Preprint, Institute of Mathematics, AS CR, Prague. 2009-8-6

On slowly growing solutions of linear functional differential systems

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Abstract. We obtain new conditions sufficient for the (unique, under an additional condition) solvability of a system of singular functional differential equations with non-increasing operators.

1. Problem setting and motivation

The aim of this note is to establish some general conditions sufficient for the existence and uniqueness of a slowly growing solution of a class of singular linear functional differential equations. More precisely, we consider the system of linear functional differential equations

$$x_i'(t) = \sum_{k=1}^{n} (l_{ik} x_k)(t) + q_i(t), \qquad t \in [a, b),\ i = 1, 2, \dots, n, \tag{1}$$

subjected to the initial conditions

$$x_i(a) = \lambda_i, \qquad i = 1, 2, \dots, n, \tag{2}$$

where $-\infty < a < b < \infty$, the functions q_i, $i = 1, 2, \dots, n$, are locally integrable, and $l_{ik} : C([a, b), \mathbb{R}) \to L_{1;\mathrm{loc}}([a, b), \mathbb{R})$, $i, k = 1, 2, \dots, n$, are linear mappings which are assumed to be positive with respect to a natural pointwise ordering. Our aim is to find conditions sufficient for the existence and uniqueness of a slowly growing solution of the initial value problem (1), (2). The "slow growth" of a solution $x = (x_i)_{i=1}^n : [a, b) \to \mathbb{R}^n$ is understood in the sense that its components satisfy the conditions

$$\sup_{t \in [a,b)} h_i(t) |x_i(t)| < +\infty, \qquad i = 1, 2, \dots, n, \tag{3}$$

*Research supported in part by AS CR, Institutional Research Plan No. AV0Z10190503.

where $h_i : [a, b) \to [0, +\infty)$, $i = 1, 2, \ldots, n$, are certain given continuous functions possessing the properties

$$\lim_{t \to b-} h_i(t) = 0, \qquad i = 1, 2, \ldots, n. \tag{4}$$

Solutions of system (1) are sought for in the class of locally absolutely continuous functions and, in particular, may be unbounded in a neighbourhood of the point b.

Definition 1. One says that a function $u : (a, b] \to \mathbb{R}$ (resp., $u : [a, b) \to \mathbb{R}$) is *locally absolutely continuous* if its restriction $u|_{[a+\varepsilon,b]}$ (resp., $u|_{[a,b-\varepsilon]}$) to the interval $[a + \varepsilon, b]$ (resp., $[a, b - \varepsilon]$) is absolutely continuous for any $\varepsilon \in (0, b - a)$.

When formulating the precise definition of a solution of this kind of equations, it is important to take into account the unpleasant circumstance that the derivative of a locally absolutely continuous vector function satisfying equations (1) may have a non-integrable singularity in a neighbourhood of the point b. For example, the function

$$x(t) = \lambda\, (1 - t)^{-3}, \qquad t \in [0, 1), \tag{5}$$

for any real λ satisfies the relations

$$x'(t) = \frac{2}{(1-t)^3} x(1 - \sqrt[3]{1-t}) + \frac{1}{(1-t)^{\frac{5}{2}}} x(1 - \sqrt{1-t}), \qquad t \in [0, 1), \tag{6}$$

$$x(0) = \lambda, \tag{7}$$

where the coefficient functions $[0, 1) \ni t \mapsto 2\,(1 - t)^{-3}$ and $[0, 1) \ni t \mapsto (1 - t)^{-\frac{5}{2}}$ are non-integrable. To introduce a proper notion of a solution of the corresponding initial value problem, a certain weight function is thus natural to be used, which would govern the behaviour of x' in a left neighbourhood of the point 1. In our case, the role of these weight functions is played by the same functions $h_i : [a, b) \to [0, +\infty)$, $i = 1, 2, \ldots, n$, that appear in the growth restriction (3), so that the corresponding definition of a solution has the following form.

Definition 2. By a *solution* of the functional differential system (1), we mean a locally absolutely continuous vector function $x = (x_i)_{i=1}^n : [a, b) \to \mathbb{R}^n$ which components possessing the properties $h_i x_i' \in L_1([a, b), \mathbb{R})$, $i = 1, 2 \ldots, n$, and satisfying equalities (1) almost everywhere on the interval $[a, b)$.

Note that in the case where h_i, $i = 1, 2, \ldots, n$, are equal identically to non-zero constants, the latter definition reduces in a natural way to the Carathéodory case, where the solution is absolutely continuous on the entire interval. Under assumption (4), however, system (1) may have solutions $(x_i)_{i=1}^n$ such that $x_i' \notin L_1([a, b), \mathbb{R})$ for some or all $i = 1, 2, \ldots, n$.

Definition 3. We say that a solution $x = (x_i)_{i=1}^n : [a, b) \to \mathbb{R}^n$ of system (1) is slowly growing if it has property (3).

In the sequel, we are interested in establishing conditions under which the initial problem (1), (2) has a unique slowly growing solution for arbitrary q_i, $i = 1, 2, \dots, n$, with $h_i q_i$ Lebesgue integrable on $[a, b)$.

Efficient and sharp conditions sufficient for the solvability of problems of type (1), (2), (3) are useful in studies of various non-linear singular problems arising in numerous applications. We note that the necessity in a systematic study of differential equations with singularities had arisen quite long ago. One can mention, e. g., the singular Cauchy problem at the point 0 for the two-dimensional system

$$\begin{aligned} u_1'(t) &= u_2(t), \\ u_2'(t) &= -\frac{2}{t} u_2(t) - (u_1(t))^\lambda, \qquad t \in (0, 1], \end{aligned}$$

which had been studied in [1] as far back as 1907. However, it is only in the second half of the last century when a general theory of such problems had been constructed. We refer the reader to the works [2, 3, 5] for more details on this subject.

It should be noted that setting (1) is rather general. In particular, any system of linear functional differential equations of the form

$$x'(t) = (lx)(t) + q(t), \qquad t \in [a, b), \tag{8}$$

determined by a linear mapping $l = (l_i)_{i=1}^n : C([a, b), \mathbb{R}^n) \to L_{1;\,\mathrm{loc}}([a, b), \mathbb{R}^n)$ is nothing but (1) with $l_{ik} : C([a, b), \mathbb{R}) \to L_{1;\,\mathrm{loc}}([a, b), \mathbb{R})$, $i, k = 1, 2, \dots, n$, defined by the formulae

$$l_{ik} v := l_i(v e_k), \qquad i, k = 1, 2, \dots, n, \tag{9}$$

for any v from $C([a, b), \mathbb{R})$, where

$$e_k := \mathrm{col}\,(0, 0, \dots, 0, 1, 0, \dots, 0), \qquad k = 1, 2, \dots, n, \tag{10}$$

with "1" on the kth place. We emphasize that l_{ik} $i, k = 1, 2, \dots, n$, are defined on the set of functions which are continuous on the half-open interval $[a, b)$ and may have discontinuities at the point b.

