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A NOTE ON FRAGMENTABILITY AND WEAK-G_δ SETS

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Abstract. In terms of fragmentability, we describe a new class of Banach spaces which may be c_0-saturated but do not contain weak-G_δ open bounded subsets. In particular, none of these spaces is isomorphic to a separable polyhedral space.

1. Introduction and Preliminaries

All Banach spaces under consideration in this note are assumed to be real and infinite-dimensional.

According to a well known theorem of Lindenstrauss and Phelps [9, Corollary 1.2], if X is a reflexive space then every closed convex and bounded body in X has uncountably many extreme points. The first named author has obtained different generalisations of this result. In particular, in [2], it is proved that every infinite-dimensional Banach space X which is not c_0-saturated does not admit a countable boundary. Moreover, if X is not c_0-saturated then [3, Corollary 3]

(a) X does not contain an open, bounded weak-G_δ set.

In [4, Theorem 3] it is shown that if a separable space X does not contain c_0 then

(b) the polar A° of any closed convex and bounded body $A \subset X$ with $0 \in \text{int}A$ contains uncountably many w^*-exposed points.

Recall that a set $B \subseteq B_{X^*}$ is said to be a *boundary* for X if, for every $x \in X$, there is $f \in B$ such that $f(x) = ||x||$. Assume that $\{f_n\}_{n=1}^\infty$ is a countable boundary

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for X (for example, X is separable and polyhedral [5, Theorem 1]), then

$$\operatorname{int} B_X = \bigcap_{n=1}^{\infty} \{x \in X : \, f_n(x) < 1\}.$$

Thus $\operatorname{int} \mathrm{B_X}$ is an open, bounded weak-G_δ set. Next let $G \subset X$ be a bounded, convex, closed body and $0 \in \operatorname{int} G$. A point $x \in \partial G$ is said to be *smooth* if the Minkowski functional p of G is Gâteaux differentiable at x. A point f of a subset $A \subset X^*$ of the dual space X^* is said to be a *w^*-exposed point* of A if there is $x \in X$ such that $f(x) > g(x)$ for every $g \in A$, $g \neq f$. Moreover, we say that this x w^*-exposes f. Let us recall also the following well known fact.

Fact 1. *A point $x \in G$ is smooth for G if and only if x w^*-exposes some point f in the polar G° of G, with $f(x) = 1$.*

In this note we describe (by means of special fragmentable sets) a new class $\mathcal{K}$ of Banach spaces X which have both properties (a) and (b) and which may be c_0-saturated.

Definition 2 (Namioka [11, p. 259])**.** *A set M in a Banach space X is said to be* fragmentable *if, for any subset A of M and any $\varepsilon > 0$, there is a weak open set V which meets A and* $\operatorname{diam}(A \cap V) < \varepsilon$*.*

Additionally, a set M is *dentable* if, for any $\varepsilon > 0$, there is an open half space H which meets M and $\operatorname{diam}(A \cap H) < \varepsilon$. Clearly, if every subset of a set M is dentable then M is fragmentable. It is known that if M is a weakly compact subset of a Banach space, or a bounded subset of a dual space of an Asplund space, then every subset of M is dentable (see e.g. [1, pp. 31, 60 and 91]).

We define the class $\mathcal{K}$ as follows.

Definition 3. *A Banach space X belongs to $\mathcal{K}$ if X contains a non-empty fragmentable set $M \subset \operatorname{int} B_X$ satisfying the following condition*

($*$) for any $\varepsilon > 0$, any weak open set V and any $x_0 \in V \cap M$, there is a finite sequence $\{x_i\}_{i=1}^n \subset V \cap M$ such that $||x_i - x_{i-1}|| < \varepsilon$, $i = 1, \dots, n$, and $||x_n|| \geq 1 - \varepsilon$.

Our main result is the following

Theorem 4. *If $X \in \mathcal{K}$ then X does not contain open bounded $w{-}G_\delta$ sets. Moreover, if $X \in \mathcal{K}$ is separable and G is a convex bounded open set then the set of all smooth points of* $\operatorname{cl} G$ *cannot be covered by a countable union of weak closed sets which does not meet G. In particular, if $0 \in G$ then the set w^*-exp G° is uncountable.*

The following corollary complements the main result from [4].

