Path and field integrals in physics: the main achievements

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I shall review what I consider as some of the important physics results in which the use of path or field integrals has played an essential role. Of course, this is by now a rather long story and I will be far from exhaustive.

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For an introductory note on path integrals in physics see, for example,

J. Zinn-Justin, *Path integral,* Scholarpedia, 4(2): 8674 (2009) (www.scholarpedia.org).

For details and more references see, for example,

J. Zinn-Justin, *Path integrals in Quantum Mechanics*, French version EDP Sciences et CNRS Editions (Les Ulis 2003), English version Oxford Univ. Press (Oxford 2005); Russian translation (Fizmatlit 2007);

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press 1989 (Oxford 4th ed. 2002).

Path integrals: the origins

The first path integral seems to have been defined by Wiener (1923), as a tool to describe the statistical properties of the Brownian motion, inspired by the well-known work of Einstein. If Wiener's work is rather well known, a less-known article of Wentzel of about the same period (1924) introduces, in the framework of quantum optics, the notions of sums over paths weighted by a phase factor, of destructive interference between paths that do not satisfy classical equations of motion, and the interpretation of the sum as a transition probability amplitude. Dirac (1933) has written a first expression of the quantum evolution operator that resembles a path integral, but he did not go beyond an approximate form with discrete time intervals. Of course, the modern history of path integrals begins with the articles of Feynman (1948) who formulates quantum evolution in terms of sums over a set of trajectories weighted by $\mathrm{e}^{iS/\hbar}$, where S is the value of the corresponding classical action (time-integral of the Lagrangian).