For the sake of simplicity, we assume throughout the paper that the above-mentioned functions h_i, $i = 1, 2, \dots, n$, possess the following properties:

The functions $h_i : [a, b) \to [0, +\infty)$, $i = 1, 2, \dots, n$ are non-increasing and such that relations (4) *hold.* (11)

Note that only the qualitative behaviour of h_i, $i = 1, 2, \dots, n$, in a neighbourhood of the point b has influence on the formulation of condition (3) and, thus, h_i, $i = 1, 2, \dots, n$, can be redefined in any suitable manner on the interval $[a, b - \delta]$ for δ small enough. Conditions (11) are satisfied, for example, by the functions

$$h_i(t) := (b - t)^{\gamma_i}, \qquad t \in [a, b),$$

where $\{\gamma_i \mid i = 1, 2, \dots, n\} \subset (0, +\infty)$. For example, one can check directly that function (5) for arbitrary $\lambda \in \mathbb{R}$ and $\varepsilon \in (0, +\infty)$ is a solution of the problem

$$\sup_{t\in[0,1)} (1-t)^{3+\varepsilon} |x(t)| < +\infty \tag{12}$$

for equation (6) with the function h given by formula

$$h(t) := (1-t)^{3+\varepsilon}, \qquad t \in [0, 1), \tag{13}$$

where ε is positive. It is obvious that assumptions (11) are satisfied in this case.

To conclude this introductionary section, we note that problem (1), (2), (3) is related to the notion of a singular Cauchy problem (see, e. g., [2]). Indeed, condition (3) yields

$$\lim_{t\to b-} \varrho_i(t)h_i(t)x_i(t) = 0, \qquad i = 1, 2, \dots, n, \tag{14}$$

for any continuous functions $\varrho_i : [a, b) \to \mathbb{R}$ with the properties $\lim_{t\to b-} \varrho_i(t) = 0$, $i = 1, 2, \dots, n$, i. e., a solution of (1), (2), (3) is also that of each of the problems (1), (2), (14) and *vice versa*.

Problems similar to (1), (2), (3), including the singular Cauchy problem for various classes of functional differential equations, are treated, in particular, in [4–12]. A problem on Carathéodory solutions of system (8) possessing properties of type (2) is studied in [13].

2. Notation

The following notation is used throughout the paper.

(1) $\mathbb{R} := (-\infty, \infty)$, $\mathbb{N} := \{1, 2, 3, \dots\}$.

(2) $\|x\| := \max_{i=1,2,\dots,n} |x_i|$ for any $x = (x_i)_{i=1}^n$ from $\mathbb{R}^n$.

(3) If $-\infty < a < b < \infty$ and $A \subseteq [a, b]$ is a measurable set, then $L_1(A, \mathbb{R}^n)$ is the Banach space of all the Lebesgue integrable vector functions $u = (u_i)_{i=1}^n : A \to \mathbb{R}^n$ with the standard norm

$$L_1(A, \mathbb{R}) \ni u \longmapsto \max_{i=1,2,\dots,n} \int_A |u_i(t)|dt.$$

(4) $L_{1;\,\mathrm{loc}}((a, b], \mathbb{R})$ is the set of functions $u : (a, b] \to \mathbb{R}$ such that $u|_{[a+\varepsilon,b]} \in L_1([a+\varepsilon, b], \mathbb{R})$ for any $\varepsilon \in (0, b-a)$.

(5) $L_{1;\,\mathrm{loc}}((a, b], \mathbb{R}^n)$ is the set of vector functions $u = (u_i)_{i=1}^n : (a, b] \to \mathbb{R}^n$ such that $u_i \in L_{1;\,\mathrm{loc}}((a, b], \mathbb{R})$ for each $i = 1, 2, \dots, n$.

(6) $C((a, b], \mathbb{R})$ is the linear manifold of all the continuous functions $u : (a, b] \to \mathbb{R}$.

(7) $\tilde{C}_{\mathrm{loc}}((a, b], \mathbb{R}^n)$ is the set of all the locally absolutely continuous vector functions $u = (u_i)_{i=1}^n : (a, b] \to \mathbb{R}^n$.

(8) $\tilde{C}_{\text{loc};\,h}((a,b],\mathbb{R})$ is the set of all the locally absolutely continuous functions $u : (a,b] \to \mathbb{R}$ such that $hu' \in L_1((a,b],\mathbb{R})$ and

$$\sup_{t\in(a,b]} h(t)\,|u(t)| < +\infty. \tag{15}$$

(9) If $h = \operatorname{diag}(h_1,\dots,h_n) : (a,b] \to \mathbb{R}^n$ is a continuous matrix-valued function, then $\tilde{C}_{\text{loc};\,h}((a,b],\mathbb{R}^n)$ is the set of all the vector functions $u = (u_i)_{i=1}^n : (a,b] \to \mathbb{R}$ such that $u_i \in \tilde{C}_{\text{loc};\,h_i}((a,b],\mathbb{R})$ for each $i = 1,2,\dots,n$.

(10) The sets $C([a,b),\mathbb{R})$, $L_{1;\,\text{loc}}([a,b),\mathbb{R}^n)$, $\tilde{C}_{\text{loc}}([a,b),\mathbb{R}^n)$, and $\tilde{C}_{\text{loc};\,h}([a,b),\mathbb{R}^n)$ are defined by analogy.

3. Existence of a slowly growing solution and its uniqueness

Our results concern the case where the right-hand sides of equations (1) are determined by linear operators which are positive in sense of the pointwise partial ordering of the linear manifold $C([a,b),\mathbb{R})$.

Definition 4. An operator $l : C([a,b),\mathbb{R}) \to L_{1;\,\text{loc}}([a,b),\mathbb{R})$ is said to be *positive* if $(lu)(t) \geq 0$ for a. e. $t \in [a,b)$ whenever u is non-negative on $[a,b)$.

For systems (1) with positive $l_{ik} : C([a,b),\mathbb{R}) \to L_{1;\,\text{loc}}([a,b),\mathbb{R})$, $i,k = 1,2,\dots,n$, we can formulate the following

Theorem 1. *Let us assume that the mappings* $l_{ik} : C([a,b),\mathbb{R}) \to L_{1;\,\text{loc}}([a,b),\mathbb{R})$*,* $i,k = 1,2,\dots,n$*, are positive and there exists a certain* $\delta \in [0,1)$ *such that*

$$\sum_{k=1}^{n} h_k l_{ik}\left(\frac{1}{h_k}\right) \in L_1([a,b),\mathbb{R}), \qquad i = 1,2,\dots,n, \tag{16}$$

and the inequality

$$\sum_{k=1}^{n} l_{ik}\left(\sum_{j=1}^{n}\int_a^{\cdot} l_{kj}\left(\frac{1}{h_j}\right)(s)\,ds\right)(t) \leq \delta \sum_{k=1}^{n} l_{ik}\left(\frac{1}{h_k}\right)(t) \tag{17}$$

is satisfied for a. e. $t \in [a,b)$ *and every* $i = 1,2,\dots,n$.

