Vasilescu-Martinelli formula for operators in Banach spaces

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Abstract: *We prove a formula for the Taylor functional calculus for functions analytic in a neighbourhood of the splitting spectrum of an n–tuple of commuting Banach space operators. This generalizes the formula of Vasilescu for Hilbert space operators and is closely related with a recent result of D. W. Albrecht.*

Let A be an n–tuple of mutually commuting operators in a Banach space X. The existence of the functional calculus for functions analytic in a neighbourhood of the Taylor spectrum is one of the most important results of the spectral theory [4], [5]. The formula giving the calculus, however, is rather inexplicit. Better situation is for commuting Hilbert space operators where an explicit formula was given by Vasilescu [6],[7].

The aim of this paper is to show that for such a formula is essential the equality between the Taylor and the splitting spectra for operators in Hilbert spaces. We generalize the Vasilescu formula for commuting Banach space operators and for functions analytic in a neighbourhood of the splitting spectrum.

The results are closely related with the paper of D. W. Albrecht [1]. He proved the Vasilescu formula under the assumption of existence of a certain "smooth generalized inverse".

We show that a smooth generalized inverse with similar properties exists everywhere in the complement of the splitting spectrum, what enables to construct the calculus. Another difference is that we do not assume the existence of the Taylor functional calculus.

Let X, Y be Banach spaces. We say that an operator $T : X \longrightarrow Y$ has a generalized inverse if there is an operator $S : Y \to X$ such that $TST = T$ and $STS = S$.

We shall use the following easy characterization (see e.g. [2]):

Proposition 1. *Let X, Y be Banach spaces, let $T : X \to Y$ be an operator. The following conditions are equivalent:*
(1) *T has a generalized inverse,*
(2) *There exists an operator $S : Y \to X$ such that $TST = T$,*
(3) $\operatorname{Im} T$ *is closed and both* $\ker T$ *and* $\operatorname{Im} T$ *are complemented subspaces of X and Y, respectively.*

Proof. Clearly (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Let $TST = T$ for some operator $S : Y \to X$. Set $S' = STS$. It is easy to check that $TS'T = T$ and $S'TS' = S'$.

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(1) $\Rightarrow$ (3): Let $TST = T$ and $STS = S$. Then $TS : Y \to Y$ is a bounded projection and $\operatorname{Im} T \supset \operatorname{Im} TS \supset \operatorname{Im} TST = \operatorname{Im} T$, so that TS is a projection onto $\operatorname{Im} T$.

Similarly ST is a bounded projection with $\ker ST = \ker T$.

(3) $\Rightarrow$ (1): Let $X = \ker T \oplus M$ and let $P \in B(Y)$ be a bounded projection onto $\operatorname{Im} T$. Then $T|M : M \to \operatorname{Im} T$ is a bijection. Set $S = (T|M)^{-1}P$. Then $TST = T(T|M)^{-1}PT = T$ and $STS = (T|M)^{-1}PT(T|M)^{-1}P = (T|M)^{-1}P = S$.

We repeat now the basic notations of Taylor [4].

Denote by $\Lambda(s)$ the complex exterior algebra generated by the indeterminates $s = (s_1, \dots, s_n)$. Then

$$\Lambda(s) = \bigoplus_{p=0}^{n} \Lambda^p(s),$$

where $\Lambda^p(s)$ is the set of all elements of degree p in $\Lambda(s)$.

Let X be a Banach space. Then we denote by $\Lambda(s, X) = X \otimes \Lambda(s)$ and $\Lambda^p(s, X) = X \otimes \Lambda^p(s)$. Thus the elements of $\Lambda^p(s, X)$ are of form

$$\sum_{1 \le i_1 < \cdots < i_p \le n} x_{i_1,\dots,i_p} s_{i_1} \wedge \cdots \wedge s_{i_p}$$

where $x_{i_1,\dots,i_p} \in X$.

Let $A = (A_1, \dots, A_n)$ be an $n-$tuple of mutually commuting operators in X. Define operator $\delta_A : \Lambda(s, X) \to \Lambda(s, X)$ by

$$\delta_A(x s_{i_1} \wedge \cdots \wedge s_{i_p}) = \sum_{j=1}^{n} (A_j x) s_j \wedge s_{i_1} \wedge \cdots \wedge s_{i_p}.$$

Denote by $\delta_A^p = \delta_A | \Lambda^p(s, X)$. Then the Koszul complex $K(A)$ is the sequence

$$0 \longrightarrow \Lambda^0(s, X) \xrightarrow{\delta_A^0} \Lambda^1(s, X) \xrightarrow{\delta_A^1} \cdots \xrightarrow{\delta_A^{n-1}} \Lambda^n(s, X) \longrightarrow 0.$$

Then $(\delta_A)^2 = 0$, i.e. $\delta_A^p \delta_A^{p-1} = 0$ for each p (for convenience we define $\Lambda^{-1}(s, X) = \Lambda^{n+1}(s, X) = 0$).

We say that the $n-$tuple $A = (A_1, \dots, A_n)$ is Taylor-regular if the Koszul complex $K(A)$ is exact (i.e. $\operatorname{Im} \delta_A = \ker \delta_A$). The Taylor spectrum $\sigma_T(A)$ is the set of all $n-$tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that $A - \lambda = (A_1 - \lambda_1, \dots, A_n - \lambda_n)$ is not Taylor-regular.

