Measuring Uncertainty with Elements of the [0,1]-Interval of Partially Ordered Rings

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Plan of Talk

1. The Algebra of Propositions
2. Axiom Systems for Uncertainty Measures
3. Ring Theorem
4. Trilemma
5. Relation to other Uncertainty Calculi
6. Open Questions / Future Research

How to represent and process Uncertainty?

- There are a lot of Uncertainty Calculi around today.
- Most of them make direct assumptions on the structure of the domain of uncertainty values (e.g. real numbers).
- But why should uncertainty not be measured by two or more numbers, or by a complex number, or matrices, or ...
- Additionally, many calculi first define a “static” uncertainty measure and then ask how to do conditionalization. In many cases, this has led to difficult questions that are the topic of ongoing research.

$\Rightarrow$ Try to directly axiomatize a “dynamic”, i.e. conditional uncertainty measures.

The Algebra of Propositions

We will call the objects to which we want to assign degrees of uncertainty Propositions.

- What is the structure of the domain of propositions?
- Probability theory assumes a σ-algebra. But σ-algebras are either finite or uncountable!
- From a foundational point of view, Boolean algebras are more suitable (every σ-algebra is a Boolean algebra).
- But there are other possible algebraic structures for propositions, most notably fuzzy algebras (e.g., no complementation laws) and quantum logic (no commutativity laws).

Terminology and Notation

A *conditional confidence measure* (CCM) for a Boolean Algebra $\mathbf{U}$ and a domain of confidence values $\mathcal{C}$ is a mapping:

$$\Gamma : \mathbf{U} \times \mathbf{U} \setminus \{\bot\} \rightarrow \mathcal{C}.$$

Instead of $\Gamma(A, B)$ we will write $\Gamma(A|B)$ and say "the confidence value of A given B (wrt. Γ)".

The domain of confidence values is partially ordered and has a greatest ($\top\!\!\top$) and a least ($\perp\!\!\perp$) element.

A *confidence space* is a triple $(\mathbf{U}, \Gamma, \mathcal{C})$.

A *collections of confidence spaces* sharing the same domain of confidence values we will call a *confidence universe*.

The Axioms System of R. T. Cox (1946)

1. $\mathcal{C} \subseteq \mathbb{R}$ (i.e., confidence values are real numbers)
2. There is a function F: $\Gamma(AB|C) = F(\Gamma(A|BC), \Gamma(B|C))$.
3. There is a function S: $\Gamma(\neg A|B) = S(\Gamma(A|B))$.
4. F, S are twice differentiable.

Cox shows that CCMs satisfying these axioms are rescalable to probability measures (but his axioms are incomplete! (see next axiom system)).

The Axiom System of J. B. Paris (1994)

1. $\mathcal{C} = [0,1]_{\mathbb{R}}$ ([0,1]-interval of the real numbers)
2. If $A \leq B$ (i.e., $AB = A$), then $\Gamma(B|A) = 1$ and $\Gamma(\neg B|A) = 0$
3. $\Gamma(AB|C) = F(\Gamma(A|BC), \Gamma(B|C))$ for some continuous function F which is strictly increasing (in both arguments) on $(0,1]^2$.
4. $\Gamma(\neg A|B) = S(\Gamma(A|B))$ for some decreasing function S.

The Axiom System of J. B. Paris (1994)

5. For any $0 \leq \alpha, \beta, \gamma \leq 1$ and $\epsilon > 0$ there are A_1, A_2, A_3, A_4 with $A_1 A_2 A_3$ consistent such that each of

$$|\Gamma(A_4|A_1 A_2 A_3) - \alpha|, \ |\Gamma(A_3|A_1 A_2) - \beta|, \ |\Gamma(A_2|A_1) - \gamma|$$

   is less than ϵ.

This axiom, along with the addidtional properties of F and S, fills the hole in the proof of Cox, but is hardly intuitive.

Paris-Cox Theorem

Given these axioms, there is a continuous, strictly increasing, surjective function $g : [0,1] \to [0,1]$ such that $\hat{\Gamma} = g \circ \Gamma$ satisfies

$$\begin{aligned} \hat{\Gamma}(\top|\cdot) &= 1 \\ \hat{\Gamma}(A \vee B|C) &= \hat{\Gamma}(A|C) + \hat{\Gamma}(B|C) \qquad \text{if } AB = \bot \\ \hat{\Gamma}(AB|C) &= \hat{\Gamma}(A|BC) \cdot \hat{\Gamma}(B|C) \end{aligned}$$

The Axiom System of S. Arnborg and G. Sjödin (2001)

Arnborg and Sjödin replace the fifth axiom of Paris by a more intuitive Refinability axiom:

For every confidence space $(\mathbf{U}, \Gamma, \mathcal{C})$, it must be possible to introduce a new subcase B of a non-false proposition A with confidence value v given to $\Gamma(B|A)$.

