A QUASI-NILPOTENT OPERATOR WITH REFLEXIVE COMMUTANT, II

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Abstract.
A new example of a non-zero quasi-nilpotent operator T with reflexive commutant is presented. Norms $\|T^n\|$ converge to zero arbitrarily fast.

Let H be a complex separable Hilbert space and let $\mathcal{B}(H)$ denote the algebra of all continuous linear operator on H. If $T \in \mathcal{B}(H)$ then $\{T\}' = \{A \in \mathcal{B}(H) \colon AT = TA\}$ is called the commutant of T. By a subspace we always mean a closed linear subspace. If $\mathcal{A} \subset \mathcal{B}(H)$ then $\operatorname{Alg} \mathcal{A}$ denotes the smallest weakly closed subalgebra of $\mathcal{B}(H)$ containing the identity I and $\mathcal{A}$, and $\operatorname{Lat} \mathcal{A}$ denotes the set of all subspaces invariant for each $A \in \mathcal{A}$. If $\mathcal{L}$ is a set of subspaces of H, then $\operatorname{Alg} \mathcal{L} = \{T \in \mathcal{B}(H) \colon \mathcal{L} \subset \operatorname{Lat}\{T\}\}$. T is said to be hyperreflexive if $\{T\}' = \operatorname{Alg} \operatorname{Lat}\{T\}'$, i.e., if the algebra $\{T\}'$ is reflexive.

It can be shown (see [1]) that if T is a nilpotent hyperreflexive operator on a separable Hilbert space then $T = 0$. This is not true for quasinilpotent operators. An example of a non-zero quasinilpotent hyperreflexive operator was given in [5] using a modification of an idea of Wogen [4]. The powers of the example converged to zero slowly, more precisely the following inequality was true for all positive integers:

$$\|T^n\|^{1/n} \geq \frac{1}{\log n}\,.$$

In [6] it was shown that the convergence of powers of T to zero can be faster, namely for each $p > 0$ there exists a non-zero hyperreflexive operator T for which

$$\|T^n\|^{1/n} \leq \frac{1}{n^p}\,.$$

The aim of this note is to show that the convergence $\|T^n\|^{1/n} \to 0$ can be arbitrarily fast:

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Theorem 1. *Let $(\beta_n)_{n\geq 1}$ be a sequence of positive numbers. Then there exists a non-zero hyperreflexive operator T on a separable Hilbert space H such that $\|T^n\|^{1/n}\leq \beta_n$ for all $n\geq 1$.*

Proof. The set of all non-negative integers will be denoted by N. Set formally $\beta_0=1$. Without loss of generality we can assume that $1=\beta_0\geq \beta_1\geq\beta_2\geq\cdots$ (if necessary, we can replace β_n by $\min\{\beta_j : 0\leq j\leq n\}$).

For $k=0,1,\dots$ set $m_k=3k(k+1)$. For $n\in N$ let $f(n)=\min\{k : m_k>n\}$. Thus $f(n)=k$ if and only if $m_{k-1}\leq n<m_k$.

Finally, set $s_0=1$ and, for $k,j\in N$, $j^2<k\leq (j+1)^2$ set

$$s_k=\min\Big\{\frac{1}{f(n)}\frac{\beta_{n+f(n)}^{n+f(n)}}{\beta_n^n} : 0\leq n\leq m_{(j+1)^2}\Big\}.$$

Clearly $1=s_0\geq s_1\geq s_2\geq\cdots$. Further $s_{j^2+1}=s_{j^2+2}=\cdots=s_{(j+1)^2}$ so that the sequence (s_n) contains constant subsequences of arbitrary length.

If $n\in N$, $f(n)=k$ and $j^2<k\leq(j+1)^2$ then $m_{k-1}\leq n<m_k\leq m_{(j+1)^2}$ so that

$$s_{f(n)}\leq\frac{1}{f(n)}\frac{\beta_{n+f(n)}^{n+f(n)}}{\beta_n^n}\qquad (n\in N)\,.$$

Now let $\mathcal{R}$ be a complex Hilbert space with $\dim\mathcal{R}=2$. Let $\{a,b\}$ be its orthonormal basis and let $c=\frac{1}{\sqrt2}(a+b)$, $d=\frac{1}{\sqrt2}(a-b)$. Note that $\{c,d\}$ is also an orthonormal basis of $\mathcal{R}$.

For $x\in\mathcal{R}$, $x\neq 0$ we denote by P_x the orthogonal projection in $\mathcal{B}(\mathcal{R})$ onto the one-dimensional space spanned by $\{x\}$. For any integer $n\geq 0$ write

$$\begin{aligned}A_n&=(I-P_a)+s_0s_1\dots s_nP_a=P_b+s_0s_1\dots s_nP_a\,,\\ B_n&=(I-P_b)+s_0s_1\dots s_nP_b=P_a+s_0s_1\dots s_nP_b\,,\\ C_n&=(I-P_c)+s_0s_1\dots s_nP_c=P_d+s_0s_1\dots s_nP_c\,.\end{aligned}$$