Closely related to the Taylor spectrum is the splitting spectrum. We say that $A = (A_1, \dots, A_n)$ is splitting-regular if $\ker \delta_A = \operatorname{Im} \delta_A$ and the space $\ker \delta_A$ is complemented in $\Lambda(s, X)$. The splitting spectrum $\sigma_s(A)$ is the set of all $\lambda \in \mathbb{C}^n$ such that $A - \lambda$ is not splitting-regular. Clearly $\sigma_T(A) \subset \sigma_s(A)$. It is well-known that the properties of the splitting spectrum are similar to those of the Taylor spectrum — it is a compact subset of $\mathbb{C}^n$ and it possesses the spectral mapping property.

The following result characterizes the splitting-regular $n-$tuples of operators.

Proposition 2. *Let $A = (A_1, \dots, A_n)$ be a Taylor-regular $n-$tuple of mutually commuting operators in a Banach space X. The following conditions are equivalent:*

(1) *A is splitting-regular,*
(2) $\ker \delta_A^p$ *is a complemented subspace of* $\Lambda^p(s,X) \quad (p=0,\dots,n-1)$,
(3) *there exist operators* $V_1, V_2 : \Lambda(s,X) \to \Lambda(s,X)$ *such that* $V_1\delta_A + \delta_A V_2 = I_{\Lambda(s,X)}$,
(4) *there exist an operator* $V : \Lambda(s,X) \to \Lambda(s,X)$ *such that* $V^2 = 0$, $V\delta_A + \delta_A V = I$ *and* $V\Lambda^p(s,X) \subset \Lambda^{p-1}(s,X) \quad (p=0,\dots,n)$ *(i.e. there are operators* $V_p : \Lambda^{p+1}(s,X) \to \Lambda^p(s,X)$ *such that* $V_{p-1}V_p = 0$ *and* $V_p\delta_A^p + \delta_A^{p-1}V_{p-1} = I_{\Lambda^p(s,X)}$ *for every* p.

Proof. (4) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1): If $V_1\delta_A + \delta_A V_2 = I$ then $\delta_A V_1 \delta_A = \delta_A$, so that δ_A has a generalized inverse, i.e. $\ker \delta_A$ is complemented.

(1) $\Rightarrow$ (2): Denote by $J_p : \Lambda^p(s,X) \to \Lambda(s,X)$ the natural embedding, $Q_p : \Lambda(s,X) \to \Lambda^p(s,X)$ the natural projection and let $P : \Lambda(s,X) \to \ker \delta_A$ be a bounded projection onto $\ker \delta_A$.

Clearly $Q_p(\ker \delta_A) = \ker \delta_A^p$. Then $Q_p P J_p$ is a bounded projection onto $\ker \delta_A^p$.

(2) $\Rightarrow$ (4): Let M_p be a subspace of $\Lambda^p(s,X)$ such that $\ker \delta_A^p \oplus M_p = \Lambda^p(s,X)$. The operator $\delta_A^p|M_p : M_p \to \operatorname{Im} \delta_A^p = \ker \delta_A^{p+1}$ is a bijection. In the decompositions $\Lambda^p(s,X) = \ker \delta_A^p \oplus M_p$, $\Lambda^{p+1}(s,X) = \ker \delta_A^{p+1} \oplus M_{p+1}$ we have

$$\delta_A^p = \begin{array}{c c} & \begin{array}{cc} \ker \delta_A^p & M_p \end{array} \\ \begin{array}{c} \operatorname{Im} \delta_A^p \\ M_{p+1} \end{array} & \begin{pmatrix} 0 & \delta_A^p|M_p \\ 0 & 0 \end{pmatrix} \end{array}.$$

Set

$$V_p = \begin{array}{c c} & \begin{array}{cc} \operatorname{Im} \delta_A^p & M_{p+1} \end{array} \\ \begin{array}{c} \ker \delta_A^p \\ M_p \end{array} & \begin{pmatrix} 0 & 0 \\ (\delta_A^p|M_p)^{-1} & 0 \end{pmatrix} \end{array}.$$

Then $V_{p-1}V_p = 0$ since $\operatorname{Im} V_p \subset M_p \subset \ker V_{p-1}$. For $x \in M_p$ we have

$$(V_p\delta_A^p + \delta_A^{p-1}V_{p-1})x = V_p\delta_A^p x = x.$$

For $x \in \ker \delta_A^p$ we have

$$(V_p\delta_A^p + \delta_A^{p-1}V_{p-1})x = \delta_A^{p-1}V_{p-1}x = x.$$

Thus $V_p\delta_A^p + \delta_A^{p-1}V_{p-1} = I_{\Lambda^p(s,X)}$ for each p. (For $p=0$ and $p=n$ this reduces to $V_0\delta_A^0 = I_{\Lambda^0(s,X)}$ and $\delta_A^{n-1}V_{n-1} = I_{\Lambda^n(s,X)}$).

