Numerical solution of 2D compressible flows in a channel

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The work deals with numerical solution of compressible viscous and inviscid 2D flows. Numerical solution was obtained by using finite volume methods, Mac Cormack predictor-corrector scheme, Lax–Wendroff scheme (Richtmyer form) and Runge-Kutta methods. Turbulent solution is modelled by algebraic model and two equations model.

1 Mathematical model

Laminar flow of **compressible and viscous** fluid is described by the system of Navier Stokes equations:

$$W_t + F_x + G_y = R_x + S_y \tag{1}$$

where

$$W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix} \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (e+p)u \end{pmatrix} \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (e+p)v \end{pmatrix} \tag{2}$$

$$R = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} + kT_x \end{pmatrix} \quad S = \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ u\tau_{xy} + v\tau_{yy} + kT_y \end{pmatrix} \tag{3}$$

$$\tau_{xx} = \frac{2}{3}\eta(2u_x - v_y) \quad \tau_{xy} = \eta(u_y + v_x) \quad \tau_{yy} = \frac{2}{3}\eta(-u_x + 2v_y) \tag{4}$$

This system is closed by the equation

$$p = (\kappa - 1)\left[e - \frac{1}{2}\rho\left(u^2 + v^2\right)\right] \tag{5}$$

ρ denotes density, u, v are components of velocity in the direction of axes x, y, e is energy density, p is pressure, $\kappa = 1.4$ and η is dynamical viscosity. Inviscid flow corresponds to ideal situation when $\eta = 0$ (Euler equations).

Above mentioned equations do not describe turbulent flow. One of the possible approaches to this modelling is using the RANS (Reynolds Averaged Navier Stokes) equations. The idea is to model the influence of turbulence by introducing new term

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into NS equations: so called turbulent viscosity η_t. RANS equations have formally the same form as classical Navier Stokes equations but their solution gives **time averaged** values of variables.

$$W_t + F_x + G_y = R_x + S_y \tag{6}$$

where

$$W = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ e \end{pmatrix} \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (e+p)u \end{pmatrix} \quad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (e+p)v \end{pmatrix} \tag{7}$$

$$R = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} + kT_x \end{pmatrix} \quad S = \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ u\tau_{xy} + v\tau_{yy} + kT_y \end{pmatrix} \tag{8}$$

but

$$\begin{aligned} \tau_{xx} &= \frac{2}{3}(\eta + \eta_t)(2u_x - v_y) \\ \tau_{xy} &= (\eta + \eta_t)(u_y + v_x) \\ \tau_{yy} &= \frac{2}{3}(\eta + \eta_t)(-u_x + 2v_y) \end{aligned} \tag{9}$$

Various turbulence models differ in the way of how they obtain values η_t.

2 Numerical methods

2.1 Boundary conditions

Inlet: $u_\infty = M_\infty$, $v_\infty = 0$, ρ_∞ and e_∞ are given, p_∞ extrapolated.
Outlet: p_2, other variables u, v, ρ are extrapolated.
Wall: $u = v = 0$.

Finite volume method was used on a structured grid of quadrilateral cells. Mac Cormack scheme was applied in solving the system of RANS equations for turbulent flow, Lax–Wendroff scheme (Richtmyer form) and Runge–Kutta methods for inviscid flow. The Jameson's artificial dissipation was added to increase numerical stability.

- Mac Cormack scheme

$$W_{i,j}^{n+\frac{1}{2}} = W_{i,j}^{n} - \frac{\Delta t}{\mu_{i,j}} \sum_{k=1}^{4} \left[\left(\tilde{F}_k^n - \frac{1}{\mathrm{Re}} R_k^n \right) \Delta y_k - \left(\tilde{G}_k^n - \frac{1}{\mathrm{Re}} S_k^n \right) \Delta x_k \right] \tag{10}$$

$$\begin{aligned} W_{i,j}^{n+1} &= \frac{1}{2}\left(W_{i,j}^{n} + W_{i,j}^{n+\frac{1}{2}} \right) \\ &- \frac{\Delta t}{2\mu_{i,j}} \sum_{k=1}^{4} \left[\left(\tilde{F}_k^{n+\frac{1}{2}} - \frac{1}{\mathrm{Re}} R_k^{n+\frac{1}{2}} \right) \Delta y_k - \left(\tilde{G}_k^{n+\frac{1}{2}} - \frac{1}{\mathrm{Re}} S_k^{n+\frac{1}{2}} \right) \Delta x_k \right]. \end{aligned} \tag{11}$$

