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Smooth bifurcation branches of solutions for a Signorini problem

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Abstract

We study a bifurcation problem for the equation $\Delta u + \lambda u + g(\lambda, u) = 0$ on a rectangle with Signorini boundary conditions on a part of one edge and mixed (zero Dirichlet and Neumann) boundary conditions on the rest of the boundary. Here $\lambda \in \mathbb{R}$ is the bifurcation parameter, and g is a small perturbation. Under certain assumptions concerning an eigenfunction u_0 corresponding to an eigenvalue λ_0 of the linearized equation with the same nonlinear boundary conditions, we prove the existence of a local smooth branch of nontrivial solutions bifurcating from the trivial solutions at λ_0 in the direction of u_0. The contact sets of these nontrivial solutions are intervals which change smoothly along the branch. The main tool of the proof is a local equivalence of the unilateral BVP to a system consisting of a corresponding classical BVP and of two scalar equations. To this system classical Crandall-Rabinowitz type local bifurcation techniques (scaling and Implicit Function Theorem) are applied.

Keywords: Smooth bifurcation; Signorini problem; Variational inequality.
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1 Introduction

Let $\Omega := (0,1) \times (0,\ell)$ be a rectangle with $\ell > 0$. Let its boundary be divided into a "Dirichlet part" $\Gamma_D := (\{0\}\times(0,\ell))\cup(\{1\}\times(0,\ell))$, a "unilateral part" $\Gamma_U := ((\gamma_1,\gamma_2)\times\{0\}) \subset ((0,1)\times\{0\})$ with $0 < \gamma_1 < \gamma_2 < 1$ and a "Neumann part" $\Gamma_N := \partial\Omega \setminus (\Gamma_D \cup \Gamma_U)$ (see Fig. 1). We will study bifurcation from the trivial solution of the Signorini boundary value problem

$$\Delta u + \lambda u + g(\lambda, u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_D, \quad \partial_\nu u = 0 \text{ on } \Gamma_N, \tag{1.2}$$

$$u \leq 0, \quad \partial_\nu u \leq 0, \quad u\partial_\nu u = 0 \quad \text{on } \Gamma_U, \tag{1.3}$$

where λ is a real positive bifurcation parameter and $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a C^1-smooth function (where $\mathbb{R}^+ := (0,\infty)$). We will assume that there are $C > 0$ and $q > 2$ such that for all $\lambda \in \mathbb{R}^+$

$$g(\lambda, 0) = \partial_u g(\lambda, 0) = 0, \tag{1.4}$$

$$|g(\lambda, u)| + |\partial_u g(\lambda, u)| \leq C(1 + |u|^q) \text{ for all } u \in \mathbb{R}. \tag{1.5}$$

Furthermore, we will assume that we are given an eigenvalue $\lambda_0 > 0$ and a corresponding eigenfunction u_0 to the (nonlinear) eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega \tag{1.6}$$

with boundary conditions (1.2), (1.3) and that the contact set

$$\mathcal{A}(u) := \{x \in (\gamma_1, \gamma_2), : \ u(x, 0) = 0\}$$

of the eigenfunction u_0 is a strong subinterval of (γ_1, γ_2):

$$\mathcal{A}(u_0) = [\alpha_0, \beta_0] \text{ with } \gamma_1 < \alpha_0 < \beta_0 < \gamma_2. \tag{1.7}$$

Our main goal is to prove that under certain natural assumptions a smooth local branch of nontrivial weak solutions of the problem (1.1)–(1.3) bifurcates at λ_0 in the direction u_0 from the branch of trivial solutions, and that this branch contains all nontrivial solutions satisfying $u \in W^{2,2}(\Omega)$, lying near $(\lambda_0, 0)$ and such that $\frac{u}{\|u\|}$ is close to u_0 (see Theorem 2.3). This branch will be parametrized as $(\hat{\lambda}(s), \hat{u}(s))$, where the function $\hat{\lambda} : [0, s_0) \to \mathbb{R}$ is C^1-smooth and the mapping $s \to \hat{u}(s)$ is C^1-smooth as the map into $L^2(\Omega)$ and continuous as a map into $W^{1,2}(\Omega)$. The contact sets $\mathcal{A}(\hat{u}(s))$ are intervals in (γ_1, γ_2) changing also C^1-smoothly along the bifurcating branch.

