Introduction to algebraic geometry

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Algebraic geometry can be thought as an approach to solve problems in (commutative) algebra by systematical constructing necessary geometric objects, e.g. we associate to the solution of a system of polynomial equations with an algebraic variety in the corresponding affine space. Further a commutative algebra is considered as an algebra of functions on some set.

In this lecture course we shall deal with algebraic varieties in affine or projective spaces. We shall introduce the following important concepts: dimension, regular functions, rational maps, and tangent spaces. Many examples and exercises shall be provided.

Prerequisite: basis knowledge on rings, ideals and modules.

Recommended textbooks: - R. Hartshortne "Algebraic Geometry, Chapter I", Springer 1997,
- J. Harris, "Algebraic Geometry, a first course ", Springer

Contents

1 Algebraic sets in affine spaces. Hilbert's Nullstellensatz

Algebraic geometry deals with spaces and varieties over arbitrary ring (schemes) but the classical algebraic geometry deals mostly with a closed field k. For the simplicity we shall take $k = \mathbb{C}$. The main philosophy is to associate appropriate geometric notions (points, sets, topology, mappings, etc.) with corresponding algebraic notions (ideals, rings, Zariski topology, morphisms, etc.) and conversely, appropriate algebraic notions with corresponding geometric notions.

1.1 Algebraic sets

We denote by $\mathbb{C}^n$ the complex n-dimensional vector space. This space is also considered as a **complex affine space**, i.e. a set with a faithful freely transitive $\mathbb{C}^n$-action. We also denote this space by $A^n_{\mathbb{C}}$ or A^n once the ground field $\mathbb{C}$ is specified.

The algebraic object associated to this affine space is the ring $\mathbb{C}[z_1, \cdots, z_n]$ which is also called **the ring of regular functions** over $\mathbb{C}^n$:

$$\mathbb{C}^n \Longleftrightarrow \mathbb{C}[z_1, \cdots, z_n].$$

A set $X \subset \mathbb{C}^n$ is called **algebraic**, if there exists a subset $T \subset \mathbb{C}[z_1, \cdots, z_n]$ such that X is the zero set of T:

$$X = Z(T),$$

i.e. for any $f \in T$ and any $(z_1, \cdots, z_n) \in X$ we have $f(z_1, \cdots, z_n) = 0$. Denote by $I(T)$ the ideal generated by T in $\mathbb{C}[z_1, \cdots, z_n]$. Then we have

$$Z(T) = Z(I(T)).$$

In this way we associate

$$\{I,\ I \text{ is an ideal in } \mathbb{C}[z_1, \cdots, z_n]\} \Longleftrightarrow \{ \text{ algebraic sets in } \mathbb{C}^n\}.$$

For any (algebraic) set X we denote by $I(X)$ the ideal in $\mathbb{C}[z_1, \cdots, z_n]$ of regulars functions which vanish on X.

1.1.1 Exercise. Show that the union of two algebraic sets is an algebraic set and the intersection of a family of algebraic sets is an algebraic set.

1.2 Zariski topology

How we can define the notion of "close" or "far away" between ideals in a ring? Rephrasing, how we can define a topology on the set of ideals?

1.2.1. Definition. The Zariski topology on $\mathbb{C}^n$ is defined by specifying the closed sets in $\mathbb{C}^n$ to be precisely the algebraic sets. Equivalently a set is said to be **open in Zariski topology**, if it is a complement of an algebraic set.

1.2.2. Example. A closed set in $A^1_{\mathbb{C}}$ is either a finite set (the roots of a polynomial $P \in \mathbb{C}[z]$), or the whole affine line $A^1_{\mathbb{C}}$ (in this case $P = 0$). Thus this topology is not Haussdorf. (A topology is called Haussdorf if it satisfies the second separateness axiom which says that for any two different points we can find their neighborhoods which have no intersection.)

1.2.3. Exercise. If A and B are topological spaces, then we can define the product topology on the space $A \times B$ by specifying the base of this product topology to be the collection of the sets $U_\alpha \times V_\beta$, where U_α and V_β are open sets in A and B respectively. Show that the usual topology on $\mathbb{C}^n$ is the product topology of the usual topology on $\mathbb{C}$ but the Zariski topology on $\mathbb{C}^2$ is not the product of the Zariski topology on $\mathbb{C}$.

Hint Examine all closed subsets in the product of the Zariski topology on $\mathbb{C}\times\mathbb{C}$.

Let us define the closure $\bar{Y}$ of a set $Y \subset \mathbb{C}^n$ to be the smallest closed set which contains Y.

1.2.4. Exercise. Show that the closure of the set $S = \{(m, n), | \; m \geq n \geq 0, m \in \mathbb{Z}, n \in \mathbb{Z}\} \subset \mathbb{C}^2$ is equal to $\mathbb{C}^2$.

Hint Let P be a polynomial on $\mathbb{C}^2$ such that S are roots of P. Examine the degree of P.

If Y is an algebraic set in $\mathbb{C}^n$ then we can define the induced Zariski topology on Y by specifying the open sets in Y to be the intersection of open sets in $\mathbb{C}^n$ with Y.

It is easy to see that the induced Zariski topology on $\mathbb{C}^1 = \{z_2 = 0\} \subset \mathbb{C}^2$ is the usual Zariski topology on $\mathbb{C}^1$.

1.3 Affine algebraic varieties

Now we shall examine the structure of algebraic sets closer. We shall decompose an algebraic set into its irreducible components.

An algebraic set Y is called **irreducible**, if it cannot be represented as the union of two algebraic sets such that each of them is a proper subset in S . For example the affine line $\mathbb{C}^1 = \{(z_2 = 0)\} \subset \mathbb{C}^2$ is an irreducible algebraic set, because any closed set in $\mathbb{C}^1$ is either a finite set or the whole line $\mathbb{C}^1$.

1.3.1. Proposition. *An algebraic set is irreducible, if and only if, its ideal is prime.*

Proof. First we show that if a set Y is irreducible, then its ideal $I(Y)$ is prime. Indeed, if $fg \in I(Y)$ then $Y \subset Z(fg) = Z(f) \cup Z(g)$. Hence we get the decomposition

$$Y = (Y \cap Z(f)) \cup (Y \cap Z(g)),$$

so that $Z(f) \cap Y$ or $Z(g) \cap Y$ must be equal to Y. Consequently, $f \in I(Y)$ or $g \in I(Y)$ which implies that $I(Y)$ is prime.

Conversely, let $I(Y)$ be prime, we shall show that Y is irreducible. If $Y = Y_1 \cup Y_2$, then $I(Y) = I(Y_1) \cap I(Y_2)$. Assume that $I(Y) \neq I(Y_1)$ i.e. there is an element $g \in I(Y_1) \setminus I(Y)$. Since $I(Y)$ is prime, and $g \cdot I(Y_2) \subset I(Y)$ we get that $I(Y_2) \subset I(Y)$. Hence $I(Y_2) = I(Y)$, i.e. Y is irreducible. $\square$

An **affine algebraic variety** (or simply affine variety) is an irreducible closed algebraic set with the induced Zariski topology of $\mathbb{C}^n$. An open subset of an algebraic variety is called **a quasi-affine variety**.

1.3.2. Example. The twisted cubic curve $C = (t, t^2, t^3 | \, t \in \mathbb{C}) \subset \mathbb{C}^3$ is an affine algebraic variety. Clearly $I(C) = ((z_1^2 - z_2), (z_1 z_2 - z_3))$. To prove that $I(C)$ is prime, it suffices to show that the quotient $A(C) = \mathbb{C}[z_1, z_2, z_3]/I(C)$ is an integral domain. But it is easy to see that $A(C) = \mathbb{C}[z]$ is an integral domain.

1.3.3. Exercise. Prove that any closed subset Y in $\mathbb{C}^n$ has a decomposition of into irreducible closed subsets and this decomposition is unique.

Hint: Any chain of decompositions of closed subsets of Y must stop at irreducible closed subsets, since the ring $\mathbb{C}[z_1, \cdots, z_n]$ is Noetherian.

1.4 Hilbert's Nullstellensatz

Let us study deeper the correspondence between algebraic sets Y_i in $\mathbb{C}^n$ and ideals $\mathfrak{a}_i$ in $\mathbb{C}[z_1, \cdots, z_n]$. The following properties are obvious

$$\mathfrak{a}_1 \subset \mathfrak{a}_2 \Longrightarrow Z(\mathfrak{a}_1) \supset Z(\mathfrak{a}_2).$$
$$Y_1 \subset Y_2 \Longrightarrow I(Y_1) \supset I(Y_2).$$
$$I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2).$$

We shall prove the following important theorem which says that the correspondence between algebraic sets in $\mathbb{C}^n$ and radical ideals in $\mathbb{C}[z_1, \cdots, z_n]$ are 1-1.

1.4.1. Hilbert's Nullstellensatz. *Let* $\mathfrak{a}$ *be an ideal in* $\mathbb{C}[z_1, \cdots, z_n]$. *Then*

$$I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}.$$

Proof. First we shall prove the following

1.4.2. Lemma. *Any maximal ideal* $\mathfrak{m} \subset \mathbb{C}[z_1, \cdots, z_n]$ *is of the form*

$$\mathfrak{m} = (z_1 - a_1, \cdots, z_n - a_n),\ a_i \in \mathbb{C}.$$

Consequently for any ideal $\mathfrak{a} \neq \mathbb{C}[z_1, \cdots, z_n]$ *we have*

$$Z(\mathfrak{a}) \neq \emptyset.$$

Proof of Lemma 1.4.2. Let $\mathfrak{m}$ be a maximal ideal in $\mathbb{C}[z_1, \cdots, z_n]$. Denote by K the residue class field $\mathbb{C}[z_1, \cdots, z_n]/\mathfrak{m}$. Clearly K contains $\mathbb{C}$ as its subfield, and K has a countable $\mathbb{C}$ -basis, since $\mathbb{C}[z_1, \cdots, z_n]$ has a countable $\mathbb{C}$-basis consisting of monomials $z_1^{k_1} \cdots z_n^{k_n}$.

If $K \neq \mathbb{C}$ then there is an element $p \in K \setminus \mathbb{C}$. Element p is transcendental over $\mathbb{C}$ because $\mathbb{C}$ is algebraic closed [1]. Hence the set

$$(\frac{1}{p - \lambda} | \ \lambda \in \mathbb{C})$$

is uncountable and their elements are linearly independent over $\mathbb{C}$, which is a contradiction. Therefore $K = \mathbb{C}$. In particular we have

$$z_i + \mathfrak{m} = a_i + \mathfrak{m} \text{ for suitable } a_i \in \mathbb{C}.$$

This proves the first statement. The second follows from the fact that $\mathfrak{a}$ must belong to some maximal ideal. □

Continuation of the proof of Hilbert's Nullstellensatz. Let f be a polynomial which vanishes on the set $Z(\mathfrak{a})$. We shall find a finite number m such that $f^m \in \mathfrak{a}$.

