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Computational comparison of the discretization and iteration errors

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1 Introduction

In this contribution we study the influence of the error in a linear algebraic solver to the precission of the finite element approximations. Since we focus on iterative solvers we call this error the *iteration* error. In particular, we are interested in the relationship between the discretization error and the iteration error.

For simplicity we restrict ourselves to the linear elliptic problems with smooth solutions solved by the lowest-order (piecewise linear) finite elements. In this case, it is well known that the discretization error behaves like $O(h)$ if the discretization parameter h tends to zero. We recall that h stands for the largest diameter of all elements. On the other hand, the condition number $\kappa(A)$ of the stiffness matrix A behaves like $O(h^2)$ for $h \to 0$. Hence, the iteration error grows for $h \to 0$ while the discretization error decreases. Naturally, starting from a certain (small enough) value of h the iteration error outweighs the discretization error and it starts to dominate.

In this paper we would like to reproduce this behavior numerically with the aim to quantify the magnitudes of the discretization and iteration errors. For this purpose, we construct a simple example for which we know the exact solution as well as the exact discrete solution. Hence, we will be able to compute the discretization error exactly and, moreover, we will be able to compute the iteration error numerically.

2 The model problem

As a model problem we choose the d-dimensional Poisson problem

$$-\Delta u = f \quad \text{in a domain } \Omega \subset \mathbb{R}^d \tag{1}$$

with the homogeneous Dirichlet boundary conditions.

To solve (1) by the finite element method [1, 3], we assume Ω to be a polytop. Further, we introduce a simplicial partion $\mathcal{T}_h$ of Ω and a space V_h of globally continuous and piecewise linear functions. The dimension of V_h is denoted by n. The finite element solution $u_h \in V_h$ is defined by

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h, \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \quad \text{and} \quad F(v) = \int_\Omega f v \,\mathrm{d}x.$$

Problem (2) is equivalent to the system of linear algebraic equations

$$Ay = b, \tag{3}$$

where the entries of the stiffness matrix $A \in \mathbb{R}^{n\times n}$ are $A_{ij} = a(\varphi_i, \varphi_j)$, $i, j = 1, 2, \ldots, n$, the entries of the load vector $b \in \mathbb{R}^n$ are $b_j = F(\varphi_j)$, $j = 1, 2, \ldots, n$, and the vector $y \in \mathbb{R}^n$ contains the expansion coefficients of the discrete solution u_h with respect to the basis $\varphi_1, \varphi_2, \ldots, \varphi_n$, i.e.,

$$u_h = \sum_{i=1}^{n} y_i \varphi_i.$$

Practically, system (3) have to be solved by a suitable numerical method. This method delivers an approximate solution $y^* \in \mathbb{R}^n$ to (3) which yields the approximate finite element solution

$$u_h^* = \sum_{i=1}^{n} y_i^* \varphi_i.$$

Hence, the approximate discrete solution u_h^* is loaded by the round-off and iteration errors while the exact discrete solution u_h is not. In this contribution we are interested not only in the discretization error $e = u - u_h$ but mainly to the iteration error $e_h = u_h - u_h^*$. It is well known that the energy norm of the discretization error behaves like $O(h)$. On the other hand much less is known about the behavior of e_h, cf. [2].

3 Numerical example

For the test purposes we use problem (1) with the simplest possible setting. Let $d = 1$, $\Omega = (0, 1)$, $f = 2$, and hence $u = x(1 - x)$. We discretize problem (1) on a uniform mesh with $n + 1$ elements. The size of each element is $h = 1/(n+1)$ and the nodal points are denoted by $x_i = ih$, $i = 0, 1, 2, \ldots, n + 1$. We define the standard finite element basis functions φ_i, $i = 1, 2, \ldots, n$ as the piecewise linear and globally continuous functions in $\overline{\Omega} = [0, 1]$ with the property

$$\varphi_i(x_j) = \delta_{ij}, \quad i = 1, 2, \ldots, n, \ j = 0, 2, \ldots, n + 1.$$

The finite element space is then spanned by these basis functions, i.e., $V_h = \text{span}\{\varphi_i, i = 1, 2, \ldots, n\}$. The finite element solution $u_h \in V_h$ is uniquely determined by (2).