If two new subcases B and B' of a proposition A are defined, they can be specified to be independent, i.e., $\Gamma(B|B'A) = \Gamma(B|A)$ and $\Gamma(B'|BA) = \Gamma(B'|A)$.

For two confidence values v, w such that $v < S(w)$, it should be possible to define two new subcases C, C' of any non-false proposition A such that $v = \Gamma(C|A)$, $w = \Gamma(C'|A)$ and $\Gamma(CC'|A) = \perp\!\!\!\perp$.

The Axiom System of S. Arnborg and G. Sjödin (2001)

Arnborg and Sjödin introduce functions F and S like Cox, but additionally a *partial* function G (only defined for confidence values v, w with $v < S(w)$) such that:

$$\Gamma(A \vee B|C) = G(\Gamma(A|C), \Gamma(B - A|C)).$$

Furthermore, in a later part of their paper, they introduce a total order assumption for the domain of confidence values.

The Axiom System of S. Arnborg and G. Sjödin (2001)

Their other axioms are very similar to the axioms of Paris, but they take the important step to drop the real value assumption and derive the structure of the domain of confidence values from their axioms.

Their main result is the following: given their axioms, the domain of confidence values can be embedded in a totally ordered field, where multiplication and addition are extensions of F and G.

Our Axiom System

As Arnborg and Sjödin, we make no assumptions on the structure of domain of confidence values (only partial order).

We have tried to apply the design principle of Separation of Concerns, so that every axiom states a definite property which is as independent as possible from the other axioms.

We divide our axioms into Connective Axioms, concerning basic properties of the interplay of connectives and confidence measures, and Infrastructure Axioms, which we think are elementary properties of every uncertainty theory.

Connective Axioms

(Not)[a] $\forall\ (\mathbf{U_1}, \Gamma_1, \mathcal{C}),\ (\mathbf{U_2}, \Gamma_2, \mathcal{C})$:

If $\Gamma_1(A_1) = \Gamma_2(A_2)$, then $\Gamma_1(\neg A_1) = \Gamma_2(\neg A_2)$.

(And$_1$) $\forall\ (\mathbf{U_1}, \Gamma_1, \mathcal{C}),\ (\mathbf{U_2}, \Gamma_2, \mathcal{C})$:

If $\Gamma_1(A_1|B_1) = \Gamma_2(A_2|B_2)$ and $\Gamma_1(B_1) = \Gamma_2(B_2)$,
then $\Gamma_1(A_1B_1) = \Gamma_2(A_2B_2)$.

(And$_2$) $\forall\ (\mathbf{U_1}, \Gamma_1, \mathcal{C}),\ (\mathbf{U_2}, \Gamma_2, \mathcal{C})$:

If $\Gamma_1(A_1B_1) = \Gamma_2(A_2B_2)$ and $\Gamma_1(B_1) = \Gamma_2(B_2) \neq \perp\!\!\!\perp$,
then $\Gamma_1(A_1|B_1) = \Gamma_2(A_2|B_2)$.

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[a] $\Gamma(A)$ is shorthand for $\Gamma(A|\top)$.

Infrastructure Axioms

(Order$_1$) $\forall\ (\mathbf{U}, \Gamma, \mathcal{C})$, and $A, B \in \mathbf{U}$:

If $A \leq B$, then $\Gamma(A) \leq \Gamma(B)$.

(Order$_2$) $\forall\ v, w \in \mathcal{C}$ with $v \leq w$ there is a confidence space $(\mathbf{U}, \Gamma, \mathcal{C})$ and $A, B \in \mathbf{U}$ with $A \leq B$, $\Gamma(A) = v$, $\Gamma(B) = w$.

We dropped in the above axioms the general conditioning on a further proposition in order to increase readability.

Infrastructure Axioms

(Extensibility) $\forall$ $(\mathbf{U_1}, \Gamma_1, \mathcal{C})$ and $(\mathbf{U_2}, \Gamma_2, \mathcal{C})$ there is a confidence space $(\mathbf{U_3}, \Gamma_3, \mathcal{C})$, so that $\mathbf{U_3} \cong \mathbf{U_1} \otimes \mathbf{U_2}$, and for all $A_1, B_1 \in \mathbf{U}_1$, $A_2, B_2 \in \mathbf{U}_2$:

$$\Gamma_3(A_1 \otimes \top_2 \,|\, B_1 \otimes B_2) = \Gamma_1(A_1|B_1),$$

$$\Gamma_3(\top_1 \otimes A_2 \,|\, B_1 \otimes B_2) = \Gamma_2(A_2|B_2).$$

Ring Theorem

The domain of confidence values $\mathcal{C}$ of a confidence universe satisfying the axioms **Not**, **And**$_1$, **And**$_2$, **Order**$_1$, **Order**$_2$ and **Extensibility** can be embedded in a partially ordered ring $(\hat{\mathcal{C}}, 0, 1, \oplus, \odot, \leq)$. Let $\hat{\cdot} : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ be the embedding map, then the following holds:

$$\hat{\perp\!\!\!\perp} = 0, \quad \hat{\top\!\!\!\top} = 1,$$

$$\forall v, w \in \mathcal{C} : \; v \leq w \;\Leftrightarrow\; \hat{v} \leq \hat{w}.$$

