuity of f).

Multiplicative linear functionals play an important role in the theory of generalized spectra (see [3],[6],[2]) in commutative Banach algebras. As generalized spectra in non-commutative Banach algebras are defined only for commuting systems of elements, it is natural to replace multiplicative linear functionals in the non-commutative case by semicharacters.

However, usually it is rather difficult to find a proper semicharacter (i.e. a semicharacter which is not a character). Note that a linear semicharacter is clearly continuous and by [5] it is already multiplicative, so that it is a character. In [4] the problem was raised whether a continuous semicharacter is already a character.

The aim of this note is to give a negative answer to this question.

Theorem: *There exist a Banach algebra B and a continuous semicharacter $f : B \to \mathbb{C}$ which is not a multiplicative linear functional.*

Proof: Denote by $\mathbb{R}_+$ the set of all positive real numbers and by $D = \{z \in \mathbb{C}, |z| < 1\}$ the open unit disc in the complex plain. Let A be the disc algebra of all functions holomorphic in D and continuous in $\bar{D}$. For $a \in A$ denote $\|a\| = \max_{z \in \bar{D}} |a(z)|$. Set $B = A \times A$. We define the norm and the algebraic operations in B by

$$\begin{aligned} &\|(a,b)\| = \|a\| + \|b\|, \\ &(a,b) + (a',b') = (a+a', b+b'), \\ &\alpha(a,b) = (\alpha a, b), \\ &(a,b)\cdot(a',b') = (aa', ab') \qquad (a,b,a',b' \in A, \quad \alpha \in \mathbb{C}). \end{aligned}$$

In this way B becomes a Banach algebra.

Let $(a,b),(a',b') \in B$. Then $(a,b)\cdot(a',b') = (aa',ab')$ and $(a',b')\cdot(a,b) = (a'a,a'b)$ so that (a,b) and $(a',b') \in B$ commute if and only if $ab' = a'b$. Thus B has only few commutative subalgebras which are easy to describe.

For $n \in \mathbb{N}, \lambda = (\lambda_1,\dots,\lambda_n) \in D^n, r = (r_1,\dots,r_n) \in \mathbb{R}_+^n$ and $s > 0$ we denote

$$F_{\lambda,r,s} = \{z \in D, |z| \le 1-s, |z-\lambda_i| \ge r_i \quad (i = 1,\dots,n)\}.$$

Clearly $F_{\lambda,r,s}$ is a closed subset of D. Let $k > 0$ and $0 < s < 1/2$. Denote by $M_{k,s}$ the set of all pairs $(a,b) \in B$ for which there exist $n \in \mathbb{N}$,$\lambda = (\lambda_1,\dots,\lambda_n) \in D^n$ and $r = (r_1,\dots,r_n) \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n r_i < s$ and

$$z \in F_{\lambda,r,s} \quad \Rightarrow \quad a(z) \neq 0 \quad \text{and} \quad \left|\frac{b(z)}{a(z)}\right| < k.$$

Clearly, if $\sum_{i=1}^n r_i < s < 1/2$ then $F_{\lambda,r,s}$ is a non-empty subset of D so that $(a,b) \in M_{k,s}$ implies $a \neq 0$. On the other hand, if $a \neq 0$ then $(a,0) \in M_{k,s}$ for every $k > 0$ and $0 < s < 1/2$. Indeed, a has only a finite number of zeros $\lambda_1, \dots, \lambda_n$ in the disc $\{z \in \mathbb{C}, |z| \leq 1-s\}$ so that for any positive numbers $r_1, \dots, r_n$ with $\sum_{i=1}^n r_i < s$ we have $z \in F_{\lambda,r,s} \Rightarrow a(z) \neq 0$.

Further, $M_{k,s} \subset M_{k',s'}$ if $k < k'$ and $s < s'$.