We denote by R the ring $\mathbb{C}[z_0, z_1, \cdots, z_n]$. Let

$$\mathfrak{b} := (\mathfrak{a}, 1 - z_0 f) \subset R.$$

Clearly $Z(\mathfrak{b}) = 0$. By Lemma 1.4.2 we get

$$\mathfrak{b} = R.$$

[1] that is why Lemma 1.4.2 does not hold for the ring $\mathbb{R}$

In particular we can find solutions $h_i, h \in R$ and $f_i \in \mathfrak{a}$ to the following equation

$$\sum h_i f_i + h(1 - z_0 f) = 1. \tag{1.4.3}$$

Now let us substitute $\frac{1}{f}$ for z_0 as a formal variable in (1.4.3). We get

$$\sum_i h_i(\frac{1}{f}, z_1, \cdots, z_n) f_i = 1. \tag{1.4.4}$$

Let m be the maximal degree of x_0 in LHS of (1.4.4). Then multiplying the both sides of (1.4.4) with f^m we get

$$\sum_i \tilde{h}_i f_i = f^m, \tag{1.4.5}$$

where $\tilde{h}_i \in \mathfrak{a}$. This completes the proof of Hilbert's Nullstellensatz. □

1.4.6. Exercise. i) Prove that a system of polynomial equations

$$f_1(z_1, \cdots, z_n) = 0,$$

$$\cdots$$

$$f_m(z_1, \cdots z_n) = 0$$

has no solution in $\mathbb{C}^n$ iff 1 can be expressed as a linear combination

$$1 = \sum p_i f_i$$

with polynomial coefficients p_i.
ii)Show that any point x in an algebraic set $X \subset \mathbb{C}^n$ is a Zariski closed set.

Hint. Use the Nullstellensatz for the first statement and use Lemma 1.3.2 for the second statement.

2 Projective varieties and graded rings

2.1 Projective spaces

We denote by $\mathbb{C}P^n$ the complex projective space whose points are complex lines in the vector space $\mathbb{C}^{n+1}$, i.e. 1-dimensional subspaces of the vector space $\mathbb{C}^n$. Equivalently

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$$

where $\mathbb{C}^*$ is the group of non-zero scalars acting on $\mathbb{C}^{n+1}$ by multiplication. This means that we consider a point of $\mathbb{C}P^n$ as an equivalence class of points in $\mathbb{C}^{n+1}$ under the action of $\mathbb{C}^*$ as follows. Two points $(z_0, \cdots, z_n)$ and $(z'_0, \cdots, z'_n)$ are equivalent, if there exists a number $\lambda \in \mathbb{C}^*$ such that

$$z_i = \lambda z'_i, \text{ for all } 0 \le i \le n.$$

The equivalent class of $(z_0, z_1, \cdots, z_n)$ will be denoted by $[z_0 : z_1 : \cdots, : z_n]$.

2.2 Homogeneous polynomials and graded rings

We can also define the dual action of $\mathbb{C}^*$ on the ring $\mathbb{C}[z_0, z_1, \cdots, z_n]$ by setting

$$(\lambda \circ P)(z_0, \cdots, z_n) := P(\lambda z_0, \cdots, \lambda z_n),$$

for any $\lambda \in \mathbb{C}^*$. Since $\mathbb{C}^*$ is abelian, the ring $\mathbb{C}[z_0, z_1, \cdots, z_n]$ considered as a vector space over $\mathbb{C}$ can be decomposed into eigen-spaces of the action of λ for all $\lambda \in \mathbb{C}$

$$\mathbb{C}[z_0, z_1, \cdots, z_n] = \oplus_k S_k. \tag{2.2.1}$$

Here S_k is an eigen-space w.r.t. weight $k \in Hom(\mathbb{C}^*, \mathbb{C}^*) : \lambda \mapsto \lambda^k$,

$$\lambda \circ P = \lambda^k \cdot P, \text{ if } P \in S_k,$$

for all $\lambda \in \mathbb{C}^*$. The splitting (2.2.1) is also called a **grading** of the ring $\mathbb{C}[z_0, \cdots, z_n]$, since we have

$$S_k \cdot S_l \subset S_{k+l}. \tag{2.2.2}$$

Elements of S_k are called **homogeneous polynomials.** The ring $\mathbb{C}[z_0, \cdots, z_n]$ provided with the splitting (2.2.1) which satisfies (2.2.2) is a **graded ring**.

An ideal $\mathfrak{a} \subset \mathbb{C}[z_0, \cdots, z_n]$ is called a **homogeneous ideal**, if

$$\mathfrak{a} = \oplus_k(\mathfrak{a} \cap S_k).$$

2.2.3. Example. A maximal ideal $\mathfrak{a} \subset \mathbb{C}[z_0, z_1, \cdots, z_n]$ is a homogeneous ideal, if and only if $Z(\mathfrak{a}) = \{0\} \in \mathbb{C}^{n+1}$.

2.2.4. Exercise. Prove that an ideal is homogeneous if and only if it can be generated by homogeneous elements. Prove that the sum, product, intersection and radical of homogeneous ideals are homogeneous.

2.3 Projective varieties and homogeneous ideals

We associate to any homogeneous polynomial $P \in V_k$ a function $\tilde{P} : \mathbb{C}P^n \to \{0,1\}$ according to the following rule

$$\tilde{P}([z_0 : z_1, \cdots, : z_n]) = 0, \text{ if } P(z_0, z_1, \cdots, z_n) = 0,$$

$$\tilde{P}([z_0 : z_1, \cdots, : z_n]) = 1, \text{ if } P(z_0, z_1, \cdots, z_n) \neq 0.$$

Clearly the function $\tilde{P}$ is well-defined. So we can define for any set T of homogeneous polynomials in $\mathbb{C}[z_0, z_1, \cdots, z_n]$ its zero set $Z(T)$ in the projective space $\mathbb{C}P^n$ by setting

$$Z(T) := \{p \in \mathbb{C}P^n | \, \tilde{P}(p) = 0 \text{ for all } P \in T\}.$$

A subset $Y \subset \mathbb{C}P^n$ is called **algebraic**, if there exists a set T of homogeneous polynomials of $\mathbb{C}[z_0, \cdots, z_n]$ such that $Y = Z(T)$.

2.3.1. Exercise. Show that the union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set.

For any subset $Y \subset \mathbb{C}P^n$ we denote by $I(Y)$ the **homogeneous ideals** of $Y \subset \mathbb{C}[z_0, \cdots, z_n]$ the ideal generated by homogeneous elements f in $\mathbb{C}[z_0, \cdots, z_n]$ such that f vanishes on Y. (This ideal is homogeneous according to Exercise 2.2.4)

The Zariski topology on $\mathbb{C}P^n$ is defined by specifying the open sets to be the complement of algebraic sets.

Once we have a topological space the notion of irreducible (not necessary algebraic) sets will apply. We say that a set Y is irreducible, if it cannot be represented as the union of two proper subsets which of them is closed in Y.

A **projective (algebraic) variety** is an irreducible algebraic set in $\mathbb{C}P^n$ with the induced topology. A **quasi projective variety** is an open subset in a projective variety.

2.3.2. Example. We denote by $H_i \subset \mathbb{C}P^n$ the zero set of the linear function z_i. Then H_i is called a hyper-plane. It is a projective variety, because $I(H_i) = (z_i)$ is a prime ideal. In fact an algebraic set $Y \subset \mathbb{C}P^n$ is irreducible, if and only if its homogeneous ideal is prime. To prove this we can repeat the proof of Proposition 3.1 or we observe that there is a correspondence between algebraic set $Y \subset \mathbb{C}P^n$ and its cone CY in $\mathbb{C}^{n+1}$ which is defined by

$$CY := \{(z_0, z_1, \cdots, z_n) | \, [z_0, z_1, \cdots, z_n] \in Y\}.$$

They have the same ideal. The property being reducible is also preserved by this correspondence. Thus our statement about the correspondence between homogeneous prime ideals and projective varieties is a consequence of the Proposition 3.1.

The following statement shows that the projective space $\mathbb{C}P^n$ is a compactification of the affine space $\mathbb{C}^n$.

2.3.3. Proposition. *The quasi-projective variety $U_i = \mathbb{C}P^n \setminus H_i$ with its induced topology is homeomorphic to the affine space $\mathbb{C}^n$ with its Zariski topology.*

Proof. We consider the map $\phi_i : U_i \to \mathbb{C}^n$

$$\phi_i([z_0 : \cdots : z_i]) = (\frac{z_0}{z_i}, \cdots_{\hat{i}}, \cdots, \frac{z_n}{z_i}).$$

Clearly ϕ_i is a bijection. We need to show that ϕ_i is a homeomorphism, i.e. ϕ_i and ϕ_i^{-1} send closed sets into closed sets.

Let Y be a closed set in U_i. Then there is a homogeneous ideal $T \subset \mathbb{C}[z_0, \cdots, z_n]$ such that $Y = Z(T) \cap U_i$. We want to find an ideal T' in $\mathbb{C}[z_0, \cdots,_{\hat{i}}, \cdot z_n]$ such that $\phi_i(Y) = Z(T')$. Let T' be the set of polynomials in $\mathbb{C}[z_0, \cdots,_{\hat{i}}, \cdots, z_n]$ obtained by restricting the set T^h of *homogeneous* elements in T to the hyper-plane $\{z_i = 1\}$ in $\mathbb{C}^{n+1}$. This map $T^h \to T'$ shall be denoted by r_i (restriction). Then we have for any homogeneous element t of degree d in T^h

$$r_i(t)(\phi_i(z)) = z_i^{-d} \cdot t(z), \text{ for all } z \in U_i. \tag{2.3.4}$$

Since ϕ_i is a bijection, it follows from (2.3.4) that $\phi_i(Y) = Z(T')$. So ϕ_i is a closed map.

Now let W be a closed set in $\mathbb{C}^n$. Then $W = Z(T')$ for some ideal $T' \subset \mathbb{C}[z_0, \cdots_{\hat{i}}, \cdots z_n]$. We shall find a homogeneous ideal $T \subset \mathbb{C}[z_0, \cdots, z_n]$ such that $\phi_i^{-1}(W) = Z(T^h) = Z(T)$, where as before T^h denotes the set of homogeneous elements in T.

Let $t' \in T'$ be a polynomial of degree d. We set, cf. (2.3.4)

$$\beta(t')(z) := z_i^d \cdot t'(\phi_i(z)) \in \mathbb{C}[z_0, \cdots, z_n]. \tag{2.3.5}$$

Clearly $\beta(t')$ is a homogeneous polynomials of degree d. Let $T := \beta(T')$. Since ϕ_i is a bijection, (2.3.5) implies that $\phi_i^{-1}(W) = Z(T) \cap U_i$. Hence ϕ_i^{-1} is also a closed map. $\square$

2.3.6. Remark. The map $\beta : T' \to T$ is not a ring homomorphism. Thus if $\{l_i\}$ generate some ideal $\mathfrak{a}$, the set $\{\beta(l_i)\}$ may not generate the ideal $\beta(\mathfrak{a})$, see the following example 2.3.7.

