Charles University in Prague
Faculty of Mathematics and Physics

DIPLOMA THESIS

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Shape optimization in problems governed by generalised Navier–Stokes equations

Mathematical Institute of Charles University

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Study program:	Mathematics
Branch:	Matematical Modelling

I am very grateful to my supervisor prof. Jaroslav Haslinger for his patient and careful guidance. Next I want to thank doc. Josef Málek for many worthy advices, which have helped me in the theoretical part of the work, and dr. Jaroslav Hron for help with the numerical computations. Finally my sincerest thank belongs to my parents who have supported me during the period of my studies.

I declare that I have worked out this thesis by myself using only the literature cited here. I agree with lending of this thesis from the library of Faculty of Mathematics and Physics.

Prague, April 16, 2004 Jan Stebel

Contents

1 Introduction **5**

2 Steady flow of a non-Newtonian fluid **8**
- 2.1 Classical formulation 8
- 2.2 Weak formulation 10
  - 2.2.1 Boundary conditions 10
  - 2.2.2 Function spaces 10
  - 2.2.3 Weak solution 15
- 2.3 Energy estimates 15
- 2.4 Existence and uniqueness 18

3 Shape optimization problem **23**
- 3.1 Formulation of the problem 23
- 3.2 Existence of an optimal solution 23

4 Numerical results **30**
- 4.1 Numerical solution of the state problem 30
- 4.2 Numerical solution of shape optimization problem 32

5 Conclusion **36**

A Auxiliary tools **38**

B Properties of the Sobolev spaces **38**

Název práce: Tvarová optimalizace v úlohách řízených zobecněnými Navier–Stokesovými rovnicemi
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Abstrakt: V práci se řeší problém optimalizace tvaru vstupní komory, která je součástí strojů na výrobu papíru a která přivádí směs "voda+dřevní hmota" do výrobního procesu. Cílem je navrhnout takový tvar, který zajišťuje a priori daný průběh rychlosti směsi na výtokové části. Z matematického hlediska se jedná o úlohu optimálního řízení, kdy řídící proměnnou je tvar oblasti, která představuje vstupní komoru, stavovou úlohou je zobecněný Navier-Stokesův systém s netriviálními okrajovými podmínkami. Cílem je teoretické studium této úlohy (důkaz existence řešení), její diskretizace a numerická realizace.
Klíčová slova: Tvarová optimalizace, vstupní komora papírenského stroje, nestlačitelná nenewtonovská tekutina, algebraický model turbulence

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Abstract: We study the shape optimization problem for the paper machine headbox which distributes a mixture of water and wood fibers in the paper making process. The aim is to find a shape which a priori ensures the given velocity profile on the outlet part. The mathematical formulation leads to the optimal control problem in which the control variable is the shape of the domain representing the header, the state problem is represented by the generalised Navier-Stokes system with nontrivial boundary conditions. The objective is to analyze theoretically this problem (proof of the existence of a solution), its discretization and the numerical realization.
Keywords: Optimal shape design, paper machine headbox, incompressible non-Newtonian fluid, algebraic turbulence model

1 Introduction

For many years paper belongs to the most used every day's tools. Thus its invention can be considered as one of the most important steps both in the technological and cultural progress. About 19 centuries ago ancient Chinese developed the paper making technique using the bark and hemp. Since that time many improvements have been made in order to reduce the costs and enhance the quality, production speed and environmental compatibility. Today's paper production has become a complex process.

Recently the paper machine technology has been achieved mostly through the experimental work in pilot plants. With increasing speeds and sophisticated machines this approach has become too expensive and time-consuming so that more effective methods must be used to bring further development. One of such methods is mathematical modelling and the numerical simulation. However the experimental research is still needed to verify the simulated results.

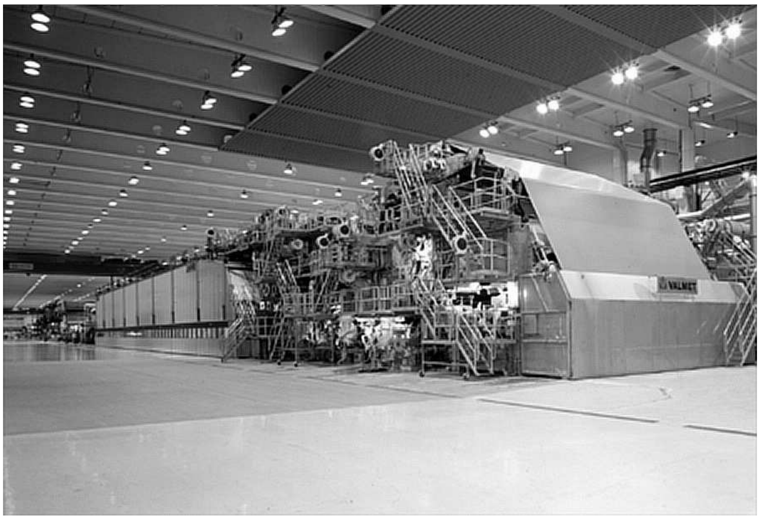

Figure 1: Paper machine viewed from the wet end (reproduced from http://www.csc.fi).

The first component in the paper making process is the headbox which is located at the wet end of a paper machine. The headbox shape and the fluid flow phenomena taking place there largely determines the quality of the produced paper. The first flow passage in the headbox is a dividing manifold, called the header. It is designed to distribute the fiber suspension on the wire so that the produced paper has an optimal basis weight and fiber orientation across the whole width of a paper machine. The aim of this work is to find an optimal shape for the back wall of the header so that the outlet flow rate distribution from the headbox results in an optimal paper quality.

The paper making pulp (also called the fibre suspension, furnish or stock) is a mixture of wood fibres, water, filler clays and various chemicals at concentration of 1% solids to 99 % water by weight. In the large-scale simulation it is sufficient to perform one-phase modelling with the fluid being an incompressible liquid. The presence of only a few solid particles in the suspension changes its properties to a non-Newtonian fluid. Flows of such fluids are mathematically described by the generalized Navier–Stokes equations (2.3). In the flow model the effect of turbulence must be taken into account as it is a desirable phenomenon in the paper making process. Here we use an algebraic turbulence model based on the so-called mixing-length.

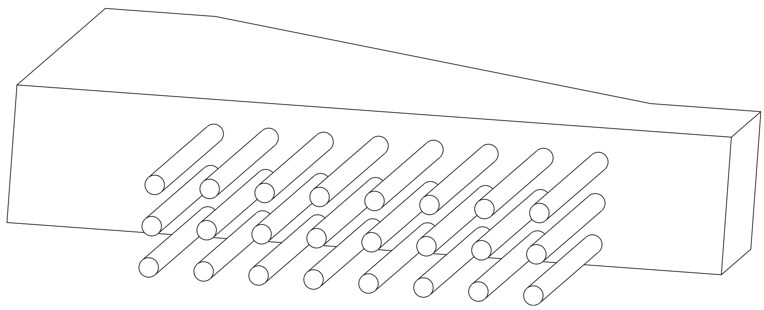

Figure 2: The header.

On Figure 2 the geometry of the header is shown. The inlet is on the left and the so-called recirculation on the right hand side. Typically about 10 % of the fluid flows out through the recirculation. The main outlet is performed by a number (usually several hundreds or thousands) of small tubes. This fact presents a difficulty in the numerical simulation and thus the complicated geometry of the tube bank is replaced by an effective medium using the homogenization technique. It introduces a nonstandard boundary condition of the form

$$T_{22} = -\sigma |u_\nu| u_\nu, \tag{1.1}$$

where T, u_ν, σ are the stress tensor, normal component of the velocity and the coefficient of suction, respectively. Another simplification is done using depth-averaged equations and two-dimensional geometry. Such model gives still satisfactory results while the computations become less time-consuming.

This work was motivated by some previous papers: The fluid flow model which is used here has been derived and studied numerically in [2]. The shape optimization problem has also been solved numerically and the results are presented in [3]. Both fluid flow model and shape optimization problem have been studied there formally without establishing existence results. Therefore our primary goal is to give the theoretical analysis of the flow equations and of the whole optimization problem, further the discretisation and finally the numerical solution.

The text is organized as follows. In Section 2 we present the fluid flow model and analyze the existence of a solution. The existence proof is based on appropriate energy estimates and the Galerkin method. The shape optimization problem is then formulated in Section 3 and the existence of an optimal shape is established. The continuous dependence of solutions to state problems with respect to shape variations is the most important result of this part. A discretization of these problems and numerical results are presented in Section 4. The elementary mathematical tools as well as notation are listed in Appendix.

2 Steady flow of a non-Newtonian fluid

2.1 Classical formulation

For describing the fluid flow we will use a two–dimensional stationary model. First we define the geometry of the problem.

Let $L_1, L_2, L_3 > 0$, $H_1 \geq H_2 > 0$, $\alpha_{max} \geq \alpha_{min} > 0$, $\gamma > 0$ be given and suppose that $\alpha \in \mathcal{U}_{ad}$, where

$$\begin{aligned}\mathcal{U}_{ad} = \Big\{\alpha \in C^{0,1}([0,L]);\ \alpha_{min} \leq \alpha \leq \alpha_{max},\\ \alpha|_{[0,L_1]} = H_1,\ \alpha|_{[L_1+L_2,L]} = H_2,\\ |\alpha'| \leq \gamma \text{ a.e. in } [0,L]\Big\}.\end{aligned} \tag{2.1}$$

Here $C^{0,1}([0,L])$ denotes the set of Lipschitz continuous functions on $[0,L]$ and $L = L_1 + L_2 + L_3$. With any $\alpha \in \mathcal{U}_{ad}$ we associate the domain $\Omega(\alpha)$:

$$\Omega(\alpha) = \Big\{(x_1,x_2) \in \mathbb{R}^2; 0 < x_1 < L, 0 < x_2 < \alpha(x_1)\Big\} \tag{2.2}$$

and introduce the system of admissible domains

$$\mathcal{O} = \big\{\Omega;\ \exists\ \alpha \in \mathcal{U}_{ad} :\ \Omega = \Omega(\alpha)\big\}.$$

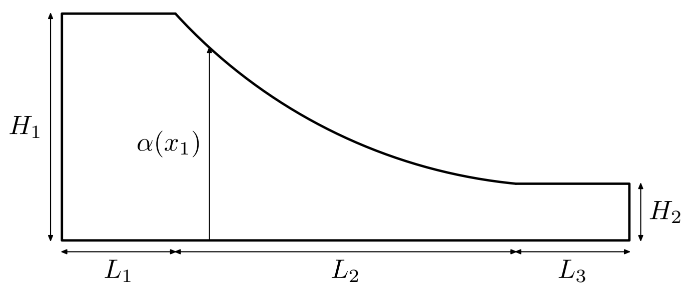

Figure 3: Geometry of the domain Ω.

Further we will need the domains

$$\widehat{\Omega} = (0,L) \times (0,\alpha_{max}),$$

$$\Omega_0 = \big((0,L_1) \times (0,H_1)\big) \cup \big((0,L) \times (0,\alpha_{min})\big) \cup \big((L_1+L_2,L) \times (0,H_2)\big)$$

for which it holds that $\Omega_0 \subset \Omega \subset \widehat{\Omega}$ for all $\Omega \in \mathcal{O}$.

Let us remark that $\Omega(\alpha) \in C^{0,1}$ for all $\alpha \in \mathcal{U}_{ad}$, where $C^{0,1}$ is the system of bounded domains with Lipschitz continuous boundaries [1]. We will denote

[1]See e.g. [5], Chapter 1 for the definition

the parts of the boundary $\partial\Omega(\alpha)$ as follows:

$$\begin{aligned}
\Gamma_D &= \Big\{(x_1,x_2)\in\partial\Omega(\alpha); x_1=0 \text{ or } x_1=L\Big\}\\
\Gamma_{out} &= \Big\{(x_1,x_2)\in\partial\Omega(\alpha); L_1\le x_1\le L_1+L_2, x_2=0\Big\}\\
\Gamma_\alpha &= \Big\{(x_1,x_2)\in\partial\Omega(\alpha); L_1\le x_1\le L_1+L_2, x_2=\alpha(x_1)\Big\}\\
\Gamma_f &= \partial\Omega(\alpha)\setminus\big(\Gamma_D\cup\Gamma_{out}\cup\Gamma_\alpha\big).
\end{aligned}$$

The components Γ_D, Γ_{out} and Γ_f are fixed for every $\alpha\in\mathcal{U}_{ad}$.

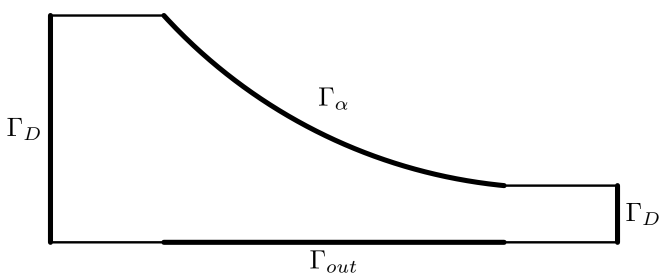

Figure 4: Parts of the boundary $\partial\Omega$.