The basic ideas are close to those from the previous paper [3]. We show that the variational inequality, which is a weak formulation of our problem, is after a suitable scaling equivalent in a neighborhood of the bifurcation point to a certain smooth operator equation in a suitable Hilbert space, and that this smooth operator equation can be solved locally by means of the Implicit Function Theorem. We show that for all solutions under consideration, the contact set is an interval $\mathcal{A}(u) = [\alpha, \beta]$. Then they are simultaneously solutions of a mixed boundary value problem, but only solutions of this boundary value problem which are in $W^{2,2}(\Omega)$ and satisfying simultaneously (1.3) are really solutions of the variational inequality. Moreover, the parts of the boundary where Dirichlet and Neumann conditions are fulfilled, that means α, β, change with λ. Therefore we transform our problem by using a diffeomorpism of the interval $[0, 1]$ onto itself in such a way that we can work with the mixed boundary value problem with fixed boundary data but with α, β in the coefficients of the transformed equation. The operator equation mentioned is in fact a weak formulation of this transformed problem together with two scalar conditions guaranteing the $W^{2,2}(\Omega)$ regularity.

Let us note that the basic ideas of the proof of our main result are explained in [4].

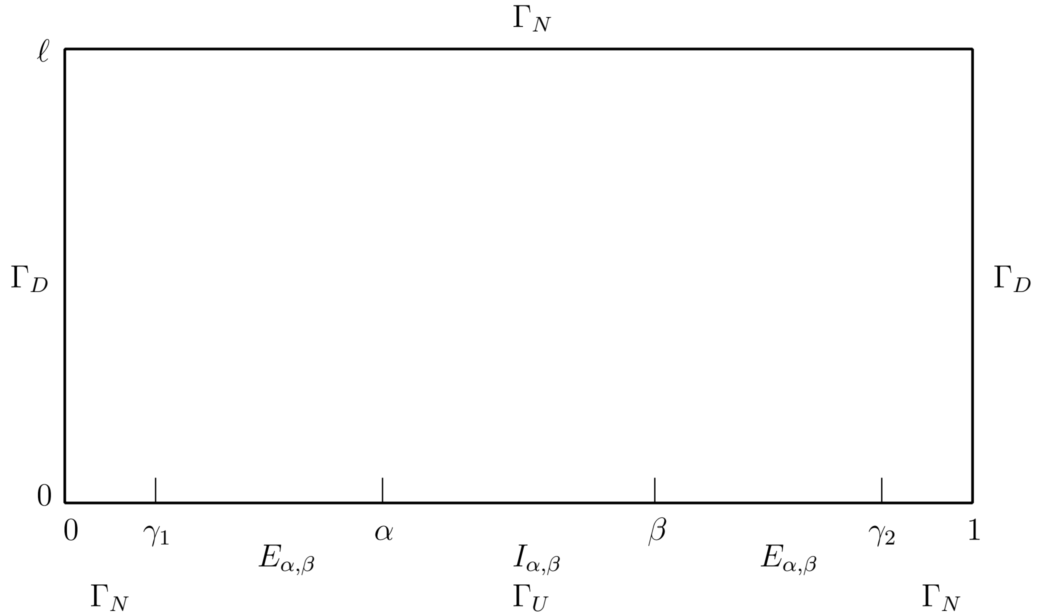

Figure 1: The domain Ω.

2 Main Results

We introduce a real Hilbert space H with scalar product $\langle \cdot, \cdot \rangle$, defined by

$$H := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_D\}, \quad \langle u, \varphi \rangle := \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}y, \; u, \varphi \in H,$$

and its closed convex subset

$$K := \{u \in H : u \le 0 \text{ on } \Gamma_U\}$$

and consider the weak formulation of (1.1)–(1.3) and (1.6), (1.2), (1.3) in terms of the variational inequalities

$$u \in K : \int_\Omega (\nabla u \cdot \nabla(\varphi - u) - (\lambda u + g(\lambda, u))(\varphi - u)) \, \mathrm{d}x \, \mathrm{d}y \ge 0 \text{ for all } \varphi \in K \tag{2.1}$$

and

$$u \in K : \int_\Omega (\nabla u \cdot \nabla(\varphi - u) - \lambda u(\varphi - u)) \, \mathrm{d}x \, \mathrm{d}y \ge 0 \text{ for all } \varphi \in K, \tag{2.2}$$

respectively. We will denote by $\| \cdot \|$ the norm in H which is equivalent on our space H to the usual Sobolev norm.