2.3.7. Projective closure of an affine variety. If $Y \subset \mathbb{C}^n$ is an affine variety then we shall say $\bar{Y} \subset \mathbb{C}P^n$ is the projective closure of of Y, if Y is the closure of $\phi_0(Y)$ in $\mathbb{C}P^n$, or equivalently $I(\bar{Y}) = \beta(I(Y))$. So $\bar{Y}$ is a projective closure of Y iff $\bar{Y} = Y(\beta(I(Y))$.

Now let us consider for example the projective closure of the twisted cubic curve $C = (t, t^2, t^3)$. The closure $\bar{C}$ has an ideal $I(\bar{C})$ generated by $\{(z_1^2 - z_0z_2), (z_1z_3 - z_2^2), (z_1z_2 - z_0z_3)\}$ but not by $\{\beta(z_2 - z_1^2) = z_0z_2 - z_1^2, \beta(z_1z_2 - z_3) = z_1z_2 - z_0z_3\}$ (see Haris, [Algebraic geometry, the first course Example 1.10]) for a proof of the last statement).

2.3.8. Exercise. Homogeneous Nullstellensatz. *If $\mathfrak{a} \subset S$ is a homogeneous ideal, and if f is a homogeneous polynomial such that $f(P) = 0$ for all $P \in Z(\mathfrak{a}) \subset \mathbb{C}P^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$.*

Hint. We use the correspondence between $Z(\mathfrak{a})$ and $CZ(\mathfrak{a}) \subset \mathbb{C}^{n+1}$ to deduce this Proposition from the Hilbert's Nullstellensatz. □

2.3.9. Exercise. We define the Serge embedding $\psi : \mathbb{C}P^r \times \mathbb{C}P^s \to \mathbb{C}P^N$ as follows. Set $N = rs + r + s$ and

$$\psi([x_0, \cdots, x_r] \times [y_0, \cdots, y_s]) = [\cdots, x_iy_j, \cdot]$$

Prove that ψ is injective and the image of ψ is a subvariety in $\mathbb{C}P^N$.

Hint. Show that $\psi(\mathbb{C}P^r \times \mathbb{C}P^s) = Z(\ker\theta)$ where $\theta : \mathbb{C}[z_{ij},\ i = \overline{0, r}, j = \overline{0, s}] \to \mathbb{C}[x_i, y_j\ i = \overline{0, r}, j = \overline{0, s}]$: $\theta(z_{ij}) = x_iy_j$.

3 Coordinate ring and the dimension of an algebraic set

The most basic idea of algebraic geometry is to consider any ring or algebra as a coordinate ring of some algebraic set.

3.1 Affine coordinate ring

We have already introduced the notion of an affine coordinate ring in example 1.3.2 for the affine twisted curve. In general case, the affine coordinate ring of an affine algebraic set $Y \subset \mathbb{C}^n$ is defined to be the quotient

$$A(Y) := \mathbb{C}[z_1, \cdots, z_n]/I(Y).$$

$A(Y)$ is called the coordinate ring, since any element $f \in A(Y)$ is the restriction of some polynomial $\tilde{f} \in \mathbb{C}[Z_1, \cdots, z_n]$ to Y, and moreover, as we shall see in Corollary 3.1.3, the values $f(x) \in \mathbb{C}, f \in A(Y)$, can distinguish different points in Y.

3.1.1. Remark. Since $\mathbb{C}[z_1, \cdots, z_n]$ is a finitely generated $\mathbb{C}$-algebra, the quotient $A(Y)$ is a finitely generated algebra. We have seen in example 1.3.2

that the affine coordinate ring $A(Y)$ is an integral domain, if Y is irreducible. Conversely, if B is a finitely generated $\mathbb{C}$-algebra which is an integral domain, then $B = \mathbb{C}[z_1, \cdots, z_n]/\mathfrak{a}$, where $\mathfrak{a}$ is simple. So B is the affine coordinate ring of the algebraic set $Z(\mathfrak{a})$.

$$\{\text{ finitely generated } \mathbb{C}\text{-algebras which are domains}\} \Longleftrightarrow \{\text{ affine varieties }\}.$$

For $y \in Y$ we set $\mathfrak{m}_y := \{f \in A |\, f(y) = 0\}$. Then $\mathfrak{m}_y$ is a maximal ideal in $A(Y)$.

3.1.2. Proposition. *(i) The correspondence $y \mapsto \mathfrak{m}_y$ is a 1-1 correspondence between points $y \in Y$ and the maximal ideals in $A(Y)$.*
(ii) There is a 1-1 correspondence between closed sets in Y and perfect (radical) ideals $\mathfrak{m}$ in $A(Y)$.

Proposition 3.1.2 says that Y as a topological space can be defined by the structure of the ring $A(Y)$.

Proof of Proposition 3.1.2.(i) Denote by p the projection $\mathbb{C}[z_1, \cdots, z_n] \to A(Y)$. Let $\mathfrak{m}$ be a maximal ideal in $A(Y)$. Then $p^{-1}(\mathfrak{m})$ is a maximal ideal in $\mathbb{C}[z_1, \cdots, z_n]$. By Hilbert's Nullstellensatz $p^{-1}(\mathfrak{m}) = (z_1 - a_1, \cdots, z_n - a_n) = \{f \in \mathbb{C}[z_1, \cdots, z_n] \,|\, f(a_1, \cdots, a_n) = 0\}$. Since $I(Y) \subset p^{-1}(\mathfrak{m})$ the point $(a_1, \cdots, a_n)$ belongs to Y. Hence $\mathfrak{m} = \{f \in A(Y) | f(a_1, \cdots, a_n) = 0\}$. Thus the correspondence $y \mapsto \mathfrak{m}_y$ is surjective. In fact this correspondence is 1-1 because there is a 1-1 correspondence between maximal ideals in $\mathbb{C}[z_1, \cdots, z_n]$ which contain $I(Y)$ and maximal ideals in $A(Y)$.

The second statement (ii) can be proved in the same way. We need to show that $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ for any $\mathfrak{a} \subset A$. From Hilbert's Nullstellensatz we get

$$p^{-1}(I(Z(\mathfrak{a}))) = \sqrt{p^{-1}(\mathfrak{a})}.$$

Hence

$$I(Y(\mathfrak{a})) = p(\sqrt{p^{-1}(\mathfrak{a})}) = \sqrt{p \circ p^{-1}(\mathfrak{a})} = \sqrt{\mathfrak{a}}.$$

□

From the proof of Proposition 3.1.2 we get immediately

3.1.3. Corollary. *For any $y \neq y' \in Y$ there exists $f \in A(Y)$ such that $f(y) = 0$ and $f(y') = 1$.*

3.1.4. Exercise. Show that a $\mathbb{C}$-algebra A is an affine coordinate ring $A(Y)$ for some algebraic set Y iff A is reduced (i.e. its only nilpotent element is 0) and finitely generated as $\mathbb{C}$-algebra.

Hint. Write $A = \mathbb{C}[z_1, \cdots, z_n]/I$ and use the Nullstellensatz.

3.2 Dimension of a topological space

Let X be a topological space. Then we define the **dimension** of X to be the supremum of all integers n such that there exists a chain $Z_0 \subset Z_1 \subset \cdots \subset Z_n$ of distinct irreducible closed subsets of X. This definition depends on the structure of all closed subsets of X but we shall see that dimension is a local property.

3.2.1. Proposition. *a) If Y is any subset of a topological space X, then* $\dim Y \leq \dim X$.
b) If X is topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.
c) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, *then* $X = Y$.

Proof. The first and last statements follow directly from the definition. Let $Z_0 \subset \cdots \subset Z_n$ be distinct closed irreducible subsets of X such that $Z_n \cap U_i \neq \emptyset$, then $\{Z_j \cap U_i | \, j = 0, n\}$ are closed subsets of U_i. They are all irreducible, since U is open: if $Z = (\bar{Z}_A \cap U) \cup (\bar{Z}_B \cap U)$ then $Z = [(Z \cap (X \setminus U)) \cup (Z \cap \bar{Z}_A)] \cup (Z \cap \bar{Z}_B)$ is not irreducible. Finally they all are distinct, since if $(Z_j \cap U) = (Z_{j+1} \cap U)$ then $Z_{j+1} = Z_j \cup (Z_{j+1} \cap (X \setminus U))$ is irreducible. This proves that $\dim X \leq \sup \dim U_i$. Combining with the first statement we get the second statement. □

3.2.2. Exercise i) Prove that $\dim \mathbb{C}^1 = 1$.
ii) Prove that if X is an affine variety in $\mathbb{C}^n$ and $Y \subset X$ is a proper closed subset then we have $\dim Y < \dim X$.

Hint For (ii) observe first that $\dim X \geq \dim Y$ by using 3.2.3 and 3.2.4 below. Assuming that $\dim X = \dim Y$ and using 3.2.4 again lead to the existence of the relation $f^k + p_1 f^{k-1} + \cdots + p_k = 0$ for any $f \in A(X)$ and $p_i \in \mathbb{C}[\tilde{h}_1, \cdots, \tilde{h}_m]$ for some basis $(\tilde{h}_i) \in A(X)$ where $m = \dim X = \dim Y$ and k is minimal. Take f such that $f_{|Y} = 0$. Then $p_k = 0$, so k is not minimal.

In a ring A the **height** of a prime ideal $\mathfrak{p}$ is the supremum of all integers n such that there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ of distinct prime ideals. The **Krull dimension** of A is defined as the supremum of the height of all prime ideals.

3.2.3. Proposition. *If Y is an affine algebraic set, then the dimension of Y is equal to the dimension of its affine coordinate ring $A(Y)$.*

Proof. By definition the dimension of Y equals the length of the longest chain of closed irreducible subsets in Y which correspond to the chain of prime ideals of $A(Y)$. □

Now we are going to state the following important theorem on the Krull dimension. We recall that the **transcendence degree** of an extension K over a field k is the maximal number of algebraically independent element in K.

3.2.4. Theorem (see e.g. Atiyah-Macdonald [Theorem 11.25] for a detailed proof). *Let k be an algebraic closed field and let B be an integral domain which is a finitely generated k-algebra. Then the dimension of B is equal to the transcendence degree of the quotient field $K(B)$ of B over k.*

We shall not prove this theorem but in the following exercise 3.2.5 we state some elementary properties of the dimension which are used in the proof of Theorem 3.2.4.