The fluid motion in $\Omega(\alpha)$ is described by the generalised Navier–Stokes system

$$\left.\begin{aligned}
-\operatorname{div}\ T(\varepsilon(u),p)+\rho(u\cdot\nabla)u &= 0\\
\operatorname{div}\ u &= 0
\end{aligned}\right\}\text{in } \Omega. \tag{2.3}$$

Here u means the velocity, p the pressure, ρ is the density of the fluid and the stress tensor T is defined by the following formulae:

$$T_{ij}(\varepsilon(u),p)=-p\delta_{ij}+2\mu(|\varepsilon(u)|)\varepsilon_{ij}(u),$$

$$\mu(|\varepsilon(u)|):=\mu_0+\mu_t(s)=\mu_0+\rho l_{m,\alpha}^2|\varepsilon(u)|,\ \ \mu_0>0,$$

where μ_0 is a constant laminar viscosity and $\mu_t(|\varepsilon(u)|)$ stands for a turbulent viscosity. The function $l_{m,\alpha}$ represents an algebraic model of turbulence and it has the following form (see [3] for more details):

$$l_{m,\alpha}(x)=\frac{1}{2}\alpha(x_1)\left[0.14-0.08\Big(1-\frac{2d_\alpha(x)}{\alpha(x_1)}\Big)^2-0.06\Big(1-\frac{2d_\alpha(x)}{\alpha(x_1)}\Big)^4\right],$$

$$d_\alpha(x)=\min\Big\{x_2,\alpha(x_1)-x_2\Big\}, x\in\Omega(\alpha).$$

The symbol $\varepsilon(u)$ means the symmetric part of the gradient of u whose components are

$$\varepsilon_{ij}(u)=\frac{1}{2}\bigg(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\bigg),\ \ i,j=1,2$$

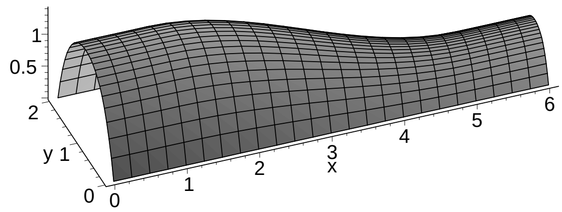

Figure 5: Graph of the function $l_{m,\alpha}$.

and $|\varepsilon(u)|$ means its norm

$$|\varepsilon(u)| = \left(\frac{1}{2}\sum_{i,j=1}^{2} \varepsilon_{ij}(u)\varepsilon_{ij}(u)\right)^{1/2}.$$

The equations are completed by the following boundary conditions:

$$\begin{array}{rcll} u &=& 0 & \text{on } \Gamma_f \cup \Gamma_\alpha, \\ u &=& u_D & \text{on } \Gamma_D, \\ u\cdot\tau = u_1 &=& 0 & \text{on } \Gamma_{out}, \\ \sum_{i,j=1}^{2} T_{ij}\nu_i\nu_j = T_{22} &=& -\sigma|u_2|u_2 & \text{on } \Gamma_{out}, \end{array} \tag{2.4}$$

where ν, τ stands for the unit normal, tangential vector, respectively and $\sigma > 0$ is a given suction coefficient.

By the classical solution we mean any velocity field $u \in \left(\mathcal{C}^2(\Omega(\alpha))\right)^2 \cap \left(\mathcal{C}^1(\overline{\Omega(\alpha)})\right)^2$ and pressure $p \in \mathcal{C}^1(\Omega(\alpha)) \cap \mathcal{C}(\overline{\Omega(\alpha)})$ satisfying (2.3) and (2.4).

2.2 Weak formulation

2.2.1 Boundary conditions

Throughout the paper we assume that there exists a function $u_0 \in \left(W^{1,3}(\Omega_0)\right)^2$, which satisfies the Dirichlet boundary conditions in the sense of traces, i.e.

$$u_0|_{\Gamma_D} = u_D, \quad u_0|_{\partial\Omega_0\setminus(\Gamma_D\cup\Gamma_{out})} = 0, \quad u_0\cdot\tau|_{\Gamma_{out}} = 0$$

and, in addition, $\operatorname{div} u_0 = 0$ in Ω_0. We extend u_0 by zero on $\widehat{\Omega}\setminus\Omega_0$. Then, due to the boundary conditions, $u_0 \in \left(W^{1,3}(\widehat{\Omega})\right)^2$ and $\operatorname{div} u_0 = 0$ in $\widehat{\Omega}$.

2.2.2 Function spaces

For any $\alpha \in \mathcal{U}_{ad}$ we denote

$$\mathcal{V}(\alpha) = \left\{\varphi \in (\mathcal{C}^\infty(\overline{\Omega(\alpha)}))^2;\ \operatorname{div}\ \varphi = 0 \text{ in } \Omega(\alpha)\right\}$$

$$\mathcal{V}_0(\alpha) = \Big\{\varphi = (\varphi_1, \varphi_2) \in \mathcal{V}(\alpha);\ \varphi_1 \in \mathcal{C}_0^\infty(\Omega(\alpha)), \\ \operatorname{dist}(\operatorname{supp}(\varphi_2), \partial\Omega(\alpha) \setminus \Gamma_{out}) > 0\Big\}$$

and define the spaces

$$\begin{aligned} W(\alpha) &= \overline{\mathcal{V}(\alpha)}^{\|\cdot\|_\alpha}, \\ W_0(\alpha) &= \overline{\mathcal{V}_0(\alpha)}^{\|\cdot\|_\alpha}, \end{aligned}$$

where the norm $\|\cdot\|_\alpha$ is defined by

$$\begin{aligned} \|v\|_\alpha &:= \|v\|_{1,2,\Omega(\alpha)} + \|M_\alpha \varepsilon(v)\|_{3,\Omega(\alpha)} \\ &= \|v\|_{1,2,\Omega(\alpha)} + \Big(\textstyle\sum_{i,j=1}^2 \big\|M_\alpha \varepsilon_{ij}(v)\big\|^3_{3,\Omega(\alpha)} \Big)^{1/3}, \\ M_\alpha(x) &:= \Big(l_{m,\alpha}(x) \Big)^{2/3}, x \in \Omega(\alpha). \end{aligned}$$

Further we define the set

$$W_{u_0}(\alpha) = \big\{ v \in W(\alpha);\ v - u_0 \in W_0(\alpha) \big\}.$$

Remark 2.1. *It is very easy to verify that the norms $\|\cdot\|_\alpha$ and $\|\cdot\|_{1,2,\Omega(\alpha)} + \|M_\alpha|\varepsilon(\cdot)|\|_{3,\Omega(\alpha)}$ are equivalent in $W(\alpha)$.*

Remark 2.2. *The seminorm*

$$|v|_\alpha := \|\nabla v\|_{2,\Omega(\alpha)} + \|M_\alpha \varepsilon(v)\|_{3,\Omega(\alpha)}$$

is due to the Friedrichs inequality a norm in $W_0(\alpha)$, which is equivalent with $\|v\|_\alpha$.

Lemma 2.1. *$W(\alpha)$ is a separable reflexive Banach space.*

Proof. We will use the fact that a closed subspace of a separable reflexive space is also separable and reflexive (for details see [6], Chapter 8). Let us define the space

$$S := \big(L^2(\Omega(\alpha))\big)^2 \times \big(L^2(\Omega(\alpha))\big)^{2\times 2} \times \big(L^3(\Omega(\alpha))\big)^{2\times 2}$$

which is a separable reflexive Banach space with the norm

$$\|(v, w, z)\|_S := \|v\|_{2,\Omega(\alpha)} + \|w\|_{2,\Omega(\alpha)} + \|z\|_{3,\Omega(\alpha)},\ (v, w, z) \in S.$$

Further define the mapping $\mathcal{I} : W(\alpha) \to S$ by the formula

$$\mathcal{I}(v) := (v, \nabla v, M_\alpha \varepsilon(v)).$$

Then $\mathcal{I}$ is an isomorphism of $W(\alpha)$ onto $S_\alpha := \mathcal{I}(W(\alpha))$ and

$$\forall v \in W(\alpha)\ \|\mathcal{I}(v)\|_S = \|v\|_\alpha.$$

We show that S_α is a closed subspace of S. Let $\{v_n\} \subset W(\alpha)$ and $\mathcal{I}(v_n) \to (v, w, z)$ in S. Then clearly $w = \nabla v$ in $\Omega(\alpha)$. Moreover

$$\forall f \in L^{3/2}(\Omega(\alpha))\ \forall i,j = 1,2\ \int_{\Omega(\alpha)} M_\alpha \varepsilon_{ij}(v_n) f\, dx \to \int_{\Omega(\alpha)} z_{ij} f\, dx,\ z = (z_{ij})_{i,j=1}^2.$$

Because $\nabla v_n \to \nabla v$ in $L^2(\Omega(\alpha))$ and $M_\alpha \in L^\infty(\Omega(\alpha))$, also

$$\int_{\Omega(\alpha)} M_\alpha \varepsilon_{ij}(v_n)\varphi\, dx \to \int_{\Omega(\alpha)} M_\alpha \varepsilon_{ij}(v)\varphi\, dx\ \forall \varphi \in \mathcal{C}^\infty(\overline{\Omega(\alpha)}).$$

Since $\mathcal{C}^\infty(\overline{\Omega(\alpha)})$ is dense in $L^{3/2}(\Omega(\alpha))$, we have $z = M_\alpha \varepsilon(v)$ in $\Omega(\alpha)$. The fact that $\operatorname{div} v = 0$ in $\Omega(\alpha)$ is readily seen. Finally, for any $\delta > 0$ there exists $v_n \in \{v_n\}$ and $\varphi_n \in \mathcal{V}(\alpha)$ such that

$$\|v - v_n\|_\alpha \leq \delta/2$$

$$\|v_n - \varphi_n\|_\alpha \leq \delta/2.$$

From this and the triangle inequality we have

$$\|v - \varphi_n\|_\alpha \leq \delta,$$

meaning that $v \in W(\alpha)$ and $(v, w, z) = \mathcal{I}(v) \in S_\alpha$. □

Definition 2.1. *Define the operator* $A_\alpha : W(\alpha) \to \big(W(\alpha)\big)^*$ *by the formula*

$$\big\langle A_\alpha(v), w \big\rangle_\alpha := \sum_{i,j=1}^{2} \int_\Omega M_\alpha^3 |\varepsilon(v)| \varepsilon_{ij}(v)\varepsilon_{ij}(w) dx;\ v, w \in W(\alpha).$$

Here $\big\langle \cdot, \cdot \big\rangle_\alpha$ *denotes the duality pairing between* $\big(W(\alpha)\big)^*$ *and* $W(\alpha)$*.*

Remark 2.3. *The fact that* $A_\alpha(v) \in \big(W(\alpha)\big)^*$ *follows from the Hölder's inequality.*

In what follows we will use the Einstein summation convention, i.e. $a_i b_i := \sum_{i=1}^n a_i b_i$.

Remark 2.4. *Since* $M_\alpha = 0$ *on* $\partial\Omega(\alpha) \setminus \Gamma_D$ *it can be extended by zero on* $\widehat{\Omega} \setminus \Omega(\alpha)$*. The resulting function, which is continuous in* $\overline{\widehat{\Omega}}$ *and which will be used in the next analysis, will be denoted by* $\tilde{M}_\alpha$*.*

The following lemma is needed in order to prove a useful relation between the functions α and $l_{m,\alpha}$.

Lemma 2.2. *Let (X_1, ρ_1), (X_2, ρ_2), (X_3, ρ_3) be metric spaces and consider functions $f_n : X_1 \to X_2$, $n \in \mathbb{N}$, $f : X_1 \to X_2$, $g : X_2 \to X_3$ such that g is uniformly continuous in X_2 and*

$$f_n \rightrightarrows f \text{ in } X_1.$$

Then

$$g \circ f_n \rightrightarrows g \circ f \text{ in } X_1.$$

Proof. Choose $\delta > 0$. Then there exists $\eta > 0$ such that for every $x, y \in X_2$, $\rho_2(x, y) < \eta$,

$$\rho_3(g(x) - g(y)) < \delta.$$

Further there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $x \in X_1$ it holds

$$\rho_2(f_n(x), f(x)) < \eta$$

from which the lemma follows. □

Lemma 2.3. *(Some properties of M_α and A_α, $\alpha \in \mathcal{U}_{ad}$)*

(i) If $\alpha_n \rightrightarrows \alpha$ in $[0, L]$ then $\tilde{M}_{\alpha_n} \rightrightarrows \tilde{M}_\alpha$ in $\overline{\hat{\Omega}}$.

(ii) A_α is monotone in $W(\alpha)$:

$$\left\langle A_\alpha(v) - A_\alpha(w), v - w \right\rangle_\alpha \geq 0 \ \forall v, w \in W(\alpha),$$

and strictly monotone in $W_0(\alpha)$, i.e. the previous inequality is sharp for $v \neq w$, where $v, w \in W_0(\alpha)$.

(iii) A_α is continuous operator in $W(\alpha)$.