It is well known [3] that for the 1D Poisson problem (1) the (exact) discrete solution u_h given by (2) equals to the exact solution u in the mesh points, i.e.,

$$y_i = u_h(x_i) = u(x_i) = ih(1 - ih) \quad \text{for all } i = 0, 1, 2, \ldots, n + 1, \tag{4}$$

where we recall that the values of the expansion coefficients y_i equal to the nodal values of the discrete solution. Moreover, in this simple setting we can use (4) to compute exactly the energy norm of the discretization error $e = u - u_h$ as follows

$$\|e\| = \sqrt{a(e, e)} = \frac{\sqrt{3}}{3} h. \tag{5}$$

To investigate the iteration error e_h, we solve the stiffness system (3) by the conjugated gradients method (CG). The stiffness matrix A is tridiagonal in this case with entries $2/h$ on the main diagonal and with entries $-1/h$ on the diagonals one above and one blow the main diagonal. A system with this matrix is very easy to solve by a direct method but we use the CG for our test purposes because CG are often used in the real-life problems.

We proceeds as follows. We assemble the tridiagonal matrix A and the load vector b. Let us remark that $b_i = 2h$ for all $i = 1, 2, \ldots, n$. Then we solve system (3) by CG to obtain the vector

y^*. We stop the CG iteration process after $n/2$ iterations or if the relative residual drops below 10^{-4}. The energy norm of the iteration error is then computed by

$$\|e_h\|^2 = a(e_h, e_h) = (y - y^*)^T A(y - y^*), \tag{6}$$

where the expansion coefficients y are computed by (4).

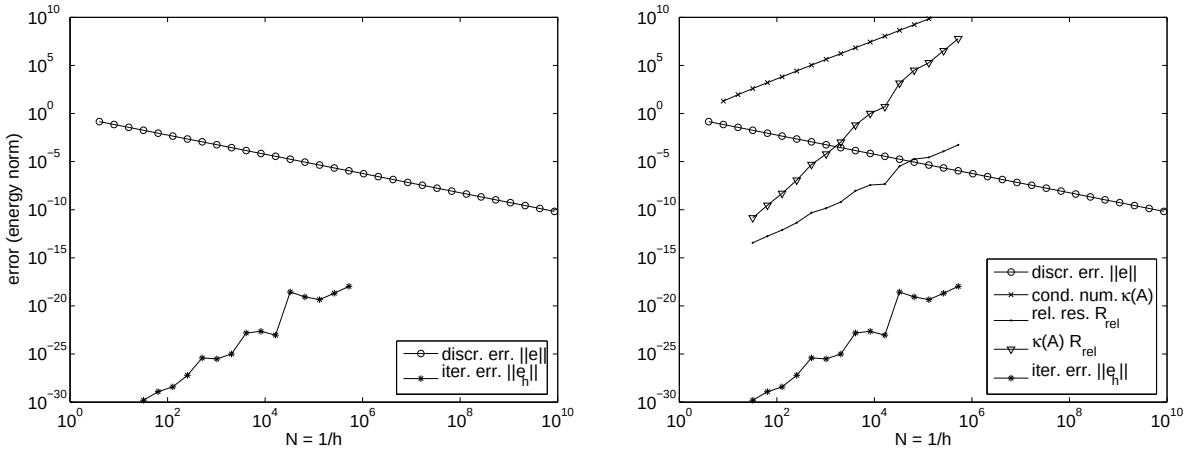

Figure 1: The behavior of the discretization error $\|e\|$ and the iteration error $\|e_h\|$ with respect to the number of degrees of freedom (left panel). The same plot with the upper bound (7) is shown in the right panel. Recall that $R_{\rm rel} = \|r^*\|_A/\|b\|_A$ with $r^* = b - Ay^*$.