3.2.5. Exercise. a) Show that the dimension of a Noetherian ring A is equal to the dimension of the its localization $A_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m} \subset A$.
b) For any prime ideal $\mathfrak{p}$ in B, we have

$$\textit{height}\, \mathfrak{p} + \dim(B/\mathfrak{p}) = \dim B.$$

c) Show that the dimension of an affine variety X is equal to the maximal number of algebraically independent elements in $A(X)$.

(Hint: A Krull chain in a Noetherian ring A is the set of distinct prime ideals $0 \subset \mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_n = A$ where $\mathfrak{a}_i$ is the maximal ideal of $\mathfrak{a}_{i+1}$. The number n is called the length of this chain. Let $l(A)$ be the shortest length of a Krull chain in A. Show that if $\mathfrak{p} \subset A$ is a prime ideal of A then $l(\mathfrak{p}) < l(A)$. Prove that $l(A)$ is equal to the dimension of A. Apply this equality to A and $A_{\mathfrak{m}}$ to get 3.2.5.a. Apply this equality to $\mathfrak{p}$ to get 3.2.5.b. To prove 3.2.5.c use the fact that $\mathbb{C}[z_1, \cdots, z_n]$ is an integral domain.)

Applying these algebraic result we shall prove

3.2.6. Proposition. *a) The dimension of $\mathbb{C}^n$ is equal to n.*
b) A variety $Y \subset \mathbb{C}^n$ has dimension $n-1$ if and only if its ideal $I(Y)$ is generated by a single non-constant irreducible polynomial f in $\mathbb{C}[z_1, \cdots, z_n]$.

Proof. The first statement of Proposition 3.2.6 is a consequence of Theorem 3.2.4. Alternatively we can prove Proposition 3.2.6.a by induction on dimension n. Using 3.2.5.b it suffices to show that the height of the principal ideal $(z_n) \subset \mathbb{C}[z_1, \cdots, z_n]$ is equal to 1. Assume the opposite. Let $\mathfrak{a}_1 \subset (z_n)$ be a prime ideal of $\mathbb{C}[z_1, \cdots, z_n]$. Let $y \in \mathfrak{a}_1 \subset (z_n)$. Then $y = z_n \cdot f$ for some $f \in \mathbb{C}[z_1, \cdots, z_n]$. Since $\mathfrak{a}_1$ is prime, either $z_n \in \mathfrak{a}_1$ or $f \in \mathfrak{a}_1$. Since $\mathfrak{a}_1 \neq (z_n)$ we get that $f \in \mathfrak{a}_1$. Repeating this argument for f and iterating further we arrive at a contradiction. Hence the height of of (z_n) is 1.

To prove 3.2.6.b we need to show that $\dim Z(f) = m - 1$ if f is irreducible, and conversely if $\dim(\mathbb{C}^n/I(Y)) = m - 1$, then $I(Y)$ is a maximal ideal generated by an irreducible polynomial f. To show that $\dim Z(f) = m - 1$ we use the same argument as in the proof of 3.2.6.a replacing z_n by f. Now assume that

$\dim(\mathbb{C}^n/I(Y)) = m-1$, where $I(Y)$ is a prime ideal. If $I(Y)$ contains an irreducible polynomials f and $I(Y) \neq (f)$ then Y is a proper closed subset of $Z(f)$ so we can apply 3.2.2.ii to get a contradiction. $\square$

3.3 Homogeneous coordinate ring

Let Y be an algebraic set in $\mathbb{C}P^n$ and $I(Y)$ be its homogeneous ideal. Then we define the **homogeneous coordinate ring of** Y to be $S(Y) = C[z_0, \cdots, z_n]/I(Y)$. For any $y \in Y$ denote by $\mathfrak{m}_y$ the set $\{f \in S(Y) | \, f(y) = 0\}$. It is easy to see that $\mathfrak{m}_y$ is a homogeneous maximal ideal of $S(Y)$. Unlike the affine case (see 3.1.2.i), not every homogeneous maximal ideal $\mathfrak{a}$ in $S(Y)$ is of the form $\mathfrak{m}_y$ for some $y \in Y$, as the following example shows. Let us consider the homogeneous ideal $S_+ = \oplus_{d>0} S_d$. Then $I(Y) \subset S_+$. The ideal $S_+/I(Y)$ is a homogeneous maximal ideal in $S(Y)$ but it does not correspond to any point $y \in Y$. In fact by using the correspondence $Y \mapsto CY$ we conclude that $S_+/I(Y)$ is the only homogeneous maximal ideal in $S(Y)$ which does not have the form $\mathfrak{m}_y$.

3.3.1. Proposition. *(i) There is a 1-1 correspondence between points y in an algebraic set $Y \subset \mathbb{C}P^n$ and homogeneous maximal ideals $\mathfrak{m}_y$ in $S(Y)$.*
(ii) $\dim S(Y) = \dim Y + 1$.

Proof. (i) This statement follows from Proposition 3.1.2.(i) and our observation about $\mathfrak{m}_y$ above.

(ii) Using the correspondence between an algebraic set Y in $\mathbb{C}P^n$ and its cone $CY \subset \mathbb{C}^{n+1}$ (see 2.3.2) we conclude that $\dim S(Y) = \dim A(CY) = \dim CY$. Clearly $\dim Y = \dim(CY \cap \{z_i = 1\})$ for some i (3.2.1.b). Hence $\dim Y = \dim[A(CY)/z_i = 1] \geq \dim CY - 1$. Now to prove that $\dim CY > \dim Y$ we use 3.2.2. (ii) and (3.2.1.b) which says that $\dim Y = \dim Y \cap U_i$. Alternatively repeat argument of the proof of 3.2.6.b). $\square$

3.3.3. Exercise. i) Prove that a projective variety $Y \subset \mathbb{C}P^n$ has dimension $(n-1)$, if and only if it is the zero set of a single irreducible homogeneous polynomial f of a positive degree.
ii) Prove that if a projective variety $Y \subset \mathbb{C}P^n$ is not a hypersurface H_i then $\dim(Y \cap H_i) = \dim Y - 1$.

Hint. Use the proof of Proposition 3.2.6. (b) or alternatively use 3.2.2.ii.

4 Morphisms

Morphisms between algebraic varieties are constructed as regular maps which are compatible with the concept of regular functions when we study small neighborhoods of a point in an algebraic variety. The notion of a regular function is defined locally, thus its definition is the same for regular functions on affine varieties and projective varieties.

4.1 Regular functions

4.1.1. Definitions. i) Let Y be a quasi-affine variety in $\mathbb{C}^n$. A function $f : Y \to \mathbb{C}$ is **regular** at a point $P \in Y$, if there is an open neighborhood U with $P \in U \subset Y$ and polynomials $g, h \in \mathbb{C}[z_1, \cdots, z_n]$ such that h is nowhere zero on U, and $f = g/h$ on U. We say that f is regular on Y if it is regular at every point of Y.

This definition includes the set of rational functions (g/h) as regular functions, since we want to include the notion of a (local) inverse function for a polynomial function.

4.1.2. Definition. Let Y be a quasi-projective variety in $\mathbb{C}P^n$. A function $f : Y \to \mathbb{C}$ is **regular** at a point $P \in Y$, if there is an open neighborhood U with $P \in U \subset Y$ and a homogeneous polynomials $g, h \in \mathbb{C}[z_1, \cdots, z_n]$ of the same degree, such that h is nowhere zero on U and $f = g/h$ on U.

The condition of "the same degree" ensures that g/h is well-defined as a function on U.

4.1.3. Lemma. *A regular function is continuous with respect to the Zariski topology.*

Proof. It suffices to prove that for each closed subset $Z \subset \mathbb{C}$ the pre-image $f^{-1}(Z)$ is a closed set in Y. Any closed subset Z of $\mathbb{C}$ is a finite set of points. Thus it suffices to prove that the pre-image of any point $z \in \mathbb{C}$ is a closed subset of Y. Let us consider the intersection $f^{-1}(z) \cap U$. For $f = g/h$ this set consists of all $y \in U$ such that $g(y) - z \cdot h(y) = 0$, so it is a closed subset of U. Hence $f^{-1}(z)$ is a closed subset in Y. $\square$

4.2 Local rings and rational functions

4.2.1. Definition. Let Y be a variety (i.e. any affine, quasi-affine, projective or quasi-projective variety). We denote by $\mathcal{O}(Y)$ the ring of all regular functions on Y. For any point $P \in Y$ we define the **local ring** of P on Y, $\mathcal{O}_{P,Y}$ (or simply $\mathcal{O}_P$) to be the ring of germs of regular functions on Y near P:

$$\mathcal{O}_P = \lim_{U \to p} \{(U, f),\ f \text{ is a regular function on } U\}.$$

4.2.2. Exercise. Prove that $\mathcal{O}_P$ is a local ring.

Hint. Show that the only maximal ideal in $\mathcal{O}_P$ is the set of germs of regular functions vanishing at P, because any other ideal contains invertible elements $f, 1/f$ for a somewhere non-vanishing f.

To any variety X we have associated a coordinate ring $A(X)$. Now we shall associate to X a field $K(X)$ which is called **the function field** of X as follows.

Any element of $K(X)$ is an equivalence class of pairs $< U, f >$ where U is a nonempty open subset of Y and f is a regular function on U. Two pairs $< U, f >$ and $< V, g >$ are equivalent, if $f = g$ on the intersection $U \cap V$. The elements of $K(X)$ is called **rational functions** on Y.

4.2.3. Remark. i) There exists a natural addition and multiplication on $K(X)$, so $K(X)$ is a ring. For any element $< U, f > \in K(X)$ with $f \neq 0$, the element $< U \setminus U \cap Z(f), 1/f >$ is an inverse for $< U, f >$. Hence $K(X)$ is a field.

ii) There exists natural maps $\mathcal{O}(X) \xrightarrow{i_p} \mathcal{O}_P \xrightarrow{j_p} K(X)$. Clearly i_p and j_p are injective. So we consider $\mathcal{O}(X)$ and $\mathcal{O}_P$ as sub-rings of $K(X)$.

4.2.4. Theorem. *Let $Y \subset \mathbb{C}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then*
i) $\mathcal{O}(Y) \cong A(Y)$.
ii) for each P the local field $\mathcal{O}_P$ is isomorphic to the localization $A(Y)_{\mathfrak{m}_P}$, where $\mathfrak{m}_P$ is the maximal ideal of functions vanishing at P (see 3.1.2), moreover $\dim \mathcal{O}_P = \dim Y$.
iii) $K(Y)$ is isomorphic to the quotient field $K(A(Y))$ of $A(Y)$ and hence the dimension of $K(Y)$ is equal to the dimension of $A(Y)$.