Note that $A_0=B_0=C_0=I$. Define the sequence $\{R_n\}_{n\geq 0}$ of operators in $\mathcal{B}(\mathcal{R})$ as follows:
$I,A_1,\ I,B_1,\ I,C_1,\ I,A_1,A_2,A_1,\ I,B_1,B_2,B_1,\ I,C_1,C_2,C_1,$
$I,A_1,A_2,A_3,A_2,\dots$

More precisely, if $i,k\in N$ then

$$R_n=\begin{cases}A_i & \text{if } n=m_k+i\,, & 0\leq i\leq k+1\,,\\ A_i & \text{if } n=m_k+2(k+1)-i\,, & 1\leq i\leq k\,,\\ B_i & \text{if } n=m_k+2(k+1)+i\,, & 0\leq i\leq k+1\,,\\ B_i & \text{if } n=m_k+4(k+1)-i\,, & 1\leq i\leq k\,,\\ C_i & \text{if } n=m_k+4(k+1)+i\,, & 0\leq i\leq k+1\,,\\ C_i & \text{if } n=m_{k+1}-i\,, & 1\leq i\leq k\,.\end{cases}$$

For $n \in N$ set $g(n) = i$ if and only if $R_n \in \{A_i, B_i, C_i\}$. By the definition of $f(n)$ we have $g(n) \le f(n)$ for all $n \ge 0$.

Note that R_n is invertible, $\|R_n\| = 1$ and

$$\|R_{n+1}R_n^{-1}\| = \max\Bigl\{1, \frac{s_0 s_1 \cdots s_{g(n+1)}}{s_0 s_1 \cdots s_{g(n)}}\Bigr\}$$

where $|g(n+1) - g(n)| = 1$. If $g(n+1) > g(n)$ then $\|R_{n+1}R_n^{-1}\| \le 1$. If $g(n+1) < g(n)$ then $\|R_{n+1}R_n^{-1}\| = \frac{1}{s_{g(n)}} \le \frac{1}{s_{f(n)}}$. Thus $\|R_{n+1}R_n^{-1}\| \le \frac{1}{s_{f(n)}}$ $(n \in N)$. For $0 \le i < j$ we have

$$\|R_j R_i^{-1}\| \le \|R_j R_{j-1}^{-1}\| \cdot \|R_{j-1} R_{j-2}^{-1}\| \cdots \|R_{i+1} R_i^{-1}\| \le \frac{1}{s_{f(j-1)} s_{f(j-2)} \cdots s_{f(i)}}.$$

Let H be the orthogonal sum of infinitely many copies of $\mathcal{R}$

$$H = R \oplus R \oplus \cdots \tag{1}$$

For $n \ge 0$ set

$$\alpha_n = s_{f(n)} \frac{\beta_{n+1}^{n+1}}{\beta_n^n} \qquad \text{and} \qquad T_n = \alpha_n R_{n+1} R_n^{-1}.$$

Let $T \in \mathcal{B}(H)$ be the weighted shift with weights T_n,

$$T(x_0 \oplus x_1 \oplus \cdots) = 0 \oplus T_0 x_0 \oplus T_1 x_1 \oplus \cdots$$

We show that T satisfies the required conditions.

Let $n \ge 1$. Then

$$T^n\Bigl(\bigoplus_{i=0}^{\infty} x_i\Bigr) = \underbrace{0 \oplus \cdots \oplus 0}_{n} \oplus \bigoplus_{i=0}^{\infty} \alpha_i \alpha_{i+1} \cdots \alpha_{i+n-1} R_{n+i} R_i^{-1} x_i.$$

Thus

$$\begin{aligned}\|T^n\| &= \sup_i \alpha_i \alpha_{i+1} \dots \alpha_{i+n-1} \|R_{n+i} R_i^{-1}\| \\ &\le \sup_i \frac{s_{f(i)} s_{f(i+1)} \dots s_{f(i+n-1)}}{s_{f(i+n-1)} \cdots s_{f(i)}} \frac{\beta_{i+1}^{i+1}}{\beta_i^i} \frac{\beta_{i+2}^{i+2}}{\beta_{i+1}^{i+1}} \cdots \frac{\beta_{i+n}^{i+n}}{\beta_{i+n-1}^{i+n-1}} \\ &\le \sup_i \frac{\beta_{i+n}^{i+n}}{\beta_i^i} \le \sup_i \frac{\beta_{i+n}^{i+n}}{\beta_{i+n}^i} = \sup_i \beta_{i+n}^n \le \beta_n^n .\end{aligned}$$

Hence

$$\|T^n\|^{1/n} \le \beta_n \qquad (n \ge 1).$$

The above defined operator-weighted shift T is reflexive since it has injective weights of dimension 2 [2, Corollary 3.5]. We shall show that $\{T\}' = \operatorname{Alg} T$ and then T is also hyperreflexive. Similarly as in [5, p. 281] let $(U_{ij})_{i,j\geq 0}$ be the matrix of an operator $U \in \{T\}'$ in the decomposition (1). Then

$$0 = TU - UT = \begin{pmatrix} -U_{01}T_0 & -U_{02}T_1 & -U_{03}T_2 & \dots \\ T_0U_{00} - U_{11}T_0 & T_0U_{01} - U_{12}T_1 & T_0U_{02} - U_{13}T_2 & \dots \\ T_1U_{10} - U_{21}T_0 & T_1U_{11} - U_{22}T_1 & T_1U_{12} - U_{23}T_2 & \dots \\ T_2U_{20} - U_{31}T_0 & T_2U_{21} - U_{32}T_1 & T_2U_{22} - U_{33}T_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since T_n's are invertible we obtain from the first row $U_{0i} = 0$ for all $i \geq 1$. Similarly we obtain by induction $U_{ij} = 0$ if $i < j$, i.e., the matrix U is lower triangular.

