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OPERATOR HYPERREFLEXIVITY OF SUBSPACE LATTICES

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ABSTRACT. We introduce and study the notion of operator hyperreflexivity of subspace lattices. This notion is a natural analogue of the operator reflexivity and is related to hyperreflexivity of subspace lattices introduced by Davidson and Harrison.

1. INTRODUCTION

Let $\mathcal{H}$ be a complex Hilbert space. By $\mathcal{B}(\mathcal{H})$ we denote the algebra of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{P}(\mathcal{H})$ the lattice of all orthogonal projections in $\mathcal{B}(\mathcal{H})$. A *subspace lattice* is a lattice which contains the trivial projections 0 and I, and is closed in the strong operator topology. Note that every subspace lattice is complete, which means that it is closed under taking arbitrary infima and suprema.

For a subspace lattice $\mathcal{L} \subseteq \mathcal{P}(\mathcal{H})$, the *reflexive hull* of $\mathcal{L}$ is defined as

$$\mathrm{Ref}\mathcal{L} = \{P \in \mathcal{P}(\mathcal{H}); \quad Px \in \overline{\mathcal{L}x}, \quad \text{for all} \quad x \in \mathcal{H}\}.$$

A subspace lattice $\mathcal{L}$ is said to be *operator reflexive* if $\mathrm{Ref}\mathcal{L} = \mathcal{L}$ (see [11]).

Recall that the classical notion of *reflexivity* of $\mathcal{L}$ means $\mathrm{Lat}\,\mathrm{Alg}\,\mathcal{L} = \mathcal{L}$, which is strictly stronger condition than operator reflexivity [11]. Note that not every subspace lattice is operator reflexive [5]. Here, for a family of operators $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$, we let $\mathrm{Lat}\,\mathcal{S} = \{P \in \mathcal{P}(\mathcal{H}); \quad SP = PSP \quad \forall\ S \in \mathcal{S}\}$ be collection of orthogonal projections onto the subspaces invariant for $\mathcal{S}$. For a subspace lattice $\mathcal{L}$, we denote by $\mathrm{Alg}\,\mathcal{L}$ the algebra of all operators $A \in \mathcal{B}(\mathcal{H})$ satisfying $\mathcal{L} \subseteq \mathrm{Lat}\,A$, i.e., operators that leave invariant the ranges of all projections in $\mathcal{L}$.

Let $\mathcal{L} \subseteq \mathcal{P}(\mathcal{H})$ be a subspace lattice, $P \in \mathcal{P}(\mathcal{H})$, and let

$$\mathrm{d}(P, \mathcal{L}) = \inf\{\|P - Q\|; \quad Q \in \mathcal{L}\} = \inf_{Q \in \mathcal{L}} \sup_{\|x\| \leq 1} \|Px - Qx\|$$

denote the usual distance between P and $\mathcal{L}$. In [4], Davidson and Harrison introduce, in analogy with the *Arveson distance* for algebras (see [1]), the following quantity for subspace lattices. Let $\mathcal{L}$ be a subspace lattice and $P \in \mathcal{P}(\mathcal{H})$. They set

$$\beta(P, \mathcal{L}) = \sup\{\|P^{\perp} A P\|; \quad A \in (\mathrm{Alg}\,\mathcal{L})_1\},$$

where $(\mathrm{Alg}\,\mathcal{L})_1$ denotes the set of all contractions in $\mathrm{Alg}\,\mathcal{L}$. It is straightforward to see that $\beta(P, \mathcal{L}) \leq 2\mathrm{d}(P, \mathcal{L})$ for every P (see [4, p. 310]). A subspace lattice $\mathcal{L}$ is said to be *hyperreflexive* if there is a positive number κ such that

$$\mathrm{d}(P, \mathcal{L}) \leq \kappa \beta(P, \mathcal{L}) \quad \text{for all} \quad P \in \mathcal{P}(\mathcal{H}). \tag{1}$$

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The infimum $\kappa(\mathcal{L})$ of all positive numbers κ satisfying (1) is called the *constant of hyperreflexivity for* $\mathcal{L}$. Every hyperreflexive subspace lattice is reflexive, however the converse does not hold, in general.