Theorem 3. *Let $A = (A_1,\dots,A_n)$ be an $n-$tuple of mutually commuting operators in a Banach space X. Let $\mu \in \mathbb{C}^n$ and suppose that A is splitting-regular, i.e.* $\ker \delta_{\mu-A} = \operatorname{Im} \delta_{\mu-A}$ *and $\delta_{\mu-A}$ has a generalized inverse. Then there exists a neighbourhood U of μ in $\mathbb{C}^n$ and an analytic function $V : U \to B(\Lambda(s,X))$ such that* $V(\lambda)\delta_{\lambda-A} + \delta_{\lambda-A}V(\lambda) = I_{\Lambda(s,X)}$ *for every $\lambda \in U$.*

Moreover, we may assume that $V(\lambda)^2 = 0 \quad (\lambda \in U)$ *and*

$$V(\lambda)\Lambda^p(s,X) \subset \Lambda^{p-1}(s,X) \quad (\lambda \in U, p=0,\dots,n).$$

Proof. By the previous proposition there exists an operator $V : \Lambda(s, X) \to \Lambda(s, X)$ such that $V^2 = 0$, $\delta_{\mu-A}V + V\delta_{\mu-A} = I_{\Lambda(s,X)}$, and $V\Lambda^p(s, X) \subset \Lambda^{p-1}(s, X)$ for every p.

For $\lambda \in \mathbb{C}^n$ denote by $H_\lambda = \delta_{\lambda-A} - \delta_{\mu-A}$. Let U be the set of all $\lambda \in \mathbb{C}^n$ such that $\|H_\lambda\| < \|V\|^{-1}$. Clearly U is a neighbourhood of μ in $\mathbb{C}^n$ and, for $\lambda \in U$, the operators $I + H_\lambda V$ and $I + VH_\lambda$ are invertible. We have $V(I + H_\lambda V) = (I + VH_\lambda)V$, so that $(I + VH_\lambda)^{-1}V = V(I + H_\lambda V)^{-1}$. For $\lambda \in U$ set $V(\lambda) = (I + VH_\lambda)^{-1}V$. Then

$$\begin{aligned}
&\delta_{\lambda-A}V(\lambda) + V(\lambda)\delta_{\lambda-A}\\
=&(\delta_{\mu-A} + H_\lambda)V(I + H_\lambda V)^{-1} + (I + VH_\lambda)^{-1}V(\delta_{\mu-A} + H_\lambda)\\
=&(I + VH_\lambda)^{-1}\big[(I + VH_\lambda)(\delta_{\mu-A} + H_\lambda)V + V(\delta_{\mu-A} + H_\lambda)(I + H_\lambda V)\big](I + H_\lambda V)^{-1}.
\end{aligned}$$

The expression in the middle is equal to

$$\begin{aligned}
&\delta_{\mu-A}V + H_\lambda V + VH_\lambda\delta_{\mu-A}V + VH_\lambda^2V + V\delta_{\mu-A} + VH_\lambda + V\delta_{\mu-A}H_\lambda V + VH_\lambda^2V\\
=&(I + VH_\lambda)(I + H_\lambda V) + V(H_\lambda\delta_{\mu-A} + \delta_{\mu-A}H_\lambda + H_\lambda^2)V\\
=&(I + VH_\lambda)(I + H_\lambda V) + V\big((\delta_{\mu-A} + H_\lambda)^2 - (\delta_{\mu-A})^2\big)V = (I + VH_\lambda)(I + H_\lambda V)
\end{aligned}$$

since $(\delta_{\mu-A})^2 = 0$ and $(\delta_{\mu-A} + H_\lambda)^2 = (\delta_{\lambda-A})^2 = 0$. Thus

$$\delta_{\lambda-A}V(\lambda) + V(\lambda)\delta_{\lambda-A} = I_{\Lambda(s,X)} \qquad (\lambda \in U).$$

Further

$$V(\lambda)^2 = (I + VH_\lambda)^{-1}V \cdot V(I + H_\lambda V)^{-1} = 0.$$

Finally $V(\lambda) = \sum_{i=0}^{\infty}(-1)^i(VH_\lambda)^iV$ where

$$(VH_\lambda)\Lambda^p(s, X) \subset \Lambda^p(s, X) \qquad (p = 0, \dots, n),$$

so that

$$V(\lambda)\Lambda^p(s, X) \subset \Lambda^{p-1}(s, X) \qquad (\lambda \in U, p = 0, \dots, n).$$

Corollary 4. *Let $A = (A_1, \dots, A_n)$ be an $n-$tuple of mutually commuting operators in a Banach space X. Denote by $G = \mathbb{C}^n - \sigma_s(A)$. Then there exists an operator-valued $C^\infty-$function $V : G \to B(\Lambda(s, X))$ such that $\delta_{\lambda-A}V(\lambda) + V(\lambda)\delta_{\lambda-A} = I_{\Lambda(s,X)}$, $V(\lambda)^2 = 0$ and*

$$V(\lambda)\Lambda^p(s, X) \subset \Lambda^{p-1}(s, X) \qquad (\lambda \in G, p = 0, \dots, n).$$

Proof. For every $\mu \in G$ there exists a neighbourhood U_μ of μ and an analytic operator-valued function $V_\mu : U_\mu \to B(\Lambda(s, X))$ such that $V_\mu(\lambda)\delta_{\lambda-A} + \delta_{\lambda-A}V_\mu(\lambda) = I_{\Lambda(s,X)}$, $V_\mu(\lambda)^2 = 0$ and

$$V_\mu(\lambda)\Lambda^p(s, X) \subset \Lambda^{p-1}(s, X) \qquad (\lambda \in U_\mu, p = 0, \dots, n).$$