Proof. (i) Let us define the function l_m by the formulae

$$l_m(x) := \frac{1}{2} x_1 \left(0.14 - 0.08 d^2(x) - 0.06 d^4(x)\right),$$

$$d(x) := \left(1 - \frac{2 \min\{x_2, x_1 - x_2\}}{x_1}\right),$$

$x \in [\alpha_{min}, \alpha_{max}] \times [0, \alpha_{max}]$. Then it is easy to verify that l_m is continuous in $[\alpha_{min}, \alpha_{max}] \times [0, \alpha_{max}]$ and that

$$l_{m,\alpha}(x) = l_m(\alpha(x_1), x_2) \quad \forall \alpha \in \mathcal{U}_{ad}.$$

From this and Lemma 2.2 it follows that

$$l_{m,\alpha_n} \rightrightarrows l_{m,\alpha} \text{ in } \overline{\hat{\Omega}}.$$

(ii) For any $B_1, B_2 \in \mathbb{R}^{2\times 2}$ we have:

$$\begin{aligned}\big(|B_1|B_1 - |B_2|B_2\big)\cdot\big(B_1 - B_2\big)& \\ = |B_1|^3 + |B_2|^3 - \big(|B_1| + |B_2|\big)B_1\cdot B_2& \\ \geq |B_1|^3 + |B_2|^3 - \big(|B_1| + |B_2|\big)|B_1||B_2|& \\ = \big(|B_1| + |B_2|\big)\big(|B_1| - |B_2|\big)^2 \geq 0.&\end{aligned} \tag{2.5}$$

Now, let us assume that everywhere in (2.5) the equality holds. Then

$$|B_1| = |B_2|$$
$$|B_1||B_1 - B_2|^2 = 0$$

and thus $B_1 = B_2$.

Let $u, v \in W(\alpha)$. Then

$$\begin{aligned}&\big\langle A_\alpha(u) - A_\alpha(v), u - v\big\rangle_\alpha \\ &\quad = \int_{\Omega(\alpha)} M_\alpha^3 \underbrace{\big(|\varepsilon(u)|\varepsilon_{ij}(u) - |\varepsilon(v)|\varepsilon_{ij}(v)\big)\varepsilon_{ij}(u-v)}_{\geq 0}\ dx \geq 0\end{aligned} \tag{2.6}$$

as follows from (2.5). To prove strict monotonicity of A_α assume that $u, v \in W_0(\alpha)$ and the expression (2.6) equals zero. Then necessarily $\varepsilon(u) = \varepsilon(v)$ a.e. in $\Omega(\alpha)$. From the Korn and the Friedrichs inequality it follows that $u = v$ a.e. in $\Omega(\alpha)$.

(iii) Let $v_n \to v$ in $W(\alpha)$. Then also $M_\alpha|\varepsilon(v_n)| \to M_\alpha|\varepsilon(v)|$ in $L^3\big(\Omega(\alpha)\big)$. We want to show that $A_\alpha(v_n) \to A_\alpha(v)$ in $\big(W(\alpha)\big)^*$. Indeed:

$$\begin{aligned}&\big|\big\langle A_\alpha(v_n) - A_\alpha(v), w\big\rangle_\alpha\big| \\ &\quad\leq \left|\int_{\Omega(\alpha)} M_\alpha^3\big(|\varepsilon(v_n)|\varepsilon_{ij}(v_n - v) + (|\varepsilon(v_n)| - |\varepsilon(v)|)\varepsilon_{ij}(v)\big)\varepsilon_{ij}(w)\ dx\right| \\ &\leq \|w\|_\alpha\Big(\|v_n\|_\alpha\|v_n - v\|_\alpha + \|v\|_\alpha\|M_\alpha\big(|\varepsilon(v_n)| - |\varepsilon(v)|\big)\|_{3,\Omega(\alpha)}\Big) \to 0\end{aligned}$$

holds for every $w \in W(\alpha)$. Therefore

$$\|A_\alpha(v_n) - A_\alpha(v)\|_{\big(W(\alpha)\big)^*} = \sup_{w\in W(\alpha)\setminus\{0\}} \frac{\big|\big\langle A_\alpha(v_n) - A_\alpha(v), w\big\rangle_\alpha\big|}{\|w\|_\alpha} \to 0$$

□

Definition 2.2. *For every $u, v, w \in \big(W^{1,2}(\Omega(\alpha))\big)^2$ we define the trilinear form b_α:*

$$b_\alpha(u, v, w) := \int_{\Omega(\alpha)} u_j \frac{\partial v_i}{\partial x_j} w_i\ dx$$

Remark 2.5. *From the imbedding of $W^{1,2}(\Omega(\alpha))$ into $L^4(\Omega(\alpha))$ it follows that b_α is continuous on $\Big[\big(W^{1,2}(\Omega(\alpha))\big)^2\Big]^3$.*

2.2.3 Weak solution

Now we are ready to give the weak formulation of the state problem. It can be formally derived by multiplying the equations (2.3) by a smooth solenoidal test function φ and integrating over $\Omega(\alpha)$ with the use of the Green theorem.

Definition 2.3. *We will say that $u := u(\alpha)$ is a weak solution of the state problem $(\mathcal{P}(\alpha))$ iff*

- $u \in W_{u_0}(\alpha)$
- *for every $\varphi \in W_0(\alpha)$ it holds:*

$$\begin{aligned} 2\mu_0 \int_{\Omega(\alpha)} \varepsilon_{ij}(u)\varepsilon_{ij}(\varphi)\ dx + 2\rho \big\langle A_\alpha(u), \varphi \big\rangle_\alpha + \\ +\rho b_\alpha(u,u,\varphi) + \sigma \int_{\Gamma_{out}} |u_2|u_2\varphi_2\ dS = 0 \end{aligned} \tag{2.7}$$

Remark 2.6. *Since $\varphi = 0$ on $\partial\Omega(\alpha) \setminus \Gamma_{out}$ and* $\operatorname{div}\varphi = 0$, *the pressure disappeared from the weak formulation.*

In the following subsections the existence of a weak solution to $(\mathcal{P}(\alpha))$ on a fixed domain $\Omega(\alpha)$, $\alpha \in \mathcal{U}_{ad}$ will be examined. Thus for simplicity of notations the letter α in the argument will be omitted (we shall write $\Omega := \Omega(\alpha)$, $W := W(\alpha)$, $b := b_\alpha$ etc. in what follows).

2.3 Energy estimates

Recall that the function u_0 is now defined in the whole $\widehat{\Omega}$ and it does not depend on $\alpha \in \mathcal{U}_{ad}$.

Theorem 2.4. *Let*

$$\|\nabla u_0\|_{3,\widehat{\Omega}} < C \qquad \text{and} \qquad \sigma > \frac{\rho}{2}, \tag{2.8}$$

where $C > 0$ is specified in (2.14) *below. Then there exists a constant $C_E := C_E(\|\nabla u_0\|_{3,\widehat{\Omega}})$ such that for any weak solution u of $(\mathcal{P}(\alpha))$ the following estimate holds*

$$\|\nabla u\|^2_{2,\Omega} + \|M|\varepsilon(u)|\|^3_{3,\Omega} + \int_{\Gamma_{out}} |u_2|^3\ dS \le C_E \tag{2.9}$$

Remark 2.7. *From the proof it will be seen that estimate* (2.9) *holds with a constant C_E independent of $\alpha \in \mathcal{U}_{ad}$. In particular, the zero extension of $u(\alpha)$ from $\Omega(\alpha)$ to $\widehat{\Omega}$ still satisfies* (2.9).

Proof of Theorem 2.4. We use $\varphi := u - u_0$ as a test function in $(\mathcal{P}(\alpha))$ and estimate each term on the left of $(\mathcal{P}(\alpha))$ from below.

(i) The first term can be estimated by means of Hölder's, Young's and Korn's inequalities:

$$\int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(u-u_0)dx \geq \frac{C_{Korn}^2}{2}\|\nabla u\|_{2,\Omega}^2 - \frac{1}{2}\|\nabla u_0\|_{2,\widehat{\Omega}}^2 \tag{2.10}$$

The Korn inequality is applied to the zero extension of u from Ω to $\widehat{\Omega}$ with the constant $C_{Korn} := C_{Korn}(\widehat{\Omega})$ which is independent of $\alpha \in \mathcal{U}_{ad}$.

(ii) The second term can be estimated by using the Hölder and the Young inequality:

$$\big\langle A(u), u-u_0 \big\rangle \geq \frac{1}{3}\|M|\varepsilon(u)|\|_{3,\Omega}^3 - \frac{1}{3}\|M|\varepsilon(u_0)|\|_{3,\widehat{\Omega}}^3 \tag{2.11}$$

(iii) The convective term can be rearranged as follows:

$$b(u,u,u-u_0) = \underbrace{b(u,u-u_0,u-u_0)}_{J_1} + \underbrace{b(u,u_0,u)}_{J_2} \underbrace{-b(u,u_0,u_0)}_{J_3}$$

Since $u = u_0$ on $\partial\Omega \setminus \Gamma_{out}$ and $\operatorname{div} u = 0$ in Ω, we have

$$\begin{aligned} J_1 &= \int_\Omega u_j \frac{\partial}{\partial x_j}\Big(\frac{|u-u_0|^2}{2}\Big)dx = \int_{\partial\Omega}(u\cdot\nu)\frac{|u-u_0|^2}{2}dS - \\ &- \int_\Omega \operatorname{div} u \frac{|u-u_0|^2}{2}dx \geq -\frac{1+\eta}{2}\int_{\Gamma_{out}}|u_2|^3\, dS - C_\eta \int_{\Gamma_{out}}|u_{02}|^3\, dS \end{aligned} \tag{2.12}$$

for any $\eta > 0$ with $C_\eta > 0$ depending on η.

The term J_2 can be estimated using the imbedding $\tilde{W}^{1,2}(\widehat{\Omega}) \hookrightarrow L^3(\widehat{\Omega})$:

$$J_2 \geq -\|\nabla u_0\|_{3,\widehat{\Omega}}\|u\|_{3,\Omega}^2 \geq -C_{Imb}^2\|\nabla u_0\|_{3,\widehat{\Omega}}\|\nabla u\|_{2,\Omega}^2 \tag{2.13}$$

where $\tilde{W}^{1,2}(\widehat{\Omega})$ is the subspace of functions from $W^{1,2}(\widehat{\Omega})$ which are equal to zero on the top of $\widehat{\Omega}$, i.e. on $\widehat{\Gamma} = (0,L)\times\{\alpha_{max}\}$ and C_{Imb} is the norm of this imbedding.

Further

$$J_3 \geq -\|u\|_{3,\Omega}\|\nabla u_0\|_{3,\widehat{\Omega}}\|u_0\|_{3,\widehat{\Omega}} \geq -\eta_1\|\nabla u\|_{2,\Omega}^2 - C_{\eta_1}\|\nabla u_0\|_{3,\widehat{\Omega}}^4$$

holds for any $\eta_1 > 0$ with $C_{\eta_1} > 0$ depending on η_1 making use of the Friedrichs inequality on $\tilde{W}^{1,2}(\widehat{\Omega})$ and the imbedding of $\tilde{W}^{1,2}(\widehat{\Omega})$ into $L^3(\widehat{\Omega})$.

(iv) Finally the boundary term can be estimated as follows:

$$\int_{\Gamma_{out}}|u_2|u_2(u_2-u_{02})\, dS \geq (1-\eta_2)\int_{\Gamma_{out}}|u_2|^3\, dS - C_{\eta_2}\int_{\Gamma_{out}}|u_{02}|^3\, dS$$

holds for any $\eta_2 > 0$ with $C_{\eta_2} > 0$ depending on η_2.

Multiplying each term by the respective physical constant and summing them up we obtain that

$$\Big(\mu_0 C_{Korn}^2 - \rho C_{Imb}^2 \|\nabla u_0\|_{3,\widehat{\Omega}} - \rho\eta_1\Big)\|\nabla u\|_{2,\Omega}^2 + \frac{2}{3}\rho\|M|\varepsilon(u)|\|_{3,\Omega}^3 + \Big((1-\eta_2)\sigma - \rho\frac{1+\eta}{2}\Big)\int_{\Gamma_{out}} |u_2|^3\, dS$$
$$\leq C_E\Big(\|\nabla u_0\|_{2,\widehat{\Omega}}, \|M|\varepsilon(u_0)|\|_{3,\widehat{\Omega}}, \int_{\Gamma_{out}} |u_{02}|^3\, dS, \|\nabla u_0\|_{3,\widehat{\Omega}}\Big)$$

holds for any $\eta, \eta_1, \eta_2 > 0$ with a constant C_E which depends on the indicated arguments. Choosing

$$\|\nabla u_0\|_{3,\widehat{\Omega}} < \frac{\mu_0}{\rho}\left(\frac{C_{Korn}}{C_{Imb}}\right)^2 \tag{2.14}$$

$$\frac{\rho}{2} < \sigma$$

we finally arrive at (2.9). Here we also used the fact that all arguments appearing in C_E can be estimated by $\|\nabla u_0\|_{3,\widehat{\Omega}}$. □

Remark 2.8. *Assume that there exists a constant $\overline{C} > 0$ such that*

$$\forall \alpha \in \mathcal{U}_{ad} \quad \|M_\alpha^{-1}\|_{6,\Omega(\alpha)} \leq \overline{C}. \tag{2.15}$$

Then Theorem 2.4 holds for any $\|\nabla u_0\|_{3,\widehat{\Omega}}$ with the constant $C_E > 0$ independent of α.