Corollary 5. *Assume that a separable Banach space X contains a subspace with the Radon-Nikodým property (e.g. reflexive or l_1). If $F \subset X$ is a bounded closed convex body with $0 \in \operatorname{int} F$ then the set w^*-exp F° is uncountable.*

Hitherto, there were no c_0-saturated Banach spaces known to satisfy the conclusions of Theorem 4. Despite the fact that the proof of Theorem 4 uses some ideas from [4], we give examples of c_0-saturated Banach spaces which admit sets satisfying the hypotheses of Theorem 4.

2. Proof of Theorem 4

The proof of the following fact is standard.

Fact 6. *Let K be a weak compact subset of a weak open subset V of a Banach space X. Then there is a non-empty weak open neighbourhood W of the origin such that $K + W \subset V$.*

The following rather technical proposition will be our main tool.

Proposition 7. *Let G be an open bounded subset of a Banach space X and assume $0 \in K \subset G$, where K is compact. Put $G_K = \{x \in X : \ x + K \subset G\}$. Assume that M is a non-empty fragmentable subset of $\mathrm{cl}G_K$, such that for any weak open set U with $U \cap M \neq \emptyset$, for any weak closed subset E with $E \cap G = \emptyset$, and for any $\varepsilon > 0$, there is $y \in M \cap U$ such that $(y + K) \cap E = \emptyset$ and $d(y, \partial G_K) < \varepsilon$. Then, for any w-F_σ set F with $F \cap G = \emptyset$, there is $x \in X$ such that $x + K \subset \mathrm{cl}G$, $(x + K) \cap F = \emptyset$, and $(x + K) \cap \partial G \neq \emptyset$.*

Proof. Let $F = \bigcup_{n=1}^{\infty} F_n$, where $\{F_n\}$ is an increasing sequence of weak closed sets. Set $F_0 = \emptyset$ and let $\{\varepsilon_n\}_{n=0}^{\infty}$ be a sequence of positive numbers tending to 0, where $\varepsilon_0 > \mathrm{diam}(G_K)$. We construct a sequence $\{x_n\}_{n=0}^{\infty} \subset M$ and decreasing sequences of w-open sets $\{U_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ with the following properties

(1) $x_n \in U_n$ and $x_n + K \subset V_n$;
(2) $w\text{-cl}V_n \cap F_n = \emptyset$;
(3) $\mathrm{diam}(U_n \cap M) < \varepsilon_n$;
(4) $d(x_n, \partial G_K) < \varepsilon_n$

for all n. To begin, let $x_0 \in M$ be arbitrary and $U_0 = V_0 = X$. Assume we have constructed x_n, U_n and V_n. By Fact 6, we can take a weak open neighbourhood W of x_n such that $W + K \subset V_n$. Since $x_n \in U_n \cap W \cap M$ and M is fragmentable, there exists weak open $U_{n+1} \subset W \cap U_n$ such that $U_{n+1} \cap M$ is non-empty and $\mathrm{diam}(U_{n+1} \cap M) < \varepsilon_{n+1}$. From our hypothesis, there exists $x_{n+1} \in U_{n+1} \cap M$ with the property that $(x_{n+1} + K) \cap F_{n+1} = \emptyset$. Since $x_{n+1} + K \subset U_{n+1} + K \subset W + K \subset V_n$ and $x_{n+1} + K \subseteq X \backslash F_{n+1}$, again by Fact 6 we can pick a weak open neighbourhood W' of x_{n+1}, satisfying $w\text{-cl}W' + K \subseteq V_n \backslash F_{n+1}$. Define $V_{n+1} = W' + K$ to complete the construction.

From the conditions above, it follows that $\{x_n\}$ is a Cauchy sequence. Let $x = \|\cdot\|\text{-}\lim x_n$. We have $x + K \subset \bigcap_{n=0}^{\infty} w\text{-cl}V_n$ and $x \in \partial G_K$. Hence $x + K \subset \mathrm{cl}G$ and $(x + K) \cap F = \emptyset$. Since K is a compact set and $x \in \partial G_K$, it follows that $(x + K) \cap \partial G \neq \emptyset$. The proof is complete. □

Recall that a Banach space X is called *polyhedral* [7, p. 265] if the unit ball of any its finite-dimensional subspace is a polytope. It was proved in [5] that a separable polyhedral space admits a countable boundary. The next assertion shows that fragmentable subsets of the unit sphere of a separable polyhedral space are quite small.