An essential part of our considerations is related to mixed boundary value problems of the following type: Take the differential equation (1.1) and accompany it with the boundary conditions (1.2) and

$$u = 0 \text{ on } I_{\alpha,\beta}, \quad \partial_y u = 0 \text{ on } E_{\alpha,\beta}, \tag{2.3}$$

where

$$I_{\alpha,\beta} := \{(x, 0) \in \Gamma_U : \alpha < x < \beta\} = (\alpha, \beta) \times \{0\},$$
$$E_{\alpha,\beta} := \{(x, 0) \in \Gamma_U : \gamma_1 < x < \alpha \text{ or } \beta < x < \gamma_2\} = \Gamma_U \setminus \overline{I_{\alpha,\beta}},$$

and α and β are parameters with $\gamma_1 < \alpha < \beta < \gamma_2$.

Let $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$ be the parameters from assumption (1.7), and set

$$\delta_0 := \frac{1}{3} \min\{\alpha_0 - \gamma_1, \beta_0 - \alpha_0, \gamma_2 - \beta_0\}, \; D := \{(\alpha, \beta) : |\alpha - \alpha_0| < \delta_0, \, |\beta - \beta_0| < \delta_0\}.$$

We introduce coordinate transformations in $\overline{\Omega}$, i.e. diffeomorphisms of $\overline{\Omega}$ onto itself, which map $I_{\alpha,\beta}$ onto I_{α_0,β_0} and $E_{\alpha,\beta}$ onto E_{α_0,β_0}. These coordinate transformations will be used to transform the mixed boundary value problem (1.1),(1.2),(2.3), which has (α, β)-independent coefficients in the equation, but (α, β)-dependent boundary conditions, into a mixed boundary value problem, which has (α, β)-dependent coefficients in the equation, but (α, β)-independent boundary conditions.

For any $(\alpha, \beta) \in D$ let $\xi_{\alpha,\beta} : [0, 1] \to [0, 1]$ be a function such that

$$\text{the map } (\alpha, \beta, x) \mapsto \xi_{\alpha,\beta}(x) \text{ is } C^\infty\text{-smooth on } D \times [0, 1], \tag{2.4}$$

$$\xi_{\alpha_0,\beta_0}(x) = x \quad \text{for all } x \in [0, 1], \tag{2.5}$$

$$\left.\begin{array}{l} \xi_{\alpha,\beta}(0) = 0, \quad \xi_{\alpha,\beta}(1) = 1, \\ \xi^{-1}_{\alpha,\beta}(x) = x + \alpha - \alpha_0 \quad \text{for } |x - \alpha_0| \le \delta_0, \\ \xi^{-1}_{\alpha,\beta}(x) = x + \beta - \beta_0 \quad \text{for } |x - \beta_0| \le \delta_0, \\ \xi_{\alpha,\beta} \text{ is a diffeomorphism of } [0, 1] \text{ onto } [0, 1], \end{array}\right\} \text{ for all } (\alpha, \beta) \in D, \tag{2.6}$$

where $\xi_{\alpha,\beta}^{-1}$ is the inverse function to $\xi_{\alpha,\beta}$. For $(\alpha,\beta)\in D$ let us define the linear bounded operator $\Phi_{\alpha,\beta}: L^2(\Omega)\to L^2(\Omega)$ by

$$(\Phi_{\alpha,\beta}f)(x,y) := f(\xi_{\alpha,\beta}(x),y) \quad \text{for any } f\in L^2(\Omega). \tag{2.7}$$

The following technical lemma will be proved in the Appendix:

Lemma 2.1 *The map $(\alpha,\beta,f)\mapsto \Phi_{\alpha,\beta}f$ is continuous from $D\times H$ into H, and for any $q>1$ it is C^1-smooth from $D\times W^{1,q}(\Omega)$ into $L^q(\Omega)$. Similarly for the families of inverse operators $\Phi_{\alpha,\beta}^{-1}$ and adjoint operators $\Phi_{\alpha,\beta}^*$.*