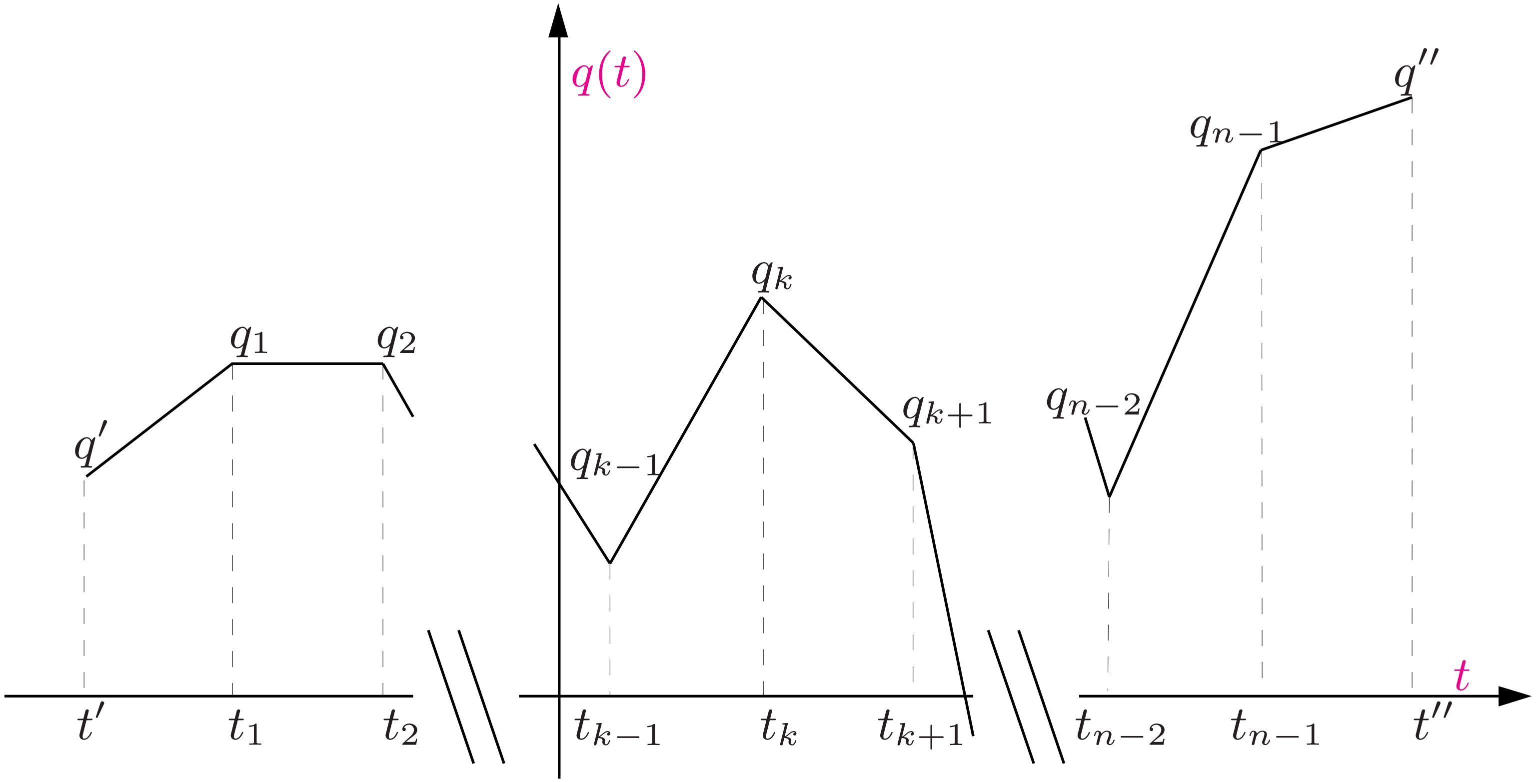

Fig. 1 A piecewise linear path contributing to the time-discretized path integral: one then integrates over $q_1, q_2, \ldots, q_{n-1}$ with weight $e^{iS(q)/\hbar}$. The continuum limit is reached by taking the limit $n \to \infty$.

The mystery of the variational principle in classical physics

Euler–Lagrange equations

Around 1750, Euler and Lagrange develop the variational calculus. Lagrange (1788) then shows that the equations of motion of Newtonian mechanics can be derived from a variational principle. He constructs a mathematical quantity, the action integral of a Lagrangian,

$$\mathcal{A}(q) = \int_{t'}^{t''} \mathrm{d}t\, \mathcal{L}\big(\mathbf{q}(t), \dot{\mathbf{q}}(t); t\big),$$

and recovers the equation of the classical motion by expressing the stationarity of the action with respect to variations of the trajectory $\mathbf{q}(t)$:

$$\delta\mathcal{A} = 0 \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}\,,$$

a form called Euler–Lagrange equations.

The simplest example is

$$\mathcal{L}(q,\dot{q}) = \tfrac{1}{2}m\dot{q}^2 - V(q) \ \Rightarrow \ m\ddot{q} = -V'(q).$$

In this framework, the action and the Lagrangian are pure mathematical quantities but, technically, this formalism happens to be very useful, for example, for systems with constraints, or to establish conservation laws generated by continuous symmetries.

The particle in a magnetic field
Later, it was discovered that the equation of motion of a particle in a static magnetic field $\mathbf{B}$, which takes the form

$$m\ddot{\mathbf{q}} = e\dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}) \quad \text{where} \quad \nabla \cdot \mathbf{B}(\mathbf{q}) = 0\,,$$

quite remarkably, can also be derived from an action principle, provided a new mathematical quantity is introduced, the vector potential:

$$\mathbf{B}(\mathbf{q}) = -\nabla \times \mathbf{A}(\mathbf{q}).$$

The Lagrangian can be written as

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} m \dot{\mathbf{q}}^2 - e\mathbf{A}(\mathbf{q}) \cdot \dot{\mathbf{q}} \, .$$

In this classical framework, the vector potential is not considered as a physical quantity since it is defined only up to gradient. Equivalent vector potentials are related by a gauge transformation:

$$\mathbf{A}(\mathbf{q}) \mapsto \mathbf{A}(\mathbf{q}) + \nabla\Omega(\mathbf{q}).$$

Electromagnetism and Maxwell's equations

Maxwell's equations (in the vacuum) can be written as

$$\nabla \cdot \mathbf{E} = \rho\,, \quad \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}\,,$$
$$\nabla \cdot \mathbf{B} = 0\,, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,,$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, resp., ρ the charge and $\mathbf{J}$ the current densities, resp..