Proof. ii) Let us first prove the second statement. Let α be the natural inclusion $A(Y) \to \mathcal{O}(Y)$. This map i descends to a map $\bar{\alpha} : A(Y)_{\mathfrak{m}_P} \to \mathcal{O}_P$. This $\bar{\alpha}$ is injective since α is injective and $\bar{\alpha}$ is surjective by definition of a regular function. So $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$. Hence

$$\dim \mathcal{O}_P = \dim A(Y)_{\mathfrak{m}} \overset{3.2.5.a}{=} \dim A(Y) \overset{3.2.3}{=} \dim Y.$$

This proves 4.2.4.ii.

iii) Using remark 4.2.3 ii) we get that the quotient field $K(\mathcal{O}_P)$ of $\mathcal{O}_P$ is a subfield of $K(Y)$, so by 4.2.4.ii $K(A(Y)) \subset K(Y)$. But any rational function is in some $\mathcal{O}_P$, so $K(Y) \subset \cup_{P\in Y} K(\mathcal{O}_P) = K(A(Y))$. This proves 4.2.4.iii.

i) Clearly

$$\mathcal{O}(Y) \subset \cap_{P\in Y} \mathcal{O}_P \overset{4.2.4.ii}{=} \cap_{\mathfrak{m}_P} A(Y)_{\mathfrak{m}_P},$$

where $\mathfrak{m}_P$ are maximal ideals. We shall show that

$$\cap_{\mathfrak{m}_P} A(Y)_{\mathfrak{m}_P} = A(Y). \tag{4.2.5}$$

It suffices to show that if $(a, x) \in A(Y)_{\mathfrak{m}}$ for all maximal ideal $\mathfrak{m}$, then $(a, x) = (\bar{a}, 1)$ for some $\bar{a} \in A(Y)$. Using induction argument we can assume that x is irreducible, and therefore the principal ideal (x) is simple.

Let $\mathfrak{m}_P$ be a maximal ideal containing x. Since $(a, x) \in \cap_Q A(Y)_{\mathfrak{m}_Q}$ there is some $y \notin \mathfrak{m}_P$ and $b \in A(Y)$ such that

$$K(A(Y)) \ni (a, x) = (b, y).$$

From $t(ay - xb) = 0$ using integrality of $A(Y)$ we get $ay \in \mathfrak{m}_P$. Since $\mathfrak{m}_P$ is maximal, and $y \notin \mathfrak{m}_P$ we get $a \in \mathfrak{m}_P$. Now using the fact that $A(Y)/(x)$ is an integral domain and $\mathfrak{m}_P/(x)$ is a maximal ideal in $A(Y)/(x)$ we easily get

$$\bigcap \{ \text{ maximal } \mathfrak{m} |\, \mathfrak{m} \supset x\} = (x).$$

Hence $a \in (x)$ and therefore $(a, x) = (\bar{a}, 1)$ for some $\bar{a}$. This proves (4.2.5). This completes the proof of (i) and Theorem 4.2.4. $\square$

4.2.6. Remark. Some authors say that f is a regular function on an affine variety X if f belong to $A(X)$. This definition does not work for a quasi-affine variety, in particular for open set in X, so we can not have the notion of local regularity.

Before stating a structure theorem for projective varieties let us introduce a new notation. For a homogeneous prime ideal $\mathfrak{p}$ in a graded ring S we denote by $S_{(\mathfrak{p})}$ the subring of elements of degree 0 in the localization of S w.r.t. the multiplicative subset T consisting of the homogeneous elements of S not in $\mathfrak{p}$. Here the degree of an element (f/g) in $T^{-1}S$ is given by $\deg f - \deg g$. Clearly $S_{(\mathfrak{p})}$ is a local ring with maximal ideal $(\mathfrak{p} \cdot T^{-1}S) \cap S_{(\mathfrak{p})}$, since any $y \in S_{(\mathfrak{p})} \setminus \{\mathfrak{p} \cdot T^{-1}S) \cap S_{(\mathfrak{p})}\}$ is invertible. In particular the localization $S_{((0))}$ is a field, if S is a domain.

4.2.7. Theorem. *Let Y be a projective variety. Then:*
i) $\mathcal{O}(Y) = \mathbb{C}$,
ii) $\mathcal{O}_P = S(Y)_{(\mathfrak{m}_P)}$, where $\mathfrak{m}_P \subset S(Y)$ is ideal generated by homogeneous elements f vanishing at P,
iii) $K(Y) \cong S(Y)_{((0))}$.

Except statement (i), which is an analog of the Louiville theorem, the other statements (ii) and (iii) of Theorem 4.2.7 are similar to that ones in Theorem 4.2.4.

Proof of Theorem 4.2.7. ii) As in the proof of Theorem 4.2.4 we begin with the second statement. This is a local statement, so we shall apply Theorem 4.2.4.ii to this situation. We cover $\mathbb{C}P^n$ by open sets $U_i = \mathbb{C}P^n \setminus H_i$ (see Proposition 2.3.3) and let $\phi : U_i \to \mathbb{C}^n$ be the homeomorphism defined in 2.3.3. Now we define $\phi^* : \mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(U_0)$ by

(4.2.7.1) $$\phi^*(f)(z) = f(\phi(z)).$$

We shall show that this definition is correct, i.e. if locally $f = g/h$, where $g, h \in \mathbb{C}[z_1, \cdots z_n]$, then $\phi^*(f) = \tilde{g}/\tilde{h}$ where $\tilde{f}, \tilde{g}$ are homogeneous polynomials of the same degree in $\mathbb{C}[z_0, \cdots, z_n]$. Using (2.3.5) and substituting t' in (2.3.5) by g and h resp. we have

$$\frac{g(\phi(z))}{h(\phi(z))} = \frac{z_0^{-deg(g)}\beta(g)(z)}{z_0^{-deg(h)}\beta(h)(z)},$$

where $\beta(g)$ (resp. $\beta(h)$) is a homogeneous polynomial of degree $deg(g)$ (resp. $deg(h)$). Hence homogeneous polynomials $\tilde{g} = \beta(g)z_0^{deg(h)}$ and $\tilde{f} = z_0^{deg(g)}\beta(f)$ satisfy the required conditions.

4.2.8. Lemma *The map $\phi^* : \mathcal{O}(\mathbb{C}^n) \to \mathcal{O}(U_i)$ is a ring isomorphism.*

Proof Clearly ϕ^* is a ring homomorphism and ϕ is injective, since $f \in \ker \phi^*$ iff $f = 0$. To see that ϕ is surjective, we observe that if $f = (\tilde{g}/\tilde{h}) \in \mathcal{O}(U_0)$, where $\tilde{g}$ and $\tilde{h}$ are homogeneous of the same degree then

$$f(z) = \frac{r(\tilde{g})(\phi(z))}{r(\tilde{h})(\phi(z))},$$

where r is defined in (2.3.4) and we replace t in (2.3.4) by $\tilde{g}$ (resp. $\tilde{h}$). So

$$f = \phi^*(\frac{r(\tilde{g})}{r(\tilde{h})}).$$

$\square$

Now let us continue the proof of Theorem 4.2.7.ii. Let $Y_i = Y \cap U_i$. We can consider Y_i as an affine variety in $U_i = \mathbb{C}^n$. Using Lemma 4.2.8 and Theorem 4.2.4.ii we get $\mathcal{O}_P \cong A(Y_i)_{\mathfrak{m}'_P}$ where $Y_i \ni P$ and $\mathfrak{m}'_P$ is the maximal ideal of $A(Y_i)$ corresponding to P. Since $z_i \notin \mathfrak{m}_P$ and $\beta^{-1}(\mathfrak{m}_P) \subset \mathfrak{m}'_P$ we can construct a map $\phi^* : A(Y_i)_{\mathfrak{m}'_p} \to S(Y)_{(\mathfrak{m}_P)}$ as follows

$$(g, h) \overset{\phi^*}{\mapsto} (z_i^{deg(h)}\beta(g), z_i^{deg(g)}\beta(h)) \qquad (4.2.8.1)$$

(cf. 4.2.7.1). Clearly ϕ^* is a ring homomorphism whose kernel is empty because $\beta^{-1}(\mathfrak{m}_P) \subset \mathfrak{m}'_P$. It is easy to check that ϕ^* is surjective, so ϕ^* is an isomorphism which proves (ii).

iii) First we note that $K(Y) = K(Y_i)$ since any pair (U, f) representing an element in $K(Y)$ is equivalent to an element $(U \cap Y_i, f_{|(U\cap Y_i)})$. By Theorem 4.2.4.iii we get that $K(Y) = K(Y_i)$ is the quotient field $K(A(Y_i))$ of $A(Y_i)$. Using the natural isomorphism ϕ^* in (4.2.8.1) which extends to an isomorphism between the quotient field $K(A(Y_i))$ and $S(Y)_{((0))}$ we prove the statement (iii).

i) Let $f \in \mathcal{O}(Y)$ be a global function. Then f is regular on Y_i and therefore, by 4.2.4.i) we have $f \in A(Y_i)$. Using the isomorphism $\phi^* : A(Y_i) = S(Y)_{(z_i)}$ (see the proof of (ii) above, here we consider $A(Y_i)$ as a subring of $A(Y_i)_{\mathfrak{m}'_p}$) we conclude that $\phi^*(f)$ has the form $g_i/x_i^{N_i}$ where $g_i \in S(Y)$ is a homogeneous polynomial of degree N_i. Recall that $S(Y)_N$ denotes the subspace of $S(Y)$ with grading N. Choose a number $N \geq \sum N_i$ and note that $S(Y)_N \cdot (\phi^*(f)) \subset S(Y)_N$. Hence we get $S(Y)_N \cdot \phi^*(f)^q \subset S(Y)_N$. In particular $x_0^N \cdot \phi^*(f)^q \in S(Y)_N \subset S(Y)$ for all q.

Thus the subring $S(Y)[\phi^*(f)] \subset K(S(Y))$ is contained in $x_0^{-N}S(Y)$. Since $S(Y)$ is a noetherian ring, $S(Y)[\phi^*(f)]$ is finitely generated $S(Y)$-module. Therefore $\phi^*(f)$ is integral over $S(Y)$, or equivalently there are $a_1, \cdots, a_m \in S(Y)$ such that

$$\phi^*(f)^m + a_1\phi^*(f)^{m-1} + \cdots + a_m = 0. \tag{4.2.9}$$

($\phi^*(f)$ is a root of the characteristic polynomial).

For the sake of convenience of the reader we shall reproduce the proof of (4.2.9) from the book of Atiyah-Macdonald.

Let $x_1, \cdots, x_k$ be a system of generators of $S(Y)[f]$. Denote by M_f the endomorphism of $S(Y)[f]$ defined by the multiplication with f. Then

$$M_f(x_i) = \sum a_{ij}x_{ij}, \, \forall i$$

$$\iff \sum_j (\delta_{ij}M_f - a_{ij})x_j = 0, \, \forall i. \tag{4.2.10}$$

Multiplying the LHS of (4.2.10) with the adjoint matrix of $(\delta_{ij}M_f - a_{ij})$ we get $\det(\delta_{ij}M_f - a_{ij})$ annihilates all x_i, so $\det(\delta_{ij}M_f - a_{ij}) = 0$. Decompose this polynomial and substituting $f = \phi^*(f)$ we get (4.2.9).