Let $\{\psi_i\}_{i=1}^\infty$ be a $C^\infty-$partition of unity subordinated to the cover $\{U_\mu, \mu \in G\}$ of G, i.e. ψ_i's are $C^\infty-$functions, $0 \le \psi_i \le 1$, $\operatorname{supp} \psi_i \subset U_{\mu_i}$ for some $\mu_i \in G$, for each $\mu \in G$ there exists a neighbourhood U of μ such that all but finitely many of ψ_i's are 0 on U and $\sum_{i=1}^\infty \psi_i(\mu) = 1$ for each $\mu \in G$. For $\lambda \in G$ set

$$P(\lambda) = \sum_{i=1}^\infty \psi_i(\lambda)\delta_{\lambda-A}V_{\mu_i}(\lambda).$$

Clearly $\operatorname{Im} P(\lambda) \subset \operatorname{Im} \delta_{\lambda-A}$ and, for $x \in \operatorname{Im} \delta_{\lambda-A}$, we have

$$P(\lambda)x = \sum_{i=1}^\infty \psi_i(\lambda)x = x,$$

since $\delta_{\lambda-A}V_{\mu_i}(\lambda)$ is a projection onto $\operatorname{Im} \delta_{\lambda-A}$ ($V_{\mu_i}(\lambda)$ is a generalized inverse of $\delta_{\lambda-A}$). Thus $P(\lambda)$ is a projection onto $\operatorname{Im} \delta_{\lambda-A} \quad (\lambda \in G)$. Further

$$P(\lambda)\Lambda^p(s,X) \subset \Lambda^p(s,X) \qquad (\lambda \in G, p = 0, \dots, n).$$

Set

$$V(\lambda) = \sum_{i=1}^\infty \psi_i(\lambda)(I - P(\lambda))V_{\mu_i}(\lambda)P(\lambda) \qquad (\lambda \in G).$$

Clearly V is a $C^\infty-$function, $V(\lambda)^2 = 0$ and

$$V(\lambda)\Lambda^p(s,X) \subset \Lambda^{p-1}(s,X) \qquad (\lambda \in G, p = 0, \dots, n).$$

It remains to show that $\delta_{\lambda-A}V(\lambda) + V(\lambda)\delta_{\lambda-A} = I_{\Lambda(s,X)}$. If $x \in \operatorname{Im} \delta_{\lambda-A}$ then

$$\big(\delta_{\lambda-A}V(\lambda) + V(\lambda)\delta_{\lambda-A}\big)x = \delta_{\lambda-A}V(\lambda)x = \sum_{i=1}^\infty \psi_i(\lambda)\delta_{\lambda-A}(I - P(\lambda))V_{\mu_i}(\lambda)P(\lambda)x$$
$$= \sum_{i=1}^\infty \psi_i(\lambda)\delta_{\lambda-A}V_{\mu_i}(\lambda)x = \sum_{i=1}^\infty \psi_i(\lambda)\big(I - V_{\mu_i}(\lambda)\delta_{\lambda-A}\big)x = \sum_{i=1}^\infty \psi_i(\lambda)x = x.$$

If $x \in \ker P(\lambda)$ then

$$\big(\delta_{\lambda-A}V(\lambda) + V(\lambda)\delta_{\lambda-A}\big)x = V(\lambda)\delta_{\lambda-A}x = \sum_{i=1}^\infty \psi_i(\lambda)(I - P(\lambda))V_{\mu_i}(\lambda)P(\lambda)\delta_{\lambda-A}x$$
$$= \sum_{i=1}^\infty \psi_i(\lambda)(I - P(\lambda))V_{\mu_i}(\lambda)\delta_{\lambda-A}x = \sum_{i=1}^\infty \psi_i(\lambda)(I - P(\lambda))\big(I - \delta_{\lambda-A}V_{\mu_i}(\lambda)\big)x$$
$$= \sum_{i=1}^\infty \psi_i(\lambda)(I - P(\lambda))x = \sum_{i=1}^\infty \psi_i(\lambda)x = x.$$

Hence

$$\delta_{\lambda-A}V(\lambda) + V(\lambda)\delta_{\lambda-A} = I_{\Lambda(s,X)} \qquad (\lambda \in G).$$

In the rest of the paper we shall fix a commuting $n-$tuple $A = (A_1, \ldots, A_n)$ of operators in a Banach space X, $G = \mathbf{C}^n - \sigma_s(A)$ and a $C^\infty-$function $V : G \to B(\Lambda(s, X))$ with properties of Corollary 4. Denote by $C^\infty(G, X)$ the space of all X-valued $C^\infty-$functions defined in G.

We shall consider the space $C^\infty(G, \Lambda(s, X))$. Clearly this space can be identified with the set $\Lambda(s, C^\infty(G, X))$.