Then the initial value problem (1), (2) *has a unique slowly growing solution for arbitrary locally integrable functions* $q_i : [a,b) \to \mathbb{R}$*,* $i = 1,2,\dots,n$*, possessing the property*

$$\{h_i q_i \mid i = 1,2,\dots,n\} \subset L_1([a,b),\mathbb{R}). \tag{18}$$

Furthermore, if q_i *and* λ_i*,* $i = 1,2,\dots,n$*, satisfy the condition*

$$-\sum_{k=1}^{n} \lambda_k (l_{ik}1)(t) \leq q_i(t), \qquad t \in [a,b),\ i = 1,2,\dots,n, \tag{19}$$

then the unique solution of problem (1), (2), (3) *has non-negative components.*

The symbol $l_{ik}1$ in (19) stands for the result of application of the operator l_{ik} to the function equal identically to 1.

Remark 1. By virtue of the positivity of the mappings l_{ik}, $i,k = 1,2,\dots,n$, conditions (19) are satisfied, in particular, if $\{\lambda_i \mid i = 1,2,\dots,n\} \subset [0,+\infty)$ and the functions q_i, $i = 1,2,\dots,n$, are non-negative almost everywhere on $[a,b)$.

In the more general case where the mappings $l_{ik} : C([a,b),\mathbb{R}) \to L_{1;\,\mathrm{loc}}([a,b),\mathbb{R})$, $i,k = 1,2,\dots,n$, in (1) are monotone decomposable, i. e., (1) has the form

$$x_i'(t) = \sum_{k=1}^{n} \big((l_{ik}^0 x_k)(t) - (l_{ik}^1 x_k)(t)\big) + q_i(t), \qquad t \in [a,b),\ i = 1,2,\dots,n, \tag{20}$$

where $l_{ik}^j : C([a,b),\mathbb{R}) \to L_{1;\,\mathrm{loc}}([a,b),\mathbb{R})$, $i,k = 1,2,\dots,n$, $j = 0,1$, are positive, a weaker assertion holds.

Theorem 2. *Let us assume that the mappings* $l_{ik}^j : C([a,b),\mathbb{R}) \to L_{1;\,\mathrm{loc}}([a,b),\mathbb{R})$*,* $i,k = 1,2,\dots,n$*,* $j = 0,1$*, are positive and there exists a certain* $\delta \in [0,1)$ *such that*

$$\sum_{k=1}^{n} h_k l_{ik}^j \left(\frac{1}{h_k}\right) \in L_1([a,b),\mathbb{R}), \qquad i = 1,2,\dots,n,\ j = 0,1, \tag{21}$$

and the inequality

$$\sum_{k=1}^{n} \bar{l}_{ik} \left(\sum_{j=1}^{n} \int_a^{\cdot} \bar{l}_{kj}\left(\frac{1}{h_j}\right)(s)\,ds \right)(t) \le \delta \sum_{k=1}^{n} \bar{l}_{ik}\left(\frac{1}{h_k}\right)(t) \tag{22}$$

is satisfied for a. e. $t \in [a,b)$ *and every* $i = 1,2,\dots,n$*, where*

$$\bar{l}_{ik} := l_{ik}^{+} + l_{ik}^{-}, \qquad i,k = 1,2,\dots,n.$$

Then the initial value problem (20)*,* (2) *has a unique slowly growing solution for arbitrary locally integrable functions* $q_i : [a,b) \to \mathbb{R}$*,* $i = 1,2,\dots,n$*, possessing property* (18)*.*

Prior to the proof of Theorems 1 and 2, which are given in Section 7, we present its corollaries for systems with argument deviations.

4. Corollaries for equations with argument deviations

Let us consider problem (1), (2), (3) for the system of differential equations with argument deviations

$$x_i'(t) = \sum_{k=1}^{n} p_{ik}(t) x_k(\omega_{ik}(t)) + q_i(t), \qquad t \in [a,b),\ i = 1,2,\dots,n, \tag{23}$$

where $-\infty < a < b < \infty$ and $\{p_{ik}, q_i \mid i,k = 1,2,\dots,n\} \subset L_{1;\,\mathrm{loc}}([a,b),\mathbb{R})$. The argument deviations ω_k, $k = 1,2,\dots,n$, in (23) are arbitrary Lebesgue measurable functions that are supposed to transform the interval $[a,b)$ to itself. It is important to note that the latter assumption does not lead one to any loss of generality (see, e. g., [6] for a detailed discussion of this subject).

We are interested in conditions guaranteeing the existence of solutions with properties (3) in the case where the coefficients of equation (23) are non-negative. As above, we assume that the functions h_i, $i = 1,2,\dots,n$, appearing in conditions (3) possess properties (11).

Corollary 1. *Assume that the functions p_{ik}, $i,k = 1,2,\dots,n$, are non-negative almost everywhere on $[a,b)$. Moreover, let*

$$\int_a^b \frac{h_k(t)p_{ik}(t)}{h_k(\omega_{ik}(t))}dt < +\infty, \qquad i,k = 1,2,\dots,n, \tag{24}$$

and there exists a certain $\delta \in [0,1)$ such that the inequality

$$\sum_{k=1}^{n} p_{ik}(t)\left(\sum_{j=1}^{n}\int_a^{\omega_{ik}(t)} \frac{p_{kj}(s)}{h_j(\omega_{kj}(s))}ds - \frac{\delta}{h_k(\omega_{ik}(t))}\right) \le 0 \tag{25}$$

is satisfied for a. e. $t \in [a,b)$ and every $i = 1,2,\dots,n$.

Then problem (23), (2), (3) *has a unique slowly growing solution for arbitrary locally integrable functions $q_i : [a,b) \to \mathbb{R}$, $i = 1,2,\dots,n$, possessing property* (18)*, and any $\{\lambda_i \mid i = 1,2,\dots,n\}$. Furthermore, if q_i and λ_i, $i = 1,2,\dots,n$, satisfy the condition*

$$-\sum_{k=1}^{n} \lambda_k p_{ik}(t) \le q_i(t), \qquad i = 1,2,\dots,n, \tag{26}$$

for almost every $t \in [a,b)$, then the unique solution of problem (23), (2), (3) *has non-negative components.*

Proof. It is clear that equation (23) can be represented in form (1), where the operators l_{ik} are defined by the equality

$$(l_{ik}x_k)(t) = p_{ik}(t)x_k(\omega_{ik}(t)), \qquad t \in [a,b),\ i,k = 1,2,\dots,n.$$

Since the functions p_{ik}, $k = 1,2,\dots,n$ are non-negative, it follows that operators l_{ik}, $k = 0,1,\dots,n$, are positive. Moreover, in view of property (24), operators l_{ik}, $k = 1,2,\dots,n$, satisfy conditions (16). Then it follows from (25) that inequalities (17) are true.