Proof. Let $v \in W$ satisfy $v|_{\{(x_1,\alpha(x_1));\ x_1\in(0,L)\}} = 0$. Then the zero extension of v belongs to $\left(\tilde{W}^{1,2}(\widehat{\Omega})\right)^2$ and one can use the Korn inequality with the same constant $C_{Korn} > 0$ as in (2.10):

$$\|\nabla v\|_{2,\Omega} \leq C_{Korn}^{-1}\|M^{-1}M\varepsilon(v)\|_{2,\Omega}.$$

The Hölder inequality then yields

$$\|\nabla v\|_{2,\Omega} \leq \frac{\overline{C}}{C_{Korn}}\|M\varepsilon(v)\|_{3,\Omega}. \tag{2.16}$$

In the proof of Theorem 2.4 the term J_2 is estimated as follows:

$$J_2 \geq -C_{Imb}^2\|\nabla u\|_{2,\Omega}^2\|\nabla u_0\|_{3,\widehat{\Omega}} \geq -\left(\frac{C_{Imb}\overline{C}}{C_{Korn}}\right)^2 \|M\varepsilon(u)\|_{3,\Omega}^2\|\nabla u_0\|_{3,\widehat{\Omega}}$$

making use of (2.16). Using the Young inequality we obtain for any $\eta > 0$:

$$J_2 \geq -\eta\|M\varepsilon(u)\|_{3,\Omega}^3 - C_\eta\|\nabla u_0\|_{3,\widehat{\Omega}}^3$$

with a constant $C_\eta > 0$ depending on η. Finally, summing up all the terms multiplied by the respective constants, the term

$$\rho\left(\frac{1}{3}-\eta\right)\|M\varepsilon(u)\|_{3,\Omega}^3$$

appears on the left, which is positive for $\eta \in (0, \frac{1}{3})$. □

Remark 2.9. *In our case unfortunately, condition* (2.15) *is not satisfied since*

$$M(x_1, x_2) = O(x_2^{2/3}),\ x_2 \to 0+$$

for $x_1 \in (0, L)$ *fixed.*

2.4 Existence and uniqueness

Theorem 2.5 (Existence). *Let the assumptions of Theorem 2.4 be satisfied. Then there exists a weak solution of* $(\mathcal{P}(\alpha))$.

Proof. Will be done in two steps (for the sake of simplicity of notations we set $2\mu_0 = 2\rho = \sigma = 1$):

(i) Galerkin approximations

Let $\{\omega^s\}_{s=1}^{\infty}$ be a dense set in W_0 of linearly independent functions and denote the finite-dimensional subspace

$$K_N := \text{span}\{\omega^1, ..., \omega^N\}.$$

For every $N = 1, 2, ...$ we solve the Galerkin problem:

Find $u^N \in W$ such that

- $u^N - u_0 \in K_N$,
- Equation (2.7) is satisfied for all $\varphi \in K_N$.

Define a mapping $P_N : \mathbb{R}^N \to \mathbb{R}^N$ by

$$P_N(d^N)_s = \int_\Omega \varepsilon_{ij}(u^N)\varepsilon_{ij}(\omega^s)dx + \langle A(u^N), \omega^s\rangle +$$
$$+\frac{1}{2}b(u^N, u^N, \omega^s) + \int_{\Gamma_{out}} |u_2^N| u_2^N \omega_2^s \, dS;\ s = 1, ...N,$$

where

$$u^N(x) := u_0(x) + \sum_{r=1}^{N} d_r^N \omega^r(x).$$

Then the Galerkin problem is equivalent to:

Find $\overline{d}^N \in \mathbb{R}^N$ such that

$$P_N(\overline{d}^N) = 0. \tag{2.17}$$

Next we show that this nonlinear algebraic system has a solution by using Brouwer's theorem (respectively Corollary A.3). Clearly the mapping P_N is continuous. To prove the existence of a solution to (2.17) we need to verify that there exists $R > 0$ such that

$$\forall d^N \in \mathbb{R}^N, |d^N| = R : \; P_N(d^N) \cdot d^N > 0.$$

Using the same technique as in the proof of the energy estimate (2.9) we obtain:

$$\begin{aligned} P_N(d^N) \cdot d^N \geq C\Big(\|\nabla u^N\|_{2,\Omega}^2 + \|M|\varepsilon(u^N)|\|_{3,\Omega}^3 + \int_{\Gamma_{out}} |u_2^N|^3 \; dS\Big) \\ - C_E(\|\nabla u_0\|_{3,\widehat{\Omega}}) \geq C\|\nabla u^N\|_{2,\Omega}^2 - C_E(\|\nabla u_0\|_{3,\widehat{\Omega}}). \end{aligned}$$

For $|d^N|$ large enough the last term is positive. Indeed:

$$\begin{aligned} &\|\nabla u^N\|_{2,\Omega}^2 = \\ &= \|u_0\|_{2,\Omega}^2 + 2\sum_{r=1}^{N} d_r^N \big(\nabla u_0, \nabla \omega^r\big)_{0,\Omega} + \sum_{r,s=1}^{N} d_r^N d_s^N \big(\nabla \omega^r, \nabla \omega^s\big)_{0,\Omega} \\ &\geq \|u_0\|_{2,\Omega}^2 + 2\sum_{r=1}^{N} d_r^N (\nabla u_0, \nabla \omega^r)_{0,\Omega} + \beta |d^N|^2 \to \infty \; as \; |d^N| \to \infty. \end{aligned}$$

Here we used that the Gramm matrix of the linearly independent system $\{\omega^s\}_{s=1}^N$ is positive definite with a constant $\beta > 0$. The notation $(\cdot,\cdot)_{0,\Omega}$ stands for the scalar product in $L^2(\Omega)$.

From Corollary A.3 the existence of $\overline{d}^N \in \mathbb{R}^N$ solving (2.17) follows.

(ii) Limit passages

Energy estimate (2.9) holds for every u^N with the same constant $C_E(\|\nabla u_0\|_{3,\widehat{\Omega}})$. From this it follows that there exists $u \in W$ such that (a chosen subsequence is denoted again by the same index N)

$$u^N \rightharpoonup u \text{ in } W, \; N \to \infty \tag{2.18}$$

and also in $\big(W^{1,2}(\Omega)\big)^2$ because $\big(W^{1,2}(\Omega)\big)^* \subset W^*$. Trivially $u \in W_{u_0}$. Further we will use the compact imbedding of $W^{1,2}(\Omega)$ into $L^q(\Omega)$, and $L^q(\partial\Omega), \; q \in [1, +\infty)$ so that

$$\begin{aligned} &u^N \to u \text{ in } L^q(\Omega) \\ &u^N \to u \text{ in } L^q(\partial\Omega), \; N \to \infty. \end{aligned} \tag{2.19}$$

Let $\varphi \in W_0$ be given. Then

$$\int_\Omega \varepsilon_{ij}(u^N)\varepsilon_{ij}(\varphi)dx \to \int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(\varphi)dx,$$

$$\int_{\Gamma_{out}} |u_2^N|u_2^N\varphi_2 \; dS \to \int_{\Gamma_{out}} |u_2|u_2\varphi_2 \; dS,$$

$$b(u^N,u^N,\varphi) = b(u^N-u,u^N,\varphi) + b(u,u^N,\varphi) \to b(u,u,\varphi), \;\; N\to\infty$$

as follows from (2.18) and (2.19).

It remains to analyze the second term in (2.7). It is sufficient to show that

$$A(u^N) \rightharpoonup A(u) \text{ in } W^*, \;\; N\to\infty.$$

From the energy estimates we obtain boundedness of $\{A(u^N)\}$ in W^*:

$$\left|\left\langle A(u^N),\varphi\right\rangle\right| \le C\|\varphi\|_\alpha,$$

where C does not depend on N and therefore $A(u^N) \rightharpoonup \chi$ in W^*. To prove that $\chi = A(u)$ we use monotonicity of A:

$$\begin{gathered}\forall\psi\in W: \;\; 0 \le \left\langle A(u^N)-A(\psi),u^N-\psi\right\rangle = \\ = \left\langle A(u^N),u^N-u_0\right\rangle - \left\langle A(\psi),u^N-\psi\right\rangle - \left\langle A(u^N),\psi-u_0\right\rangle.\end{gathered} \tag{2.20}$$

Since $u^N - u_0 \in K_N$, the term $\left\langle A(u^N),u^N-u_0\right\rangle$ can be expressed by the remaining terms of the Galerkin identity. Therefore (2.20) reads as follows:

$$\begin{gathered}\int_\Omega \varepsilon_{ij}(u^N)\varepsilon_{ij}(u^N-u_0)dx \le -\frac{1}{2}b(u^N,u^N,u^N-u_0)- \\ -\int_{\Gamma_{out}} |u_2^N|u_2^N(u_2^N-u_{02}) \; dS - \left\langle A(\psi),u^N-\psi\right\rangle - \left\langle A(u^N),\psi-u_0\right\rangle.\end{gathered}$$

Letting $N\to\infty$ and using weak lower semi-continuity of the term on the left of the previous inequality and continuity of the remaining terms we obtain:

$$\begin{gathered}\int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(u-u_0)dx \le -\frac{1}{2}b(u,u,u-u_0)- \\ -\int_{\Gamma_{out}} |u_2|u_2(u_2-u_{02}) \; dS - \left\langle A(\psi),u-\psi\right\rangle - \left\langle \chi,\psi-u_0\right\rangle.\end{gathered} \tag{2.21}$$

Further, we use $u^L - u_0, \; L \le N$ as a test function in the Galerkin identity for u^N. Passing to the limit with $N\to\infty$ and then with $L\to\infty$ we have:

$$\begin{gathered}\int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(u^L-u_0)dx + \left\langle \chi,u^L-u_0\right\rangle + \\ +\frac{1}{2}b(u,u,u^L-u_0) + \int_{\Gamma_{out}} |u_2|u_2(u_2^L-u_{02}) \; dS = 0,\end{gathered}$$

$$
\begin{aligned}
&\int_\Omega \varepsilon_{ij}(u)\varepsilon_{ij}(u-u_0)dx + \big\langle \chi, u-u_0\big\rangle + \\
&+\frac{1}{2}b(u,u,u-u_0) + \int_{\Gamma_{out}} |u_2|u_2(u_2-u_{02})\; dS = 0,
\end{aligned} \tag{2.22}
$$

respectively. From (2.21) and (2.22) we arrive at the inequality

$$
0 \le \big\langle \chi - A(\psi), u-\psi \big\rangle \tag{2.23}
$$

We now use the so-called Minty trick. Instead of ψ we insert into (2.23) a function $u \pm \lambda\xi$, where $\lambda > 0,\ \xi \in W$:

$$
0 \le \big\langle \chi - A(u \pm \lambda\xi), \mp\lambda\xi \big\rangle.
$$

Dividing this inequality by λ we obtain for $\lambda \to 0$:

$$
0 \le \pm\big\langle \chi - A(u), \xi \big\rangle \ \forall \xi \in W,
$$

making use of continuity of A. Thus $\chi = A(u)$.

It remains to verify that u is a weak solution: Choose $\varphi \in W_0$. This function can be approximated by a sequence $\{\varphi^L\}$, $\varphi^L \in K_L$:

$$
\varphi^L \to \varphi \text{ in } W_0,\ L \to \infty.
$$

Substituting φ^L into the Galerkin identity for u^N, $N \ge L$ and letting $N \to \infty$ and next $L \to \infty$, we see that $(\mathcal{P}(\alpha))$ is satisfied for every $\varphi \in W_0$. Therefore u is a weak solution.

□

Theorem 2.6 (Uniqueness). *Let all the assumptions of Theorem 2.4 be satisfied and $\|\nabla u_0\|_{3,\widehat{\Omega}}$ be small enough. Then there exists exactly one solution to $(\mathcal{P}(\alpha))$.*

Proof. Let u and v be two solutions of $(\mathcal{P}(\alpha))$. We subtract the weak formulations for u and v with $\varphi = u - v \in W_0$ as a test function. We obtain:

$$
\begin{aligned}
2\mu_0\|\varepsilon(u-v)\|_{2,\Omega}^2 + \underbrace{2\rho\big\langle A(u)-A(v), u-v\big\rangle}_{\ge 0} + \sigma\int_{\Gamma_{out}} \underbrace{(|u_2|u_2 - |v_2|v_2)(u_2-v_2)}_{\ge 0}\; dS \\
= \rho b(v-u,v,v-u) + \rho b(u,v-u,v-u).
\end{aligned}
$$

We estimate the terms on the right hand side, making use of the Hölder inequality, the imbedding of $\tilde{W}^{1,2}(\widehat{\Omega})$ into $L^4(\widehat{\Omega})$ and the energy estimates:

$$
b(v-u,v,v-u) \le \|\nabla v\|_{2,\Omega}\|u-v\|_{4,\Omega}^2 \le C_E C_{Imb}^2 \|\nabla(u-v)\|_{2,\Omega}^2,
$$

$$b(u, v-u, v-u) \leq \|u\|_{4,\Omega}\|\nabla(u-v)\|_{2,\Omega}\|u-v\|_{4,\Omega} \leq C_E C_{Imb}^2 \|\nabla(u-v)\|_{2,\Omega}^2,$$

where C_{Imb} is the norm of the respective imbedding and $C_E = C_E\big(\|\nabla u_0\|_{3,\widehat{\Omega}}\big)$ is the constant from the energy estimates. Applying the Korn inequality on the left hand side, we finally obtain

$$\mu_0 C_{Korn}^2 \|\nabla(u-v)\|_{2,\Omega}^2 \leq 2\rho C_E C_{Imb}^2 \|\nabla(u-v)\|_{2,\Omega}^2,$$

from which it follows that $u = v$ a.e. in Ω if $2C_E < \frac{\mu_0}{\rho}\left(\frac{C_{Korn}}{C_{Imb}}\right)^2$.

□

Remark 2.10. *Let us observe that the bound guaranteeing uniqueness of the solution to* $(\mathcal{P}(\alpha))$ *is independent of* $\alpha \in \mathcal{U}_{ad}$.