Function $V : G \to B(\Lambda(s, X))$ induces naturally the operator (denoted by the same symbol) $V : C^\infty(G, \Lambda(s, X)) \to C^\infty(G, \Lambda(s, X))$ by

$$(Vy)(\mu) = V(\mu)y(\mu) \qquad \big(\mu \in G, y \in C^\infty(G, \Lambda(s, X))\big).$$

Similarly we define operator $\delta : C^\infty(G, \Lambda(s, X)) \to C^\infty(G, \Lambda(s, X))$ by

$$(\delta y)(\mu) = \delta_{\mu - A}\, y(\mu) \qquad \big(\mu \in G, y \in C^\infty(G, \Lambda(s, X))\big).$$

Clearly $V^2 = 0$, $\delta^2 = 0$, $V\delta + \delta V = I_{\Lambda(s, C^\infty(G,X))}$ and both V and δ are "graded", i.e.

$$\begin{aligned} &V\Lambda^p(s, C^\infty(G, X)) \subset \Lambda^{p-1}(s, C^\infty(G, X)) \qquad \text{and} \\ &\delta\Lambda^p(s, C^\infty(G, X)) \subset \Lambda^{p+1}(s, C^\infty(G, X)). \end{aligned}$$

Consider now another indeterminates $\mathrm{d}\bar z = (\mathrm{d}\bar z_1, \ldots, \mathrm{d}\bar z_n)$ and the set $\Lambda(s, \mathrm{d}\bar z, C^\infty(G, X))$. We define the operator

$$\bar\partial : \Lambda(s, \mathrm{d}\bar z, C^\infty(G, X)) \to \Lambda(s, \mathrm{d}\bar z, C^\infty(G, X))$$

by

$$\bar\partial f s_{i_1} \wedge \ldots \wedge s_{i_p} \wedge \mathrm{d}\bar z_{j_1} \wedge \ldots \wedge \mathrm{d}\bar z_{j_q} = \sum_{k=1}^n \frac{\partial f}{\partial \bar z_k} \mathrm{d}\bar z_k \wedge s_{i_1} \wedge \ldots \wedge s_{i_p} \wedge \mathrm{d}\bar z_{j_1} \wedge \ldots \wedge \mathrm{d}\bar z_{j_q}.$$

Clearly $\bar\partial^2 = 0$.

Operators V and δ can be "lifted" from $\Lambda(s, C^\infty(G, X))$ to $\Lambda(s, \mathrm{d}\bar z, C^\infty(G, X))$ by

$$\begin{aligned} &V(y \wedge \mathrm{d}\bar z_{i_1} \wedge \ldots \mathrm{d}\bar z_{i_p}) = (Vy) \wedge \mathrm{d}\bar z_{i_1} \wedge \ldots \mathrm{d}\bar z_{i_p} \qquad \text{and} \\ &\delta(y \wedge \mathrm{d}\bar z_{i_1} \wedge \ldots \mathrm{d}\bar z_{i_p}) = (\delta y) \wedge \mathrm{d}\bar z_{i_1} \wedge \ldots \mathrm{d}\bar z_{i_p} \qquad \big(y \in \Lambda(s, C^\infty(G, X))\big). \end{aligned}$$

Clearly the properties of V and δ are preserved: $V^2 = 0$, $V\delta + \delta V = I$ and both V and δ are graded. Note also that $\delta\bar\partial = -\bar\partial\delta$ and if U is an open subset of G and $\eta \in \Lambda(s, C^\infty(G, X))$ $\big(= C^\infty(G, \Lambda(s, X)\big)$ with $\eta|U \equiv 0$, then $\bar\partial\eta|U \equiv 0$, $\delta\eta|U \equiv 0$ and $V\eta|U \equiv 0$.

Theorem 5. *There exists an operator $W : \Lambda(s, \mathrm{d}\bar z, C^\infty(G, X)) \to \Lambda(s, \mathrm{d}\bar z, C^\infty(G, X))$ such that $W^2 = 0$, $W(\delta + \bar\partial) + (\delta + \bar\partial)W = I$ and*

$$W\Lambda^p(s, \mathrm{d}\bar z, C^\infty(G, X)) \subset \Lambda^{p-1}(s, \mathrm{d}\bar z, C^\infty(G, X)) \qquad (p = 0, \ldots, 2n)$$

(i.e. W "splits" $\delta + \bar\partial$).

Proof. Clearly $V : \Lambda(s, \mathrm{d}\bar{z}, C^\infty(G,X)) \to \Lambda(s, \mathrm{d}\bar{z}, C^\infty(G,X))$ decreases by 1 the degree in $s_1, \dots, s_n$ and $\bar\partial$ does not decrease this degree. Thus $(\bar\partial V)^{n+1} = 0$. Hence $(I + \bar\partial V)^{-1}$ exists and $(I + \bar\partial V)^{-1} = \sum_{j=0}^{n} (-1)^j (\bar\partial V)^j$. Similarly $(I + V\bar\partial)^{-1} = \sum_{j=0}^{n} (-1)^j (V\bar\partial)^j$. Since $V(I+\bar\partial V) = (I+V\bar\partial)V$ we have $(I+V\bar\partial)^{-1}V = V(I+\bar\partial V)^{-1}$.