- Lax–Wendroff scheme (Richtmyer form)

$$
\begin{aligned}
W_{i,j}^{n+\frac{1}{2}} &= W_{i,j}^{n} - \frac{\Delta t}{2\mu_{i,j}} \sum_{k=1}^{4} \left[\left(\tilde{F}_k^n - \frac{1}{\mathrm{Re}} R_k^n \right) \Delta y_k - \left(\tilde{G}_k^n - \frac{1}{\mathrm{Re}} S_k^n \right) \Delta x_k \right] && (12)\\
W_{i,j}^{n+1} &= W_{i,j}^{n} && (13)\\
&- \frac{\Delta t}{\mu_{i,j}} \sum_{k=1}^{4} \left[\left(\tilde{F}_k^{n+\frac{1}{2}} - \frac{1}{\mathrm{Re}} R_k^{n+\frac{1}{2}} \right) \Delta y_k - \left(\tilde{G}_k^{n+\frac{1}{2}} - \frac{1}{\mathrm{Re}} S_k^{n+\frac{1}{2}} \right) \Delta x_k \right].
\end{aligned}
$$

- multistage Runge–Kutta scheme

$$
\begin{aligned}
RezW_{i,j}^{n} &= \frac{1}{\mu_{i,j}} \left[\sum_{k=1}^{4} (\tilde{F}_k^n - \tilde{R}_k^n)\Delta y_k - (\tilde{G}_k^n - \tilde{S}_k^n)\Delta x_k \right] && (14)\\
W_{i,j}^{(0)} &= W_{i,j}^{n}\\
W_{i,j}^{(r+1)} &= W_{i,j}^{(0)} - \alpha_r \Delta t RezW_{i,j}^{(r)} \text{ where r} = 0, 1, ..., \mathrm{m} - 1\\
W_{i,j}^{n+1} &= W_{i,j}^{(m)}
\end{aligned}
$$

- artificial dissipation

$$
\begin{aligned}
W_{i,j}^{n+1} = W_{i,j}^{n+1} &+ k_1 \nu_i \left[W_{i+1,j}^n - 2W_{i,j}^n + W_{i-1,j}^n \right] && (15)\\
&+ k_2 \nu_j \left[W_{i,j+1}^n - 2W_{i,j}^n + W_{i,j-1}^n \right],
\end{aligned}
$$

$$
\nu_i = \frac{\left| p_{i+1,j}^n - 2p_{i,j}^n + p_{i-1,j}^n \right|}{\left| p_{i+1,j}^n \right| + \left| p_{i,j}^n \right| + \left| p_{i-1,j}^n \right|}, \ \nu_j = \frac{\left| p_{i,j+1}^n - 2p_{i,j}^n + p_{i,j-1}^n \right|}{\left| p_{i,j+1}^n \right| + \left| p_{i,j}^n \right| + \left| p_{i,j-1}^n \right|}. \qquad (16)
$$

3 Numerical results

Authors present two types, subsonic and transonic flows computed by Lax–Wendroff (Richtmyer form) and Runge–Kutta schemes (inviscid flows), see Figs. $1, 2$ and $3, 4$ for $M_\infty = 0.4$ and $M_\infty = 0.675$ in the GAMM channel.

Next presented results show comparison of numerical results for $M_\infty = 0.35$ and $M_\infty = 0.5$ for $Re = 5 \cdot 10^6$ using algebraic (Baldwin–Lomax) and $k - \omega$ turbulence models. We can see very good results, but the physics of separated region using algebraic and $k - \omega$ model is not the same. Figs. $5, 6$ and 7 show result of $Re = 5 \cdot 10^5$ and $M_\infty = 0.35$ using smaller artificial viscosity (smaller coefficients). Here authors did not achieved expected steady state flow but unsteady flows (see unsteady residual behaviour).