Thus, all conditions of Theorem 1 hold and, therefore, problem (23), (2), (3) has a unique solution for arbitrary locally integrable functions q_i, $i = 1,2,\dots,n$, with properties (18). □

This statement implies, in particular, the following

Corollary 2. *Let p_{ik}, $i, k = 1, 2, \dots, n$, be non-negative, possess properties* (24)*, and be such that the condition*

$$\operatorname*{ess\,sup}_{t\in[a,b)} h_k(\omega_{ik}(t)) \sum_{j=1}^{n} \int_a^{\omega_{ik}(t)} \frac{p_{kj}(s)}{h_j(\omega_{kj}(s))} ds < 1 \tag{27}$$

holds for all $i, k = 1, 2, \dots, n$.

Then, for any locally integrable functions $q_i : [a, b) \to \mathbb{R}$, $i = 1, 2, \dots, n$, possessing property (18) *and arbitrary real λ_i, $i = 1, 2, \dots, n$, the initial value problem* (23)*,* (2) *has a unique solution possessing property* (3)*. Furthermore, under the additional condition* (26)*, this solution has non-negative components.*

Proof. The statement is an immediate consequence of Corollary 1. □

It is worth pointing out that, under assumptions of Corollaries 1 and 2, some or all equations of system (23) may not have the Volterra property, i. e., it may happen that

$$\operatorname{mes}\{t \in [a, b) \mid \omega_{ik}(t) > t\} > 0 \tag{28}$$

or

$$\operatorname{mes}\{t \in [a, b) \mid \omega_{ik}(t) < t\} > 0, \tag{29}$$

or both (28) and (29) may hold simultaneously for some i and k.

Remark 2. Condition (27) of Corollary 2 is unimprovable in the sense that it cannot be replaced by the corresponding non-strict inequality

$$\operatorname*{ess\,sup}_{t\in[a,b)} h_k(\omega_{ik}(t)) \sum_{j=1}^{n} \int_a^{\omega_{ik}(t)} \frac{p_{kj}(s)}{h_j(\omega_{kj}(s))} ds \le 1 \tag{30}$$

even for a single pair of indices i and k, because after such a replacement the assertion of Corollary 2 is not true any more. In order to show this, it is sufficient to consider the simplest scalar functional differential equation

$$x'(t) = p(t)x(\tau), \qquad t \in [a, b), \tag{31}$$

where $a \le \tau < b$, $p : [a, b) \to [0, +\infty)$ is such that $hp \in L_1([a, b), \mathbb{R})$, and the function $h : [a, b) \to [0, +\infty)$ is non-increasing and such that $\lim_{t\to b-} h(t) = 0$. It is easy to verify that, under the condition

$$\int_a^{\tau} p(t)dt < 1, \tag{32}$$

the homogeneous problem

$$x(a) = 0, \tag{33}$$

$$\sup_{t\in[a,b)} h(t)\,|x(t)| < +\infty \tag{34}$$

for equation (31) has no non-trivial solutions. This circumstance agrees with the assumptions of Corollary 2 because condition (24) for this problem means the integrability of p with the weight h, whereas (27) coincides with (32). However, if

$$\int_a^\tau p(t)dt = 1, \tag{35}$$

then the homogeneous initial value problem (31), (33) has a one-parametric family of solutions

$$x(t) = \lambda \int_a^t p(s)ds, \qquad t \in [a,b),\ \lambda \in (-\infty,+\infty),$$

each of which satisfies condition (34) because h is non-increasing, hp is integrable, and, therefore,

$$\operatorname*{ess\,sup}_{t\in[a,b)} h(t) \int_a^t p(s)ds \le \operatorname*{ess\,sup}_{t\in[a,b)} \int_a^t h(s)p(s)ds < +\infty.$$

By virtue of (35), condition (30) is satisfied in this case but (27) is not.

Corollary 3. *Let p_{ik}, $i,k = 1,2,\dots,n$, satisfy relations* (24) *and the condition*

$$\operatorname*{ess\,sup}_{t\in[a,b)} h_k(\omega_{ik}(t)) \sum_{j=1}^n \int_a^{\omega_{ik}(t)} \frac{|p_{kj}(s)|}{h_j(\omega_{kj}(s))} ds < 1 \tag{36}$$

for all $i,k = 1,2,\dots,n$.

Then, for any locally integrable functions $q_i : [a,b) \to \mathbb{R}$, $i = 1,2,\dots,n$, possessing property (18) *and arbitrary real λ_i, $i = 1,2,\dots,n$, the initial value problem* (23)*,* (2) *has a unique solution possessing property* (3)*.*

It should be noted that, under the assumptions of the last corollary, the unique solution of problem (23), (2), (3) may not be non-negative even under condition (26).

Proof. The statement is an immediate consequence of Theorem 2. □

5. Examples

As an illustration, let us consider the case where $a = 0$, $b = 1$, and both the weight functions and the argument deviations in problem (23), (2), (3) are powers of the independent variable, i. e., we have the problem

$$x_i'(t) = \sum_{k=1}^n p_{ik}(t)x_k\big(1-(1-t)^{\beta_{ik}}\big) + q_i(t), \qquad t \in [0,1),\ i = 1,2,\dots,n, \tag{37}$$

$$\sup_{t\in[0,1)} (1-t)^{\gamma_i} |x_i(t)| < +\infty, \qquad i = 1,2,\dots,n, \tag{38}$$

$$x_i(0) = \lambda_i, \qquad i = 1,2,\dots,n, \tag{39}$$

where $\{p_{ik}, q_i \mid i,k = 1,2,\dots,n\} \subset L_{1;\,\mathrm{loc}}([0,1),\mathbb{R})$ and $\{\beta_{ik}, \gamma_i \mid i,k = 1,2,\dots,n\} \subset [0,+\infty)$.