Set $W = (I + V\bar\partial)^{-1}V = V(I + \bar\partial V)^{-1} = \sum_{j=0}^{n-1} (-1)^j V(\bar\partial V)^j$. Clearly

$$W^2 = (I + V\bar\partial)^{-1}V \cdot V(I + \bar\partial V)^{-1} = 0$$

and W decreases the (total) degree by 1. It remains to prove that $(\delta+\bar\partial)W + W(\delta+\bar\partial) = I$, i.e.

$$(\delta + \bar\partial)V(I + \bar\partial V)^{-1} + (I + V\bar\partial)^{-1}V(\delta + \bar\partial) = I.$$

It is sufficient to show

$$(I + V\bar\partial)(\delta + \bar\partial)V + V(\delta + \bar\partial)(I + \bar\partial V) = (I + V\bar\partial)(I + \bar\partial V)$$

or

$$\delta V + \bar\partial V + V\bar\partial\delta V + V\delta + V\bar\partial + V\delta\bar\partial V = I + V\bar\partial + \bar\partial V.$$

The last equality follows from the relations $\delta V + V\delta = I$ and $\bar\partial\delta + \delta\bar\partial = 0$.

Denote by P the natural projection $P : \Lambda(s, \mathrm{d}\bar{z}, C^\infty(G,X)) \to \Lambda(\mathrm{d}\bar{z}, C^\infty(G,X))$. Let $M : X \to \Lambda^{n-1}(\mathrm{d}\bar{z}, C^\infty(G,X))$ be the operator defined by

$$Mx = (-1)^{n-1} PWxs,$$

where we write shortly $s = s_1 \wedge \cdots \wedge s_n$. Since

$$W = V \cdot \sum_{i=0}^{n-1} (-1)^i (\bar\partial V)^i = V - V\bar\partial V + \cdots + (-1)^{n-1} V(\bar\partial V)^{n-1},$$

$\bar\partial$ does not decrease the degree in $(s_1, \dots, s_n)$ and V decreases it by 1, we can see that

$$Mx = V(\bar\partial V)^{n-1} xs.$$

Proposition 6. *$\bar\partial Mx = 0$ for every $x \in X$.*

Proof. We have $(\delta + \bar\partial)xs = 0$ so that

$$(\delta + \bar\partial)Wxs = \big[(\delta + \bar\partial)W + W(\delta + \bar\partial)\big]xs = xs.$$

Let $Wxs = PWxs + \eta$, where $\eta \in \Lambda(s, \mathrm{d}\bar{z}, C^\infty(G,X))$ consists of terms of degree at least 1 in $s_1, \dots, s_n$.

Thus

$$(\delta + \bar\partial)Wxs = \big[(\delta + \bar\partial)\eta + \delta PWxs\big] + \bar\partial PWxs$$

where $\bar\partial PWxs$ consists of terms of degree 0 in $s_1, \dots, s_n$. Thus

$$0 = Pxs_n = P(\delta + \bar\partial)Wxs = \bar\partial PWxs.$$

Let U be a neighbourhood of $\sigma_s(A)$. It is possible to find an open subset Δ containing $\sigma_s(A)$ such that $\bar{\Delta}$ is compact, $\bar{\Delta} \subset U$ and the boundary $\partial\Delta$ is a smooth surface. Let f be a function analytic in U. Define the operator $f(A)$ by

$$f(A)x = \frac{1}{(2\pi i)^n} \int_{\partial\Delta} Mf(z)x \wedge dz \qquad (x \in X), \tag{1}$$

where $\mathrm{d}z$ stands for $\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$. By the Stokes formula

$$f(A)x = \frac{1}{(2\pi i)^n} \int \bar{\partial}\varphi Mf(z)x \wedge dz$$

where φ is $C^\infty-$ function equal to 0 on a neighbourhood of $\sigma_s(A)$ to 1 on $\mathbb{C}^n - \Delta$.

To show the correctness of the definition of $f(A)$ we need the following simple proposition (see [6]).

Proposition 7. *Let $\eta \in \Lambda^n(s, d\bar{z}, C^\infty(G, X))$ be a differential form with a compact support disjoint with $\sigma_s(A)$ such that $(\delta + \bar{\partial})\eta = 0$. Then*

$$\int P\eta \wedge dz = 0.$$

Proof: Set $\xi = W\eta$. Then $(\delta + \bar{\partial})\xi = \eta$ and

$$P\eta = P(\delta + \bar{\partial})\xi = P\bar{\partial}\xi = \bar{\partial}P\xi.$$

Hence, for a suitable surface Σ we have

$$\int P\eta dz = \int \bar{\partial}P\xi dz = \int_\Sigma P\xi dz = 0.$$

We show now that the definition of $f(A)$ does not depend on the particular choice of φ. Indeed, if φ_1 and φ_2 are two $C^\infty-$ function with required properties, then $(\delta + \bar{\partial})(\varphi_1 - \varphi_2)Wf(z)xs$ satisfies the properties of Proposition 7. Thus

$$0 = \int P(\delta + \bar{\partial})(\varphi_1 - \varphi_2)f(z)Wxs \wedge dz = \int P\bar{\partial}(\varphi_1 - \varphi_2)f(z)Wxs \wedge dz =$$

$$= (-1)^{n-1} \int \bar{\partial}(\varphi_1 - \varphi_2)f(z)Mx \wedge dz.$$

This means also that $f(A)$ does not depend on the choice of the set Δ.