Further, for $i \geq j \geq 1$, we have $T_{i-1}U_{i-1,j-1} - U_{ij}T_{j-1} = 0$ so that

$$U_{ij} = T_{i-1}U_{i-1,j-1}T_{j-1}^{-1}.$$

Thus for $i, n \geq 0$ we have by induction

$$\begin{aligned} U_{n+i,n} &= T_{n+i-1}T_{n+i-2}\cdots T_iU_{i0}T_0^{-1}\cdots T_{n-1}^{-1} \\ &= (T_{n+i-1}T_{n+i-2}\cdots T_0)S_i(T_{n-1}T_{n-2}\cdots T_0)^{-1} \\ &= \alpha_n\alpha_{n+1}\cdots\alpha_{n+i-1}R_{n+i}S_iR_n^{-1}\,, \end{aligned}$$

where $S_i = (T_{i-1}T_{i-2}\dots T_0)^{-1}U_{i0}$.

We are going to show now that each S_i is a scalar multiple of identity. Fix $i \geq 0$. Suppose that $S_ia = \lambda_ia + \mu_ib$. To show that $\mu_i = 0$ find $k \in N$, $k > i$ such that $s_k = s_{k-1} = \dots = s_{k-i}$. Let $n = m_{k-1} + k$. Then $R_n = A_k$, $R_{n+i} = A_{k-i}$, $f(n) = f(n+1) = \dots = f(n+i) = k$ and we have

$$\begin{aligned} \|U\| &\geq \|U_{n+i,n}\| \geq \|U_{n+i,n}a\| = \alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}\left\|R_{n+i}S_iR_n^{-1}a\right\| \\ &= \frac{\alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}}{s_0s_1\dots s_k}\|A_{k-i}(\lambda_ia + \mu_ib)\| \\ &= \frac{\alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}}{s_0s_1\dots s_k}\|s_0s_1\dots s_{k-i}\lambda_ia + \mu_ib\| \\ &\geq |\mu_i|\frac{\alpha_n\alpha_{n+1}\dots\alpha_{n+i-1}}{s_0s_1\dots s_k} = |\mu_i|\frac{s_k^i}{s_0\dots s_k}\frac{\beta_{n+i}^{n+i}}{\beta_n^n} \\ &\geq |\mu_i|\frac{s_k^i}{s_{k-i}\dots s_k}\frac{\beta_{n+i}^{n+i}}{\beta_n^n} = |\mu_i|\frac{1}{s_k}\frac{\beta_{n+i}^{n+i}}{\beta_n^n} \\ &\geq |\mu_i|k\frac{\beta_n^n}{\beta_{n+k}^{n+k}}\frac{\beta_{n+i}^{n+i}}{\beta_n^n} = |\mu_i|k\left(\frac{\beta_{n+i}}{\beta_{n+k}}\right)^{n+i}\frac{1}{\beta_{n+k}^{k-i}} \geq |\mu_i|k\,. \end{aligned}$$

Since k could have been chosen arbitrarily large, we conclude that $\mu_i = 0$. Thus $S_i a = \lambda_i a$. Similarly (for $n = m_{k-1} + 3k$ and $n = m_{k-1} + 5k$, respectively) we can prove that $S_i b = \lambda_i' b$ and that $S_i c = \lambda_i'' c$ for some complex numbers λ_i', λ_i''. Thus

$$\frac{1}{\sqrt{2}}\lambda_i''(a+b) = \lambda_i'' c = S_i c = S_i\Big(\frac{1}{\sqrt{2}}(a+b)\Big) = \frac{1}{\sqrt{2}}\lambda_i a + \frac{1}{\sqrt{2}}\lambda_i' b.$$

Thus $\lambda_i = \lambda_i'' = \lambda_i'$, i.e., $S_i = \lambda_i I$. Hence $U_{n+i,n} = \lambda_i T_{n+i-1} T_{n+i-2} \dots T_n$ for all $i, n \geq 0$.

Observe that the only non-zero entries of the matrix of the operator T^i are $(T^i)_{n+i,n} = T_{n+i-1}T_{n+i-2}\dots T_n$ for $n = 0, 1, 2, \dots$ and so formally $U = \sum \lambda_i T^i$.

The rest of the proof is exactly the same as that of Lemma 2.3 in [3]. The operator U can be written as a formal power series $\sum \lambda_i T^i$. The series need not converge but its Cesaro means converge to U strongly. So the commutant of T coincides with $\operatorname{Alg} T$ and therefore it is reflexive. This finishes the proof of Theorem 1.

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