In this paper we introduce another quantity related to a subspace lattice which seems to be a more natural analog of the Arveson distance. Our idea is based on the definition of the Arveson distance for general spaces of operators.

Let $\mathcal{L}$ be a subspace lattice and $P \in \mathcal{P}(\mathcal{H})$. Then we set

$$\alpha(P, \mathcal{L}) = \sup\{\mathrm{d}(Px, \mathcal{L}x); \quad \|x\| \le 1\} = \sup_{\|x\|\le 1} \inf_{Q\in\mathcal{L}} \|Px - Qx\|.$$

It is obvious from the definition that $\alpha(P, \mathcal{L}) \le \mathrm{d}(P, \mathcal{L})$. We say that a subspace lattice $\mathcal{L}$ is *operator hyperreflexive* if there exists a constant $c > 0$ such that

$$\mathrm{d}(P,\, \mathcal{L}) \le c\, \alpha(P, \mathcal{L}), \quad \text{for all} \quad P \in \mathcal{P}(\mathcal{H}). \tag{2}$$

The infimum $c(\mathcal{L})$ of all positive numbers c satisfying (2) is called the *constant of operator hyperreflexivity for* $\mathcal{L}$. It is clear that every operator hyperreflexive lattice is operator reflexive.

The goal of this paper is to study operator hyperreflexivity for subspace lattices. In Section 2 we show that hyperreflexivity implies operator hyperreflexivity. The opposite implication is not true. It is shown in Section 3 that every finite subspace lattice is operator hyperreflexive. We also show some basic properties of operator hyperreflexive subspace lattices. In the last section an example of a subspace lattice that is operator reflexive but not operator hyperreflexive is given.

The following diagram summarizes the relations among these properties of a subspace lattice:

$$\begin{array}{ccc} \text{reflexivity} & \Longrightarrow & \text{operator reflexivity} \\ \Uparrow & & \Uparrow \\ \text{hyperreflexivity} & \Longrightarrow & \text{operator hyperreflexivity} \end{array}$$

All the implications are strict.

2. Hyperreflexivity vs. operator hyperreflexivity

In this section we compare operator hyperreflexivity with hyperreflexivity of subspace lattices.

Theorem 2.1. *Every hyperreflexive subspace lattice is operator hyperreflexive. Moreover, if $\mathcal{L}$ is a hyperreflexive subspace lattice with constant of hyperreflexivity $\kappa(\mathcal{L})$, then the constant of operator hyperreflexivity for $\mathcal{L}$ is at most $4\kappa(\mathcal{L})$.*

Proof. Let $\mathcal{L}$ be a subspace lattice and $P \in \mathcal{P}(\mathcal{H})$ be arbitrary. We claim that $\beta(P, \mathcal{L}) \le 4\alpha(P, \mathcal{L})$. To see this, let $A \in (\mathrm{Alg}\,\mathcal{L})_1$ and $x \in \mathcal{H}$, $\|x\| \le 1$, be arbitrary. Then, for every $Q \in \mathcal{L}$, one has

$$\begin{aligned} |\langle P^\perp APx, x\rangle| &= |\langle (P^\perp AP - Q^\perp AQ)x, x\rangle| \le |\langle (P^\perp - Q^\perp)APx, x\rangle| + |\langle Q^\perp A(P-Q)x, x\rangle| \\ &= |\langle APx, (P-Q)x\rangle| + |\langle (P-Q)x, A^*Q^\perp x\rangle| \le 2\|(P-Q)x\|. \end{aligned}$$

It follows $|\langle P^\perp APx, x\rangle| \le 2\inf\{\|(P-Q)x\|; \quad Q \in \mathcal{L}\}$ and consequently

$$\sup\{|\langle P^\perp APx, x\rangle|; \quad \|x\| = 1\} \le 2\sup\{\inf\{\|(P-Q)x\|; \quad Q \in \mathcal{L}\}; \quad \|x\| = 1\}.$$