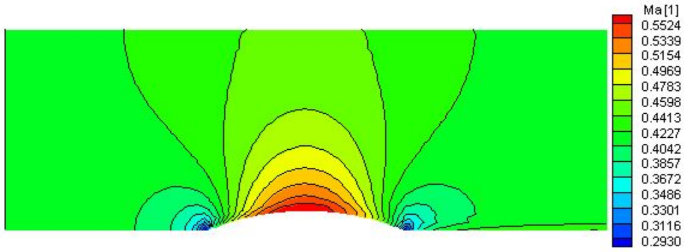

Figure 1: $M_\infty = 0.4$, compressible inviscid flow, Lax–Wendroff scheme (Richtmyer form), Mach number isolines

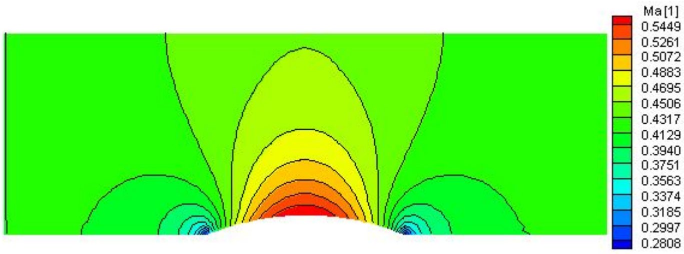

Figure 2: $M_\infty = 0.4$, compressible inviscid flow, Runge–Kutta scheme, Mach number isolines

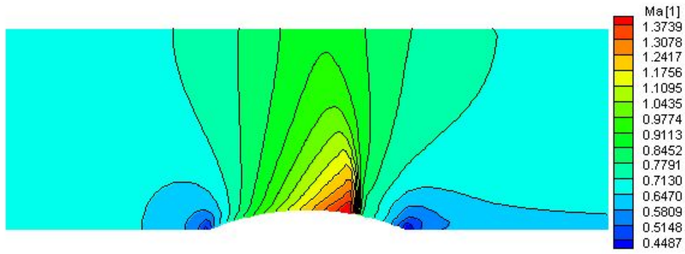

Figure 3: $M_\infty = 0.675$, compressible inviscid flow, Lax–Wendroff scheme (Richtmyer form), Mach number isolines

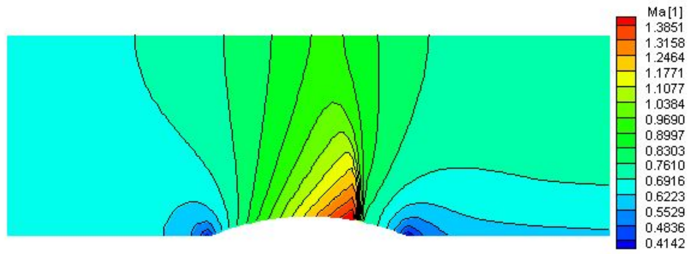

Figure 4: $M_\infty = 0.675$, compressible inviscid flow, Runge–Kutta scheme, Mach number isolines

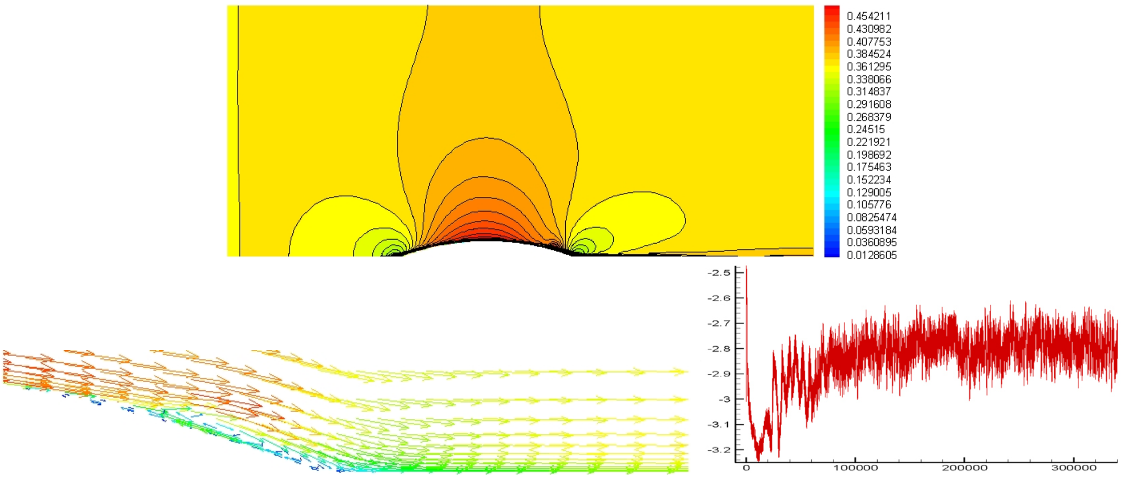

Figure 5: $Re = 5\,000\,000$, $M_\infty = 0.35$, compressible viscous flow, $k - \omega$ model, Mach number isolines, velocity vectors near separation, L_2 logarithmic residual