In quadri-covariant notation where $(i, j = 1, 2, 3)$

$$t \equiv x_0\,,\ F_{i0} = E_i\,,\ F_{ij} = -\sum_k \epsilon_{ijk} B_k\,,\ J_0 = \rho\,,$$

they take the form

$$\sum_{\mu=0}^{3} \partial_\mu F^{\mu\nu} = J^\nu \ \Rightarrow\ \sum_{\nu=0}^{3} \partial_\nu J^\nu = 0\,.$$

These equations imply that the tensor $F_{\mu\nu}$ can be expressed in terms of a vector potential, or gauge field, $A_\mu(x)$ under the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, .$$

The gauge is defined only up to an Abelian gauge transformation

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \Omega(x).$$

Then again, remarkably enough, with the introduction of this new mathematical quantity, Maxwell's equations can be derived from an action principle with the Lagrangian density

$$\mathcal{L}(\mathbf{A}, \dot{\mathbf{A}}) = -\frac{1}{4} \sum_{\mu,\nu} F^{\mu\nu} F_{\mu\nu} - \sum_{\mu} J^\mu A_\mu \text{ with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \, ,$$

and gauge invariant action,

$$\mathcal{A} = \int \mathrm{d}^4 x \, \mathcal{L}(\mathbf{A}, \dot{\mathbf{A}}).$$

General Relativity

In Einstein's relativistic theory of gravitation (or General Relativity), the equations of motion can also be derived from an action principle.

For example, in the absence of matter, in terms of metric tensor $g_{ij}(x)$ they read

$$R_{ij}\big(\mathbf{g}(x)\big) - \tfrac{1}{2}R\big(\mathbf{g}(x)\big)g_{ij} = 0\,,$$

where R is the scalar curvature and R_{ij} the Ricci tensor.

These equations can be derived from Einstein–Hilbert's action,

$$\mathcal{A}(\mathbf{g}) = \int \mathrm{d}^4x\,\big(-g(x)\big)^{1/2} R\big(\mathbf{g}(x)\big),$$

where $g(x)$ is the determinant of the metric tensor. This property still holds in presence of a cosmological constant and matter.

The question then arises: why can all fundamental classical equations be derived by expressing the stationarity of a local action?

At first sight, quantum mechanics in its Hamiltonian formulation, does not give a direct answer. It should be considered as a major success of quantum mechanics in the path integral formulation, quantum field theory in the field integral formulation, that it provides a very simple explanation to this property. According to Feynman, quantum evolution is given by a path integral of the form

$$\langle q''|\, U(t'',t')\, |q'\rangle = \mathcal{N} \int_{q(t')=q'}^{q(t'')=q''} [\mathrm{d}q(t)]\, \mathrm{e}^{i\mathcal{A}(q)/\hbar},$$

where $U(t'',t')$ is the evolution operator, $\mathcal{A} = \int \mathrm{d}t\, \mathcal{L}$ is the classical action, time-integral of the classical Lagrangian, and one sums over all possible trajectories $q(t)$ satisfying the boundary conditions at times t' and t''.

In the classical limit, for $\hbar \to 0$, the path integral can be calculated by the stationary phase method and thus is dominated by paths that leave the action stationary: these are precisely the classical paths. This property generalizes to the relativistic quantum field theory.

As a potential non-trivial consequence, one can conclude that since the classical equations of General Relativity follow from a variational principle, the field integral over metrics (or, more generally, spin connection) of $\mathrm{e}^{i\mathcal{S}_{\mathrm{EH}}/\hbar}$, involving Einstein–Hilbert's action $\mathcal{S}_{\mathrm{EH}}$, properly regularized at short distance (of course, a non-trivial issue), should be directly relevant to quantum gravity. It is certainly the first term of an effective theory.

Covariance of the relativistic quantum field theory

The standard Hamiltonian formulation of relativistic quantum theory, is not explicitly covariant. As first noticed by Dirac in a time-discretized form, the corresponding field integral formulation, which by contrast involves the Lagrangian, is explicitly covariant.