Note that the number on the left side of the last inequality is the numerical radius $w(P^\perp AP)$ of the operator $P^\perp AP$ and that the number on the right hand side is $2\alpha(P, \mathcal{L})$. By the Lumer's formula, one has $\|P^\perp AP\| \le 2w(P^\perp AP)$, which gives $\|P^\perp AP\| \le 4\alpha(P,\mathcal{L})$, and we may conclude that $\beta(P,\mathcal{L}) \le 4\alpha(P,\mathcal{L})$. It is obvious now that for a hyperreflexive subspace lattice $\mathcal{L}$ one has $c(\mathcal{L}) \le 4\kappa(\mathcal{L})$, which in particular means that every hyperreflexive subspace lattice is operator hyperreflexive. □

In [4], several classes of subspace lattices were proved to be hyperreflexive. So we have the following immediate corollary of Theorem 2.1.

Corollary 2.2. *(i) Every nest $\mathcal{N}$ is operator hyperreflexive with constant of operator hyperreflexivity not exceeding* 4.

(ii) Let $\mathcal{A}$ be a hyperreflexive von Neumann algebra with hyperreflexivity constant a. Then the projection lattice $\mathcal{L}$ of $\mathcal{A}$ is operator hyperreflexive with operator hyperreflexivity constant not exceeding $4a$.

(iii) If $\mathcal{L}$ is a commutative subspace lattice, then it is operator hyperreflexive with operator hyperreflexivity constant not exceeding 20*.*

Proof. By [4, Theorem 3.1], every nest is hyperreflexive with hyperreflexivity constant 1. Hence, by Theorem 2.1, (i) follows. Clauses (ii) and (iii) follow similarly by Theorem 4.1, respectively by Theorem 5.1, in [4]. □

As the following example shows, hyperreflexivity is a condition strictly stronger than operator hyperreflexivity.

Example 2.3. Let $\mathcal{H}$ be a two-dimensional Hilbert space. Assume that $P_1, P_2, P_3 \in \mathcal{P}(\mathcal{H})$ are of rank one and that $(P_i\mathcal{H}) \cap (P_j\mathcal{H}) = \{0\}$ and $(P_i\mathcal{H}) \vee (P_i\mathcal{H}) = \mathcal{H}$ hold for all $i,j = 1,2,3$, $i \neq j$. Denote by $\mathcal{L}$ the lattice $\{0, P_1, P_2, P_3, I\}$. It is easy to see that $\operatorname{Alg}\mathcal{L}$ is trivial, i.e., it consists only of scalar multiples of the identity operator. Thus, $\beta(P,\mathcal{L}) = 0$ for every $P \in \mathcal{P}(\mathcal{H})$ which means that $\mathcal{L}$ is not hyperreflexive. On the other hand, it will be shown later, see Theorem 3.2, that every finite subspace lattice is operator hyperreflexive.

3. Basic results

We start this section by showing that every finite subspace lattice is operator hyperreflexive which is not the case for hyperreflexivity, see Example 2.3. We need the following lemma, cf. [9, Theorem 37.17].

Lemma 3.1. *Let $T_1, \ldots, T_n \in \mathcal{B}(\mathcal{H})$ be arbitrary operators and assume that $\alpha_1, \ldots, \alpha_n$ are positive numbers such that $\sum_{i=1}^n \alpha_i^2 < 1$. Then there exists $x \in \mathcal{H}$, $\|x\| = 1$, such that $\|T_i x\| \ge \alpha_i \|T_i\|$, for every $i = 1, \ldots, n$.*

Proof. Without loss of the generality we can assume that every operator T_i is non-zero. Choose $\varepsilon > 0$ such that $\sum \alpha_i^2 < 1 - \varepsilon$. For $i = 1, \ldots, n$, set $\alpha_i' = \frac{\alpha_i}{\sqrt{1-\varepsilon}}$. Then $\sum (\alpha_i')^2 < 1$. For every i choose $y_i \in \mathcal{H}$, $\|y_i\| = 1$, such that $\|T_i^* y_i\| > \sqrt{1-\varepsilon}\,\|T_i^*\| = \sqrt{1-\varepsilon}\,\|T_i\|$. Set $u_i = \|T_i^* y_i\|^{-1} T_i^* y_i$, so that $\|u_i\| = 1$. By [2], there exists a vector $x \in \mathcal{H}$ of norm 1 such that $|\langle x, u_i\rangle| \ge \alpha_i'$, for all $i = 1, \ldots, n$. Hence $\|T_i x\| \ge |\langle T_i x, y_i\rangle| = |\langle x, T_i^* y_i\rangle| = |\langle x, \|T_i^* y_i\| u_i\rangle| \ge \alpha_i' \|T_i^* y_i\| \ge \sqrt{1-\varepsilon}\,\alpha_i' \|T_i\| = \alpha_i \|T_i\|$. □