Ring Theorem

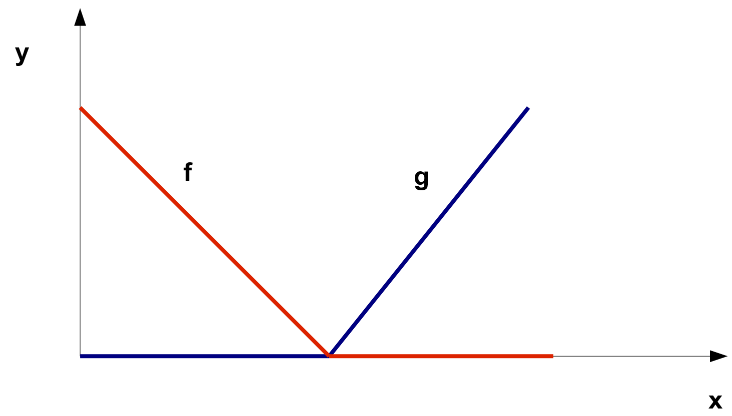
Furthermore, all confidence measures Γ of the confidence universe satisfy:

$$
\begin{aligned}
\hat{\Gamma}(\top | \cdot) &= 1 \\
\hat{\Gamma}(A \vee B | C) &= \hat{\Gamma}(A|C) \oplus \hat{\Gamma}(B|C) \qquad \text{if } AB = \bot \\
\hat{\Gamma}(AB|C) &= \hat{\Gamma}(A|BC) \odot \hat{\Gamma}(B|C)
\end{aligned}
$$

Function Rings

The set of functions from $\mathbb{R}$ to $\mathbb{R}$ form a ring: α, $1-\alpha$, $\max(0,\alpha)$, ...

$\Rightarrow$ contains zero divisors ($f \neq 0$, $g \neq 0$, $fg = 0$)



Proof Sketch of Ring Theorem :

- **Not** implies existence of S.
- $\textbf{And}_1$ implies existence of F.
- F is associative and commutative (**Extensibility** plays a decisive role here)
- F is a cancellative monoid.
- Extend F to a group.
- Define the G function with F, S, F^{-1}.
- The structure (F, G) is now a semi-field.
- Extend G to a group, then the structure (F, G) is a ring.

The G-Function

Define with S, F, and F^{-1} the following function:

$$G(x, y) = S(F(S(F(x, F^{-1}(S(y)))), S(y)))$$

In order to illustrate this definition, we note that G essentially has to solve the problem to represent addition with the functions $x * y$, $1 - x$, and $1/x$. Using these functions, G becomes:

$$1 - (1 - \tfrac{x}{1-y})(1 - y)$$

which reduces to addition.

Total Order Theorem

A lattice-ordered domain of confidence values $\mathcal{C}$ of a confidence universe satisfying the axioms **Not**, $\mathbf{And}_1$, $\mathbf{And}_2$, $\mathbf{Order}_1$, $\mathbf{Order}_2$ and **Extensibility** is totally ordered.

Trilemma

- In general lattice order is far from total order.
- But in the context of our axioms, this weak property has a strong implication.

This results is the following trilemma: the following three properties can't be satisfied all at the same time:

- partiality of the order of confidence values
- lattice property of the order of confidence values
- field structure of the domain of confidence values

$\Rightarrow$ One has to decide which of these three properties is the most dispensable (we opt for the field structure).

Relation to other uncertainty calculi

1. Non-monotonic Logic:

Set $\mathcal{C} = [0, 1]$ of the hyperreal numbers, containing infinitesimal elements (ϵ). Use ϵ and $1 - \epsilon$ as confidence values instead of 0 or 1. Then one can reproduce the revision mechanism of non-monotonic logic.

2. Lower Probability:

Set $\mathcal{C} = [0, 1]$ of the field of rational functions over $\mathbb{R}$. Then

$$\underline{P}(A) := \inf\ \Gamma(A)$$

is a lower probability function.

Open Questions / Future Research

- Is there a universal embedding "confidence ring"?
- Drop axiom $\mathbf{And}_2$. Do we still get a ring theorem?
- What could be a candidate for an efficient evaluable confidence ring which contains uncomparable elements?
- Connection to Decision Theory (especially the Continuity axiom)?
- Clarify the connection to other Uncertainty Calculi.