In order to calculate the $L^2(\Omega)$-adjoint operators $\Phi_{\alpha,\beta}^*$ we use change of integration variable $\bar{x}=\xi_{\alpha,\beta}(x)$ and get

$$\int_\Omega (\Phi_{\alpha,\beta}^* f)(x,y)\varphi(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x,y)\varphi(\xi_{\alpha,\beta}(x),y)\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f(\xi_{\alpha,\beta}^{-1}(x),y)\varphi(x,y)}{\xi_{\alpha,\beta}'(\xi_{\alpha,\beta}^{-1}(x))}\,\mathrm{d}x\,\mathrm{d}y,$$

i.e.

$$(\Phi_{\alpha,\beta}^* f)(x,y) = \frac{f\left(\xi_{\alpha,\beta}^{-1}(x),y\right)}{\xi_{\alpha,\beta}'(\xi_{\alpha,\beta}^{-1}(x))}. \tag{2.8}$$

In order to calculate the transformed differential operators we calculate $\partial_x(\Phi_{\alpha,\beta}u)(x,y) = \partial_x u(\xi_{\alpha,\beta}(x),y)\xi_{\alpha,\beta}'(x)$ and $\partial_{xx}^2(\Phi_{\alpha,\beta}u)(x,y) = \partial_{xx}^2 u(\xi_{\alpha,\beta}(x),y)\left(\xi_{\alpha,\beta}'(x)\right)^2 + \partial_x u(\xi_{\alpha,\beta}(x),y)\xi_{\alpha,\beta}''(x)$. Denoting

$$\Delta_{\alpha,\beta} := \partial_x\left(\xi_{\alpha,\beta}'\left(\xi_{\alpha,\beta}^{-1}(x)\right)\partial_x\right) + \frac{\partial_{yy}^2}{\xi_{\alpha,\beta}'(\xi_{\alpha,\beta}^{-1}(x))},\ \nabla_{\alpha,\beta} := \left(\sqrt{\xi_{\alpha,\beta}'\left(\xi_{\alpha,\beta}^{-1}(x)\right)}\partial_x, \frac{\partial_y}{\sqrt{\xi_{\alpha,\beta}'\left(\xi_{\alpha,\beta}^{-1}(x)\right)}}\right),$$

we get $\Delta_{\alpha,\beta} = \Phi_{\alpha,\beta}^*\Delta\Phi_{\alpha,\beta}$ and

$$\int_\Omega \nabla_{\alpha,\beta}u\cdot\nabla_{\alpha,\beta}v\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \nabla\Phi_{\alpha,\beta}u\cdot\nabla\Phi_{\alpha,\beta}v\,\mathrm{d}x\,\mathrm{d}y \tag{2.9}$$

for $u\in W^{1,p}(\Omega)$, $v\in W^{1,q}(\Omega)$ and $p,q>1$ with $1/p+1/q=1$. Moreover, it follows from (2.6) that

$$\Delta_{\alpha,\beta}=\Delta,\ \nabla_{\alpha,\beta}=\nabla \text{ for } |x-\alpha_0|<\delta_0 \text{ or } |x-\beta_0|<\delta_0. \tag{2.10}$$

Next, let us choose a smooth cut-off function $\chi:[0,\infty)\to[0,1]$ such that

$$\chi(r)=1 \text{ for } 0\le r\le \delta_0/2,\ \chi(r)=0 \text{ for } r\ge\delta_0, \tag{2.11}$$

and let us define functions $X^{(-1/2)}, Y^{(-1/2)}:\Omega\to\mathbb{R}$ by

$$\begin{aligned} X^{(-1/2)}(\alpha_0+r\cos\omega, r\sin\omega) &:= \chi(r)r^{-1/2}\sin\tfrac{\omega}{2},\\ Y^{(-1/2)}(\beta_0+r\cos\omega, r\sin\omega) &:= \chi(r)r^{-1/2}\sin\tfrac{\omega}{2}. \end{aligned} \tag{2.12}$$

Here r is the distance of $(x,y) \in \Omega$ from $(\alpha_0, 0)$, ω is the angle measured anticlockwise from the segment $\overline{(x,y),(\alpha_0,0)}$ to I_{α_0,β_0} in the definition of $X^{(-1/2)}$, while r is the distance of $(x,y) \in \Omega$ from $(\beta_0, 0)$, ω is the angle measured clockwise from the segment $\overline{(x,y),(\beta_0,0)}$ to I_{α_0,β_0} in the definition of $Y^{(-1/2)}$. The following lemma (see [3], Lemma 2.4) states the main properties of the functions $X^{(-1/2)}$ and $Y^{(-1/2)}$:

Lemma 2.2 *(i) We have* $X^{(-1/2)}, Y^{(-1/2)} \in L^q(\Omega)$ *for all* $1 \le q < 4$, $X^{(-1/2)}, Y^{(-1/2)} \in W^{1,q}(\Omega)$ *for all* $1 \le q < \frac{4}{3}$ *and* $\Delta X^{(-1/2)}, \Delta Y^{(-1/2)} \in C^\infty(\overline{\Omega})$.