The basic remark is as follows: in quantum mechanics, starting from first principles, one derives a path integral representation of the matrix elements of the evolution operator $U(t'',t')$ between times t' and t'' of the form

$$\langle q'' \left| U(t'',t') \right| q' \rangle = \int_{q(t')=q'}^{q(t'')=q''} [\mathrm{d}p(t)\mathrm{d}q(t)] \exp\left(\frac{i}{\hbar}\mathcal{A}(p,q)\right),$$

where p and q are phase space variables (position and conjugate momentum), and $\mathcal{A}(p,q)$ the classical action in the Hamiltonian formalism:

$$\mathcal{A}(p,q) = \int_{t'}^{t''} \left[p(t)\dot{q}(t) - H\big(p(t),q(t);t\big)\right] \mathrm{d}t\,.$$

When the classical Hamiltonian H is quadratic form in p, like $p^2/2m+V(q)$, the integral over p is Gaussian and can be performed explicitly:

$$\int [\mathrm{d}p(t)] \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} \mathrm{d}t \big(p(t)\dot{q}(t) - p^2(t)/2m\big)\right] \propto \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} \mathrm{d}t\, \tfrac{1}{2}m\dot{q}^2(t)\right].$$

The integration amounts to replacing $p(t)$ by the solution $m\dot{q}(t)$ of the classical equation and thus generates the Lagrangian:

$$\langle q'' \left| U(t'', t') \right| q' \rangle = \int_{q(t')=q'}^{q(t'')=q''} [\mathrm{d}q(t)] \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} \mathrm{d}t\, \mathcal{L}(q, \dot{q})\right)$$

with

$$\mathcal{L}(q, \dot{q}) = \tfrac{1}{2}m\dot{q}^2 - V(q).$$

In the relativistic quantum theory, the Lagrangian formulation is explicitly relativistic covariant, in contrast with the Hamiltonian formulation.

Critical phenomena and quantum field theory

Following Wilson, it was realized that universal critical properties of a large class of statistical models could be described by an Euclidean quantum or statistical field theory. The field integral formulation allows establishing a relation between classical statistical physics and quantum field theory.

For example, the critical properties of the d-dimensional Ising model

$$\mathcal{Z} = \sum_{\{S_i\}=\pm 1} \exp\left(J \sum_{i,j \text{ n.n.}} S_i S_j \right)$$

are described by the ϕ^4 quantum field theory (in imaginary time)

$$\mathcal{Z} = \int [\mathrm{d}\phi] \exp\left[-\mathcal{S}(\phi)\right],$$

where one now integrates over all fields $\phi(x)$, $x \in \mathbb{R}^d$ and $\mathcal{S}(\phi)$ is the Euclidean action:

$$\mathcal{S}(\phi) = \int \mathrm{d}^d x \left[\tfrac{1}{2} \textstyle\sum_\mu \left(\partial_\mu \phi(x)\right)^2 + \tfrac{1}{2}\phi^2(x) + \tfrac{1}{4!} g \phi^4(x) \right].$$

This property generalizes to classical spin models with $O(N)$ symmetry like

$$\mathcal{Z} = \sum_{\{|\mathbf{S}_i|\}=1} \exp\left(J \sum_{i,j\ \mathrm{n.n.}} \mathbf{S}_i \cdot \mathbf{S}_j\right).$$

In addition path integral techniques allow proving directly that the $N = 0$ limit describes the statistical properties of polymers.

It allows using the quantum field theory renormalization group to calculate universal critical properties of classical statistical systems near a continuous phase transition.

Most **reliable** *critical exponents from $O(N)$ symmetric $(\phi^2)^2_3$ field theory*

(Le Guillou and Z.-J. (1980) updated by Guida and Z.-J. (1998))