Corollary 4. *Let the functions p_{ik}, $i,k = 1,2,\ldots,n$, be non-negative, satisfy the conditions*

$$\int_0^1 p_{ik}(t)\,(1-t)^{\gamma_k(1-\beta_{ik})}\,dt < +\infty, \qquad i,k = 1,2,\ldots,n, \tag{40}$$

and, moreover, be such that

$$\sup_{t\in[0,1)} (1-t)^{\gamma_k\beta_{ik}} \sum_{j=1}^{n} \int_0^{1-(1-t)^{\beta_{ik}}} p_{kj}(s)\,(1-s)^{-\gamma_j\beta_{kj}}\,ds < 1, \qquad i,k = 1,2,\ldots,n. \tag{41}$$

Then, for arbitrary locally integrable functions $q_i : [0,1) \to \mathbb{R}$, $i = 1,2,\ldots,n$, possessing properties (18) *and any real λ_i, $i = 1,2,\ldots,n$, problem* (37), (38), (39) *has a unique slowly growing solution. Furthermore, if q_i and λ_i, $i = 1,2,\ldots,n$, satisfy the additional condition* (26) *for almost every $t \in [0,1)$, then the unique solution of problem* (37), (38), (39) *has non-negative components.*

Proof. It is sufficient to apply Corollary 2 with $a = 0$, $b = 1$, $\omega_{ik}(t) = 1-(1-t)^{\beta_{ik}}$ and $h_i(t) = (1-t)^{\gamma_i}$, $t \in [0,1)$, $i,k = 1,2,\ldots,n$. □

In the case of a scalar equation of the form

$$x'(t) = \frac{m}{(1-t)^{\alpha}} x\big(1-\theta\,(1-t)^{\beta}\big) + q(t), \qquad t \in [0,1), \tag{42}$$

with the additional condition

$$\sup_{t\in[0,1)} (1-t)^{\gamma}|x(t)| < +\infty, \tag{43}$$

where $\{m,\alpha,\beta,\gamma\} \subset [0,+\infty)$, $0 < \theta \le 1$, we arrive immediately at the following

Corollary 5. *Let us assume that the inequality*

$$\gamma\,(1-\beta) - \alpha > -1 \tag{44}$$

holds and, moreover,

$$\max\left(\frac{1}{\theta^{\gamma}}\gamma^{\frac{\gamma}{\kappa}}(\gamma+\kappa)^{-\frac{\gamma+\kappa}{\kappa}},\ \frac{1}{\kappa}(1-\theta^{\kappa})\right) < \frac{1}{m}, \tag{45}$$

where $\kappa := 1-\alpha-\gamma\beta$.

Then, for an arbitrary locally integrable function $q \in L_{1;\mathrm{loc}}([0,1),\mathbb{R})$ such that $[0,1) \ni t \mapsto (1-t)^{\gamma}q(t)$ is Lebesgue integrable and any real λ problem (42), (43) *has a unique solution satisfying the additional condition*

$$x(0) = \lambda. \tag{46}$$

Furthermore, if q and λ satisfy the inequality

$$q(t) \ge -\frac{m\lambda}{(1-t)^{\alpha}} \tag{47}$$

for almost every $t \in [0, 1)$*, then the unique solution of problem* (42)*,* (43)*,* (46) *is non-negative.*

Proof. It is sufficient to apply Corollary 4 in the case where $n = 1$, $p(t) = m\,(1-t)^{-\alpha}$, $\omega(t) = 1 - \theta(1-t)^{\beta}$ and $h(t) = (1-t)^{\gamma}$, $t \in [0, 1)$. □

Remark 3. It follows from property (44) that coefficient function in (42) may have a non-integrable singularity at the right-hand end of the interval if $0 < \beta < 1$.

6. Auxiliary statements

In the sequel we need an abstract theorem on operators in partially ordered normed spaces [14, Theorem 16.2]. In order to state it, we first formulate definitions. We use [14] as the main reference (see also [15]).

Let E be a normed space over $\mathbb{R}$ and P be a cone [15] in E, i. e., a non-empty closed subset of E possessing the properties $P \cap (-P) = \{0\}$ and $\alpha_1 P + \alpha_2 P \subseteq P$ for all $\{\alpha_1, \alpha_2\} \subset [0, +\infty)$. A cone P generates a natural partial ordering of E. As usual, we shall write $u \leqq_P v$ and $v \geqq_P u$ if and only if $v - u \in P$.

Let us recall that a cone $P \subset E$ in a Banach space $\langle E, \|\cdot\|_E \rangle$ is *normal* if and only if the relation

$$\inf\{\gamma \in (0, +\infty) \mid \|x\|_E \le \gamma \|y\|_E \ \forall \{x, y\} \subset P : \ y - x \in P\} < +\infty$$

is true. By definition, the cone P is *reproducing* in E if and only if an arbitrary element x from E can be represented in the form $x = u - v$, where u and v belong to P (see, e. g., [14, 15]).

Definition 5. An operator $T : E \to E$ is said to be *positive* if $TP \subset P$.

Definition 6. Let α be an element of the cone $P \subset E$. An operator $T : E \to E$ is said to be *α-bounded from above* (along P) if for an arbitrary $u \in P$ one can specify some $m_u \in \mathbb{N}$ and $c_u \in (0, +\infty)$ such that

$$-T^{m_u} u + c_u \alpha \in P. \tag{48}$$

Theorem 3 ([14]). *Let linear operator* $T : E \to E$ *be positive with respect to the cone* P*,* α*-bounded from above and the inequality*

$$T\alpha \leqq_P \delta\alpha, \qquad \alpha \in P, \tag{49}$$

is satisfied. Furthermore, let the cone P *be normal and reproducing. Then the estimate*

$$r(T) \le \delta \tag{50}$$

is true.

For any x from $C([a,b),\mathbb{R})$, let us put

$$(Sx)(t) := x(a+b-t), \qquad t \in (a,b].$$

Lemma 1. *A function $x : [a,b) \to \mathbb{R}$ is a solution of problem* (1), (2), (3) *if and only if the function $u : (a,b] \to \mathbb{R}$ defined by the equality $u := Sx$ is a solution of the problem*

$$u_i'(t) = \sum_{k=1}^{n} (\tilde{l}_{ik} u_k)(t) + \tilde{q}_i(t), \qquad t \in (a,b],\ i = 1,2,\ldots,n, \tag{51}$$

$$\sup_{t\in(a,b]} \tilde{h}_i(t)|u_i(t)| < +\infty, \qquad i = 1,2,\ldots,n, \tag{52}$$

$$u_i(b) = \lambda_i, \qquad i = 1,2,\ldots,n,, \tag{53}$$

where $\tilde{q}_i := -Sq_i$, $\tilde{h}_i := Sh_i$, and $\tilde{l}_{ik} : C((a,b],\mathbb{R}) \to L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$, are the linear mappings given by the formula

$$\tilde{l}_{ik} := -Sl_{ik}S \tag{54}$$

for $i,k = 1,2,\ldots,n$.