Theorem 3.2. *Let $\mathcal{L} = \{L_1, \ldots, L_n\} \subset \mathcal{P}(\mathcal{H})$ be a finite subspace lattice. Then $\mathcal{L}$ is operator hyperreflexive and $c(\mathcal{L}) \le \sqrt{n}$.*

Proof. Let $P \in \mathcal{P}(\mathcal{H})$ and $\varepsilon > 0$. Consider the operators $P - L_1, \ldots, P - L_n$. By Lemma 3.1, there exists $x \in \mathcal{H}$ with $\|x\| = 1$ and

$$\|(P - L_j)x\| \geq \left(\frac{1}{\sqrt{n}} - \varepsilon\right) \|P - L_j\|$$

for all $j = 1, \ldots, n$. So

$$\alpha(P, \mathcal{L}) = \sup_{\|y\|=1} \min_{1\leq j\leq n} \|(P - L_j)y\| \geq (\tfrac{1}{\sqrt{n}} - \varepsilon) \min_{1\leq j\leq n} \|P - L_j\| = (n^{-1/2} - \varepsilon)\, \mathrm{d}(P, \mathcal{L}).$$

Since $\varepsilon > 0$ was arbitrary we have $\mathrm{d}(P, \mathcal{L}) \leq \sqrt{n} \cdot \alpha(P, \mathcal{L})$. □

Proposition 3.3. *Let $\mathcal{M}$ and $\mathcal{L}$ be subspace lattices with $\mathcal{L} \subseteq \mathcal{M}$. Suppose that $\mathcal{M}$ is operator hyperreflexive with constant a and that $\mathrm{d}(M, \mathcal{L}) \leq b\, \alpha(M, \mathcal{L})$ holds for all $M \in \mathcal{M}$. Then $\mathcal{L}$ is operator hyperreflexive with constant at most $a + b + ab$.*

Proof. Let $P \in \mathcal{P}(\mathcal{H})$. Then for every $\varepsilon > 0$ there is $M_0 \in \mathcal{M}$ such that $\|P - M_0\| \leq \mathrm{d}(P, \mathcal{M}) + \varepsilon$. Since $\mathcal{L} \subset \mathcal{M}$ one has $\mathrm{d}(Px, \mathcal{M}x) \leq \mathrm{d}(Px, \mathcal{L}x)$, for every $x \in \mathcal{H}$. Hence $\alpha(P, \mathcal{M}) \leq \alpha(P, \mathcal{L})$. Note that for every $L \in \mathcal{L}$ and $x \in \mathcal{H}$ one has $\|M_0x - Lx\| \leq \|M_0x - Px\| + \|Px - Lx\|$, which means that $\alpha(M_0, \mathcal{L}) \leq \sup_{\|x\|=1} \|M_0x - Px\| + \alpha(P, \mathcal{L}) = \|M_0 - P\| + \alpha(P, \mathcal{L})$. Therefore

$$\begin{aligned}
\mathrm{d}(P, \mathcal{L}) &\leq \|P - M_0\| + \mathrm{d}(M_0, \mathcal{L}) \leq \mathrm{d}(P, \mathcal{M}) + \varepsilon + \mathrm{d}(M_0, \mathcal{L}) \\
&\leq a\, \alpha(P, \mathcal{M}) + \varepsilon + b\, \alpha(M_0, \mathcal{L}) \leq a\, \alpha(P, \mathcal{L}) + \varepsilon + b\, (\|M_0 - P\| + \alpha(P, \mathcal{L})) \\
&\leq a\, \alpha(P, \mathcal{L}) + \varepsilon + b\, (\mathrm{d}(P, \mathcal{M}) + \varepsilon) + b\, \alpha(P, \mathcal{L}) \leq (a + b)\alpha(P, \mathcal{L}) + \varepsilon + b\, (a\alpha(P, \mathcal{M}) + \varepsilon) \\
&\leq (a + b + ab)\alpha(P, \mathcal{L}) + \varepsilon + b\varepsilon.
\end{aligned}$$