Now we observe that $deg\, \phi^*(f) = 0$, so (4.2.9) still valid if we replace a_i by their homogeneous component of degree 0, i.e. we can assume that $a_i \in \mathbb{C}$. Thus $\phi^*(f)$ is algebraic over $\mathbb{C}$, so $\phi^*(f) \in \mathbb{C}$, hence $f \in \mathbb{C}$. □

4.3 Morphisms between varieties

We have met and used the notion of isomorphism between two particular varieties in Lemma 4.2.8. In general, a **morphism** $\phi : X \to Y$ is a continuous map such that for every open set $V \subset Y$ we have $\phi^*(\mathcal{O}(V)) \subset \mathcal{O}(\phi^{-1}(V))$, i.e. ϕ preserves the structure sheaf. We denote by $Mor(X, Y)$ the set of all morphisms from X to Y.

4.3.1. Proposition. *Let X be a variety and let Y be an affine variety. Then there is a natural bijective map of sets*

$$\alpha : Mor(X, Y) \to Hom(A(Y), \mathcal{O}(X)).$$

Proof. A morphism $\phi \in Mor(X,Y)$ defines a homomorphism $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. Since Y is affine, by (4.2.4.i) this natural transformation defines a map α. We first show that map α is injective, i.e. if ϕ_1 and ϕ_2 are two different morphisms, then ϕ_1^* and ϕ_2^* are different homomorphisms.

Any map $\phi : X \to Y \subset \mathbb{C}^n$ can be written in the following form

$$\phi(P) = (\xi_1(P), \cdots, \xi_n(P)) \in Y \subset \mathbb{C}^n. \tag{4.3.2}$$

Clearly $\mathcal{O}(X) \ni \xi_i = \phi^*(\bar{z}_i)$ where $\bar{z}_i$ the image of z_i in $A(Y) = \mathbb{C}[z_1, \cdots, z_n]/I(Y)$.

From (4.3.2) we see immediately that α is injective.

Now we shall show that α is surjective. Let $\bar{\phi}$ be a homomorphism from $A(Y)$ to $\mathcal{O}(X)$. Let $\xi_i = \bar{\phi}(\bar{z}_i) \in \mathcal{O}(X)$. We shall define a continuous map $\phi : X \to \mathbb{C}^n$ by (4.3.2). To complete the proof it suffices to show that $\phi(P) \in Y$ and $\phi^* = \bar{\phi}$. First we shall show that for any $f \in I(Y)$ we have $f(\phi(P)) = 0$ which shall imply that $\phi(P) \in Y$. Since $\bar{\phi}$ is a homomorphism of $\mathbb{C}$-algebras we have

$$f(\phi(P)) = f(\xi_1(P), \cdots, \xi_n(P)) = f(\bar{\phi}(\bar{z}_1(P)), \cdots, \bar{\phi}(\bar{z}_n(P))) = \bar{\phi}(f(\bar{z}_1, \cdots \bar{z}_n))(P) = 0.$$

The second statement $\phi^* = \bar{\phi}$ follows by checking

$$\phi^*(\bar{z}_i)(P) \stackrel{def}{=} \bar{z}_i(\phi(P)) = \bar{z}_i(\xi_1(P), \cdots, \xi_n(P)) = \bar{\phi}(\bar{z}_i)(P).$$

□

Now we shall say that a morphism $(\phi, \phi^*) : X \to Y$ is an **isomorphism**, if ϕ and ϕ^* admit inverse. In the category of differentiable manifolds with structure sheaf consisting of differentiable functions we can replace the global condition of invertibility of ϕ^* by the local invertibility of the tangent map $D\phi$. Analogously in the category of (complex algebraic) varieties we can replace the condition of global invertibility of ϕ^* by invertibility of the induced homomorphism $\phi_P^* : \mathcal{O}_{\phi(P),Y} \to \mathcal{O}_{P,X}$ for all $P \in X$.

4.3.3. Example. Let $H_d \subset \mathbb{C}P^n$ be a hyper-surface defined by a homogeneous polynomial P^d of degree d. We shall show that $\mathbb{C}P^n \setminus H_d$ is isomorphic to an affine variety. First we shall find an embedding $\phi_d : \mathbb{C}P^n \to \mathbb{C}P^N$ such that $\phi_d(H_d)$ lies in some hyper-plane $\{z_j = 0\}$ in $\mathbb{C}P^N$. Then we shall show that ϕ_d^* induces an isomorphism of local rings $\mathcal{O}_{\phi(P),\phi(\mathbb{C}P^n)}$ and $\mathcal{O}_{P,\mathbb{C}P^n}$ for all $P \in \mathbb{C}P^d$. This shall imply that $\phi_d(\mathbb{C}P^n \setminus H_d)$ is isomorphic to an affine variety $\phi^d(\mathbb{C}P^n \setminus H_d) \subset \mathbb{C}^N = \mathbb{C}P^N \setminus \{z_j = 0\}$ with the induced ring of regular functions. In particular $\mathcal{O}_{\phi(P),\phi(\mathbb{C}P^n)} = j^*\mathcal{O}_{P,\mathbb{C}P^N}$, where j is the restriction map.

The map ϕ_d can be chosen as a Veronese map of degree d

$$\phi_d : \mathbb{C}P^n \to \mathbb{C}P^N$$

$$[z_0, \cdots z_N] \mapsto [\cdots X^I \cdots]$$

where z^I ranges over all monomials of degree d in $z_0, \cdots, z_n$. Clearly ϕ_d is an embedding. Since P^d can be written as a linear combination of z^I, this proves the first statement. To show that ϕ_d^* induces a local isomorphism for all P it suffices to do it for any $P \in U_0 \subset \mathbb{C}P^n$. In this case $\mathcal{O}_{P,\mathbb{C}P^n} = \mathbb{C}[z_1, \cdots, z_n]_{\mathfrak{m}_P}$ and it is easy to check that $\phi_d^*(\mathcal{O}_{\phi(P),\mathbb{C}P^N}) = \mathcal{O}_{P,\mathbb{C}P^n}$, so $\phi_d^* : \mathcal{O}_{\phi(P),\phi(\mathbb{C}P^n)} \to \mathcal{O}_{P,\mathbb{C}P^n}$ is surjective. The kernel of ϕ_d^* at P consists of regular functions $g/h \in \mathcal{O}_{P,\mathbb{C}P^N}$ such that $(g/h)(\phi(U_P)) = 0$ for some neighborhood $P \in U_P \subset \mathbb{C}P^n$, hence $g \in I(\phi(U_P))$, so ϕ_d^* is injective.

4.3.4. Exercise. (i) Let $X \subset \mathbb{C}^n$ be an affine variety and $f \in \mathcal{O}(X)$. Define the open set $X_f \subset X$ by

$$X_f := X \setminus Z(f) = \{x \in X | \, f(x) \neq 0\}.$$

Prove that $\mathcal{O}(X_f) = \mathcal{O}(X)_{|X_f}[1/f]$. Using this show that $(X_f, \mathcal{O}(X_f))$ is an affine variety.

(ii) Prove that on any variety Y there is a base for the topology consisting of open affine subsets.

Hint. (i) Let $\tilde{X} := Z(I(X), f \cdot z_{n+1} - 1) \subset \mathbb{C}^{n+1}$. Show that the projection from $\mathbb{C}^{n+1} \to \mathbb{C}^n$ maps $\tilde{X}$ bijectively onto X_f. Show that the inverse of this projection pull z_{n+1} to f^{-1}.

(ii) If Y is an affine variety or quasi-affine variety, then reduce (ii) to (i). If Y is projective or quasi-projective, use the fact that Y can be covered by quasi-affine varieties (see Proposition 2.3.3 and consider the intersection $(U_i \cap Y)$.

4.3.5. Exercise. Let $f : X \to Y$ be a morphism between affine varieties. Prove that the image $\phi(X)$ is also an affine variety.

Hint. Extend ϕ to a morphism $e \circ \phi : X \to \mathbb{C}^n$ where $e : Y \to \mathbb{C}^n$ is the canonical embedding. Show that $I(e \circ \phi(X)) = \ker(e \circ \phi)^* : \mathbb{C}[z_1, \cdots, z_n] \to A(X)$.

4.4 Rational maps

The notion of a rational map is an extension of the notion of a rational function. A rational map is a morphism which is only defined on some open subset of a variety.

4.4.1. Definition. Let X, Y be varieties. **A rational map** $\phi : X \to Y$ is an equivalence class of pairs $< U, \phi_U >$ where U is a nonempty open subset of X, ϕ_U is a morphism of U to Y, and $< U, \phi_U >$ is said to be equivalent to $< V, \phi_V >$ if ϕ_U and ϕ_V agree on $U \cap V$.

The rational map ϕ is **dominant**, if for some pair $< U, \phi_U >$ the image of ϕ_U is dense in Y.

4.4.1.a. Example Let $Y = \{(z_1, z_2) \in \mathbb{C}^2 | \, z_1 z_2 = 1\}$. Define a map $\phi : Y \to \mathbb{C}$ by setting : $\phi(z_1, z_2) = z_1$. Then ϕ is a dominant rational map.

A **birational map** $\phi : X \to Y$ is a rational map which admits an inverse, i.e. there is a rational map $\psi : Y \to X$ such that $\psi \circ \phi = Id_X$ and $\phi \circ \psi = Id_Y$. If there is a birational map from X to Y, we say that X and Y are **birational equivalent**, or simply **birational**.

The equivalence notion of rational maps is very strong, since any open set is dense in Zariski topology.

4.4.2. Lemma. *Let X and Y be varieties and let ϕ and ψ be two morphisms from X to Y such that there is a nonempty open subset $U \subset X$ with $\phi_{|U} = \psi_{|U}$. Then $\phi = \psi$.*

Proof. Morphisms ϕ and ψ can be composed further with any morphism χ from Y to another variety Z leaving U unchanged. Therefore we can assume that $Z = \mathbb{C}P^n = Y$. We consider the map

$$(\phi \times \psi) : X \to \mathbb{C}P^n \times \mathbb{C}P^n.$$

Using the Serge embedding (3.2.9) we can provide $\mathbb{C}P^n \times \mathbb{C}P^n$ with a structure of a projective variety. Denote by $\triangle$ the diagonal in $\mathbb{C}P^n \times \mathbb{C}P^n$. Then $\triangle$ is a closed subset of $\mathbb{C}P^n \times \mathbb{C}P^n$. By assumption we have $\phi \times \psi(U) \subset \triangle$. But any open set U is dense, hence $(\phi \times \psi)(X) \subset \triangle$. □

The following theorem can be seen as an extension of Theorems 4.3.1. If X, Y are affine, then $Mor(X, Y) = Hom(A(Y), A(X)) \supset Hom(K(Y), K(X))$. Denote by $Mor_d(X, Y)$ the subset of dominant rational maps from X to Y.