(ii) We have $X^{(-1/2)} = Y^{(-1/2)} = 0$ *on* $\Gamma_D \cup I_{\alpha_0,\beta_0}$ *and* $\partial_\nu X^{(-1/2)} = \partial_\nu Y^{(-1/2)} = 0$ *on* $\Gamma_N \cup E_{\alpha_0,\beta_0}$.

(iii) For any $(\alpha,\beta) \in D$ *it holds* $\nabla_{\alpha,\beta} X^{(-1/2)} = \nabla X^{(-1/2)}$, $\Delta_{\alpha,\beta} X^{(-1/2)} = \Delta X^{(-1/2)}$, $\nabla_{\alpha,\beta} Y^{(-1/2)} = \nabla Y^{(-1/2)}$, $\Delta_{\alpha,\beta} Y^{(-1/2)} = \Delta Y^{(-1/2)}$,

$$\left.\begin{array}{l} \int_\Omega \Delta_{\alpha,\beta} X^{(-1/2)} \varphi \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega \Delta \Phi_{\alpha,\beta} X^{(-1/2)} \Phi_{\alpha,\beta} \varphi \, \mathrm{d}x \, \mathrm{d}y, \\ \int_\Omega \Delta_{\alpha,\beta} Y^{(-1/2)} \varphi \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega \Delta \Phi_{\alpha,\beta} Y^{(-1/2)} \Phi_{\alpha,\beta} \varphi \, \mathrm{d}x \, \mathrm{d}y, \end{array}\right\} \text{ for all } \varphi \in L^1(\Omega) \tag{2.13}$$

and

$$\left.\begin{array}{r} \int_\Omega \Delta_{\alpha,\beta} X^{(-1/2)} \varphi \, \mathrm{d}x \, \mathrm{d}y = -\int_\Omega \nabla_{\alpha,\beta} X^{(-1/2)} \cdot \nabla_{\alpha,\beta} \varphi \, \mathrm{d}x \, \mathrm{d}y, \\ \int_\Omega \Delta_{\alpha,\beta} Y^{(-1/2)} \varphi \, \mathrm{d}x \, \mathrm{d}y = -\int_\Omega \nabla_{\alpha,\beta} Y^{(-1/2)} \cdot \nabla_{\alpha,\beta} \varphi \, \mathrm{d}x \, \mathrm{d}y, \\ \text{for all } \varphi \in W^{1,p}(\Omega) \text{ with } \varphi = 0 \text{ on } \Gamma_D \cup I_{\alpha_0,\beta_0} \text{ and } p > 4. \end{array}\right\} \tag{2.14}$$

Define a closed subspace $H_0 \subset H$ by $H_0 := \{u \in H : u = 0 \text{ on } I_{\alpha_0,\beta_0}\}$. Further, for $(\alpha,\beta) \in D$ let $X_{\alpha,\beta}, Y_{\alpha,\beta} \in H_0$ be defined by

$$\begin{aligned} &\int_\Omega \left(\nabla_{\alpha,\beta} X_{\alpha,\beta} \cdot \nabla_{\alpha,\beta} \varphi - \Delta X^{(-1/2)} \varphi\right) \mathrm{d}x \, \mathrm{d}y = \\ &\quad = \int_\Omega \left(\nabla_{\alpha,\beta} Y_{\alpha,\beta} \cdot \nabla_{\alpha,\beta} \varphi - \Delta Y^{(-1/2)} \varphi\right) \mathrm{d}x \, \mathrm{d}y = 0 \text{ for all } \varphi \in H_0. \end{aligned} \tag{2.15}$$