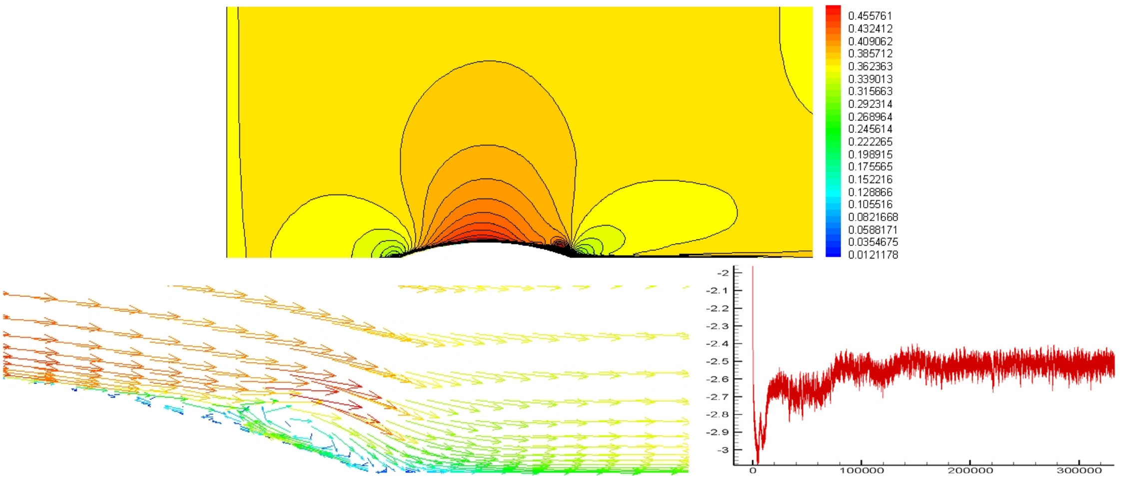

Figure 6: $Re = 5\,000\,000$, $M_\infty = 0.35$, compressible viscous flow, Baldwin-Lomax model, $k_1 = 1.35$, $k_2 = 0.9$, Mach number isolines, velocity vectors near separation, L_2 logarithmic residual

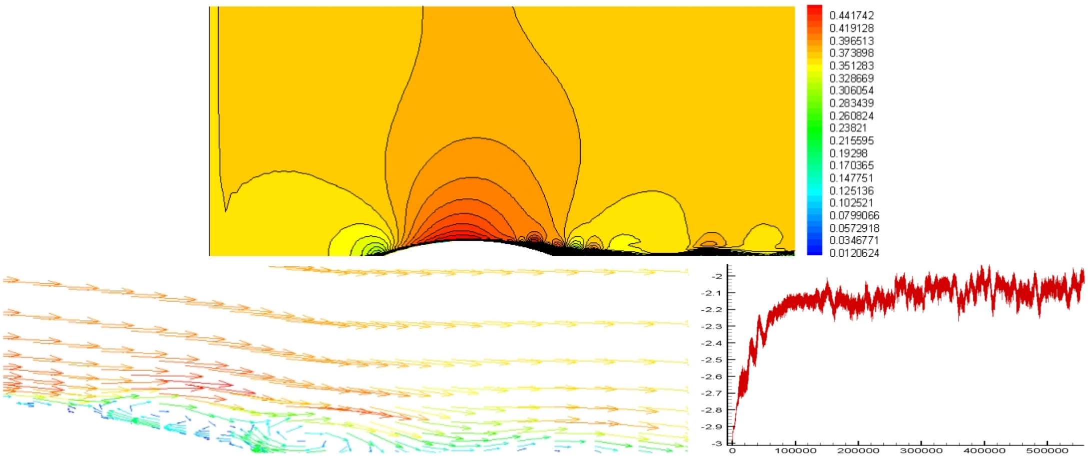

Figure 7: $Re = 5\,000\,000$, $M_\infty = 0.35$, compressible viscous flow, Baldwin-Lomax model, $k_1 = 0.27$, $k_2 = 0.18$, Mach number isolines, velocity vectors near separation, L_2 logarithmic residual