Proof. This statement is obtained immediately by carrying out the substitution

$$u(t) = x(a+b-t), \qquad t \in (0,1], \tag{55}$$

in relations (1), (2), and (3). □

Note that, in view of (11), the functions $\tilde{h}_i$, $i = 1,2,\ldots,n$, possess the following properties:

The functions $\tilde{h}_i : (a,b] \to (0,+\infty)$, $i = 1,2,\ldots,n$ are non-decreasing and such that

$$\lim_{t\to a+} \tilde{h}_i(t) = 0, \qquad i = 1,2,\ldots,n. \tag{56}$$

The following simple lemma concerns the solvability of problem (51), (52),(53) and corresponding semi-homogeneous problem (51), (52) and

$$u_i(b) = 0, \qquad i = 1,2,\ldots,n. \tag{57}$$

Lemma 2. *If the semi-homogeneous problem* (51), (52), (57), *is uniquely solvable for arbitrary locally integrable functions $\tilde{q}_i : (a,b] \to \mathbb{R}$, $i = 1,2,\ldots,n$, possessing the properties*

$$\{\tilde{h}_i \tilde{q}_i \mid i = 1,2,\ldots,n\} \subset L_1((a,b],\mathbb{R}), \tag{58}$$

then the same is true for problem (51), (52), (53) *for arbitrary $\tilde{q}_i : (a,b] \to \mathbb{R}$, $i = 1,2,\ldots,n$, possessing the properties* (58) *and arbitrary $\{\lambda_i \mid i = 1,2,\ldots,n\} \subset \mathbb{R}$. Furthermore, if $\tilde{q}_i$ and λ_i, $i = 1,2,\ldots,n$, satisfy the condition*

$$\sum_{k=1}^{n} \lambda_k (\tilde{l}_{ik} 1)(t) \le -\tilde{q}_i(t), \qquad i = 1,2,\ldots,n, \tag{59}$$

then the solution of problem (51), (52), (53) *has non-negative components.*

Proof. Consider the inhomogeneous problem (51), (52), (53) and perform there the change of variable according to the formula

$$v := u - \lambda. \tag{60}$$

If $u = (u_i)_{i=1}^n$ is a solution of (51), (52), (53), then the function $v = (v_i)_{i=1}^n$ satisfies the relations

$$v_i'(t) = \sum_{k=1}^n (\tilde{l}_{ik} v_k)(t) + \tilde{q}_i(t) + \sum_{k=1}^n (\tilde{l}_{ik} \lambda_k)(t), \qquad t \in (a,b],\ i = 1,2,\ldots,n, \tag{61}$$

$$\sup_{t \in (a,b]} \tilde{h}_i(t)|v_i(t)| < +\infty, \qquad i = 1,2,\ldots,n, \tag{62}$$

$$v_i(b) = 0, \qquad i = 1,2,\ldots,n, \tag{63}$$

and *vice versa*. Thus, if there exists a unique solution $v = (v_i)_{i=1}^n : (a,b] \to \mathbb{R}^n$ of a semi-homogeneous problem (61), (62), (63) for all $\tilde{q}_i : (a,b] \to \mathbb{R}$, $i = 1,2,\ldots,n$, possessing the property (58), then the unique solution $u = (u_i)_{i=1}^n$ of problem (51), (52), (53) is determined from relation (60).

It follows from the positivity of operators $l_{ik} : C([a,b),\mathbb{R}) \to L_{1;\,\mathrm{loc}}([a,b),\mathbb{R})$, $i,k = 1,2,\ldots,n$, that the operators $\tilde{l}_{ik} : C((a,b],\mathbb{R}) \to L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$, $i,k = 1,2,\ldots,n$, defined by formulae (54) are negative. Then, in view of inequalities (59), the right-hand side terms of the equations (61) are non-positive and, thus, the derivatives of the functions v_i, $i,k = 1,2,\ldots,n$, are non-positive. This means that the solution of problem (51), (52), (53) has non-negative components. □

Let us set $\tilde{h} := \operatorname{diag}(\tilde{h}_1,\ldots,\tilde{h}_n)$ and introduce a linear manifold X in $C((a,b],\mathbb{R}^n)$ by putting

$$X := \tilde{C}_{\mathrm{loc};\,h}((a,b],\mathbb{R}^n). \tag{64}$$

Lemma 3. *The set X is a Banach space with respect to the norm*

$$X \ni u = (u_i)_{i=1}^n \longmapsto \max_{i=1,2,\ldots,n} \left(\int_a^b \tilde{h}_i(s)\left|u_i'(s)\right| ds + \sup_{\xi \in (a,b]} \tilde{h}_i(\xi)\left|u_i(\xi)\right| \right). \tag{65}$$

The *proof* of Lemma 3 for $n = 1$ can be found in [16]. The case where $n > 1$ is treated in a similar manner.

Lemma 4. *The set*

$$X_0 := \{u \in X \mid u(b) = 0\} \tag{66}$$

is a closed linear subspace in X.

Proof. This statement is obvious from (64) and (65). □

Let us consider the set

$$K := \Big\{ u = (u_i)_{i=1}^n \in X \mid \inf_{t\in(a,b]} u_i(t) \geq 0 \text{ and } \operatorname*{ess\,sup}_{t\in(a,b]} u_i'(t) \leq 0 \text{ for all } i = 1, 2, \dots, n \Big\}. \tag{67}$$

Lemma 5. *The following assertions are true:*

(1) *The set K is a normal and reproducing cone in the space X;*

(2) *The set*

$$K_0 := K \cap X_0 \tag{68}$$

is a normal and reproducing cone in the space X_0.

Proof. Assertion 1. It follows from [16, Lemma 5.8] that K is a regular cone in X. In particular, K is normal (see, e. g., [15, Theorem 1.6]). It thus remains to show that the cone K is reproducing in X.

We shall argue by analogy to the proof of the classical Jordan theorem (see, e. g., [17]). Let us choose an arbitrary $u \in X$ and consider the functions $\nu_j = (\nu_{ji})_{i=1}^n : (a, b] \to \mathbb{R}^n$, $j = 0, 1$, where

$$(a, b] \ni t \longmapsto \nu_{0i}(t) := \operatorname*{Var}_{[t,b]} u_i + |u_i(b)|, \tag{69}$$

$$(a, b] \ni t \longmapsto \nu_{1i}(t) := \operatorname*{Var}_{[t,b]} u_i + |u_i(b)| - u_i(t) \tag{70}$$

for any $i = 1, 2, \dots, n$.

Let us show that $\nu_0 \in X$. Indeed, using the well-known property of the total variation

$$\operatorname*{Var}_{[t,b]} u_i = \int_t^b \left|u_i'(s)\right| ds, \qquad t \in (a, b],$$

we obtain that

$$\int_a^b \tilde{h}_i(t) \left|\nu_{0i}'(t)\right| dt \leq \int_a^b \tilde{h}_i(t) \left|u_i'(t)\right| dt < +\infty, \qquad i = 1, 2, \dots, n, \tag{71}$$

because $u \in X$ and, in particular, $\tilde{h}u' \in L_1((a, b], \mathbb{R}^n)$. It follows from (71) that the derivative of function (69) is integrable with the weight $\tilde{h}_i$. Furthermore, since $\tilde{h}_i$, $i = 1, 2, \dots, n$, are non-decreasing (see assumptions (56)), we have

$$\begin{aligned} \sup_{t\in(a,b]} \tilde{h}_i(t) \operatorname*{Var}_{[t,b]} u_i &= \sup_{t\in(a,b]} \tilde{h}_i(t) \int_t^b \left|u_i'(s)\right| ds \\ &\leq \sup_{t\in(a,b]} \int_t^b \tilde{h}_i(s) \left|u_i'(s)\right| ds = \int_a^b \tilde{h}_i(t) \left|u_i'(t)\right| dt < +\infty \end{aligned}$$

for $i = 1, 2, \dots, n$, whence it follows that $\nu_0 \in X$ and, by virtue of (70), the function ν_1 is also an element of X.