N	0	1	2	3
$\tilde{g}^*$	1.413 ± 0.006	1.411 ± 0.004	1.403 ± 0.003	1.390 ± 0.004
g^*	26.63 ± 0.11	23.64 ± 0.07	21.16 ± 0.05	19.06 ± 0.05
γ	1.1596 ± 0.0020	1.2396 ± 0.0013	1.3169 ± 0.0020	1.3895 ± 0.0050
ν	0.5882 ± 0.0011	0.6304 ± 0.0013	0.6703 ± 0.0015	0.7073 ± 0.0035
η	0.0284 ± 0.0025	0.0335 ± 0.0025	0.0354 ± 0.0025	0.0355 ± 0.0025
β	0.3024 ± 0.0008	0.3258 ± 0.0014	0.3470 ± 0.0016	0.3662 ± 0.0025
α	0.235 ± 0.003	0.109 ± 0.004	-0.011 ± 0.004	-0.122 ± 0.010
ω	0.812 ± 0.016	0.799 ± 0.011	0.789 ± 0.011	0.782 ± 0.0013
$\omega\nu$	0.478 ± 0.010	0.504 ± 0.008	0.529 ± 0.009	0.553 ± 0.012

Relation between classical and quantum statistical physics

It can be shown that for a scalar field ϕ in d space dimensions at temperature $T = 1/\beta$, the quantum partition function reads

$$\mathcal{Z} = \int [\mathrm{d}\phi] \exp\left[-\int_0^\beta \mathrm{d}t \int \mathrm{d}^d x\, \mathcal{S}(\phi)\right],$$

where $\mathcal{S}$ is the Euclidean (imaginary time) action, and the (Bose) fields satisfy the periodic boundary conditions

$$\phi(0, x) = \phi(\beta, x).$$

However, this field integral representation immediately shows that the same partition function has the interpretation of a classical partition function in $(d+1)$ space dimensions with finite size β and periodic boundary conditions in one space direction.

This remark plays a very important role in the theory of continuous phase transitions, relating a class of classical transitions in $(d+1)$ dimensions and quantum transitions at zero temperature $(\beta = \infty)$ in d dimensions.

More generally, the relation between classical and quantum statistical physics maps finite temperature quantum effects to finite size effects in the classical theory. This is most useful from the renormalization group viewpoint.

Finite temperature QFT, finite size effects in Statitiscal Field Theory and dimensional reduction

In particular, in this framework, high temperature is associated to dimensional reduction. Technically, one expands the periodic field in Fourier (Matsubara) modes

$$\phi(t,x)=\sum_{\nu}\mathrm{e}^{i2\pi\nu t/\beta}\,\phi_\nu(x).$$

At high temperature, near a continuous phase transition, when the correlation length is much larger than the thermal wave length $\lambda\!\!\!/=\hbar\sqrt{2\pi/mT}$, only the zero-mode is critical. One can then integrate perturbatively over all non-zero modes:

$$\mathrm{e}^{-\mathcal{S}_{\mathrm{eff.}}(\phi_0)}=\int\prod_{\nu\neq 0}[\mathrm{d}\phi_\nu]\,\mathrm{e}^{-\mathcal{S}(\phi)}\ \text{ with }\mathcal{Z}=\int[\mathrm{d}\phi_0]\,\mathrm{e}^{-\mathcal{S}_{\mathrm{eff}}(\phi_0)},$$

but must treat the zero-mode ϕ_0 non-perturbatively.

The dilute (thus weakly interacting) Bose gas

As an example, the technique has been applied to the dilute Bose gas. The initial field integral over fields ψ^*, ψ periodic in Euclidean time reads

$$\mathcal{Z} = \int [\mathrm{d}\psi(t,x)\mathrm{d}\psi^*(t,x)]\, \mathrm{e}^{-\mathcal{S}(\psi^*,\psi)/\hbar},$$

Since one is interested only in long wavelength phenomena, the two-body potential can be replaced by a delta-function and parametrized in terms of the s-wave scattering length a (positive because the interaction is assumed repulsive).