In other words, $X_{\alpha,\beta}$ and $Y_{\alpha,\beta}$ are the weak solutions to the mixed boundary value problems

$$\begin{aligned} -\Delta_{\alpha,\beta} u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D \cup I_{\alpha_0,\beta_0}, \\ \partial_\nu u &= 0 \quad \text{on } \Gamma_N \cup E_{\alpha_0,\beta_0} \end{aligned}$$

with $f = \Delta X^{(-1/2)}$ and $f = \Delta Y^{(-1/2)}$, respectively. Standard results about smooth dependence of weak solutions to linear elliptic boundary value problems on the coefficients yield that the map $(\alpha,\beta) \in D \mapsto X_{\alpha,\beta} \in W^{1,2}(\Omega)$ is C^∞-smooth. Finally, denote

$$\overline{X}_{\alpha,\beta} := X_{\alpha,\beta} + X^{(-1/2)}, \quad \overline{Y}_{\alpha,\beta} := Y_{\alpha,\beta} + Y^{(-1/2)}, \tag{2.16}$$

and

$$\begin{aligned}
a_{11} &:= \int_\Omega \left(\overline{X}_{\alpha_0,\beta_0}\partial_x u_0 \partial_\alpha \xi_{\alpha_0,\beta_0} + u_0\left(\partial_\alpha X_{\alpha_0,\beta_0} + \overline{X}_{\alpha_0,\beta_0}\partial_\alpha \xi'_{\alpha_0,\beta_0}\right)\right) \mathrm{d}x\,\mathrm{d}y,\\
a_{12} &:= \int_\Omega \left(\overline{X}_{\alpha_0,\beta_0}\partial_x u_0 \partial_\beta \xi_{\alpha_0,\beta_0} + u_0\left(\partial_\beta X_{\alpha_0,\beta_0} + \overline{X}_{\alpha_0,\beta_0}\partial_\beta \xi'_{\alpha_0,\beta_0}\right)\right) \mathrm{d}x\,\mathrm{d}y,\\
a_{21} &:= \int_\Omega \left(\overline{Y}_{\alpha_0,\beta_0}\partial_x u_0 \partial_\alpha \xi_{\alpha_0,\beta_0} + u_0\left(\partial_\alpha Y_{\alpha_0,\beta_0} + \overline{Y}_{\alpha_0,\beta_0}\partial_\alpha \xi'_{\alpha_0,\beta_0}\right)\right) \mathrm{d}x\,\mathrm{d}y,\\
a_{22} &:= \int_\Omega \left(\overline{Y}_{\alpha_0,\beta_0}\partial_x u_0 \partial_\beta \xi_{\alpha_0,\beta_0} + u_0\left(\partial_\beta Y_{\alpha_0,\beta_0} + \overline{Y}_{\alpha_0,\beta_0}\partial_\beta \xi'_{\alpha_0,\beta_0}\right)\right) \mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{2.17}$$

Our main result is the following

Theorem 2.3 *Let (λ_0, u_0) be a solution to (2.2) with $\|u_0\| = 1$ and (1.7) such that there exists $d > 0$ with*

$$\partial_y u_0 > 0 \text{ on } I_{\hat\alpha(s),\hat\beta(s)} \cup ((0,1)\times(0,d)), \tag{2.18}$$

that

$$\lambda_0 \text{ is simple as an eigenvalue of the BVP (1.6), (1.2), (2.3) with } (\alpha,\beta) = (\alpha_0,\beta_0), \tag{2.19}$$

and that

$$\det\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \neq 0. \tag{2.20}$$

Further, assume that g is C^1-smooth and (1.4), (1.5) hold.

Then there exist $s_0 > 0$ and mappings $\hat\lambda, \hat\alpha, \hat\beta : [0, s_0) \to \mathbb{R}$ and $\hat u : [0, s_0) \to H$ with $\hat\lambda(0) = \lambda_0$, $\hat u(0) = 0$, $\hat\alpha(0) = \alpha_0$ and $\hat\beta(0) = \beta_0$ such that the following holds:

(i) For all $s \in (0, s_0)$ the pair $(\lambda, u) = \left(\hat\lambda(s), \hat u(s)\right)$ is a solution to (2.1) with $\mathcal{A}(\hat u(s)) = \left[\hat\alpha(s), \hat\beta(s)\right]$, and $\hat u(s) \in W^{2,p}(\Omega)$ for all $p \geq 2$, and there exists $\varepsilon > 0$ such that

$$\partial_y \hat u(s) > 0 \text{ on } I_{\hat\alpha(s),\hat\beta(s)} \cup ((0,1)\times(0,\varepsilon)).$$