1) *If $k > 0$ and $0 < s < 1/2$ then $M_{k,s}$ is an open subset of B.*

Proof: Let $(a,b) \in M_{k,s}$. Let $\lambda = (\lambda_1, \dots, \lambda_n) \in D^n$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ satisfy $\sum_{i=1}^n r_i < s$ and $z \in F_{\lambda,r,s} \Rightarrow a(z) \neq 0$ and $\left|\frac{b(z)}{a(z)}\right| < k$. Denote by

$$\begin{aligned} k_0 &= \max_{z \in F_{\lambda,r,s}} \left|\frac{b(z)}{a(z)}\right| < k, \\ k_1 &= \max\{\|a\|, \|b\|\} \qquad \text{and} \\ k_2 &= \min_{z \in F_{\lambda,r,s}} |a(z)| > 0. \end{aligned}$$

Set $\delta = \min\{k_2/2, (k-k_0)k_2^2/2k_1\} > 0$. Let $(a',b') \in B, \|(a,b)-(a',b')\| < \delta$, i.e. $\|a-a'\| + \|b-b'\| < \delta$. Then, for $z \in F_{\lambda,r,s}$, we have

$$|a'(z)| \geq |a(z)| - \delta \geq k_2 - \frac{k_2}{2} = \frac{k_2}{2} > 0$$

and

$$\left|\frac{b'(z)}{a'(z)}\right| \leq \left|\frac{b(z)}{a(z)}\right| + \left|\frac{b'(z)}{a'(z)} - \frac{b(z)}{a(z)}\right| \leq k_0 + \left|\frac{a(z)(b'(z)-b(z)) + b(z)(a(z)-a'(z))}{a'(z)a(z)}\right| <$$

$$< k_0 + \frac{k_1\delta}{k_2(k_2-\delta)} \leq k_0 + \frac{2k_1\delta}{k_2^2} \leq k.$$

Thus $(a',b') \in M_{k,s}$ and $M_{k,s}$ is an open subset of B.

2) *Let $(a,b) \in M_{k,s}$ and let $(a',b') \in B$ satisfy $a' \neq 0$ and $a'b = b'a$. Then $(a',b') \in M_{k,s}$.*

Proof: Let $\lambda = (\lambda_1, \dots, \lambda_n) \in D^n$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ satisfy $\sum_{i=1}^n r_i < s$ and

$$z \in F_{\lambda,r,s} \quad \Rightarrow \quad a(z) \neq 0 \quad \text{and} \quad \left|\frac{b(z)}{a(z)}\right| < k.$$

The function a' has only a finite number of zeros $\lambda'_1, \dots, \lambda'_m$ in the disc $\{z \in \mathbb{C}, |z| \leq 1-s\}$. Choose positive numbers $r'_1, \dots, r'_m$ such that $\sum_{j=1}^m r'_j < s - \sum_{i=1}^n r_i$. Consider the set

$$F = \{z \in D, |z| \leq 1-s, |z-\lambda_i| \geq r_i \quad (i=1,\dots,n), |z-\lambda'_j| \geq r'_j \quad (j=1,\dots,m)\}.$$

Then $\sum_{i=1}^n r_i + \sum_{j=1}^m r'_j < s$ and

$$z \in F \quad \Rightarrow \quad a'(z) \neq 0 \quad \text{and} \quad \left|\frac{b'(z)}{a'(z)}\right| = \left|\frac{b(z)}{a(z)}\right| < k.$$

Hence $(a', b') \in M_{k,s}$.

3) *Let k, k', s, s' be positive numbers such that $k < k'$ and $s < s' < 1/2$. Then $\overline{M_{k,s}} \cap \{(a,b) \in B, a \neq 0\} \subset M_{k',s'}$.*

Proof: Let $(a, b) \in \overline{M_{k,s}}$ and $a \neq 0$. The function a has only a finite number of zeros $\lambda'_1, \ldots, \lambda'_m$ in the disc $\{z \in \mathbb{C}, |z| \leq 1 - s'\}$. Choose positive numbers $r'_1, \ldots, r'_m$ such that $\sum_{j=1}^n r'_j < s' - s$. Consider the set

$$F_{\lambda',r',s'} = \{z \in D, |z| \leq 1 - s', |z - \lambda'_j| \geq r'_j \quad (j = 1, \ldots, m)\}.$$

Denote

$$k_1 = \max\{\|a\|, \|b\|\} \qquad \text{and}$$
$$k_2 = \min_{z \in F_{\lambda',r',s'}} |a(z)| > 0.$$