Hence $\mathcal{L}$ is operator hyperreflexive with constant at most $a + b + ab$. □

Proposition 3.4. *For each $i \in \mathbb{N}$, let $\mathcal{L}_i \subseteq \mathcal{P}(\mathcal{H}_i)$ be an operator hyperreflexive subspace lattice with constant a_i. If $a = \sup_{i\in\mathbb{N}} a_i < \infty$, then $\mathcal{L} = \oplus\mathcal{L}_i$ is operator hyperreflexive with constant at most $16 + 17a$. Conversely, if $\mathcal{L} = \oplus\mathcal{L}_i$ is operator hyperreflexive with constant a, then all $\mathcal{L}_i$ are operator hyperreflexive with constant at most a.*

Proof. If $P = \oplus P_i \in \mathcal{P}(\oplus\mathcal{H}_i)$, then $\mathrm{d}(P, \mathcal{L}) = \sup_{i\in\mathbb{N}} \mathrm{d}(P_i, \mathcal{L}_i) \leq a \sup_{i\in\mathbb{N}} \alpha(P_i, \mathcal{L}_i)$. Let $\tilde{x}_i = (0, \ldots, 0, x_i, 0 \ldots) \in \oplus\mathcal{H}_i$. Then

$$\sup_{i\in\mathbb{N}} \alpha(P_i, \mathcal{L}_i) = \sup_{i\in\mathbb{N}} \sup_{\|x_i\|\leq 1} \mathrm{d}(P_ix_i, \mathcal{L}_ix_i) = \sup_{i\in\mathbb{N}} \sup_{\|\tilde{x}_i\|\leq 1} \mathrm{d}(P\tilde{x}_i, \mathcal{L}\tilde{x}_i) \leq \alpha(P, \mathcal{L}).$$

On the other hand, $\oplus\mathcal{P}(\mathcal{H}_i)$ is the projection lattice of the injective von Neumann algebra $\oplus\mathcal{B}(\mathcal{H}_i)$, which is hyperreflexive with constant at most 4, by [3] and [10]. By Corollary 2.2 (ii), $\oplus\mathcal{P}(\mathcal{H}_i)$ is operator hyperreflexive with constant at most 16. Now Proposition 3.3 gives that $\mathcal{L}$ is operator hyperreflexive with constant at most $16 + 17a$.

Assume now that $\mathcal{L} = \oplus\mathcal{L}_i$ is operator hyperreflexive with constant a and take a projection $P = 0 \oplus 0 \cdots \oplus P_i \oplus \cdots \oplus 0$, where $P_i \in \mathcal{P}(\mathcal{H}_i)$. It is easy to see that $\mathrm{d}(P, \mathcal{L}) = \mathrm{d}(P_i, \mathcal{L}_i)$ and $\alpha(P, \mathcal{L}) = \alpha(P_i, \mathcal{L}_i)$. Hence by hyperreflexivity of $\mathcal{L}$ we have $\mathrm{d}(P_i, \mathcal{L}_i) = \mathrm{d}(P, \mathcal{L}) \leq a\, \alpha(P, \mathcal{L}) = a\, \alpha(P_i, \mathcal{L}_i)$. □

4. Non operator hyperreflexive lattice which is operator reflexive

Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with an orthonormal basis $e_1, e_2, \dots$. For $k \in \mathbb{N}$, let $\mathcal{H}_k = \bigvee\{e_1, \dots, e_k\}$. Denote by $S_{\mathcal{H}}$ the unit sphere of $\mathcal{H}$. Let $0 < \varepsilon < \frac{1}{64}$ and fix a sequence $(x_n)_{n=1}^{\infty}$ which is an ε-net in $S_{\mathcal{H}}$. Moreover, we may assume that all the vectors x_n have finite support in the sense that $x_n \in \bigcup\limits_{k\in\mathbb{N}} \mathcal{H}_k$ for each $n \in \mathbb{N}$. Fix a sequence $(t_n)_{n=1}^{\infty} \subset (0,1)$ consisting of mutually distinct numbers.