For $d = 3$, the effective Euclidean action of the system may then be written as (μ is the chemical potential)

$$\mathcal{S}(\psi^*,\psi) = -\int_0^\beta \mathrm{d}t \int \mathrm{d}^3x \left[\psi^*(t,x)\left(\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla_x^2 + \mu\right)\psi(t,x) \right.$$
$$\left. + \frac{2\pi\hbar^2 a}{m}\left(\psi^*(t,x)\psi(t,x)\right)^2\right].$$

The reduced partition function, at leading order, takes the form of the field integral

$$\mathcal{Z} = \int [\mathrm{d}\boldsymbol{\phi}(x)] \exp\left[-\mathcal{S}(\boldsymbol{\phi})\right]$$

with

$$\mathcal{S}(\boldsymbol{\phi}) = \int \left\{ \frac{1}{2} \left[\partial_\mu \boldsymbol{\phi}(x)\right]^2 + \frac{1}{2} r \boldsymbol{\phi}^2(x) + \frac{u}{4!} \left[\boldsymbol{\phi}^2(x)\right]^2 \right\} \mathrm{d}^d x \,,$$

where $r = -2mT\mu$ and, for $d = 3$, $u = 96\pi^2 a/\lambda^2$.

The Euclidean action reduces to the ordinary $O(2)$ symmetric $(\boldsymbol{\phi}^2)^2$ field theory, which also describes the universal properties of the superfluid Helium transition.

Numerical simulations in quantum field theory

The relation between classical and quantum partition function has led to the application of statistical methods to the non-perturbative study of quantum field theories. The idea is to replace the continuum field integral by a lattice regularized form. Then, non-perturbative numerical techniques become available, like strong coupling expansions or Monte-Carlo type simulations.

An outstanding example is QCD. In the absence of matter, simulations are based on Wilson's plaquette partition function:

$$\mathcal{Z}=\int\prod_{\text{links}\{ij\}}\mathrm{d}\mathbf{U}_{ij}\;\mathrm{e}^{-\beta_p\mathcal{S}(\mathbf{U})},\quad \mathcal{S}(\mathbf{U})=-\sum_{\text{plaquettes}}\operatorname{tr}\mathbf{U}_{ij}\mathbf{U}_{jk}\mathbf{U}_{kl}\mathbf{U}_{li}$$

where U_{ij} is a group element associated to links and $\mathcal{S}$ the plaquette action. The lattice formulation also yields a non-perturbative definition of QCD.

Quantization of non-Abelian gauge theories

By contrast with QED, the quantization of non-Abelian gauge theories, even without matter fields, does not follow from simple heuristic methods.

The gauge field $\mathbf{A}_\mu(x)$ belongs to the Lie algebra $\mathfrak{L}(G)$ of a group G and transforms under the adjoint representation of the group as

$$\mathbf{A}_\mu(x) \mapsto \mathbf{g}(x)\mathbf{A}_\mu(x)\mathbf{g}^{-1}(x) + \mathbf{g}(x)\partial_\mu \mathbf{g}^{-1}(x).$$

For matter fields, gauge invariance is enforced by replacing derivatives by covariant derivatives: $\mathbf{D}_\mu = \mathbf{1}\,\partial_\mu + \mathbf{A}_\mu$.

For the gauge field action, the associated curvature

$$\mathbf{F}_{\mu\nu}(x) = [\mathbf{D}_\mu, \mathbf{D}_\nu] = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu]\,,$$

is a tensor for gauge transformations:

$$\mathbf{F}_{\mu\nu}(x) \mapsto \mathbf{g}(x)\mathbf{F}_{\mu\nu}(x)\mathbf{g}^{-1}(x).$$

The local gauge action

$$\mathcal{A}(\mathbf{A}) = \frac{1}{4g^2} \int \mathrm{d}^4 x \ \mathrm{tr}\, \mathbf{F}_{\mu\nu}(x) \mathbf{F}^{\mu\nu}(x),$$

then is gauge-invariant.

However, due to gauge invariance, not all components of the gauge field are dynamical and simple canonical quantization is impossible. Tricks that worked for QED, like the direct elimination of the auxiliary components in the Coulomb gauge fail. All concepts required to quantize non-Abelian gauge theories, like the so-called Faddeev–Popov trick and ghosts are based on field integrals. BRST symmetry has emerged from this formalism.