Let $\delta = \min\{k_2/2, (k' - k)k_2^2/2k_1\} > 0$. Then there exists $(a', b') \in M_{k,s}$ such that $\|(a', b') - (a, b)\| = \|a - a'\| + \|b - b'\| < \delta$. This means that there exist $n \in \mathbb{N}$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in D^n$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n r_i < s$ and

$$z \in F_{\lambda,r,s} \quad \Rightarrow \quad a'(z) \neq 0 \quad \text{and} \quad \left|\frac{b'(z)}{a'(z)}\right| < k.$$

Then for $z \in F_{\lambda,r,s} \cap F_{\lambda',r',s'}$ we have $a(z) \neq 0$ and

$$\left|\frac{b(z)}{a(z)}\right| \leq \left|\frac{b'(z)}{a'(z)}\right| + \left|\frac{b(z)}{a(z)} - \frac{b'(z)}{a'(z)}\right| < k + \left|\frac{b(z)(a'(z) - a(z)) + a(z)((b(z) - b'(z))}{a(z)a'(z)}\right| <$$

$$< k + \frac{k_1 \delta}{k_2(k_2 - \delta)} \leq k + \frac{2k_1\delta}{k_2^2} \leq k'.$$

Hence $(a, b) \in M_{k',s'}$.

Denote $B_0 = \{(a, b) \in B, a \neq 0\}$.

4) *There exists a non-constant continuous function $\varphi : B_0 \to \langle 0, 1/2 \rangle$ such that*

$$(a, b), (a', b') \in B_0, ab' = ba' \quad \Rightarrow \quad \varphi(a, b) = \varphi(a', b').$$

Proof: For $(a, b) \in B_0$ define

$$\varphi(a, b) = \begin{cases} \frac{1}{2} & \text{if } (a, b) \notin \cup_{0<s<1/2} M_{s,s} \\ \inf\{s, (a, b) \in M_{s,s}\} & \text{otherwise.} \end{cases}$$

Clearly, by 2), $\varphi(a, b) = \varphi(a', b')$ if $ab' = a'b$. The function φ is non-constant since $\varphi(1, 0) = 0$ and $\varphi(1, 1) = 1/2$. The proof of continuity of φ is standard. Let $s_0 \in (0, 1/2\rangle$. Then

$$\{(a, b) \in B_0, \varphi(a, b) < s_0\} = \bigcup_{s<s_0} M_{s,s},$$

which is an open subset of B_0. If $s_0 \in \langle 0, 1/2)$ then

$$\{(a,b) \in B_0, \varphi(a,b) \le s_0\} = \bigcap_{s>s_0} M_{s,s} = \bigcap_{s>s_0} \left(\overline{M_{s,s}} \cap B_0\right),$$

which is a closed subset of B_0. Thus φ is a continuous function.

Define a function $f : B \to \mathbb{C}$ by

$$f(a,b) = \begin{cases} 0 & \text{if } a = 0 \\ a(\varphi(a,b)) & \text{if } a \neq 0. \end{cases}$$

We show that f is a proper continuous semicharacter.

5) *Let $x = (a,b) \in B$ and $\alpha \in \mathbb{C}$. Then $f(\alpha x) = \alpha f(x)$.*
Proof: This is clear if $\alpha = 0$ or $a = 0$. If $a \neq 0$ and $\alpha \neq 0$, then $\varphi(x) = \varphi(\alpha x) = t_0$ so that $f(\alpha x) = f(\alpha a, \alpha b) = \alpha \cdot a(t_0) = \alpha f(x)$.

6) *Let $x = (a,b), x' = (a',b') \in B$ be commuting elements. Then $f(x+x') = f(x) + f(x')$ and $f(xx') = f(x) \cdot f(x')$.*
Proof: We have $ab' = a'b$. We distinguish several cases:

a) If $a = 0$ and $b = 0$, then $f(x) = 0 = f(xx')$ so that the statement is clear.
b) If $a = 0$ and $b \neq 0$, then $a' = 0$ so that $f(x) = f(x') = f(x+x') = f(xx') = 0$.
c) If $a' = 0$, then the statement can be proved analogously.
d) The remaining case is $a \neq 0, a' \neq 0$. Then

$$\varphi(a,b) = \varphi(a',b') = \varphi(aa',ab') = t_0,$$

so that

$$f(xx') = (aa')(t_0) = a(t_0)a'(t_0) = f(x) \cdot f(x').$$

Further either $a = -a'$ so that $b = -b'$ and $f(x+x') = f(x) + f(x') = 0$, or $a + a' \neq 0$ so that $\varphi(a+a', b+b') = t_0$ and

$$f(x+x') = (a+a')(t_0) = a(t_0) + a'(t_0) = f(x) + f(x').$$

Hence f is a semicharacter.