3 Shape optimization problem

The aim of this part is to formulate a shape optimization problem and to prove the existence of a solution.

3.1 Formulation of the problem

We proved that on every domain $\Omega(\alpha) \in \mathcal{O}$ there exists at least one weak solution of the state problem $(\mathcal{P}(\alpha))$. Let $\mathcal{G}$ be the graph of the control–to–state (generally multi-valued) mapping:

$$\mathcal{G} := \{(\alpha, u);\ \alpha \in \mathcal{U}_{ad},\ u \text{ is a weak solution of } (\mathcal{P}(\alpha))\}\,.$$

Further let us define the cost functional $J : \mathcal{G} \to \mathbb{R}$ by

$$J : (\alpha, u) \mapsto \int_{\tilde{\Gamma}} |u_2 - z_D|^2 \; dS,\ u = (u_1, u_2), \tag{3.1}$$

where $z_D \in L^2(\tilde{\Gamma})$ is a given function representing the desired outlet velocity profile and $\tilde{\Gamma} \subset \Gamma_{out}$.

We now formulate the following problem:

$$\begin{aligned} &Find\ (\alpha^*, u^*) \in \mathcal{G}\ so\ that \\ &J(\alpha^*, u^*) \leq J(\alpha, u)\ \forall(\alpha, u) \in \mathcal{G}. \end{aligned} \tag{$\mathbb{P}$}$$

In the next definition we introduce convergence of a sequence of domains.

Definition 3.1. *Let $\{\Omega(\alpha_n)\}$, $\alpha_n \in \mathcal{U}_{ad}$ be a sequence of domains. We will say that $\{\Omega(\alpha_n)\}$ converges to $\Omega(\alpha)$, shortly $\Omega(\alpha_n) \rightsquigarrow \Omega(\alpha)$, iff $\alpha_n \rightrightarrows \alpha$ in $[0, L]$.*

Lemma 3.1. *System $\mathcal{O}$ is compact with respect to convergence introduced in Definition 3.1.*

Proof. Functions from $\mathcal{U}_{ad}$ are uniformly bounded and equicontinuous, which means (by Arzelà-Ascoli's theorem A.4) that $\mathcal{U}_{ad}$ is compact in the sense of uniform convergence. □

3.2 Existence of an optimal solution

First let us recall that the function u_0 which realizes the boundary conditions is the same for all domains $\Omega \in \mathcal{O}$.

We now rewrite $(\mathcal{P}(\alpha))$, $\alpha \in \mathcal{U}_{ad}$ using the formulation on the fixed domain $\widehat{\Omega}$:

$$\begin{aligned} &2\mu_0 \int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}(\alpha))\varepsilon_{ij}(\tilde{\varphi}) \; dx + 2\rho\langle \tilde{A}_\alpha(\tilde{u}(\alpha)), \tilde{\varphi}\rangle_{\widehat{\Omega}} + \\ &+ \rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}) + \sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)|\tilde{u}_2(\alpha)\tilde{\varphi}_2 \; dS = 0 \quad \forall\varphi \in W_0(\alpha), \end{aligned} \tag{$\widehat{\mathcal{P}}(\alpha)$}$$

where the symbol ˜ stands for the zero extension of functions from $\Omega(\alpha)$ on $\widehat{\Omega}$,

$$\langle \tilde{A}_\alpha(\tilde{u}(\alpha)), \tilde{\varphi}\rangle_{\widehat{\Omega}} := \int_{\widehat{\Omega}} \tilde{M}_\alpha^3 |\varepsilon(\tilde{u}(\alpha))| \varepsilon_{ij}(\tilde{u}(\alpha)) \varepsilon_{ij}(\tilde{\varphi})\, dx,$$

$$b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}) := \int_{\widehat{\Omega}} \tilde{u}_j(\alpha) \frac{\partial \tilde{u}_i(\alpha)}{\partial x_j} \tilde{\varphi}_i \, dx.$$

Further let $\widehat{W}(\alpha) = W^{1,2}_{div}(\alpha) \cap W^{1,3}_M(\alpha)$, where

$$W^{1,2}_{div}(\alpha) := \Big\{ v \in \big(W^{1,2}(\Omega(\alpha))\big)^2;\ \operatorname{div} v = 0 \text{ in } \Omega(\alpha) \Big\},$$
$$W^{1,3}_M(\alpha) := \Big\{ v \in \big(W^{1,2}(\Omega(\alpha))\big)^2;\ M_\alpha |\varepsilon(v)| \in L^3(\Omega(\alpha)) \Big\}$$

and define

$$\widehat{W}_{u_0}(\alpha) := \Big\{ v \in \widehat{W}(\alpha);\ v \text{ satisfies the Dirichlet conditions } (2.4)_1 - (2.4)_4 \text{ on } \partial\Omega(\alpha) \Big\}.$$

Remark 3.1. *It holds that $W_{u_0}(\alpha) \subseteq \widehat{W}_{u_0}(\alpha)$. The question is, if these spaces are identical. This is in fact the problem of density and at this moment we do not know the answer.*

From the part dealing with the existence of a solution to $(\mathcal{P}(\alpha))$ we can use the energy estimate

$$\|\nabla \tilde{u}(\alpha)\|^2_{2,\widehat{\Omega}} + \|\tilde{M}_\alpha |\varepsilon(\tilde{u}(\alpha))|\|^3_{3,\widehat{\Omega}} + \int_{\Gamma_{out}} |u_2(\alpha)|^3 \, dS \le C_E(\|\nabla u_0\|_{3,\widehat{\Omega}}) \quad (3.2)$$

which holds for every $(\alpha, u(\alpha)) \in \mathcal{G}$ with the constant $C_E(\|\nabla u_0\|_{3,\widehat{\Omega}})$ independent of α.

Theorem 3.2. *Let $\alpha_n \rightrightarrows \alpha$ in $[0, L]$, $\alpha_n, \alpha \in \mathcal{U}_{ad}$ and $u_n := u(\alpha_n)$ be a solution of $(\mathcal{P}(\alpha_n))$. Then there exists $\widehat{u} \in \big(W^{1,2}(\widehat{\Omega})\big)^2$ and a subsequence of $\{\tilde{u}_n\}$ (denoted by the same symbol) such that*

$$\begin{aligned} &\tilde{u}_n \rightharpoonup \widehat{u} \text{ in } \big(W^{1,2}(\widehat{\Omega})\big)^2 \\ &\tilde{M}_{\alpha_n} \varepsilon(\tilde{u}_n) \rightharpoonup \tilde{M}_\alpha \varepsilon(\widehat{u}) \text{ in } \big(L^3(\widehat{\Omega})\big)^{2\times 2},\ n \to \infty. \end{aligned} \quad (3.3)$$

In addition, the function $u(\alpha) := \widehat{u}|_{\Omega(\alpha)}$ solves $(\mathcal{P}(\alpha))$ provided that $u(\alpha) \in W_{u_0}(\alpha)$.

Proof. Let us denote $\tilde{M}_n := \tilde{M}_{\alpha_n}$, $\Omega_n := \Omega(\alpha_n)$, $\langle \cdot, \cdot \rangle_n := \langle \cdot, \cdot \rangle_{\alpha_n}$ etc.

From energy estimate (3.2) it follows that

$$\|\tilde{u}_n\|_{1,2,\widehat{\Omega}} \leq C$$

$$\|\tilde{M}_n \varepsilon(\tilde{u}_n)\|_{3,\widehat{\Omega}} \leq C,$$

where $C > 0$ does not depend on n. Therefore we can choose a subsequence of $\{\tilde{u}_n\}$ (denoted again by the same symbol) so that

$$\begin{aligned} \tilde{u}_n &\rightharpoonup \widehat{u} \text{ in } \left(W^{1,2}(\widehat{\Omega})\right)^2 \\ \tilde{M}_{\alpha_n} \varepsilon(\tilde{u}_n) &\rightharpoonup \widehat{z} \text{ in } \left(L^3(\widehat{\Omega})\right)^{2\times 2},\ n \to \infty. \end{aligned} \tag{3.4}$$

The following properties of $\widehat{u}$ and $\widehat{z}$ are easily verified:

(i) $\widehat{u} = 0$ in $\widehat{\Omega} \setminus \overline{\Omega}(\alpha)$, $\widehat{z} = 0$ in $\widehat{\Omega} \setminus \overline{\Omega}(\alpha)$,

(ii) $\widehat{z} = \tilde{M}_\alpha \varepsilon(\widehat{u})$ in $\widehat{\Omega}$,

(iii) $\operatorname{div} \widehat{u} = 0$ in $\widehat{\Omega}$,

(iv) $\widehat{u}$ satisfies the required Dirichlet boundary conditions on $\partial\Omega(\alpha)$.

We prove (ii). Since $\mathcal{C}^\infty(\overline{\widehat{\Omega}})$ is dense in $L^{3/2}(\widehat{\Omega})$, it is sufficient to show that

$$\int_{\widehat{\Omega}} \tilde{M}_n \varepsilon_{ij}(\tilde{u}_n)\psi_{ij}\ dx \to \int_{\widehat{\Omega}} \tilde{M}_\alpha \varepsilon_{ij}(\widehat{u})\psi_{ij}\ dx,\ n \to \infty,\ i,j = 1,2$$

holds for every $\psi \in \left(\mathcal{C}^\infty(\overline{\widehat{\Omega}})\right)^{2\times 2}$. Indeed:

$$\begin{aligned} &\left| \int_{\widehat{\Omega}} \left(\tilde{M}_n \varepsilon_{ij}(\tilde{u}_n)\psi_{ij} - \tilde{M}_\alpha \varepsilon_{ij}(\widehat{u})\psi_{ij} \right)\ dx \right| \leq \\ &\leq \int_{\widehat{\Omega}} |\tilde{M}_n - \tilde{M}_\alpha||\varepsilon_{ij}(\tilde{u}_n)\psi_{ij}|\ dx + \left| \int_{\widehat{\Omega}} \tilde{M}_\alpha \left(\varepsilon_{ij}(\tilde{u}_n) - \varepsilon_{ij}(\widehat{u})\right)\psi_{ij}\ dx \right| \to 0, \end{aligned}$$

making use that $\tilde{M}_n \rightrightarrows \tilde{M}_\alpha$ in $\widehat{\Omega}$, $(3.4)_1$ and the fact that $\tilde{M}_\alpha \psi_{ij} \in L^2(\widehat{\Omega})$.

Let $u(\alpha) := \widehat{u}|_{\Omega(\alpha)}$. Then (i)-(iv) implies that $u(\alpha) \in \widehat{W}_{u_0}(\alpha)$. Next we prove that $u(\alpha)$ solves $(\mathcal{P}(\alpha))$ provided that $u(\alpha) \in W_{u_0}(\alpha)$. We start from the definition of $(\widehat{\mathcal{P}}(\alpha_n))$:

$$\begin{aligned} 2\mu_0 \int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}_n)\varepsilon_{ij}(\tilde{\varphi})\ dx + 2\rho\langle \tilde{A}_n(\tilde{u}_n), \tilde{\varphi}\rangle_{\widehat{\Omega}} + \rho b_{\widehat{\Omega}}(\tilde{u}_n, \tilde{u}_n, \tilde{\varphi}) \\ + \sigma \int_{\Gamma_{out}} |\tilde{u}_{n2}|\tilde{u}_{n2}\tilde{\varphi}\ dS = 0 \quad \forall \varphi \in W_0(\alpha_n). \end{aligned} \tag{3.5}$$

Let $\varphi \in \mathcal{V}_0(\alpha)$ be an arbitrary function. Then $\tilde{\varphi}|_{\Omega_n} \in \mathcal{V}_0(\alpha_n)$ for n sufficiently large so that it can be used as a test function in (3.5). The limit

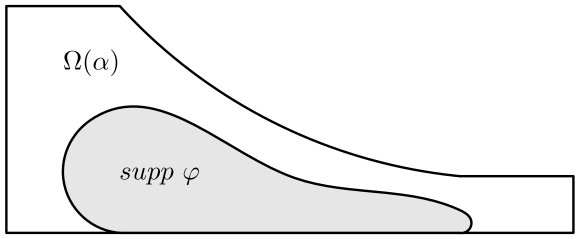

Figure 6: Support of the test function φ.

passage in the first, third and fourth term is a classical one:

$$\int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}_n)\varepsilon_{ij}(\tilde{\varphi})\ dx \to \int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}(\alpha))\varepsilon_{ij}(\tilde{\varphi})\ dx,$$

$$\begin{aligned} & b_{\widehat{\Omega}}(\tilde{u}_n, \tilde{u}_n, \tilde{\varphi}) = b_{\widehat{\Omega}}(\tilde{u}_n - \tilde{u}(\alpha), \tilde{u}_n, \tilde{\varphi}) + b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}_n, \tilde{\varphi}) \\ & \to b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}), \end{aligned} \tag{3.6}$$

$$\int_{\Gamma_{out}} |\tilde{u}_{n2}|\tilde{u}_{n2}\tilde{\varphi}_2\ dS \to \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)|\tilde{u}_2(\alpha)\tilde{\varphi}_2\ dS,\ n \to \infty.$$

The most difficult is to handle the second term. Let $B_n \in \big(W_0(\alpha)\big)^*$ be the functional defined by

$$\begin{aligned} 2\rho\big\langle B_n, \psi\big\rangle_\alpha := -2\mu_0 \int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}_n)\varepsilon_{ij}(\tilde{\psi})\ dx - \rho b_{\widehat{\Omega}}(\tilde{u}_n, \tilde{u}_n, \tilde{\psi}) - \\ - \sigma \int_{\Gamma_{out}} |\tilde{u}_{n2}|\tilde{u}_{n2}\tilde{\psi}\ dS \quad \forall \psi \in W_0(\alpha). \end{aligned}$$