Figure 1 shows the behavior of the discretization and iteration errors if the discretization parameter h decreases. The left panel contains the log-log plot of the energy norms $\|e\|$ and $\|e_h\|$ as functions of the number of degrees of freedom $n = 1/h - 1$. The right panel shows, in addition, the condition number $\kappa(A)$, the relative residual $R_{\rm rel} = \|b - Ay^*\|_A/\|b\|_A$ and the upper bound $\kappa(A)R_{\rm rel}$ on the relative error, see estimate (7) below.

The energy norm of the discretization error $\|e\|$ can be computed analytically by (5) for any h. However, to compute the energy norm of the iteration error $\|e\|$ we have to run the CG iterations. Here we are limited by the computer speed and the amount of the memory. Using Matlab, we were able to reach n about 10^6. For these values of n, however, the iteration error is about 10^{-18} while the discretization error is about 10^{-7}. Hence, there is still no danger of pollution of the computed finite element solution u_h^* by the iteration error.

This is a bit surprising because theoretical estimates indicate much worst behavior, cf. Figure 1 (right). Indeed, for our simple problem, we may compute the following quantities explicitely $\|u_h\|^2 = (1 - h^2)/3$, $\kappa(A) = (1 + \cos h\pi)/(1 - \cos h\pi)$, $\|b\|_A^2 = 8h$, where $\|b\|_A^2 = b^T Ab$ stands for the discrete energy norm induced by the symmetric and positive definite stiffness matrix A. We employ these relations in the well known estimate of the relative error of y^* in terms of the spectral condition number $\kappa(A)$ and the relative residual:

$$\frac{\|e_h\|}{\|u_h\|} = \frac{\|y - y^*\|_A}{\|y\|_A} \le \kappa(A)\frac{\|b - Ay^*\|_A}{\|b\|_A} = \kappa(A)\frac{\|r^*\|_A}{\|b\|_A}, \tag{7}$$

where $r^* = b - Ay^*$. Due to the stopping criterion we imposed to the CG iterations we may assume that $\|r^*\|_A/\|b\|_A \approx 10^{-4}$. Summarizing these results, we may find from (7) such h that $\|e_h\| \approx \|e\| = \sqrt{3}h/3$, see (5). More precisely, we arrive at the following nonlinear equation for h

$$\frac{h}{\sqrt{1 - h^2}} = \frac{1 + \cos h\pi}{1 - \cos h\pi} 10^{-4}.$$

Solving this equation, we obtain $h \approx 3 \times 10^{-2}$. For this value of h we would expect the discretization error and iteration errors to be comparable but the numerical results show that for this h the iteration error is smaller then the discretization error by more then 20 orders of magnitude. The conclusion from this simple analysis is that the estimate (7) overestimates the relative error by several orders of magnitude in this case.

4 Conclusions

Figure 1 clearly shows the expected trend of growing iteration error. We may extrapolate and say that for h about 10^{-8} the discretization and iteration errors will be both of the order 10^{-10}. More precise finite element solution u_h^* can not be computed in this way because further decrease of h would lead to the increase of the iteration error which dominates over the discretization error.

We remark that much more interesting would be to generalize the above one dimensional numerical test to the higher spatial dimension. This can be done but we lose the important property (4). For the Poisson problem in a square, we may obtain an analytic formula for the exact discrete solution. This formula is based on the known eigenvectors of the corresponding stiffness matrix. The computation of the exact discrete solution, however, requires multiplication of a matrix with 25 % of nonzero entries by a vector. This multiplication is prohibitively slow for large problems and, moreover, it leads to a cumulation of the round-off errors.

The deeper study of the influence of the iteration error to the precission of the finite element solutions with emphasis to the higher spatial dimensions will be subjected to our future research.

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