Faddeev–Popov trick

The goal is to factorize the integration over gauge transformations. One starts from a non-gauge invariant equation for the space-dependent group element $\mathbf{g}(x)$, for example,

$$F(\mathbf{A}_\mu^g) \equiv \partial_\mu \mathbf{A}_\mu^g(x) - \boldsymbol{\nu}(x) = 0\,,$$

where $\mathbf{A}_\mu^g$ is the gauge transform by $\mathbf{g}$ of $\mathbf{A}_\mu$ and $\boldsymbol{\nu}(x)$ an arbitrary field.

The variation of the equation with respect to $\mathbf{g}$: $\delta\mathbf{g}(x) = \boldsymbol{\omega}(x)\mathbf{g}(x)$, $\boldsymbol{\omega}(x)$ belonging to the Lie algebra, has the form

$$\delta F(\mathbf{A}_\mu^g) = [\mathbf{M}(\mathbf{A}_\mu^g)\boldsymbol{\omega}](x), \quad \mathbf{M} = \partial_\mu \mathbf{D}_\mu\,.$$

One then introduces spinless fermions $\bar{\mathbf{C}}$ and $\mathbf{C}$, the Faddeev–Popov 'ghosts', and a boson field $\boldsymbol{\lambda}$ all transforming under the adjoint representation.

One then uses the identity

$$1 = \int [\mathrm{d}\mathbf{g}\, \mathrm{d}\bar{\mathbf{C}}\, \mathrm{d}\mathbf{C}\, \mathrm{d}\boldsymbol{\lambda}] \exp\left[-\mathcal{S}_{\text{gauge}}(\mathbf{A}^g_\mu, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}, \boldsymbol{\nu})\right]$$

with

$$\mathcal{S}_{\text{gauge}} = \int \mathrm{d}^d x\ \mathrm{tr}\left\{\boldsymbol{\lambda}(x)\left[F(\mathbf{A}_\mu)(x) - \boldsymbol{\nu}(x)\right] + \mathbf{C}(x)\mathbf{M}(\mathbf{A})\bar{\mathbf{C}}(x)\right\}.$$

This uses the notion of integration over Grassmann fields. Introducing the identity in the formal representation of the partition function, one obtains

$$\begin{aligned}\mathcal{Z} = \int & [\mathrm{d}\mathbf{g}\, \mathrm{d}\bar{\mathbf{C}}\, \mathrm{d}\mathbf{C}\, \mathrm{d}\boldsymbol{\lambda}\, \mathrm{d}\mathbf{A}_\mu] \\ & \times \exp\left[\frac{1}{4g^2}\int \mathrm{d}^d x\ \mathrm{tr}\,\mathbf{F}^2_{\mu\nu}(x) - \mathcal{S}_{\text{gauge}}(\mathbf{A}^g_\mu, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda}, \boldsymbol{\nu})\right].\end{aligned}$$

After the change variables $\mathbf{A}^g_\mu \mapsto \mathbf{A}_\mu$, the integration over $\mathbf{g}(x)$ factorizes and yields an infinite multiplicative constant.

After a few additional simple manipulations, one obtains the quantized partition function

$$\mathcal{Z} = \int \left[\mathrm{d}\mathbf{A}_\mu \,\mathrm{d}\bar{\mathbf{C}}\,\mathrm{d}\mathbf{C}\,\mathrm{d}\boldsymbol{\lambda}\right] \exp\left[-\mathcal{S}(\mathbf{A}_\mu, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda})\right],$$

where $\mathcal{S}$, in the covariant gauge $F = \partial_\mu \mathbf{A}_\mu$, is the local action:

$$\mathcal{S}(\mathbf{A}_\mu, \bar{\mathbf{C}}, \mathbf{C}, \boldsymbol{\lambda})$$
$$= \int \mathrm{d}^d x \; \mathrm{tr}\left[-\frac{1}{4e^2}\mathbf{F}^2_{\mu\nu} + \frac{\xi e^2}{2}\boldsymbol{\lambda}^2(x) + \boldsymbol{\lambda}(x)\partial_\mu \mathbf{A}_\mu(x) + \mathbf{C}(x)\partial_\mu \mathbf{D}_\mu \bar{\mathbf{C}}(x)\right].$$

It was later noticed that this quantized action has a fermion-like symmetry, the BRST symmetry. Its generalization is supersymmetry.