From the energy estimate it follows:

$$\|B_n\|_{\big(W_0(\alpha)\big)^*} \leq C \quad \forall n \in \mathbb{N}.$$

Thus there exists $B \in \big(W_0(\alpha)\big)^*$ such that

$$B_n \rightharpoonup B. \tag{3.7}$$

In addition, if $\psi \in \mathcal{V}_0(\alpha)$ then $\tilde{\psi}|_{\Omega_n} \in \mathcal{V}_0(\alpha_n)$ for n sufficiently large and

$$\big\langle B_n, \psi\big\rangle_\alpha = \big\langle \tilde{A}_n(\tilde{u}_n), \tilde{\psi}\big\rangle_{\widehat{\Omega}}. \tag{3.8}$$

We use monotonicity of A_n on $W(\alpha_n)$. For any $\psi \in W(\alpha_n)$ we have

$$\begin{aligned} 0 \leq \big\langle A_n(u_n) - A_n(\psi), u_n - \psi\big\rangle_n = \big\langle A_n(u_n), u_n - u_0\big\rangle_n - \\ - \big\langle A_n(\psi), u_n - \psi\big\rangle_n - \big\langle A_n(u_n), \psi - u_0\big\rangle_n. \end{aligned} \tag{3.9}$$

In what follows we use ψ of the form

$$\tilde{\psi} = u_0 + \tilde{\varphi},$$

where $\varphi \in \mathcal{V}_0(\alpha)$ is fixed. Then

$$\langle A_n(u_n), \psi - u_0 \rangle_n = \langle \tilde{A}_n(\tilde{u}_n), \tilde{\varphi} \rangle_{\widehat{\Omega}} = \langle B_n, \varphi \rangle_\alpha \tag{3.10}$$

provided that n is large enough making use of (3.8). Since $u_n \in W_{u_0}(\alpha_n)$, the definition of $(\widehat{\mathcal{P}}(\alpha_n))$, (3.9) and (3.10) yield:

$$\begin{aligned} 2\mu_0 \int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}_n)\varepsilon_{ij}(\tilde{u}_n - u_0)\, dx &\le -\rho b_{\widehat{\Omega}}(\tilde{u}_n, \tilde{u}_n, \tilde{u}_n - u_0) \\ -\sigma \int_{\Gamma_{out}} |\tilde{u}_{n2}|\tilde{u}_{n2}(\tilde{u}_{n2} - u_{02})\, dS &- \langle A_n(\psi), u_n - \psi \rangle_n - \langle B_n, \varphi \rangle_\alpha . \end{aligned} \tag{3.11}$$

Letting $n \to \infty$ in (3.11) we obtain:

$$\begin{aligned} 2\mu_0 \int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}(\alpha))\varepsilon_{ij}(\tilde{u}(\alpha) - u_0)\, dx &\le -\rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{u}(\alpha) - u_0) \\ -\sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)|\tilde{u}_2(\alpha)(\tilde{u}_2(\alpha) - u_{02})\, dS &- \langle A_\alpha(\psi), u(\alpha) - \psi \rangle_\alpha - \langle B, \varphi \rangle_\alpha . \end{aligned} \tag{3.12}$$

Here we use the fact that

$$\langle A_n(\psi), u_n - \psi \rangle_n \to \langle A_\alpha(\psi), u(\alpha) - \psi \rangle_\alpha = \langle \tilde{A}_\alpha(\tilde{\psi}), \tilde{u}(\alpha) - \tilde{\psi} \rangle_{\widehat{\Omega}}.$$

Indeed: From $(3.4)_2$ and (ii) we know that

$$\tilde{M}_n \varepsilon(\tilde{u}_n) \rightharpoonup \tilde{M}_\alpha \varepsilon(\tilde{u}(\alpha)) \text{ in } \big(L^3(\widehat{\Omega})\big)^{2\times 2} \tag{3.13}$$

using that $\widehat{u} = \tilde{u}(\alpha)$. Further

$$\tilde{M}_n^2 |\varepsilon(\tilde{\psi})| \varepsilon(\tilde{\psi}) \to \tilde{M}_\alpha^2 |\varepsilon(\tilde{\psi})| \varepsilon(\tilde{\psi}) \text{ in } \big(L^{3/2}(\widehat{\Omega})\big)^{2\times 2}$$

since $M_n \rightrightarrows M_\alpha$ in $\widehat{\Omega}$ and $\tilde{\psi} \in \Big(W^{1,3}(\widehat{\Omega})\Big)^2$. From this and (3.13) we obtain that

$$\langle A_n(\psi), u_n \rangle_n \to \langle \tilde{A}_\alpha(\tilde{\psi}), \tilde{u}(\alpha) \rangle_\alpha .$$

The limit passage $\langle A_n(\psi), \psi \rangle_n \to \langle A_\alpha(\psi), \psi \rangle_\alpha$ is trivial.
By assumption there exists $w(\alpha) \in W_0(\alpha)$ such that $u(\alpha) = u_0 + w(\alpha)$. Then there exists a sequence $\{w_k\}$, $w_k \in \mathcal{V}_0(\alpha)$ such that

$$w_k \to w(\alpha) \text{ in } W(\alpha),\ k \to \infty. \tag{3.14}$$

Let $k \in \mathbb{N}$ be fixed. Then $\tilde{w}_k|_{\Omega_n} \in \mathcal{V}_0(\alpha_n)$ for n large enough. Therefore $\tilde{w}_k|_{\Omega_n}$ can be used as a test function in $(\mathcal{P}(\alpha_n))$. Inserting $\tilde{w}_k$ in $(\widehat{\mathcal{P}}(\alpha_n))$ and passing to the limit with $n \to \infty$ and then $k \to \infty$ we obtain:

$$2\mu_0 \int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}(\alpha))\varepsilon_{ij}(\tilde{w}(\alpha))\, dx + \langle B, w(\alpha)\rangle_\alpha + \rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{w}(\alpha)) + \sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)|\tilde{u}_2(\alpha)\tilde{w}_2(\alpha)\, dS = 0 \quad (3.15)$$

making use of (3.6), (3.7) and (3.14). From (3.12) and (3.15) we have:

$$-\langle \tilde{A}_\alpha(\tilde{\psi}), \tilde{u}(\alpha) - \tilde{\psi}\rangle_{\widehat{\Omega}} - \langle B, \varphi\rangle_\alpha + \langle B, w(\alpha)\rangle_\alpha \geq 0 \quad (3.16)$$

using that $\tilde{w}(\alpha) = \tilde{u}(\alpha) - u_0$. Since $\tilde{u}(\alpha) - \tilde{\psi} = \tilde{w}(\alpha) - \tilde{\varphi}$ we see that (3.16) can be written as follows:

$$\langle B - A_\alpha(\psi), w(\alpha) - \varphi\rangle_\alpha \geq 0 \quad \forall \varphi \in \mathcal{V}_0(\alpha). \quad (3.17)$$

From (3.17), density of $\mathcal{V}_0(\alpha)$ in $W_0(\alpha)$, continuity of A_α and the fact that $\psi = u_0|_{\Omega(\alpha)} + \varphi$, $\varphi \in \mathcal{V}_0(\alpha)$, we obtain

$$\langle B - A_\alpha(u_0 + z), w(\alpha) - z\rangle_\alpha \geq 0 \quad \forall z \in W_0(\alpha). \quad (3.18)$$

Let $z \in W_0(\alpha)$ be of the form $z = w(\alpha) \pm \lambda\theta$, $\lambda > 0$, where $\theta \in W_0(\alpha)$ is arbitrary. Then

$$\langle B - A_\alpha(u_0 + w(\alpha) + \lambda\theta), \pm\lambda\theta\rangle_\alpha \geq 0.$$

Dividing by λ and passing to the limit $\lambda \to 0+$ we finally obtain

$$B = A_\alpha(u_0 + w(\alpha)) = A_\alpha(u(\alpha)). \quad (3.19)$$

This, together with $(3.6)_1$-$(3.6)_3$ leads to

$$2\mu_0 \int_{\widehat{\Omega}} \varepsilon_{ij}(\tilde{u}(\alpha))\varepsilon_{ij}(\tilde{\varphi})\, dx + 2\rho\langle \tilde{A}_\alpha(\tilde{u}(\alpha)), \tilde{\varphi}\rangle_{\widehat{\Omega}} + \rho b_{\widehat{\Omega}}(\tilde{u}(\alpha), \tilde{u}(\alpha), \tilde{\varphi}) + \sigma \int_{\Gamma_{out}} |\tilde{u}_2(\alpha)|\tilde{u}_2(\alpha)\tilde{\varphi}_2\, dS = 0, \quad (3.20)$$

for every $\varphi \in \mathcal{V}_0(\alpha)$ and consequently also for $\varphi \in W_0(\alpha)$. □

Remark 3.2. *Under the assumptions which guarantee uniqueness of the solution to $(\mathcal{P}(\alpha))$ the whole sequence $\{\tilde{u}_n\}$ converges to $\tilde{u}(\alpha)$ in the sense of Theorem 3.2.*

Remark 3.3. *Let us comment on the condition $u(\alpha) \in W_{u_0}(\alpha)$. We mention two special cases when this condition is satisfied:*

(i) $W_{u_0}(\alpha) = \widehat{W}_{u_0}(\alpha)$,

(ii) it holds that $\alpha_n \leq \alpha$ *for every* $n \in \mathbb{N}$.

Proof. The case (i) is evident.

(ii) Let $\alpha_n \rightrightarrows \alpha$ in $[0, L]$, $\alpha_n \leq \alpha$ $\forall n \in \mathbb{N}$ and $\widehat{u}$ be a limit function of the sequence $\{\tilde{u}_n\}$ in the sense of Theorem 3.2. We denote

$$\begin{aligned} &w_n := u_n - u_0|_{\Omega(\alpha_n)} \in W_0(\alpha_n) \\ &w := \widehat{u}|_{\Omega(\alpha)} - u_0|_{\Omega(\alpha)} \in \widehat{W}_0(\alpha) := \widehat{W}_{u_0}(\alpha) \text{ setting } u_0 = 0. \end{aligned}$$

Since $\Omega(\alpha_n) \subset \Omega(\alpha)$ $\forall n \in \mathbb{N}$ we see that $\tilde{w}_n|_{\Omega(\alpha)} \in W_0(\alpha)$. From (3.3) it also holds that

$$\begin{aligned} &\tilde{w}_n \rightharpoonup w \text{ in } \left(W^{1,2}(\Omega(\alpha))\right)^2 \\ &\tilde{M}_{\alpha_n}\varepsilon(\tilde{w}_n) \rightharpoonup M_\alpha \varepsilon(w) \text{ in } \left(L^3(\Omega(\alpha))\right)^{2\times 2},\ n \to \infty. \end{aligned}$$

Using Theorem A.5 we know that there exists a sequence $\{\psi_n\}$ of convex combinations of $\{\tilde{w}_n\}$, i.e. $\psi_n = \sum_{k=1}^n a_k^n \tilde{w}_k$, $\sum_{k=1}^n a_k^n = 1$, $a_k^n \geq 0$, tending strongly to w in the norm of $W_0(\alpha)$. Therefore $w \in W_0(\alpha)$ and $\widehat{u}|_{\Omega(\alpha)} = u(\alpha) \in W_{u_0}(\alpha)$.

□

Theorem 3.3 (Existence of an optimal shape). *Let there exist a minimizing sequence* $\{(\alpha_n, u_n)\}$, $(\alpha_n, u_n) \in \mathcal{G}$, *of* $(\mathbb{P})$ *with an accumulation point* $(\alpha^*, u(\alpha^*))$ *such that* $u^*|_{\Omega(\alpha^*)} \in W_{u_0}(\alpha^*)$. *Then* $(\alpha^*, u^*|_{\Omega(\alpha^*)})$ *is an optimal pair for* $(\mathbb{P})$.

Proof. Without loss of generality we may assume that $\alpha_n \rightrightarrows \alpha^*$ in $[0, L]$. From the assumptions on the sequence $\{(\alpha_n, u_n)\}$ it follows that there exists its accumulation point (α^*, u^*) such that $(\alpha^*, u^*|_{\Omega(\alpha^*)}) \in \mathcal{G}$. Further

$$q = \inf_{(\alpha, u(\alpha)) \in \mathcal{G}} J(\alpha, u(\alpha)) = \lim_{n \to \infty} J(\alpha_n, u_n) = J(\alpha^*, u^*|_{\Omega(\alpha^*)}) \geq q$$

making use of continuity of J. □

4 Numerical results

4.1 Numerical solution of the state problem

Numerical computations of the flow problem are done by the open software Featflow, which is designed for solving Navier–Stokes-like problems. Description of this program can be found on its web page `http://www.featflow.de` and in [8]. We start with the formulation of the discrete version of $(\mathcal{P}(\alpha))$. Since the shape is fixed, the letter α will be dropped.