(ii) There exists a C^1-smooth map $\hat v : [0, s_0) \to H$ such that $\hat v(0) = 0$ and

$$\hat u(s) = s\Phi_{\hat\alpha(s),\hat\beta(s)}\left(u_0 + \hat v(s)\right) \text{ for all } s \in (0, s_0).$$

(iii) The functions $\hat\lambda$, $\hat\alpha$, $\hat\beta$ are C^1-smooth from $[0, s_0)$ into $\mathbb{R}$ and the map $\hat u$ is continuous from $[0, s_0)$ into H and C^1-smooth from $[0, s_0)$ into $L^2(\Omega)$. If, moreover, g is C^k-smooth with some $k \geq 2$ and

$$|\partial_u^j g(\lambda, u)| \leq C(1 + |u|^q) \text{ for all } j = 1, \dots k,\ \lambda \in \mathbb{R}^+,\ u \in \mathbb{R}, \tag{2.21}$$

then $\hat\lambda$, $\hat\alpha$, $\hat\beta$ and $\hat v$ are C^k-smooth.

(iv) There exists $\eta > 0$ such that for any solution $(\lambda, u) \in \mathbb{R} \times (H \setminus \{0\})$ to (2.1) with $u \in W^{2,2}(\Omega)$ and $|\lambda - \lambda_0| + \|u\| + \left\|\frac{u}{\|u\|} - u_0\right\| < \eta$ there exists $s \in (0, s_0)$ with $u = \hat u(s)$ and $\lambda = \hat\lambda(s)$.

Let us comment on the main assumptions of Theorem 2.3.

The conditions (2.19) and (2.20) are generically fulfilled. They have to be verified numerically.

One could ask why we don't need any simplicity or isolatedness assumption concerning λ_0 as an eigenvalue of (2.2). The eigenvalues of a variational inequality need not be isolated and they can cover even an interval in general (even in a finite dimensional case, see [20]). Moreover, an eigenvalue of a variational inequality can have more than one normalized eigenvector, and also those normalized eigenvectors can be isolated or not. The next corollary shows that under the assumptions of Theorem 2.3 a certain isolatedness property of (λ_0, u_0) as as eigenpair to (2.2) is necessarily satisfied:

Corollary 2.4 *Let (λ_0, u_0) be a solution to (2.2) with $\|u_0\| = 1$, (1.7), (2.19), (2.20) and (2.18). Then there exists $\eta > 0$ such that there is no couple (λ, u) satisfying (2.2), $\|u\| \neq 0$, $u \in W^{2,2}(\Omega)$ and $0 < |\lambda - \lambda_0| + \|\frac{u}{\|u\|} - u_0\| < \eta$.*

Example 2.5 *Let us consider the eigenvalue problem (1.6) with the boundary conditions*

$$u = 0 \text{ on } \Gamma_D, \ \partial_\nu u = 0 \text{ on } \Gamma_N \cup \Gamma_U. \tag{2.22}$$

The eigenvalues and eigenfunctions of this problem are

$$\lambda_{m,n} = (m\pi)^2 + (\frac{n\pi}{\ell})^2, \ u_{m,n}(x,y) = \sin m\pi x \cdot \cos \frac{n\pi}{\ell} y, \ m = 1, 2, \dots, \ n = 0, 1, 2, \dots,$$

respectively. If $\ell < 2^{3/2}$ then the first four eigenvalues are $\lambda_{1,0} < \lambda_{2,0} < \lambda_{3,0} < \lambda_{1,1}$. Let us assume that $1/3 < \gamma_1 < \gamma_2 < 2/3$. Then

$$u_{3,0} > 0, \ u_{1,1} > 0 \text{ on } \Gamma_U. \tag{2.23}$$

It follows from [20] that there exists an eigenvalue λ_0 of the variational inequality (2.2) between any two eigenvalues of the problem (1.6), (2.22) having eigenfunctions which are positive on $\overline{\Gamma_U}$. Numerical simulation (see Fig. 2) shows that the corresponding eigenfunction u_0 fulfills the condition (2.18). The method developed in [14] shows that there is at least one eigenvalue $\lambda_0 \in (\lambda_{3,0}, \lambda_{1,1})$ of (2.2) with the corresponding eigenvector which arises by a certain deformation of $u_{3,0}$ (which is pressed down on Γ_U in a certain way) such that the resulting function u_0 satisfies (2.18). (Variational inequalities on cones K with nonempty interiors are considered in [14] but the method works also for general cones, the interior being replaced by a pseudo-interior considered in [20], cf. [15]). This method can show the existence of couples λ_0, u_0 satisfying the assumption (2.18) and lying also between other particular couples of eigenvalues $\lambda_{m,n}, \lambda_{k,l}$.