Quantization of the non-linear σ-model

The non-linear σ-model is a model with global $O(N)$ symmetry with an N-component scalar field $\boldsymbol{\phi}(x)$ that lives on the sphere S_{N-1}:

$$\boldsymbol{\phi}^2(x) = 1 \,.$$

In terms of $\boldsymbol{\phi}$, the action takes the form of a free action,

$$\mathcal{S}(\phi) = \tfrac{1}{2} \int \mathrm{d}^d x \, \left[\partial_\mu \boldsymbol{\phi}(x)\right]^2 ,$$

but the constraint generates interactions. Within the perturbative expansion, the $O(N)$ symmetry is realized in the phase of spontaneous symmetry breaking and the dynamical fields correspond to Goldstone modes. First calculations seem to indicate that the $O(N)$ symmetry was explicitly broken by the perturbative corrections. Within the canonical formulation, a complicated calculation showed that the breaking term actually cancelled.

However, the field integral representation gave both the correct quantized form to all orders and the geometric explanation of the problem as due to forgetting the $O(N)$ invariant measure (Meetz and Honerkamp)

$$\mathcal{Z} = \int [\mathrm{d}\phi] \prod_x \delta(\boldsymbol{\phi}^2(x) - 1) \exp\left[-\mathcal{S}(\boldsymbol{\phi})/g\right].$$

Moreover, to give a meaning to the model beyond perturbation theory, one can introduce a lattice regularization and this yields an $O(N)$ lattice spin model. In this way, one can establish a connection between the non-linear σ-model and the $(\boldsymbol{\phi}^2)^2$ statistical field theory.

Large N techniques

In quantum field theories with $O(N)$ or $U(N)$ symmetries and fields in the vector representation, physical quantities can be calculated in the large N limit, leading to non-perturbative results. At leading order, the same results can be obtained by summing Feynman diagrams, but field integral techniques are much simpler and can be extended to arbitrary orders in $1/N$. Applications include the study of the $(\boldsymbol{\phi}^2)^2$ theory (and the calculation of critical exponents), the Gross–Neveu model....

The basic idea is to introduces into the $\boldsymbol{\phi}$-field integral the identity

$$1 = \int [\mathrm{d}\lambda \mathrm{d}\rho] \exp\left\{ i \int \mathrm{d}^d x \, \lambda(x) \left[\rho(x) - \boldsymbol{\phi}^2(x)\right] \right\}.$$

For a recent review see

M. Moshe, J. Zinn-Justin, *Quantum field theory in the large N limit: a review*, **Phys. Rept. 385 (2003) 69** [hep-th/0306133].

Instantons, vacuum instability and large order behaviour

In simple quantum mechanics, barrier penetration effects can evaluated in the semi-classical limit by WKB methods. Alternatively, they can be determined in the path integral framework by looking for finite action solutions of Euclidean (imaginary time) equations of motion (instantons). However, the latter methods generalizes simply to quantum field theory, unlike methods based on Schrödinger equation. Important physics phenomena, like the periodic structure of QCD vacuum and the strong CP problem, the solution of the $U(1)$ problem are related to instantons.

Also, instantons lead to a determination of the behaviour of the perturbation expansion at large orders. An important application is the summation of the perturbative expansion to determine critical exponents from the ϕ^4 field theory.

Instantons and the problem of non-Borel summability

In the case of potentials with degenerate classical minima, instanton calculus applied to the large order behaviour indicates that the perturbative expansion is non-Borel summable, that is, does not determine unique functions. In simple quantum mechanics with analytic potentials, the problem can be studied systematically and it can be shown that all multi-instanton configurations must be taken into account and a generalized summation procedure introduced.

J. Zinn-Justin, *Multi-instanton contributions in quantum mechanics*, **Nucl. Phys. B192, 125-140 (1981)**.

U.D. Jentschura and J. Zinn-Justin, *Higher-order corrections to instantons*, **J. Phys. A34 (2001) L1–L6.**; *Multi-instantons and exact results I, II: conjectures, WKB expansions, and instanton interactions*, **Ann. Phys. 313 (2004) 197;** *ibidem*, **(2004) 269-325.**

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