Let W_h, W_{0h} be finite dimensional spaces approximating W, W_0 respectively, $\{\omega_h^n\}$ be a basis of W_{0h} and $u_{0h} \in W_h$ be an approximation of u_0. We formulate the discrete problem as follows:

Find $u_h \in W_h$ such that $u_h - u_{0h} \in W_{0h}$ and

$$\begin{aligned} &2\mu_0 \int_\Omega \varepsilon_{ij}(u_h)\varepsilon_{ij}(\omega_h^n)\, dx + 2\rho\langle A(u_h), \omega_h^n\rangle + \rho b(u_h, u_h, \omega_h^n) + \\ &+ \sigma \int_{\Gamma_{out}} |u_{h2}|u_{h2}\omega_{h2}^n\, dS = 0, \quad n = 1, \dots, n_h, \end{aligned} \tag{P_h}$$

where n_h denotes the dimension of W_{0h}.

The discrete problem has 3 nonlinear terms. The second and third term are handled by the above mentioned program. To treat the nonlinearity arising from the boundary condition we use the fixed-point approach. Having the k-th iteration u_h^k at our disposal we approximate the boundary term

$$\int_{\Gamma_{out}} |u_{h2}^{k+1}|u_{h2}^{k+1}\omega_{h2}^n\, dS \approx \int_{\Gamma_{out}} |u_{h2}^k|u_{h2}^k\omega_{h2}^n\, dS,$$

which leads to the modified problem for u_h^{k+1}:

Find $u_h^{k+1} \in W_h$ such that $u_h^{k+1} - u_{0h} \in W_{0h}$ and

$$\begin{aligned} &2\mu_0 \int_\Omega \varepsilon_{ij}(u_h^{k+1})\varepsilon_{ij}(\omega_h^n)\, dx + 2\rho\langle A(u_h^{k+1}), \omega_h^n\rangle + \rho b(u_h^{k+1}, u_h^{k+1}, \omega_h^n) = \\ &= -\sigma \int_{\Gamma_{out}} |u_{h2}^k|u_{h2}^k\omega_{h2}^n\, dS, \quad n = 1, \dots, n_h. \end{aligned} \tag{4.1}$$

This modified problem is solved by Featflow using either the fixed-point or the Newton method. The algorithm reads as follows:

(1) Choose u_h^0 such that $u_h^0 - u_{0h} \in W_{0h}$ and $\varepsilon_{tol} > 0$, $k := 0$.

(2) Find u_h^{k+1} by solving (4.1).

(3) If $\|u_h^{k+1} - u_h^k\|_{2,\Gamma_{out}} \le \varepsilon_{tol}\|u_h^{k+1}\|_{2,\Gamma_{out}}$ then go to (5).

(4) $k := k + 1$, go to (2).

(5) Stop.

Alternatively we solved (P_h) using the fixed-point method updating the right hand side after each iteration.

In the first algorithm the nonlinearity arising from the boundary term is updated when (4.1) was solved "exactly". Unlike to this, the second algorithm updates this term inside of (4.1) and thus it should be more efficient. In the following table both methods are compared.

Algorithm and used method	Iterations	Time in seconds
1 - Fixed-point	12	863
1 - Newton	12	848
2 - Fixed-point	7	101

Table 1: Comparison of the used algorithms, stopping criterion $\varepsilon_{tol} = 10^{-6}$.

Let us mention that Featflow uses the mixed velocity-pressure formulation. For the discretisation of the function spaces the finite element method with a nonconforming finite element pair $\tilde{Q}_1/Q_0$ is used. Let Ω_h be a polygonal approximation of Ω and T_h a partition of Ω_h into quadrilaterals. For each $T \in T_h$ we denote by $\psi_T : \hat{T} \to T$ the bilinear transformation of $\hat{T} = [-1, 1]^2$ onto T and set

$$\tilde{Q}_1(T) := \left\{ q \circ \psi_T^{-1};\ q \in span\left\{1, \xi, \eta, \xi^2 - \eta^2\right\}\right\},$$
$$Q_0(T) := span\left\{1\right\},$$

where (ξ, η) denotes the local coordinate system in $\hat{T}$. The degrees of free-

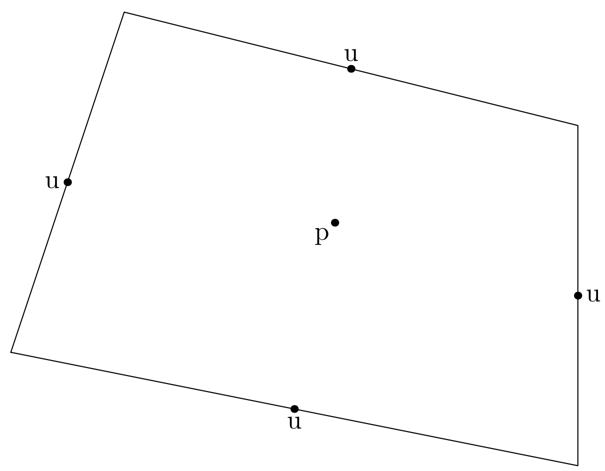

Figure 7: The finite element $\tilde{Q}_1/Q_0$

dom are given by the values at the midpoints of each edge for $\tilde{Q}_1$, and by the mean value for Q_0. This element pair satisfies the Babuška–Brezzi condition and enjoys the approximation property (we refer to [8], Chapter 3.1 for further details).

The function u_{0h} is given by the values of u_0 at the midpoints of edges belonging to the Dirichlet boundary and by zero elsewhere. Due to the velocity-pressure formulation this function does not need to be divergence-free.

In the model example the following size parameters (in meters) were chosen: $H_1 = 1.0$, $H_2 = 0.1$, $L_1 = 1.0$, $L_2 = 8.0$, $L_3 = 0.5$. The back wall Γ_α corresponds to the traditional linearly tapering design. These parameters in fact do not correspond to any existing headbox design.

The physical parameters are chosen as follows: the density $\rho = 1000$, the laminar viscosity $\mu_0 = 0.001$, the coefficient of the outflow boundary condition $\sigma = 1000$. The inflow and recirculation velocity (in m/s) is $u_D|_{\{0\}\times(0,H_1)} = (4(1 - (\frac{2}{H_1}x_2 - 1)^8), 0)$, $u_D|_{\{L\}\times(0,H_2)} = (1 - (\frac{2}{H_2}x_2 - 1)^8, 0)$, respectively.

The computational domain is discretized using 12288 elements. On Figure 8 the used computational mesh is shown. Size of the computed velocity field, the pressure filed and the viscosity is shown on Figure 9, 10, 11, respectively. Near the right end of Γ_{out} large shear and pressure changes occur, therefore the mesh is refined locally to get more accurate numerical results. On Figure 11 one can notice that the viscosity reaches its maximum in the middle of Ω while near $\partial\Omega \setminus \Gamma_D$ it diminishes to the value μ_0 due to the presence of the weight function.

4.2 Numerical solution of shape optimization problem

The set $\mathcal{U}_{ad}$ will be discretized by using Bézier functions.

Definition 4.1. *Let $\beta_0, \ldots, \beta_n \in \mathbb{R}$ be given. The expression*

$$P_n(\xi) = \sum_{i=0}^{n} \Theta_{i,n}\left(\frac{\xi - L_1}{L_2}\right)\beta_i, \ \xi \in [L_1, L_1 + L_2], \tag{4.2}$$

where

$$\Theta_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \ t \in [0,1],$$

is called a Bézier function of the n-th order on the interval $[L_1, L_1 + L_2]$. The points $C_i = \left(L_1 + \frac{i}{n}L_2, \beta_i\right), i = 0, \ldots, n$ are termed control points of P_n.

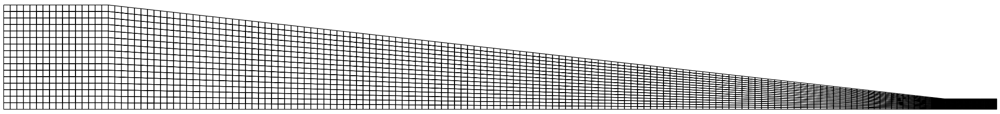

Figure 8: The computational mesh (3072 elements displayed).

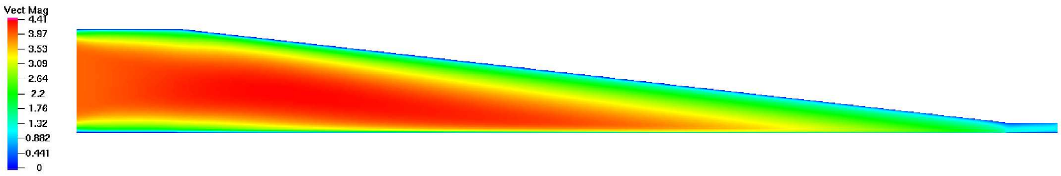

Figure 9: Size of the velocity vector.

Figure 10: Pressure field p/ρ.

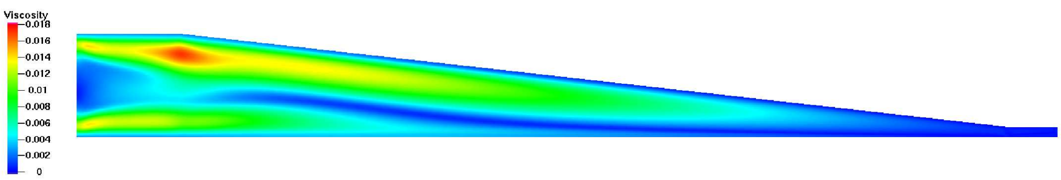

Figure 11: Kinematic viscosity μ/ρ.

We now define the set of admissible values of $\{\beta_i\}_{i=0}^n$:

$$\Delta_n = \Big\{\boldsymbol{\beta} = (\beta_0, \dots, \beta_n);\ \beta_0 = H_1,\ \beta_n = H_2, \\ \alpha_{min} \le \beta_i \le \alpha_{max},\ i = 1, \dots, n-1, \\ |\beta_i - \beta_{i-1}| \le \gamma \frac{L_2}{n},\ i = 1, \dots, n\Big\}. \tag{4.3}$$

With any $\boldsymbol{\beta} \in \Delta_n$ we associate the Bézier function P_n and denote by α_n the function

$$\alpha_n(\xi) = \begin{cases} H_1, & \xi \in [0, L_1] \\ P_n(\xi), & \xi \in (L_1, L_1 + L_2) \\ H_2, & \xi \in [L_1 + L_2, L]. \end{cases} \tag{4.4}$$

Then the discrete system $\mathcal{U}_{ad}^n$ of admissible functions is given by

$$\mathcal{U}_{ad}^n = \Big\{\alpha_n : [0, L] \to [\alpha_{min}, \alpha_{max}];\ \alpha_n \text{ is defined by (4.4) and } \boldsymbol{\beta} \in \Delta_n\Big\} \tag{4.5}$$

Due to the properties of Bézier's functions it satisfies the inclusion

$$\mathcal{U}_{ad}^n \subset \mathcal{U}_{ad}.$$

The system $\mathcal{O}$ of admissible domains will be now replaced by $\mathcal{O}_n = \{\Omega(\alpha_n);\ \alpha_n \in \mathcal{U}_{ad}^n\}$. Since all $\Omega(\alpha_n) \in \mathcal{O}_n$ are domains with a curved part of the boundary, functions $\alpha_n \in \mathcal{U}_{ad}^n$ will be replaced by their piecewise linear approximations $r_h\alpha_n$. Discrete state problems will be formulated and solved in polygonal domains $\Omega(r_h\alpha_n)$ whose shapes are still uniquely defined by $\boldsymbol{\beta} \in \Delta_n$. Starting from the design vector $\boldsymbol{\beta} \in \Delta_n$ we have the following chain of mappings:

$$\boldsymbol{\beta} \mapsto \Omega(\alpha_n) \mapsto \Omega(r_h\alpha_n) \mapsto u_h(\boldsymbol{\beta}) \mapsto J(\boldsymbol{\beta}),$$

where $u_h(\boldsymbol{\beta})$ is the solution of $(\mathcal{P}_h(r_h\alpha_n))$ and

$$J(\boldsymbol{\beta}) = \int_{\tilde{\Gamma}} |u_{h2}(\boldsymbol{\beta}) - z_D|^2 dS.$$

The discrete shape optimization problem then reads as follows:

$$\textit{Find } \boldsymbol{\beta}^* \in \Delta_n \textit{ such that } J(\boldsymbol{\beta}^*) \le J(\boldsymbol{\beta}) \ \ \forall \boldsymbol{\beta} \in \Delta_n. \tag{$\mathbb{P}_n$}$$

The minimization of J in $(\mathbb{P}_n)$ is realized by means of the student version of the optimization software KNITRO, which uses the trust-region method and the interior barier method for unconstrained, constrained optimization,

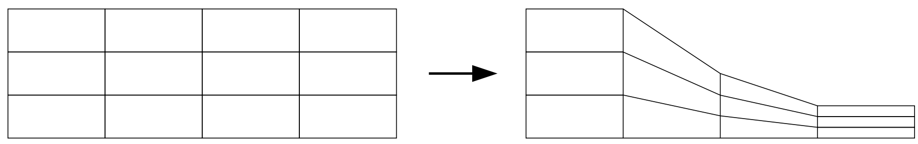

Figure 12: Creation of the mesh from the template.

respectively. The trust-region algorithm needs the gradient of J. We approximate it by the forward finite differences:

$$\frac{\partial J(\boldsymbol{\beta})}{\partial \beta_i} \approx \frac{J(\beta_0,\ldots,\beta_{i-1},\beta_i+\delta,\beta_{i+1},\ldots,\beta_n) - J(\boldsymbol{\beta})}{\delta},\ \delta > 0.$$

It is known that computations using this approach are sensitive with respect to the choice of δ. On the other hand the algorithm can be easily parallelized.