Lemma 4.1. *There exist subspaces $\mathcal{M}_n \subset \mathcal{H}$ $(n \in \mathbb{N})$ such that:*

(i) $\mathcal{M}_n \cap \mathcal{M}_m = \{0\} \quad (m, n \in \mathbb{N}, m \neq n)$;

(ii) $\mathcal{M}_n \vee \mathcal{M}_m = \mathcal{H} \quad (m, n \in \mathbb{N}, m \neq n)$;

(iii) $\|P_{\mathcal{M}_n} x_n - (1-\varepsilon)\langle x_n, e_1\rangle e_1\| < 3\sqrt{\varepsilon}$, $\quad \|P_{\mathcal{M}_n} e_j\| < \frac{1}{n}$, $\quad$ for $j = 2, \dots, n$, $\quad$ and $\|P_{\mathcal{M}_n} e_1 - P_{\mathcal{M}_s} e_1\| > \frac{\sqrt{\varepsilon}}{4} \qquad (s \neq n)$, where $P_{\mathcal{M}}$ denotes the orthogonal projection on a subspace $\mathcal{M} \subseteq \mathcal{H}$;

(iv) there is an increasing sequence $(k_n)_{n=1}^{\infty} \subseteq \mathbb{N}$ such that $k_n > \max\{2k_{n-1}, (n+1)^2\}$ and $\mathcal{M}_n$ can be written as $\mathcal{M}_n = \mathcal{F}_n \oplus \vee\{e_{2j+1} + t_n e_{2j+2}; \quad j \geq k_n\}$, where $\mathcal{F}_n \subset \mathcal{H}_{2k_n}$ is a k_n-dimensional subspace.

Proof. We construct the numbers k_n and subspaces $\mathcal{M}_n$ by induction on n. Let $n \in \mathbb{N}$ and suppose that the numbers $k_1, \dots, k_{n-1}$ and subspaces $\mathcal{M}_1, \dots, \mathcal{M}_{n-1}$ satisfying (i)–(iv) have already been constructed. Choose $k_n > \max\{2k_{n-1}, (n+1)^2\}$ such that $x_n \in \mathcal{H}_{k_n}$. Let $\mathcal{E}_s = \mathcal{M}_s \cap \mathcal{H}_{2k_n}$ for $s = 1, \dots, n-1$. By assumptions (i) and (iv), we have $\dim \mathcal{E}_s = k_n$ and $\mathcal{E}_s \cap \mathcal{E}_{s'} = \{0\}$ for all $s \neq s'$, $1 \leq s, s' \leq n-1$.

Let $u_n = (1-\varepsilon)e_1 + \sum_{j=2}^{k_n} \sqrt{\frac{2\varepsilon - \varepsilon^2}{k_n - 1}} e_j$. Then $\|u_n\| = 1$. Let $\mathcal{L}_n \subset \mathcal{H}_{2k_n}$ be the subspace spanned by the vectors $u_n, e_{k_n+2}, e_{k_n+3}, \dots, e_{2k_n}$. Clearly, $\dim \mathcal{L}_n = k_n$.

By [5, Lemma 2], there exists a subspace $\mathcal{L}'_n \subset \mathcal{H}_{2k_n}$ such that $\|P_{\mathcal{L}_n} - P_{\mathcal{L}'_n}\| < \varepsilon/n$ and $\mathcal{L}'_n \cap \mathcal{E}_s = \{0\}$ for $s = 1, \dots, n-1$. Define $\mathcal{M}_n = \mathcal{L}'_n \oplus \bigvee\{e_{2j+1} + t_n e_{2j+2}; \quad j \geq k_n\}$.

Suppose that the subspaces $\mathcal{M}_n \quad (n \in \mathbb{N})$ have been constructed in the above described way. As in [5], conditions (i), (ii) and (iv) are satisfied. So it is sufficient to show (iii).