7) *f is a continuous semicharacter.*
Proof: Let $x = (0,b)$. Then $f(x) = 0$. If $x' = (a',b') \in B$ then either $a' = 0$ so that $f(x') = 0$, or $a' \neq 0$ so that $|f(x')| = |a'(\varphi(x'))| \le \|a'\|$. In both cases we have $|f(x') - f(x)| \le \|x' - x\|$, hence f is continuous at $x = (0,b)$.

Let $x = (a,b)$ where $a \neq 0$ and let $\epsilon > 0$. Find $\delta > 0$ such that $|t - \varphi(x)| < \delta \Rightarrow |a(t) - a(\varphi(x))| < \epsilon/2$. From the continuity of φ it is possible to find a positive number $\delta_1 < \epsilon/2$ such that

$$\|x' - x\| < \delta_1 \quad \Rightarrow \quad x' \in B_0 \quad \text{and} \quad |\varphi(x') - \varphi(x)| < \delta.$$

For $x' = (a', b') \in B, \|x' - x\| < \delta_1$ we have

$$|f(x') - f(x)| = |a'(\varphi(x')) - a(\varphi(x))| \leq |a'(\varphi(x')) - a(\varphi(x'))| + |a(\varphi(x')) - a(\varphi(x))| \leq$$

$$\leq \|a' - a\| + \epsilon/2 \leq \|x' - x\| + \epsilon/2 < \epsilon.$$

Hence f is a continuous semicharacter.

It remains to show that f is not a multiplicative linear functional.To this end consider $x = (1, 0)$ and $x' = (z, z)$. Then $x'x = (z, 0), \varphi(x) = 0, \varphi(x') = 1/2$ and $\varphi(x'x) = 0$ so that $f(x) = 1, f(x') = 1/2$ and $f(x'x) = 0 \neq f(x) \cdot f(x')$.

Remark 1: The above constructed algebra B has no unit element. If we consider its unital extension $B_1 = B \oplus \{\mathbb{C}e\}$ then $f : B \to \mathbb{C}$ can be extended to a proper continuous semicharacter $f_1 : B_1 \to \mathbb{C}$ by $f_1(x + \lambda e) = f(x) + \lambda \quad (x \in B, \lambda \in \mathbb{C})$.

Problem: *Suppose that f is a uniformly continuous semicharacter on a Banach algebra A, i.e., for some constant k we have $|f(x) - f(x')| \leq k \cdot \|x - x'\| \quad (x, x' \in A)$. Does it follow that f is a multiplicative linear functional?*

Remark 2: If f is a semicharacter on a Banach algebra A such that $z \to f(a+bz)$ is a holomorphic function for every $a, b \in A$, then f is already a multiplicative linear functional. Indeed, function $\varphi : z \to f(a + bz) - f(a) - z \cdot f(b)$ is holomorphic and $\varphi(0) = 0$ so that

$$\varphi_1 : z \to \frac{\varphi(z)}{z} = f\Big(b + \frac{a}{z}\Big) - \frac{f(a)}{z} - f(b) \quad (z \neq 0)$$

extends to an entire function and $\lim_{z\to\infty} \varphi_1(z) = 0$. Thus $\varphi_1(z) = 0$ for every $z \in \mathbb{C}$. In particular,

$$0 = \varphi_1(1) = f(a + b) - f(a) - f(b)$$

so that f is a linear functional, i.e. a semicharacter.

Remark 3: A notion analogous to semicharacters is that of a quasilinear functional on a Banach algebra A (= a bounded function which is linear on each commutative subalgebra of A). This notion, which is motivated by quantum physics, has been studied intensively in the context of C^*-algebras, see [1].

References

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