In the model example we used the Bézier functions of the 8-th order. The desired outlet velocity profile was chosen to be constant $z_D = -0.45$ on the part of the boundary $\tilde{\Gamma} = [1.5, 8.5] \times \{0\}$, the bounds $\alpha_{min} = 0.09$, $\alpha_{max} = 1.2$ and the parameter for the gradient computation $\delta = 10^{-3}$. The constraint on the derivatives of α_n was not used since

$$|\alpha_n'| \leq \frac{n}{L_2}(\alpha_{max} - \alpha_{min}) \text{ a.e. in } [0, L]\ \forall \alpha_n \in \mathcal{U}_{ad}^n.$$

The computational mesh on the domain $\Omega(r_h\alpha_n)$, $\alpha_n \in \mathcal{O}_n$ was created from a template on the reference domain by stretching in the vertical direction (see Figure 12). After 8 iterations the cost functional decreased from 3.98×10^{-2} to 1.12×10^{-3} (the convergence history is shown on Figure 15). Figure 13 shows the velocity profiles on Γ_{out} for the initial shape given by the linearly tapering header and for the optimized shape. The initial and optimized shape are depicted on Figure 14, where the vertical scale is $3\times$ enlarged in order to notice the difference better.

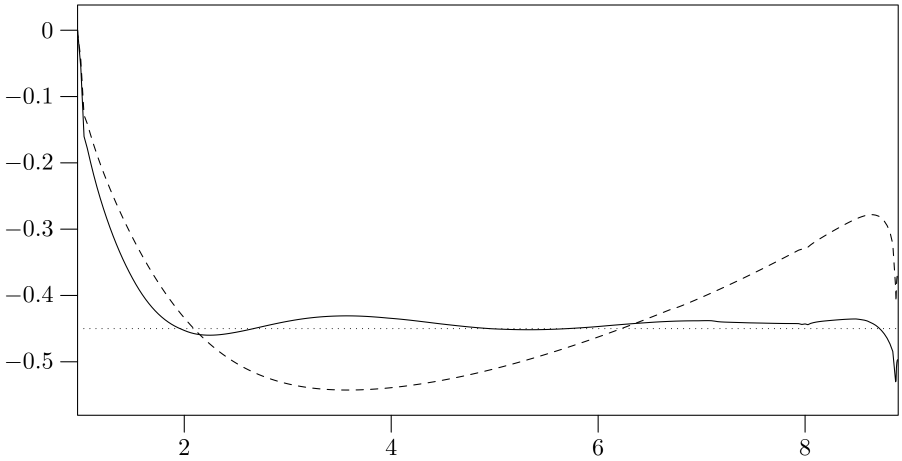

Figure 13: The original and the optimized velocity profile.

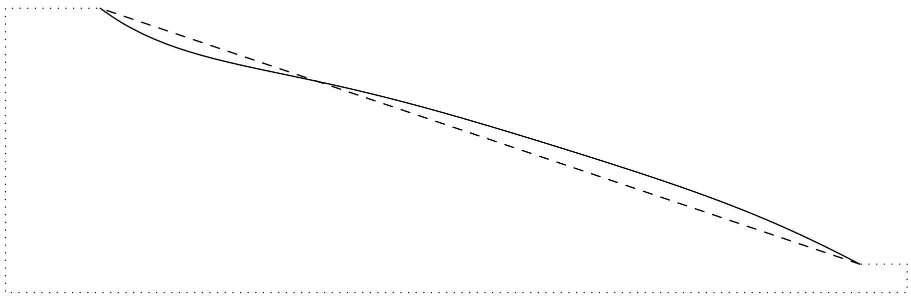

Figure 14: The original and the optimized domain shape.

5 Conclusion

The work consists of three parts. The first one deals with the existence proof for the generalised steady-state Navier–Stokes system. In the second part the shape optimization problem with the Navier–Stokes system as a state constraint is studied. Finally, the third part is devoted to the discretization of previous problems and presents numerical results.

Due to an algebraic turbulence model the weak formulation of the state problem involves the weighted Sobolev spaces. The existence and uniqueness of a solution is proved for small data and with a constraint imposed on the model parameters. The existence proof is based on energy estimates and the Galerkin method.

The key result in the shape optimization part is the proof of the continuous dependence of solutions on boundary variations. This property is proved under an additional assumption, namely that a limit function of a minimizing sequence belongs to an appropriate space.

The numerical results revealed that even a small change of the geome-

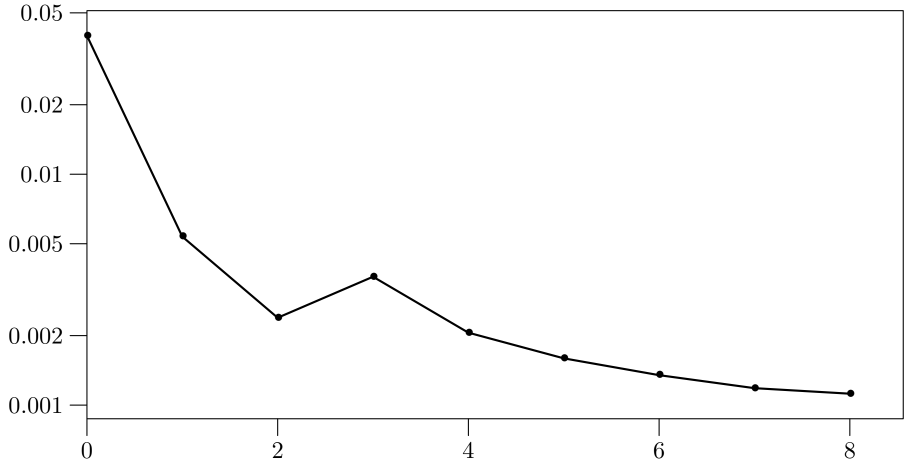

Figure 15: The convergence history.

try has a great influence on the fluid flow properties. This emphasizes the importance of state problem computations as the numerical inaccuracy can devalue the whole optimization process. The finite difference approximation of the cost functional gradient turned out to provide sufficiently exact information for the optimization algorithm. However more efficient methods may be used after performing the sensitivity analysis.

A Auxiliary tools

By $\mathbb{R}$ we denote the field of real numbers.

Theorem A.1 (Young's inequality). *Let $a, b \geq 0$, $r, s > 1$, $\frac{1}{r} + \frac{1}{s} = 1$. Then*

$$ab \leq \frac{a^r}{r} + \frac{b^s}{s} \tag{A.1}$$

Theorem A.2 (Brouwer's fixed-point theorem). *Let B denote a closed ball in $\mathbb{R}^d$ and $P : B \to B$ be a continuous mapping. Then there exists a point $x \in B$ such that $P(x) = x$.*

Corollary A.3. *Let $P : \mathbb{R}^d \to \mathbb{R}^d$ be continuous and let for some $R > 0$*

$$P(x) \cdot x > 0 \ \forall x \in \mathbb{R}^d, |x| = R.$$

Then there exists a point $x \in B_R$ such that $P(x) = 0$, where B_R is the closed ball of radius R.

Theorem A.4 (Arzelà-Ascoli). *Let (S, ρ) be a compact metric space and $\mathcal{C}(S)$ the Banach space of real- or complex-valued continuous functions f in S normed by $\|f\| = \sup_{s \in S} |f(s)|$. Then a sequence $\{f_n\} \subset \mathcal{C}(S)$ is relatively compact in $\mathcal{C}(S)$ if the following two conditions are satisfied:*

(i) f_n is equibounded, i.e. $\sup_{n \geq 1} \sup_{s \in S} |f_n(s)| < \infty$,

(ii) f_n is equicontinuous, i.e.

$$\lim_{\delta \searrow 0} \sup_{\substack{n \geq 1 \\ \rho(s,s') < \delta}} |f_n(s) - f_n(s')| = 0.$$

Proof. See [9], Chapter III.3. ☐

Theorem A.5 (Mazur). *Let X be a Banach space and $x_n \rightharpoonup x$ (weakly) in X. Then for every $\varepsilon > 0$ there exists a convex combination $\sum_{j=1}^n a_j^{(n)} x_j$ $\left(a_j^{(n)} \geq 0,\ \sum_{j=1}^n a_j^{(n)} = 1\right)$ of x_j's such that $\|x - \sum_{j=1}^n a_j^{(n)} x_j\| \leq \varepsilon$.*

Proof. See [9], Chapter V.1, Theorem 2. ☐

B Properties of the Sobolev spaces

In what follows we assume that Ω is a bounded domain in $\mathbb{R}^d$ with the Lipschitz boundary. We denote for k integer and $r \in [1, \infty)$ the Sobolev space

$$W^{k,r}(\Omega) := \left\{ v \in L^r(\Omega);\ D^\alpha v \in L^r(\Omega),\ |\alpha| \leq k \right\}$$

with the norm

$$\|v\|_{k,r,\Omega} := \left(\sum_{|\alpha| \le k} \|D^\alpha v\|_{r,\Omega}^r \right)^{1/r},$$

where $L^r(\Omega)$ is the Lebesgue space endowed with the norm $\|\cdot\|_{r,\Omega}$.

Theorem B.1 (Hölder's inequality). *Let $r, s \in (1,\infty)$ such that $\frac{1}{r} + \frac{1}{s} = 1$, $f \in L^r(\Omega)$ and $g \in L^s(\Omega)$. Then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{1,\Omega} \le \|f\|_{r,\Omega} \|g\|_{s,\Omega}.$$

Theorem B.2 (Imbedding theorem). *Let $r \in (1,d)$ and $s \in [1, \frac{dr}{d-r}]$. Then there exists a positive constant $C_{Imb} := C_{Imb}(\Omega, r, s)$ such that for all $v \in W^{1,r}(\Omega)$ it holds*

$$\|v\|_{s,\Omega} \le C_{Imb} \|v\|_{1,r,\Omega}.$$

For $s < \frac{dr}{d-r}$ this imbedding is compact.

We denote by $\operatorname{Tr} v$ the trace of $v \in W^{1,r}(\Omega)$. The symbol $L^r(\partial\Omega)$ stands for the Lebesgue space of traces with the norm $\|\cdot\|_{r,\partial\Omega}$.

Theorem B.3 (Properties of traces). *Let $r \in (1,d)$ and $s \in [1, \frac{dr-r}{d-r}]$. Then there exists a positive constant $C_{Tr} := C_{Tr}(\Omega, r, s)$ such that for all $v \in W^{1,r}(\Omega)$ it holds*

$$\|\operatorname{Tr} v\|_{s,\partial\Omega} \le C_{Tr} \|v\|_{1,r,\Omega}.$$

For $s < \frac{dr-r}{d-r}$ the operator $\operatorname{Tr} : W^{1,r}(\Omega) \to L^s(\partial\Omega)$ is compact.

Theorem B.4 (Friedrichs' inequality). *Let $r \in (1,\infty)$ and Γ be a nonempty and open part of $\partial\Omega$. Then there exists a positive constant $C_{Fr} := C_{Fr}(\Omega, \Gamma, r)$ such that for all $v \in W^{1,r}(\Omega)$ it holds*

$$\|v\|_{1,r,\Omega} \le C_{Fr} \left(\|v\|_{r,\Gamma} + \|\nabla v\|_{r,\Omega} \right).$$

Theorem B.5 (Green's theorem). *Let $u \in W^{1,r}(\Omega)$, $v \in W^{1,s}(\Omega)$, $\frac{1}{r} + \frac{1}{s} = 1$ and $i \in \{1, \ldots, d\}$. Then*

$$\int_\Omega u \frac{\partial v}{\partial x_i} \, dx = \int_{\partial\Omega} u \; v \; \nu_i \; dS - \int_\Omega \frac{\partial u}{\partial x_i} v \; dx,$$

where ν_i denotes the i-th component of the unit outward normal vector to $\partial\Omega$.

Theorem B.6 (Korn's inequality). *Let $r \in [1,\infty)$ and Γ be a nonempty and open part of $\partial\Omega$. Then there exists a positive constant $C_{Korn} := C_{Korn}(\Omega, \Gamma, r) > 0$ such that*

$$C_{Korn} \|\nabla u\|_{r,\Omega} \le \|\varepsilon(u)\|_{r,\Omega} \tag{B.1}$$

for all $u \in \left(W^{1,r}(\Omega)\right)^d$ such that $u|_\Gamma = 0$.

Lemma B.7. *Let* $u \in \left(W^{1,r}(\Omega)\right)^d$*,* $r \in [1, \infty)$*. Then*

$$\|\varepsilon(u)\|_{r,\Omega} \leq \|\nabla u\|_{r,\Omega} \tag{B.2}$$

Proof. From the triangle inequality we obtain

$$\left\|\varepsilon_{ij}(u)\right\|_{r,\Omega} \leq \frac{1}{2}\left(\left\|\frac{\partial u_i}{\partial x_j}\right\|_{r,\Omega} + \left\|\frac{\partial u_j}{\partial x_i}\right\|_{r,\Omega}\right)$$

and therefore

$$\left\|\varepsilon(u)\right\|_{r,\Omega} = \left(\sum_{i,j} \|\varepsilon_{ij}(u)\|_{r,\Omega}^r\right)^{1/r} \leq \sum_{i,j} \|\varepsilon_{ij}(u)\|_{r,\Omega} \leq \|\nabla u\|_{r,\Omega} \tag{B.3}$$

□

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