For $j \in \{2, \dots, n\}$, one has

$$\|P_{\mathcal{M}_n} e_j\| = \|P_{\mathcal{L}'_n} e_j\| \leq \|P_{\mathcal{L}_n} e_j\| + \|P_{\mathcal{L}_n} - P_{\mathcal{L}'_n}\| \leq \|\langle e_j, u_n\rangle u_n\| + \frac{\varepsilon}{n}$$
$$\leq \sqrt{\frac{2\varepsilon - \varepsilon^2}{k_n - 1}} + \frac{\varepsilon}{n} \leq \frac{1}{2\sqrt{k_n - 1}} + \frac{1}{2n} < \frac{1}{n}.$$

It follows

$$\begin{aligned}
\|P_{\mathcal{M}_n}x_n-(1-\varepsilon)\langle x_n,e_1\rangle e_1\| &= \|P_{\mathcal{L}_n'}x_n-(1-\varepsilon)\langle x_n,e_1\rangle e_1\|\\
&\le \|P_{\mathcal{L}_n}x_n-(1-\varepsilon)\langle x_n,e_1\rangle e_1\|+\|P_{\mathcal{L}_n}-P_{\mathcal{L}_n'}\|\\
&\le \frac{\varepsilon}{n}+\|\langle x_n,u_n\rangle u_n-(1-\varepsilon)\langle x_n,e_1\rangle e_1\|\\
&\le \frac{\varepsilon}{n}+\|\langle x_n,u_n\rangle u_n-\langle x_n,u_n\rangle e_1\|+\|\langle x_n,u_n\rangle e_1-(1-\varepsilon)\langle x_n,e_1\rangle e_1\|\\
&\le \frac{\varepsilon}{n}+\|u_n-e_1\|+\|u_n-(1-\varepsilon)e_1\|\\
&\le \frac{\varepsilon}{n}+\sqrt{\varepsilon^2+\sum_{j=2}^{k_n}\frac{2\varepsilon-\varepsilon^2}{k_n-1}}+\sqrt{\sum_{j=2}^{k_n}\frac{2\varepsilon-\varepsilon^2}{k_n-1}}\\
&= \frac{\varepsilon}{n}+\sqrt{2\varepsilon}+\sqrt{2\varepsilon-\varepsilon^2}\le\sqrt{\varepsilon}(\frac{\sqrt{\varepsilon}}{n}+2\sqrt{2})\le 3\sqrt{\varepsilon}.
\end{aligned}$$

Finally, for $s<n$, we have

$$\begin{aligned}
\|P_{\mathcal{M}_n}e_1-P_{\mathcal{M}_s}e_1\| &= \|P_{\mathcal{L}_n'}e_1-P_{\mathcal{L}_s'}e_1\|\\
&\ge \|P_{\mathcal{L}_n}e_1-P_{\mathcal{L}_s}e_1\|-\|P_{\mathcal{L}_n}-P_{\mathcal{L}_n'}\|-\|P_{\mathcal{L}_s}-P_{\mathcal{L}_s'}\|\\
&\ge \|\langle e_1,u_n\rangle u_n-\langle e_1,u_s\rangle u_s\|-\frac{\varepsilon}{n}-\frac{\varepsilon}{s}\\
&= (1-\varepsilon)\|u_n-u_s\|-\frac{\varepsilon}{n}-\frac{\varepsilon}{s}\\
&\ge (1-\varepsilon)\|(P_{\mathcal{H}_{k_n}}-P_{\mathcal{H}_{k_s}})u_n\|-2\varepsilon\\
&\ge (1-\varepsilon)\sqrt{\sum_{j=k_s+1}^{k_n}\frac{2\varepsilon-\varepsilon^2}{k_n-1}}-2\varepsilon\ge\frac{\sqrt{\varepsilon}}{2}-2\varepsilon>\frac{\sqrt{\varepsilon}}{4}.
\end{aligned}$$

□

Corollary 4.2. *Let $0<\varepsilon<\frac{1}{64}$. Then there exists an operator reflexive lattice such that the operator hyperreflexivity constant is greater than $\frac{1}{4\sqrt{\varepsilon}}$.*

Proof. Fix $\varepsilon>0$ and let $\mathcal{M}_n$ be the subspaces constructed in Lemma 4.1. Let $\mathcal{L}=\{0,I,P_{\mathcal{M}_n};\ n=1,2,\dots\}$. By conditions (i) and (ii) in Lemma 4.1, $\mathcal{L}$ is a lattice.