Clearly, the function ν_0 is non-negative. Using the additivity of the total variation (see, e. g., [17]), it is easy to show that function (69) is non-increasing. According to the definition (67) of the set K, this means that ν_0 belongs to K.

The function ν_1 belongs to K as well. Indeed, let s and t be arbitrary points from $(a, b]$ such that $s \geq t$. By using the additivity of the total variation, we obtain

$$\begin{aligned}\nu_{1i}(s) - \nu_{1i}(t) &= \operatorname*{Var}_{[s,b]} u_i - u_i(s) + u_i(t) - \operatorname*{Var}_{[t,s]} u_i - \operatorname*{Var}_{[s,b]} u_i \\ &= -(u_i(s) - u_i(t)) - \operatorname*{Var}_{[t,s]} u_i \end{aligned} \tag{72}$$

for $i = 1, 2, \dots, n$. It follows from the definition of the total variation that

$$\operatorname*{Var}_{[t,s]} u_i \geq |u_i(s) - u_i(t)| \tag{73}$$

and, therefore, relation (72) yields the estimate

$$\nu_{1i}(s) - \nu_{1i}(t) \leq -(u_i(s) - u_i(t)) - |u_i(s) - u_i(t)|,$$

which implies that $\nu_{1i}(s) \leq \nu_{1i}(t)$ for all $i = 1, 2, \dots, n$. Considering the arbitrariness of s and t, we conclude that ν_{1i}, $i = 1, 2, \dots, n$, are non-increasing. Furthermore, applying (73) at the point b, we get

$$\begin{aligned}\nu_{1i}(t) &\geq |u_i(t) - u_i(b)| + |u_i(b)| - u_i(t) \\ &\geq (|u_i(t)| - |u_i(b)|) + |u_i(b)| - u_i(t) \geq 0\end{aligned}$$

for any $t \in (a, b]$. We have thus shown that $\nu_1 \in K$. Finally, the obvious identity

$$u = \nu_0 - \nu_1,$$

in view of the arbitrariness of u, completes the proof of the fact that K is reproducing.

Assertion 2. The proof of this assertion is a repetition, with obvious modifications, of the argument given above. □

Let us put

$$(T_i u)(t) := -\sum_{k=1}^{n} \int_t^b (\tilde{l}_{ik} u_k)(s)\, ds, \qquad t \in (a, b], \qquad i = 1, 2, \dots, n, \tag{74}$$

for any function u from X_0. Using mappings (74), we define an operator $T : X_0 \to \tilde{C}_{\mathrm{loc};\,h}((a, b], \mathbb{R}^n)$ by setting

$$(Tu)(t) := \sum_{i=1}^{n} e_i\, (T_i u)(t), \qquad t \in (a, b], \tag{75}$$

for an arbitrary function u from X_0, where e_k, $k = 1, 2, \dots, n$, are the vectors given by formulae (10).

Lemma 6. *If the mappings $\tilde{l}_{ik} : C((a,b],\mathbb{R}) \to L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$, $i,k = 1,2,\dots,n$, are negative , then for any $u = (u_k)_{k=1}^n$ from X, there exists a certain $\mu_u \in [0,+\infty)$ such that the estimate*

$$\left|(\tilde{l}_{ik}u_k)(t)\right| \le \mu_u \left|\tilde{l}_{ik}\left(\frac{1}{\tilde{h}_k}\right)(t)\right|, \qquad t \in (a,b], \tag{76}$$

holds for all $i,k = 1,2,\dots,n$.

The *proof* of Lemma 6 follows from that of [16, Lemma 5.10].

Lemma 7. *If the mappings $\tilde{l}_{ik} : C((a,b],\mathbb{R}) \to L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$, $i,k = 1,2,\dots,n$, are negative and possess properties*

$$\sum_{k=1}^{n} \tilde{h}_k \tilde{l}_{ik}\left(\frac{1}{\tilde{h}_k}\right) \in L_1((a,b],\mathbb{R}), \qquad i = 1,2,\dots,n, \tag{77}$$

then the following assertions are true:

(1) *$T(X_0) \subseteq X_0$;*

(2) *$T(K_0) \subseteq K_0$;*

(3) *The operator $T : X_0 \to X_0$ is α-bounded from above, where $\alpha = (\alpha_i)_{i=1}^n$ is a vector function from X_0 given by the formulae*

$$\alpha_i(t) := -\sum_{k=1}^{n} \int_t^b \left(\tilde{l}_{ik}\left(\frac{1}{\tilde{h}_k}\right)\right)(s)ds, \qquad t \in (a,b]. \tag{78}$$

Proof. Assertion 1. It is proved in [16, Lemma 5.10] that, under assumption (16), $T(X) \subseteq X$. On the other hand, it is obvious from (74) that

$$(Tu)(b) = 0$$

for any u from X. Using these facts and recalling definition (66) of X_0, we arrive at the desired inclusion.

Assertion 2 follows from definitions of the operator T and the cone K_0 and negativity of the mappings $\tilde{l}_{ik} : C((a,b],\mathbb{R}) \to L_{1;\,\mathrm{loc}}((a,b],\mathbb{R})$, $i,k = 1,2,\dots,n$.

To prove Assertion 3, let us choose an arbitrary vector function $u = (u_k)_{k=1}^n$ from K_0 and consider the functions $T_i u$, $i = 1,2,\dots,n$. By virtue of Lemma 6, one can specify a constant $\mu_u \in [0,+\infty)$ such that estimate (76) is true. Considering (74) and (78), we obtain

$$\begin{aligned}(T_i u)'(t) = \sum_{k=1}^{n}(\tilde{l}_{ik}u_k)(t) &\ge -\mu_u \sum_{k=1}^{n}\left|\tilde{l}_{ik}\left(\frac{1}{\tilde{h}_k}\right)\right|(t) \\ &= \mu_u \sum_{k=1}^{n}\left(\tilde{l}_{ik}\left(\frac{1}{\tilde{h}_k}\right)\right)(t) = \mu_u \alpha_i'(t)\end{aligned} \tag{79}$$

and

$$(T_i u)(t) = -\sum_{k=1}^{n} \int_t^b (\tilde{l}_{ik} u_k)(s)\, ds \le -\mu_u \sum_{k=1}^{n} \int_t^b \left(\tilde{l}_{ik} \left(\frac{1}{\tilde{h}_k} \right) \right) (s)\, ds = \mu_u \alpha_i(t) \tag{80}$$

for any $i = 1, 2, \ldots, n$ and $t \in (a, b]$, In view of (67) and (68), relations (79) and (80) mean that inclusion (48) holds with $P := K_0$, $m_u := 1$, and $c_u := \mu_u$ for all $u \in K_0$. Therefore, it remains to show that $\alpha \in K_0$.