4.4.3. Theorem. *For any variety X and Y there is a bijection B between sets*

$$Mor_d(X, Y) \cong Hom(K(Y), K(X)).$$

Proof. Let $\phi \in Mor_d(X, Y)$ be a dominant rational map represented by $< U, \phi_U >$. Let $f \in K(Y)$ be a rational function, represented by $< V, f >$, where V is an open set in Y and f is a regular function on V. We define B by

$$B(\phi) < V, f > := < \phi^{-1}(V), \phi^*(f) > .$$

Clearly $B(\phi)$ is a homomorphism from $K(Y)$ to $K(X)$.

Now we shall construct an inverse B^{-1}. Let $\theta : K(Y) \to K(X)$ be a homomorphism of $\mathbb{C}$-algebras. We shall reduce the construction $B^{-1}\theta$ in $Mor_d(X, Y)$

to the case that Y is an affine variety and then use Proposition 4.3.1 where such case has been treated.

To define an element ϕ in $Mor_d(X, Y)$ it suffices to define a map ϕ from X to an open set U_Y of Y. By Exercise 4.3.4 (ii) Y can be covered by affine varieties, so we shall choose U_Y being one of them. We have $A(U_Y) \subset K(Y)$ so we shall use the restriction of θ to $A(U_Y)$ to construct $B^{-1}(\theta) \in Mor_d(X_Y)$ and prove that it is a dominant rational map.

Let $y_1, \cdots, y_k$ be generators of $A(U_Y)$. Then $\theta(y_i)$ are rational functions on X. Let U_X be an open set in X where all $\theta(y_i)$ are regular functions on U_X. This implies that θ defines a homomorphism from $A(U_Y)$ to $\mathcal{O}(U_X)$ whose kernel is empty since θ is a homomorphism of the quotient field. Since U_Y is an affine variety, Proposition 4.3.1 yields that θ gives rise to an element $\tilde{B}(\theta) \in Mor(U_X, U_Y)$. Since θ is injective on $A(U_Y)$ the image $\tilde{B}(U_X)$ cannot be contained in an algebraic set in U_Y, hence $\tilde{B}(\theta)$ is a dominant rational map from X to Y. The proof of Proposition 4.3.1 yields that $\tilde{B}$ is inverse of B restricted to $A(U_Y)$, and hence $\tilde{B} = B^{-1}$. □

4.4.4. Corollary. *Two varieties X and Y are birationally equivalent, if and only if $K(X)$ is isomorphic to $K(Y)$ as $\mathbb{C}$-algebras.*

Proof. Suppose that X and Y are birational equivalent, i.e. there are rational map $\phi : X \supset U \to Y$ and $\psi : Y \supset V \to X$ which are inverse to each other. We shall find two open dense sets $U_1 \subset X$ and $V_1 \subset Y$ such that U_1 isomorphic to U_1. Then $\psi \circ \phi$ is represented by $< \phi^{-1}(V), \psi \circ \phi >$. By assumption the composition $\phi \circ \psi$ is the identity on $\psi^{-1}(U)$. Now let $U_1 = \phi^{-1}(\psi^{-1}(U))$ and $V_1 = \psi^{-1}(\phi^{-1}((V))$. It is easy to see that U_1 and V_1 isomorphic via ϕ and ψ. Hence $K(X) = K(U_1) = K(V_1) = K(Y)$.

The second statement follows from theorem 4.4.2 directly. □

4.4.5. Exercise. Prove that the quadratic surface $Q : xy = zw$ in $\mathbb{C}P^3$ is birational to $\mathbb{C}P^2$ but not isomorphic to $\mathbb{C}P^2$.

Hint. Show that Q is isomorphic to the Serge embedding of $\mathbb{C}P^1 \times \mathbb{C}P^1$, so it is birational equivalent to $\mathbb{C}P^2$.

4.4.6. Remark. We should mention here a famous fact that every irreducible variety X is birational to a hypersurface in $\mathbb{C}P^n$. There are two ways to see this. The first one relies on the statement that if X is a projective variety in $\mathbb{C}P^n$, a general projection $\pi_p : X \to \mathbb{C}P^{n-1}$ gives a birational isomorphism from X to its image $\bar{X}$. Iterating this projection we arrive in the end at a birational isomorphism of X to a hypersurface (see J. Harris 7.15 and 11.23 for more details).

Alternatively we can simply invoke the primitive element theorem to say that if $x_1, \cdots, x_k$ is a transcendence base for the function field of X, then $K(X)$ is generated over $k(x_1, \cdots, x_k)$ by a single element x_{k+1} satisfying an irreducible polynomial relation

$$F(x_{k+1}) = a_d(x_1, \cdots x_k) - x_{k+1}^d + \cdots + a_0(x_1, \cdots, x_k)$$

with coefficient in $K(x_1, \cdots x_k)$. Clearing denominators we may take F to be an irreducible polynomial in all $(k+1)$ variable. So by 4.4.4 X is birational to the hypersurface in $\mathbb{C}^{n+1}$ given by this polynomial. See Hartshorn 4.9 for more details.

5 Smoothness and tangent spaces

5.1 Zariski tangent spaces

We shall start with the affine case. Suppose that $X \subset \mathbb{C}^n$ is an affine variety. **A tangent vector** δ_{x_0} at a point $x_0 \in X$ is a "rule" to differentiate regular functions in x_0, i.e. it is a $\mathbb{C}$-linear map $\delta : \mathcal{O}(X) \to \mathbb{C}$ satisfying the Leibniz rule

$$\delta_{x_0}(f \cdot g) = f(x_0)\delta_{x_0}(g) + g(x_0)\delta_{x_0}(f),$$

for all $f, g \in \mathcal{O}(X)$. Such a map is called **derivation of $\mathcal{O}(X)$ in x_0.** It follows that $\delta_{x_0}(f^n) = nf^{n-1}(x_0)\delta_{x_0}(f)$ and so, for any polynomial $F = F(y_1, \cdots, y_m)$ we get

$$\delta_{x_0}(F(f_1, \cdots, f_m)) = \sum_{j=1}^{m} \frac{\partial F}{\partial y_j}(f_1(x_0), \cdots, f_m(x_0))\delta(f_j).$$

This implies that a derivation at x_0 is completely determined by its values on a generating set of the algebra $\mathcal{O}(X)$. As a consequence the set of all derivations in x_0 is a finite dimensional subspace of $Hom(\mathcal{O}(X), \mathbb{C})$.

5.1.1. Definition. The **Zariski tangent space** T_{x_0} of a variety X at a point x_0 is defined to be the set of all tangent vectors at x_0: $T_{x_0}X := Der_{x_0}(\mathcal{O}(X))$.

$T_{x_0}X$ is a finite dimensional linear subspace of $Hom(\mathcal{O}(X), \mathbb{C})$.

5.1.2. Exercise. Let δ be a tangent vector in x. Prove that
(i) $\delta(c) = 0$ for every constant $c \in \mathcal{O}(X)$.
(ii) If $f \in \mathcal{O}(X)$ is invertible, then $\delta(f^{-1}) = -\frac{\delta f}{f(x)^2}$.

Since $\mathcal{O}(X) = \mathbb{C} \oplus \mathfrak{m}_x$ for all $x \in X$ we see that any element $\delta \in T_xX$ is determined by its restriction to $\mathfrak{m}_x$. The Leibniz formula shows that the restrition to $\mathfrak{m}_x^2$ vanishes. Hence δ induces a linear map $\bar{\delta} : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{C}$.

5.1.3. Lemma. *Given an affine variety X and a point $x \in X$ there is a canonical isomorphism*

$$T_xX \to Hom(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C}),$$

given by $\delta \mapsto \bar{\delta} := \delta_{|\mathfrak{m}_x}$.

Proof. We have seen that $\delta \mapsto \bar{\delta}$ is injective. Let $\lambda \in Hom(\mathfrak{m}_x/\mathfrak{m}_x^2, \mathbb{C})$. Let C be a complement of $\mathfrak{m}_x^2$ in $\mathfrak{m}_x$, so $\lambda : C \to \mathbb{C}$ is a linear map. Now we extend λ to a linear map $\delta : \mathcal{O}(X) = \mathbb{C} \oplus C \oplus \mathfrak{m}_x^2 \to \mathbb{C}$ by putting $\delta_{|\mathbb{C}\oplus\mathfrak{m}_x^2} = 0$. □

5.1.4. Lemma. *For all* $z \in \mathbb{C}^n$ *we have* $T_z\mathbb{C}^n = \{\frac{\partial}{\partial z_i}_{|z}\}, i = 1, n$.

Proof. Let $z = (a_1, \cdots, a_n)$. The maximal ideal in $\mathbb{C}[z_1, \cdots, z_n]$ corresponding to z is $\mathfrak{m}_z = (z_1 - a_1, \cdots z_n - a_n)$. We define the derivation map

$$D : \mathfrak{m}_z/(\mathfrak{m}_z)^2 \to \mathbb{C}^n : f \mapsto (\frac{\partial f}{\partial z_i}_{|z}, i = 1, n).$$

Clearly $\{D(z_i - a_i),\ i = 1, n\}$ form a basis of $\mathbb{C}^n$, hence D is an isomorphism . Now Lemma 5.1.3 follows immediately from Lemma 5.1.3. □

5.1.5. Exercise. If $Y \subset X$ are affine varieties in $\mathbb{C}^n$ and $x \in Y$ then $\dim T_xY \leq \dim T_xX$.

Hint. The surjection $A(X)\mathcal{O}(X) \to \mathcal{O}(Y) = A(Y)$ induces a surjection $\mathfrak{m}_{x,X}/\mathfrak{m}_{x,X}^2 \to \mathfrak{m}_{x,Y}/\mathfrak{m}_{x,Y}^2$.

The space $(\mathfrak{m}_x/\mathfrak{m}_x^2)$ is called the **cotangent space** of X at x.

5.1.6. Definition. Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. We say that A is a **regular local ring**, if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

5.2 Nonsingular varieties

5.2.1. Definition. Let $Y \subset \mathbb{C}^n$ be an affine variety and let $f_1, \cdots, f_l \in \mathbb{C}[z_1, \cdots, z_n]$ be a set of generators for the ideal of Y. We say that Y is **nonsingular at a point** $P \in Y$ if the rank of the matrix $[(\partial f_i/\partial x_j)]_P$ at P is $n - r$ where r is the dimension of Y. We say that Y is **nonsingular**, if it is nonsingular at every point. The following theorem explains that the notion of nonsingularity does not depend on the choice of $(f_1, \cdots, f_n)$, i.e. on the choice of embedding $Y \to \mathbb{C}^n$.