Finally we show that $f(A)$ does not depend on the choice of the generalized inverse V which determines W and M.

Suppose that W_1, W_2 are two operators satisfying

$$(\delta + \bar{\partial})W_i + W_i(\delta + \bar{\partial}) = I \qquad (i = 1, 2).$$

Then $(\delta+\bar\partial)W_i f(z)xs = f(z)xs$. For those z where $\varphi\equiv 1$ we have

$$(\delta+\bar\partial)\varphi(W_1-W_2)f(z)xs=0$$

so that the form $(\delta+\bar\partial)\varphi(W_1-W_2)f(z)xs$ satisfies the conditions of Proposition 7. Hence

$$0=\int P(\delta+\bar\partial)\varphi(W_1-W_2)f(z)xs\wedge dz=\int P\bar\partial\varphi(W_1-W_2)f(z)xs\wedge dz=$$

$$=\int \bar\partial\varphi P(W_1-W_2)f(z)xs\wedge dz=(-1)^{n-1}\int\bar\partial\varphi f(z)(M_1-M_2)x\wedge dz$$

where

$$M_ix=(-1)^{n-1}PW_ixs \qquad (i=1,2).$$

Clearly $f(A)$ is a bounded linear operator and the mapping $f\mapsto f(A)$ is linear. To show that $f\mapsto f(A)$ is the functional calculus it is necessary to prove that

$$\begin{array}{lll} f(A) = I & \text{if } f\equiv 1, & \\ f(A) = A_i & \text{if } f(z)=z_i & (i=1,\dots,n) \end{array}$$

and the multiplicativity of the mapping $f\mapsto f(A)$.

As the proof is rather technical and it is described elsewhere (see [6], [3]), we just outline the main steps.

1) If $n=1$ then M is just the inverse $Mx=(\lambda-A_1)^{-1}x$, so that the described calculus coincides with the ordinary calculus for one operator.

Set

$$\overline{W}=\frac{1}{(2\pi i)^n}(-1)^{n-1}\big[(\delta+\bar\partial)\varphi W-I\big], \tag{2}$$

so that

$$f(A)x=\int f(z)P\overline{W}xs\wedge dz.$$

2) Let $(A,B)=(A_1,\dots,A_n,B_1,\dots,B_m)$ be a commuting $(n+m)-$tuple of operators in X, let Δ,Δ' be open neighbourhood of $\sigma_s(A),\sigma_s(B)$ with compact closures and with smooth boundaries. Let f be a function analytic in a neighbourhood of $\bar\Delta\times\bar\Delta'$. Let $\overline{W}^n,\overline{W}^m,\overline{W}^{n+m}$ be operators defined by (2) for tuples A,B and (A,B), respectively. Then

$$\int f(z,w)P(\overline{W}^{n+m}-\overline{W}^m\overline{W}^n)xs\wedge t\wedge dz\wedge dw=0,$$

where $t=(t_1,\dots,t_m), dw=(dw_1,\dots,dw_m)$ are indeterminates corresponding to B. This follows from considerations similar to the proof of Proposition 7.

3) If $f(z,w)=f_1(z)\cdot f_2(w)$ then, by the Fubini theorem and by 2), $f(A,B)=f_1(A)f_2(B)$.

4) Consider the $n-$tuple $A = (A_1, \dots, A_n)$ and the identity function $f : \mathbb{C}^n \to \mathbb{C}$, $f \equiv 1$. Then 3) together with 1) gives $f(A) = I$.
Similarly $f(A) = A_i$ for $f(z) = z_i \qquad (i = 1, \dots, n)$.

5) Consider the $2n-$tuple $(A, A) = (A_1, \dots, A_n, A_1, \dots, A_n)$. Let f, g be functions analytic in a neighbourhood of $\sigma_s(A)$. Then

$$f(A)g(A)x = \int f(z)P\overline{W}^z \Big(\int g(w)P\overline{W}^w xt \wedge dw \Big) \wedge s \wedge dz =$$

$$= \int f(z)g(w)P\overline{W}^z\overline{W}^w xs \wedge t \wedge dz \wedge dw = \int f(z)g(w)P\overline{W}^{z,w} xs \wedge t \wedge dz \wedge dw$$

and, by 2),

$$(fg)(A) = (fg)(A) \cdot \ \mathrm{id}(A) = \int f(z)g(z)P\overline{W}^{z,w} xs \wedge t \wedge dz \wedge dw.$$

Thus it is sufficient to show

$$\int f(z)(g(z) - g(w))P\overline{W}^{z,w} xs \wedge t \wedge dz \wedge dw = 0.$$

Since $g(z) - g(w) = \sum_{i=1}^n (z_i - w_i)h_i(z, w)$ for some analytic functions $h_i(z, w)$, the previous integral is equal to

$$\sum_{i=1}^n \int f(z)h_i(z, w)(z_i - w_i)P\overline{W}^{z,w} xs \wedge t \wedge dz \wedge dw =$$

$$= \sum_{i=1}^n \int f(z)h_i(z, w)(z_i - A_i)P\overline{W}^{z,w} xs \wedge t \wedge dz \wedge dw -$$

$$- \sum_{i=1}^n \int f(z)h_i(z, w)(w_i - A_i)P\overline{W}^{z,w} xs \wedge t \wedge dz \wedge dw.$$