Indeed, the functions $\tilde{h}_i \alpha_i'$, $i = 1, 2, \ldots, n$, are integrable in view of assumption (77). Furthermore, since $\tilde{h}_i$, $i = 1, 2, \ldots, n$, are non-negative and non-decreasing, it follows from equality (78) and the negativity of $\tilde{l}_{ik}$, $i, k = 1, 2, \ldots, n$, that

$$\begin{aligned} \sup_{t \in (a,b]} \tilde{h}_i(t) \alpha_i(t) &= \sup_{t \in (a,b]} \left(-\tilde{h}_i(t) \sum_{k=1}^{n} \int_t^b \left(\tilde{l}_{ik} \left(\frac{1}{\tilde{h}_k} \right) \right) (s) ds \right) \\ &\le \sup_{t \in (a,b]} \sum_{k=1}^{n} \int_t^b \tilde{h}_i(s) \left| \tilde{l}_{ik} \left(\frac{1}{\tilde{h}_k} \right) \right| (s) ds \\ &\le \sup_{t \in (a,b]} \sum_{k=1}^{n} \int_a^b \tilde{h}_i(s) \left| \tilde{l}_{ik} \left(\frac{1}{\tilde{h}_k} \right) \right| (s) ds \end{aligned} \tag{81}$$

for all $i = 1, 2, \ldots, n$, and the integrals in (81) are finite due to assumption (77). Since the inequalities

$$(-1)^j \alpha_i^{(j)}(t) \ge 0, \qquad t \in (a, b],\ j = 0, 1,$$

are an immediate consequence of (78), we have thus proved that α belongs to K_0. It remains to recall Definition 6. □

Lemma 8. *A vector function $u = (u_i)_{i=1}^n : (a, b] \to \mathbb{R}^n$ from X_0 is a solution of problem* (51)*,* (52)*,* (57)*, if and only if it is a solution of the functional equation*

$$u = Tu + z, \tag{82}$$

where $z = (z_i)_{i=1}^n$,

$$z_i(t) = -\int_t^b \tilde{q}_i(s) ds, \qquad i = 1, 2, \ldots, n. \tag{83}$$

Proof. This last lemma is an immediate consequence of relations (74) and the definition of the set X_0. □

7. Proofs

7.1. Proof of Theorem 1. We are going to use Theorem 3. First of all, according to Lemma 1, we replace the original problem (1), (2), (3) by problem (51), (52), (53).

By virtue of Lemma 3, 4 and 5, the set K_0 given by (68) forms a normal and reproducing cone in the Banach space X_0. Lemma 7 guarantees that the operator $T : X_0 \to X_0$ defined by formula (74) is positive with respect to the cone K_0 and α-bounded from above, where $\alpha = (\alpha_i)_{i=1}^n$ given by the formula (78). Therefore, in order to be able to apply Theorem 3, we need to establish inequality (49), i. e., to show that $T\alpha \le \delta\alpha$ and $(T\alpha)' \ge \delta\alpha'$ pointwise on the given interval.

Carrying out the change of variables in the inequality (17) according to formula (55), we get the relation

$$\sum_{k=1}^{n} \tilde{l}_{ik} \left(\sum_{j=1}^{n} \int_{\cdot}^{b} \tilde{l}_{kj} \left(\frac{1}{\tilde{h}_j} \right) (s)\, ds \right) (t) \le -\delta \sum_{k=1}^{n} \tilde{l}_{ik} \left(\frac{1}{\tilde{h}_k} \right) (t) \tag{84}$$

for a. e. $t \in (a, b]$ and every $i = 1, 2, \dots, n$. In view of (78), it follows immediately from (84) that

$$\sum_{k=1}^{n} (\tilde{l}_{ik} \alpha_k)(t) \ge \delta \sum_{k=1}^{n} \tilde{l}_{ik} \left(\frac{1}{\tilde{h}_k} \right) (t), \qquad i = 1, 2, \dots, n. \tag{85}$$

Recalling that the functions α_i, $i = 1, 2, \dots, n$, are given by formulae (78), we obtain

$$\alpha_i'(t) = \sum_{k=1}^{n} \tilde{l}_{ik} \left(\frac{1}{\tilde{h}_k} \right) (t), \qquad i = 1, 2, \dots, n,$$

and

$$(T_i \alpha)'(t) = \sum_{k=1}^{n} (\tilde{l}_{ik} \alpha_k)(t), \qquad i = 1, 2, \dots, n, \qquad t \in (a, b],$$

whence, due to (85), we conclude that

$$(T_i \alpha)'(t) \ge \delta \alpha_i'(t), \qquad t \in [a, b], \qquad i = 1, 2, \dots, n.$$

Then, taking into account the definition of (78), the linearity and negativity of the operator $\tilde{l}$, and using inequalities (84), we have

$$\begin{aligned}
(T_i \alpha)(t) &= -\sum_{k=1}^{n} \int_{t}^{b} (\tilde{l}_{ik} \alpha_k)(s) ds \\
&= -\int_{t}^{b} \sum_{k=1}^{n} \tilde{l}_{ik} \left(-\sum_{j=1}^{n} \int_{s}^{b} \tilde{l}_{kj} \left(\frac{1}{\tilde{h}_j} \right) (\xi) d\xi \right) (s) ds \\
&\le -\delta \sum_{k=1}^{n} \int_{t}^{b} \tilde{l}_{ik} \left(\frac{1}{\tilde{h}_k} \right) (s) ds = \delta \alpha_i(t), \qquad t \in [a, b],\ i = 1, 2, \dots, n.
\end{aligned}$$

Thus, all the conditions of Theorem 3 are satisfied and, therefore, we get estimate (48) for the spectral radius of operator (75). Since $\delta < 1$, it follows that equation (82) is uniquely solvable and, therefore, in view of the Lemma 8, the semi-homogeneous problem (51), (52), (57) has a unique solution for arbitrary locally integrable functions $\tilde{q}_i : (a, b] \to \mathbb{R}$, $i = 1, 2, \dots, n$, possessing the property (58). Finally, to obtain the assertion required, it remains to refer to Lemmata 1 and 2. Finally, properties (77), (58), (59) are obtained immediately by carrying out substitution (55) in relations (16), (18), (19) of Theorem 1.

7.2. Proof of Theorem 2. Theorem 2 is proved similarly to Theorem 1 by using [14, Theorem 16.5].

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