5.2.2. Theorem. *Let* $Y \subset \mathbb{C}^n$ *be an affine variety. Let* $P \in Y$ *be a point. Then* Y *is nonsingular at* P*, if and only if the local ring* $\mathcal{O}_{P,Y}$ *is a regular local ring.*

Proof. Let $I(Y) \subset \mathbb{C}[z_1, \cdots, z_n]$ be the ideal of Y and let $f_1, \cdots f_l$ be a set of generators of $I(Y)$. Denote by $I(Y)_P$ the image of $I(Y)$ in the local ring $\mathfrak{m}_{P,\mathbb{C}^n}$. Then the rank of the Jacobian matrix $J_P = ||(\partial f_i/\partial x_j)||_P$ is the dimension of the space $D(I(Y)_P) \subset \mathbb{C}^n$, where $D : \mathfrak{m}_{P,\mathbb{C}^n} \to \mathbb{C}^n$ is defined in Lemma 5.1.4. Since D is an isomorphism we have

$$(5.2.3.a) \qquad rank\, J = \dim D(I(Y)_P) = \dim((I(Y)_P + \mathfrak{m}^2_{P,\mathbb{C}^n})/\mathfrak{m}^2_{P,\mathbb{C}^n}).$$

Denote by j the surjection $\mathbb{C}^n[z_1, \cdots, z_n] \to \mathcal{O}(Y) = A(Y)$ and by j_P the induced surjective map from $\mathfrak{m}_{P,\mathbb{C}^n} \to \mathfrak{m}_{P,Y}$ (see also 5.1.5). The kernel of j is $I(Y)$ and the kernel of j_P is $I(Y)_P$. Thus

$$(5.2.3.b) \qquad \frac{\mathfrak{m}_{P,Y}}{\mathfrak{m}^2_{P,Y}} = \frac{\mathfrak{m}_{P,\mathbb{C}^n}/(\ker j_P)}{(\mathfrak{m}_{P,\mathbb{C}^n}/\ker j_P)^2} = \frac{\mathfrak{m}_{P,\mathbb{C}^n}}{I(Y)_P + \mathfrak{m}^2_{P,\mathbb{C}^n}}.$$

Now since $\dim(\mathfrak{m}_{P,\mathbb{C}^n}/\mathfrak{m}^2_{P,\mathbb{C}^n}) = n$, taking into account (5.2.3.a) and (5.2.3.b) we get

$$(5.2.4) \qquad \dim(\mathfrak{m}_{P,Y}/\mathfrak{m}^2_{P,Y}) + rank\, J = n.$$

Let $\dim Y = r$. Then according to Theorem 4.2.4 ii) $\mathcal{O}_P$ is a local ring of dimension r. By definition $\mathcal{O}_p$ is regular if $\dim \mathfrak{m}/\mathfrak{m}^2 = r$. From (5.2.4) we get that this relation is equivalent to the relation $rank\, J = n - r$. □

5.2.5. Exercise. Let $X \subset \mathbb{C}^n$ be an affine subvariety. Prove that

$$T_{x_0}X = \{\delta \in T_{x_0}\mathbb{C}^n | \, \delta(f) = 0 \text{ for all } f \in I(X)\} \subset T_{x_0}\mathbb{C}^n = \mathbb{C}^n.$$

Hint Compare with 5.1.5.

Theorem 5.2.2 motivates us to give the following definition of (non)singularity of a variety. Let Y be a variety (not necessary affine). Then a point $P \in Y$ is **nonsingular** if the local ring $\mathcal{O}_{P,Y}$ is a regular local ring. Y is **nonsingular** if it is nonsingular at every point. Y is singular, if it is not nonsingular.

5.2.6. Example. Let $H := Z(f) \subset \mathbb{C}^n$ be a hypersurface where $f \in \mathbb{C}[z_1, \cdots, z_n]$ is an irreducible polynomial, hence $I(H) = (f)$. Then the tangent space in a point $x_0 \in H$ is given by (see 5.2.5)

$$T_{x_0} := \{a = (a_1, \cdots, a_n) | \sum a_i \frac{\partial f}{\partial x_i}(x_0) = 0\}. \qquad (5.2.6.1)$$

Let Y be a singular point of H. Then by definition the ring $\mathcal{O}_{P,H}$ is not regular, i.e. $\dim(\mathfrak{m}_{Y,H}/(\mathfrak{m}_{Y,H})^2) \neq \dim \mathcal{O}_{P,H}$. But $\mathcal{O}_{P,H} = A(H)_{\mathfrak{m}_{P,H}}$ and then using 3.2.5.a and 3.2.6.b we get

$$\dim \mathcal{O}_{P,H} = \dim A(H) = n - 1.$$

Thus Y is singular, iff $\dim T_{x_0}H \neq n-1$. Using (5.2.6.1) we se that the set H_{sing} of singular points of H is given by

$$H_{sing} = Z(f, \frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n}) \subset H.$$

5.2.7. Proposition. *Let X be an irreducible affine variety. Then the set X_{sing} of singular points is a proper closed subset of X whose complement is dense.*

Proof. We can assume that X is an irreducible closed subvariety in $\mathbb{C}^n$ of dimension d. Let $f_1, \cdots, f_l$ be a set of generators of $I(X)$. By Theorem 5.2.2

$$X_{sing} = \{x \in X | \, rk[\frac{\partial f_j}{\partial z_i}(x)] < n-d\}$$

is a closed subset defined by vanishing of all $(n-d)\times(n-d)$ minors of the Jacobian matrix J.

To show that X_{sing} is a proper subset of X we apply 4.4.5 to get X birational to a hypersurface $H \subset \mathbb{C}P^n$. Since birational maps preserve the dimension of variety and they map singular points/nonsingular points to singular points/nonsingular points, applying 5.2.6 we get 5.2.7. $\square$

5.3 Projective tangent spaces

Consider now a projective variety $X \subset \mathbb{C}P^n$. We may also associate to it a projective tangent space at each point $p \in X$, denoted T_pX which is a projective subspace of $\mathbb{C}P^n$. One way to do this is to choose an affine open subset $U \cong \mathbb{C}^n \subset \mathbb{C}P^n$ containing p and define the projective tangent space to X to be the closure in $\mathbb{C}P^n$ of the tangent space at p of the affine variety $X \cap U \subset U = \mathbb{C}^n$.

There is another way to describe the projective tangent space to a variety $X \subset \mathbb{C}P^n$ at a point $p \in X$. Let $\tilde{X} \subset \mathbb{C}^{n+1}$ be the cone over X and $\tilde{p} \in \tilde{X}$ be point lying over p. Then the projective tangent space T_pX is the subspace of $\mathbb{C}P^n$ corresponding to the Zariski tangent space $T_{\tilde{p}}\tilde{X} \subset T_{\tilde{p}}\mathbb{C}^{n+1} = \mathbb{C}^{n+1}$.

6 Completion

6.1 What is the completion of a ring?

Let R be an abelian group and let $R = \mathfrak{m}_0 \supset \mathfrak{m}_1 \cdots$ be a sequence of subgroups (a descending filtration). We define **the completion $\hat{R}$ of R w.r.t. the $\mathfrak{m}_i$** to be the inverse limit of the factor groups $R/\mathfrak{m}_i$ which is by definition a subgroup of the direct product

$$\hat{R} := \lim_{\leftarrow} R/\mathfrak{m}_i$$

$$:= \{g = (g_1, g_2, \cdots) \in \prod_i R/\mathfrak{m}_i | g_j \cong g_i(\mod \mathfrak{m}_i) \text{ for all } j > i\}.$$

If R is a ring and all $\mathfrak{m}_i$ are ideals then each of $R/\mathfrak{m}_i$ is a ring. Hence $\hat{R}$ is also a ring.

If moreover $\mathfrak{m}_i = \mathfrak{m}^i$ for some ideal $\mathfrak{m} \subset R$ then

$$\hat{\mathfrak{m}}_i := \{g = (g_1, g_2, \cdots) \in \hat{R} | \; g_j = 0 \text{ for all } j \leq i\},$$

is called **the $\mathfrak{m}$-adic filtration of** R. The corresponding completion $\hat{R}$ is denoted by $\hat{R}_{\mathfrak{m}}$. We write $\hat{\mathfrak{m}} = \mathfrak{m}_1$.

6.1.1. Exercise. *If $\mathfrak{m}$ is a maximal ideal, then $\hat{R}_{\mathfrak{m}}$ is a local ring with maximal ideal $\hat{\mathfrak{m}}$.*

Hint. Show that $\hat{R}/\hat{R}_{\mathfrak{m}} = R/\mathfrak{m}$ which is a field.

6.1.2. Example. If $R = \mathbb{C}[z_1, ..., z_n]$ and $\mathfrak{m} = (z_1, ..., z_n)$, then the completion with respect to $\mathfrak{m}$ is the formal power series ring $\hat{R}_{\mathfrak{m}} = \mathbb{C}[[z_1, ..., z_n]]$. Indeed, from the map $\mathbb{C}[[z_1, ..., z_n]] \to R/\mathfrak{m}_i$ sending f to $f + \mathfrak{m}_i$ we get a map $\mathbb{C}[[z_1, ..., z_n]] \to \hat{R}_{\mathfrak{m}}$ sending

$$f \mapsto (f + \mathfrak{m}, f + \mathfrak{m}^2, \cdots) \in \hat{R}_{\mathfrak{m}} \subset \prod R/\mathfrak{m}_i.$$

The inverse map is given as follows

$$\hat{R}_{\mathfrak{m}} \ni (f_1 + \mathfrak{m}, f_2 + \mathfrak{m}^2, \cdots) \mapsto (f_1 + (f_2 - f_1) + (f_3 - f_2) + \cdots . \tag{6.1.3}$$

Here the condition $f_j \cong f_i(\mod \mathfrak{m}^i)$ for $j > i$ implies that $deg\,(f_{i+1} - f_i) \geq i + 1$. Thus the RHS of (6.1.3) is a well-defined formal power series.

6.1.3. Definition. If the natural map $R \to \hat{R}_{\mathfrak{m}}$ is an isomorphism we call R **complete w.r.t.** $\mathfrak{m}$.

When $\mathfrak{m}$ is a maximal ideal, we say that R is a **complete local ring**.

6.2 Why to use the completion of a ring?

In the algebraic geometry we don't have a version of the implicit function theorem, since the inverse of a polynomial map is not a polynomial map. But the inverse can be represented by a formal power series which is a case of complete rings. The analog of the implicit function theorem for complete rings is the following Hensel's Lemma.

6.2.1. Theorem (Hensel's Lemma). *Let R be a ring that is complete w.r.t. the ideal $\mathfrak{m}$, and let $f(x) \in R[x]$ be a polynomial. If a is an approximate root of f in the sense that*

$$f(a) \cong 0 (\mod f'(a)^2\mathfrak{m})$$

then there is a root b of f near a in the sense that

$$f(b) = 0 \text{ and } b \cong a (\mod f'(a)\mathfrak{m}).$$

If $f'(a)$ is a nonzero-divisor in R, then b is unique.