Claim. For each $x\in\mathcal{H}$ the set $\{Lx;\ L\in\mathcal{L}\}$ is closed.

Proof. For $j\ge 2$ we have $\lim_{n\to\infty}\|P_{\mathcal{M}_n}e_j\|=0$. Consequently, $\lim_{n\to\infty}\|P_{\mathcal{M}_n}y\|=0$ for each $y\in\bigvee\{e_j;\ j\ge 2\}$.

Let $x\in\mathcal{H}$, $x=\alpha e_1+y$ for some $\alpha\in\mathbb{C}$, $y\in\bigvee\{e_j;\ j\ge 2\}$. For $\alpha=0$ the statement was shown above, so assume that $\alpha\ne 0$. By property (iii), we have $\|P_{\mathcal{M}_n}(\alpha e_1)-P_{\mathcal{M}_s}(\alpha e_1)\|\ge|\alpha|\cdot\frac{\sqrt{\varepsilon}}{4}$, for all $n\ne s$. So $\|P_{\mathcal{M}_n}x-P_{\mathcal{M}_s}x\|\ge\frac{|\alpha|\sqrt{\varepsilon}}{8}$ for all $n\ne s$ large enough. Hence the set $\{Lx;\ L\in\mathcal{L}\}$ is closed. It follows from [11] that $\mathcal{L}$ is operator reflexive; in particular, it is strongly closed.

Consider now the orthogonal projection $Q \in \mathcal{P}(\mathcal{H})$ onto the 1-dimensional subspace $\mathbb{C}e_1$. Clearly $\mathrm{d}(Q,\mathcal{L}) = 1$. Let $x \in \mathcal{H}$, $\|x\| = 1$. Then there exists $n \in \mathbb{N}$ with $\|x - x_n\| \le \varepsilon$. We have

$$\begin{aligned}
\|Qx - P_{\mathcal{M}_n}x\| &\le \|Qx - Qx_n\| + \|Qx_n - P_{\mathcal{M}_n}x_n\| + \|P_{\mathcal{M}_n}x_n - P_{\mathcal{M}_n}x\| \\
&\le 2\varepsilon + \|Qx_n - P_{\mathcal{L}'_n}x_n\| \le 2\varepsilon + \|Qx_n - P_{\mathcal{L}_n}x_n\| + \|P_{\mathcal{L}_n} - P_{\mathcal{L}'_n}\| \\
&\le 2\varepsilon + \|\langle x_n, e_1\rangle e_1 - \langle x_n, u_n\rangle u_n\| + \frac{\varepsilon}{n} \\
&\le 3\varepsilon + \|\langle x_n, e_1\rangle e_1 - \langle x_n, e_1\rangle u_n\| + \|\langle x_n, e_1\rangle u_n - \langle x_n, u_n\rangle u_n\| \\
&\le 3\varepsilon + 2\|e_1 - u_n\| \le 3\varepsilon + 2\sqrt{2\varepsilon} \le 4\sqrt{\varepsilon}.
\end{aligned}$$

Hence $\alpha(Q,\mathcal{L}) \le 4\sqrt{\varepsilon}$ and the operator hyperreflexivity constant of $\mathcal{L}$ is greater or equal to $\frac{1}{4\sqrt{\varepsilon}}$. □

Corollary 4.3. *There exists an operator reflexive subspace lattice which is not operator hyperreflexive.*

Proof. Let $(c_n)_{n=1}^{\infty}$ be a sequence of positive numbers tending to ∞. For each n find a Hilbert space $\mathcal{H}_n$ and an operator reflexive subspace lattice $\mathcal{L}_n$ in $\mathcal{P}(\mathcal{H}_n)$ such that the operator hyperreflexivity constant of $\mathcal{L}_n$ is greater than c_n. Let $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ and $\mathcal{L} = \bigoplus_{n=1}^{\infty} \mathcal{L}_n$. Then $\mathcal{L}$ is operator reflexive subspace lattice that is not operator hyperreflexive, by Proposition 3.4. □

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