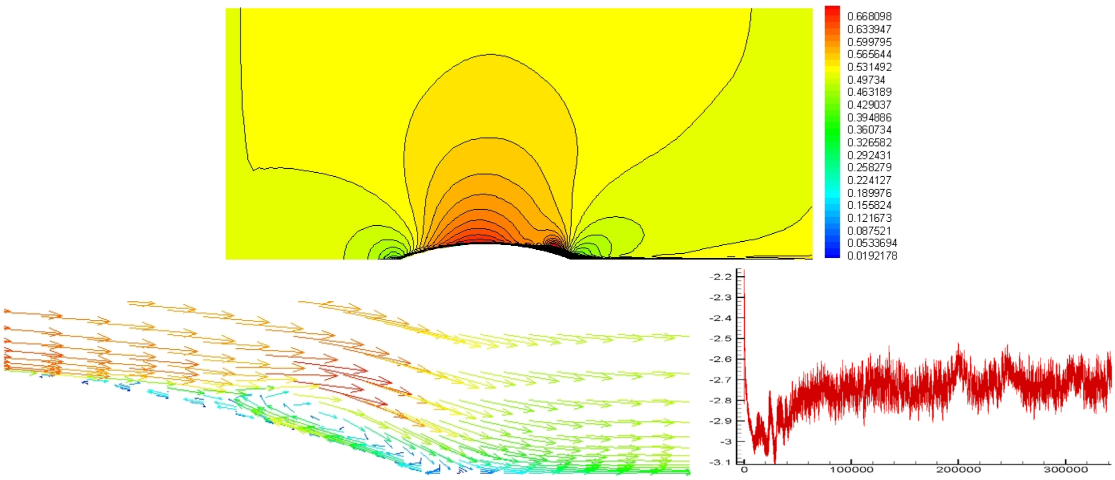

Figure 8: $Re = 5\,000\,000$, $M_\infty = 0.50$, compressible viscous flow, $k-\omega$ model, Mach number isolines, velocity vectors near separation, L_2 logarithmic residual

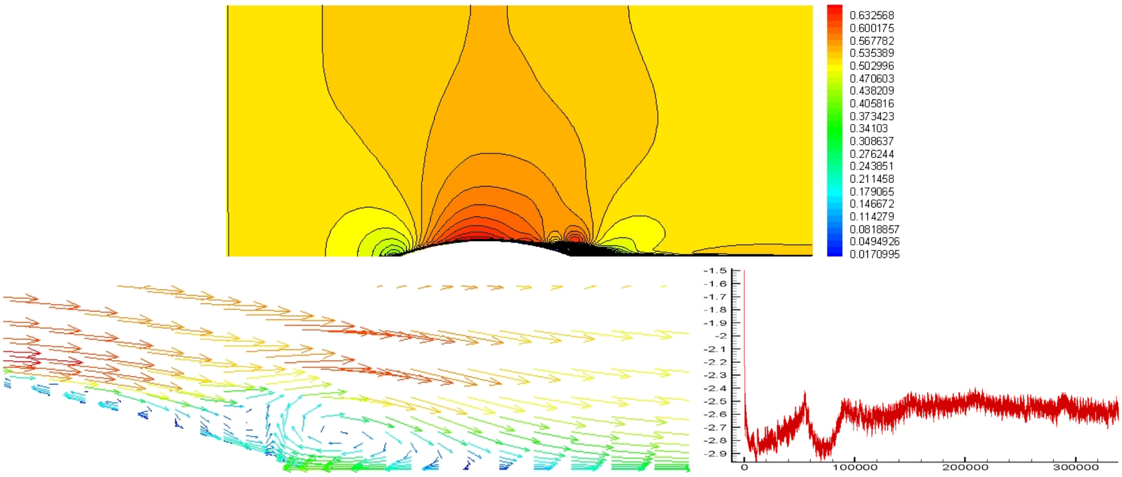

Figure 9: $Re = 5\,000\,000$, $M_\infty = 0.50$, compressible viscous flow, Baldwin-Lomax model, Mach number isolines, velocity vectors near separation, L_2 logarithmic residual

4 Closure

The article presents deveploment of own software for simulation of inviscid and viscous compressible turbulent flows. Numerical results of turbulent flows are computed by algebraic (Baldwin–Lomax) and $k - \omega$ models and both models give reliable solutions.

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