Corollary 8. *Let G be an open bounded subset of a Banach space X, and let $M \subset \partial G$ be a fragmentable set such that for any weak open set U with $U \cap M \neq \emptyset$, and for any weak closed set F with $F \cap G = \emptyset$, we have $(U \cap M) \setminus F \neq \emptyset$. Then G is not a weak G_δ set. In particular, if X is polyhedral then, for every fragmentable set $M \subset S_X$, there is a weak open set U which meets M and a finite number of hyperplanes $\{H_i\}_{i=1}^m$ in X, such that $U \cap M \subset \bigcup_{i=1}^m H_i$.*

Proof. We can assume that $0 \in G$. If we put $K = \{0\}$ and apply Proposition 7, we see that G is not a weak G_δ set. If X is polyhedral then [5, Theorem 1] it has a countable boundary and hence there is a sequence of hyperplanes $\{H_i\}_{i=1}^\infty$ in X with $S_X \subset \bigcup_{i=1}^\infty H_i$. Setting $F_n = \bigcup_{i=1}^n H_i$ for $n \geq 1$, using the proof of Proposition 7, we find a weak open set U and $m \in N$ such that $(M \cap U) \setminus F_m = \emptyset$ and $M \cap U \neq \emptyset$. □

Proof of Theorem 4. Let $M \subset X$ be as in Definition 3. It will help to assume that $0 \in M$. If necessary, this can be done by replacing M with the set

$$\left(\frac{M - z}{1 - ||z||}\right) \cap \mathrm{int} B_X$$

where $z \in M$ is arbitrary. Assume that $G \subset X$ is an open bounded set and $0 \in K \subset G$, with K a compact set which we specify later. We will check the conditions of Proposition 7. First of all $0 \in M \cap G_K$. Now let G_K, U, E, and $\varepsilon > 0$, be as in Proposition 7. Pick $x_0 \in U \cap M \cap G_K$ and by using the condition $(*)$, find $\{x_i\}_{i=1}^n \subset U \cap M$ with $\|x_i - x_{i-1}\| < \varepsilon$, $i = 1, \ldots, n$, $\|x_n\| \geq 1 - \varepsilon$. Assume that $x_n \in M \cap G_K$. Then since $||x_n|| \geq 1 - \varepsilon$ and $x_n \in M \cap G_K \subset G_K \subset G \subset B_X$, it follows that $d(x_n, \partial G) < \varepsilon$. If $x_n \notin M \cap G_K$ then there is $m < n$ with $x_m \in M \cap G_K$ and $x_{m+1} \notin M \cap G_K$. Since $||x_m - x_{m+1}|| < \varepsilon$, it follows that $d(x_m, \partial G_K) < \varepsilon$. Set $y = x_m$ if $x_n \notin M \cap G_K$, and $y = x_n$ otherwise. Hence $d(y, \partial G_K) < \varepsilon$. Since $y \in M \cap G_K \subset G_K$ we get that $y \in G$. Having in mind that $E \cap G = \emptyset$, we get $(y + K) \cap E = \emptyset$.

Now assume to contrary that G is a weak G_δ set. Put $F = X \setminus G$. Then F is a weak F_σ set and by Proposition 7 there is $x \in X$ such that $x + K \subset \mathrm{cl}G$, $(x + K) \cap F = \emptyset$, and $(x + K) \cap \partial G \neq \emptyset$, contradicting $\partial G \subset F$.

The proof of the second part of the theorem uses an idea from the proof of [4, Theorem 2]. Given a separable Banach space X and a convex, bounded open set G with $0 \in G$, we let $K = T(B(\ell_2))$, where $T : \ell_2 \to X$ is a linear, compact operator with dense range, and chosen so that K is contained in the interior of G. If F is a weak F_σ set with $F \cap G = \emptyset$ then by Proposition 7 we obtain $x \in$ clG satisfying

$$x + K \subset \mathrm{cl}G, \quad (x + K) \cap \partial G \neq \emptyset, \quad (x + K) \cap \partial G \cap F = \emptyset. \tag{2.1}$$

Now assume to the contrary that w^*-exp G° is countable. Then by Fact 1 the set $\mathrm{sm}(\mathrm{cl}G)$ of all smooth points of $\mathrm{cl}G$ is $w - F_\sigma$. Put $F = \mathrm{sm}(\mathrm{cl}G)$ and apply (2.1). We get a point $z \in (x + K) \cap (\partial G \setminus F)$. However by using that $K = T(B(\ell_2))$ and $\mathrm{cl\,span}K = X$ it is easy to see that $z \in F$, a contradiction. The proof is complete. □

3. Examples

Let X be a Banach space with a normalized shrinking basis $\{e_i\}$, such that there is a sequence of numbers $\{t_i\}$, $\lim_i t_i = 0$, with two further properties:

(a) $\sup_n ||\sum_{i=1}^n t_i e_i|| = \infty$;

(b) for any subsequence $\{t_{i_k}\}$ such that $\sup_n ||\sum_{k=1}^n t_{i_k} e_{i_k}|| < \infty$, the series $\sum_{k=1}^\infty t_{i_k} e_{i_k}$ converges.