Thus it is sufficient to show that

$$\int f(z)h(z, w)(z_i - A_i)P\overline{W}^{z,w} xs \wedge t \wedge dz \wedge dw = 0$$

for every analytic function $h(z, w)$. The last integral is equal to (up to multiplication by a constant)

$$\int h(z_i - A_i)P(\delta + \bar{\partial})\varphi W xs \wedge t \wedge dz \wedge dw = \int h\bar{\partial}\varphi P(z_i - A_i)W xs \wedge t \wedge dz \wedge dw.$$

By checking the definition of W it is possible to show that

$$P(z_i - A_i)W xs \wedge t \wedge dz \wedge dw = \bar{\partial}\xi$$

for some ξ so that

$$\int h\bar{\partial}\varphi\bar{\partial}\xi = \int \bar{\partial}\varphi\bar{\partial}h\xi = \int_{\partial\Delta} \bar{\partial}h\xi = 0.$$

Concluding remarks

1) If X is a Hilbert space, then $\Lambda(s, X)$ can be given naturally a Hilbert space structure, so that the splitting spectrum coincide with the Taylor spectrum. For $\lambda \notin \sigma_s(A)$ the operator $(\delta_{\lambda-A} + \delta^*_{\lambda-A}) : \Lambda(s, X) \to \Lambda(s, X)$ is invertible and
$$(\delta_{\lambda-A} + \delta^*_{\lambda-A})^{-1}\delta_{\lambda-A} + \delta_{\lambda-A}(\delta_{\lambda-A} + \delta^*_{\lambda-A})^{-1} = I_{\Lambda(s,X)}.$$
Clearly the function $\lambda \mapsto (\delta_{\lambda-A} + \delta^*_{\lambda-A})^{-1}$ is C^∞ and although it does not satisfy all the conditions of Corollary 4, it is possible to take it instead of the operator $V : \Lambda(s, C^\infty(G, X)) \to \Lambda(s, C^\infty(G, X))$. The remaining conditions of Corollary 4 ($V^2 = 0$ and that V is "graded") are not essential for the construction of the functional calculus and only make the considerations easier. On the other hand the formula obtained for $f(A)$ using the function $\lambda \mapsto (\delta_{\lambda-A} + \delta^*_{\lambda-A})^{-1}$ is quite explicit (see [6], [7]).

2) Let
$$V : \Lambda(s, C^\infty(G, X)) \to \Lambda(s, C^\infty(G, X))$$
be an operator with the properties of Corollary 4. Then $(\delta + V)^{-1} = \delta + V$ and
$$P(\delta + V)\big[\bar{\partial}(\delta + V)\big]^{n-1}xs = PV(\bar{\partial}V)^{n-1}xs$$
so that the functional calculus constructed here coincides with the construction of Albrecht [1].

3) If $A = (A_1 \ldots, A_n)$ has a real Taylor spectrum, $\sigma_T(A) \subset \mathbb{R}^n$, then it is possible to show that $\sigma_s(A) = \sigma_T(A)$. Indeed, if $\lambda \in \mathbb{C}^n - \sigma_T(A)$ it is possible to find a point $\mu \in \mathbb{C}^n - \big(\sigma_T(A) \cup \{\lambda\}\big)$ and a rational function $f(z) = \frac{1}{(z_1-\mu_1)} \cdots \frac{1}{(z_n-\mu_n)}$ such that $|f(\lambda)| > \max\{|f(z)|, z \in \sigma_T(A)\}$. Consider the operator $f(A)$. If $\lambda \in \sigma_s(A)$ then, by the spectral mapping theorems for σ_T and σ_s, we have
$$\max\{|z|, z \in \sigma_T(f(A))\} < \max\{|z|, z \in \sigma_s(f(A))\},$$
which contradicts to the fact that σ_T and σ_s coincide for single operators. Thus the functional calculus for functions analytic in a neighbourhood of the splitting spectrum coincide with the Taylor functional calculus.

4) In general $\sigma_T(A) \subset \sigma_s(A)$. It is an open problem whether it is possible to find an $n-$tuple $A = (A_1, \ldots, A_n)$ of mutually commuting operators in a Banach space X such that $\sigma_T(A) \neq \sigma_s(A)$.

5) The Taylor functional calculus can be constructed similarly as the calculus for the splitting spectrum constructed here. It is well-known that the sequence
$$\cdots \xrightarrow{\delta+\bar{\partial}} \Lambda^p(s, d\bar{z}, C^\infty(G, X)) \xrightarrow{\delta+\bar{\partial}} \Lambda^{p+1}(s, d\bar{z}, C^\infty(G, X)) \xrightarrow{\delta+\bar{\partial}} \cdots$$
is exact (see e.g. [8], Propositions III.2.4, 2.5, 2.8). If f is a function analytic in a neighbourhood of $\sigma_T(A)$, it is possible to take instead of Wxs in formula (1) the form $\xi \in \Lambda^{n-1}(s, d\bar{z}, C^\infty(G, X))$ such that $(\delta + \bar{\partial})\xi = xs$. It is not possible to see at the first glance that the operator $f(A)$ defined in this way is bounded. This can be shown by choosing ξ not too big in the norm (cf. [3]).

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