We show there exists a relatively weakly compact subset $M \subset B_X$, satisfying condition $(*)$ of Theorem 4.

Let $\{e_i^*\}$ be the biorthogonal sequence for $\{e_i\}$ and

$$P_n x = \sum_{i=1}^n e_i^*(x) e_i, \quad x \in X, \quad n = 1, 2, \ldots$$

Denote

$$M = \{x = \sum_{i \in \sigma} t_i e_i : \ \sigma \subset \mathbb{N}, \ |\sigma| < \infty, \ ||P_n x|| \leq 1, \ n = 1, 2, \ldots\}.$$

Now pick $x_0 = \sum_{i \in \sigma_0} t_i e_i \in M$, $||x_0|| < 1 - \varepsilon$, and a weak open set V containing x_0. Find $\delta > 0$ and $m \in \mathbb{N}$ such that

$$x_0 \in U = \{u \in X : \ |e_i^*(x_0 - u)| < \delta : \ i = 1, \ldots, m\} \subset V.$$

Given $\varepsilon > 0$, find $l \in \mathbb{N}$ such that $|t_i| < \varepsilon$ for $i > l$. Denote $i_0 = \max \sigma$ and pick $j > \max\{i_0, l, m\}$. Set

$$x_{k+1} = x_0 + \sum_{i=j}^{j+k} t_i e_i, \quad k = 0, 1, \ldots$$

Clearly, $\{x_k\} \subset U$, $||x_k - x_{k+1}|| < \varepsilon$, $k = 0, 1, \ldots$, and $\lim_k ||x_k|| = \infty$. Let n be the minimal index for which $||x_n|| < 1$ and $||x_{n+1}|| \geq 1$. Then $||x_k|| < 1$, $x_k \in M$, $k = 1, \ldots, n$, and $||x_n|| \geq ||x_{n+1}|| - ||x_n - x_{n+1}|| > 1 - \varepsilon$.

Next we show that M is relatively weakly compact. Given a sequence $\{y_l\} \subset M$, we have finite $\sigma_l \subset \mathbb{N}$ such that $y_l = \sum_{i \in \sigma_l} t_i e_i$ and $\sup_n ||P_n y_l|| \leq 1$ for each n and l. By taking a subsequence, we can find $\sigma \subset \mathbb{N}$ such that $\lim_l \sigma_l = \sigma$ in the pointwise topology of the power set of $\mathbb{N}$. We enumerate σ as a strictly increasing sequence $\{i_k\}$. Clearly $\lim_l P_{i_n} y_l = \sum_{k=1}^n t_{i_k} e_{i_k}$ for each n, so by (b), $y = \sum_{k=1}^\infty t_{i_k} e_{i_k}$ converges in X. Since $\{e_i\}$ is shrinking, it is evident that $w\text{-}\lim_l y_l = y$.

Example 9. There is a separable Banach space X with shrinking basis which is c_0-saturated but does not contain a bounded, open weak-G_δ set. Moreover, for any equivalent norm $|||\cdot|||$ on X, the set $\exp B_{(X,|||\cdot|||)^*}$ is uncountable.

Indeed, in [8, Theorem 8] a non-degenerate Orlicz function M is constructed such that there is a sequence $\{t_i\}$, $\lim_i t_i = 0$, with

$$\sup_i \frac{M(Kt_i)}{M(t_i)} < \infty,$$

for any $K > 0$, and

$$\alpha_M = \sup\{q : \sup_{0<\lambda,t\leq 1} \frac{M(\lambda t)}{M(\lambda)t^q} < \infty\} = \infty.$$

From [10, p. 143], it follows that the space h_M is c_0-saturated. By repeating some of the t_i if necessary, we may assume that $\sum_i M(t_i) = \infty$. Then the unit vector basis $\{e_i\}$ of h_M and the sequence $\{t_i\}$ satisfy the conditions (a) and (b). Let us mention that in [8], it is shown that such h_M has no countable boundary for any equivalent norm.

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