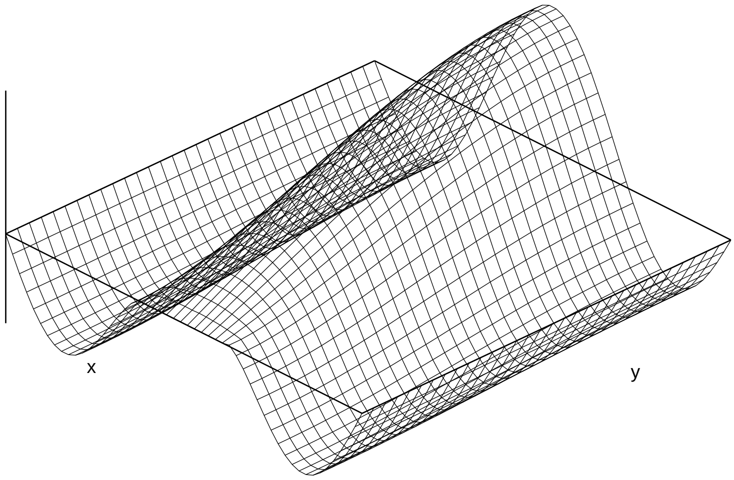

Figure 2: The eigenfunction u_0 with $\ell = 0.27$, $\lambda_0 = 99.8$, $\alpha_0 = 0.38$ and $\beta_0 = 0.62$.

3 Equivalence of the Variational Inequality to an Operator Equation

One of the grounding of the proof of our main result is a certain equivalence of our variational inequality (2.1) in a neighbourhood of a bifurcation point $(\lambda_0, 0)$ to an operator equation. In order to formulate this result, let us define a mapping $F : \mathbb{R} \times H \to H$ by

$$\langle F(\lambda, u), \varphi \rangle := -\int_\Omega \nabla u \nabla \varphi - [\lambda u + g(\lambda, u)]\varphi \, \mathrm{d}x \, \mathrm{d}y \quad \text{for all } \varphi \in H. \tag{3.1}$$

For $(\alpha, \beta) \in D$ let us denote

$$H_{\alpha,\beta} := \{u \in H \, : \, u = 0 \text{ on } I_{\alpha,\beta}\},$$

$$H_0 := H_{\alpha_0,\beta_0} = \{u \in H \, : \, u = 0 \text{ in } I_{\alpha_0,\beta_0}\}, \tag{3.2}$$

$$H_1 := \operatorname{im} \partial_u F(\lambda_0, 0). \tag{3.3}$$

In particular, we have

$$\int_\Omega \nabla u_0 \nabla \varphi - \lambda_0 u_0 \varphi \, \mathrm{d}x \, \mathrm{d}y = 0 \quad \text{for all } \varphi \in H_0. \tag{3.4}$$

We use the notation from Section 2. Moreover, let us define

$$v_{\alpha,\beta} := \Phi_{\alpha,\beta}(X_{\alpha,\beta} + X^{(-1/2)}), \quad w_{\alpha,\beta} := \Phi_{\alpha,\beta}(Y_{\alpha,\beta} + Y^{(-1/2)}). \tag{3.5}$$

Theorem 3.1 *Let the couple* (λ_0, u_0) *satisfy (2.2), (1.7) and (2.18) with some* $d \in (0, \ell)$. *Further, assume (1.4), (1.5). Then the following assertions hold.*

(i) *For any* $\eta > 0$ *there exists* $\zeta > 0$ *such that if* $(s, \lambda, v, \alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H_1 \times D$ *satisfies*

$$\langle F(\lambda, s\Phi_{\alpha,\beta}(u_0 + v)), \Phi_{\alpha,\beta}\varphi\rangle = 0 \quad \text{for any } \varphi \in H_0, \tag{3.6}$$

$$\begin{aligned} &\int_\Omega [\lambda s\Phi_{\alpha,\beta}(u_0 + v) + g(\lambda, s\Phi_{\alpha,\beta}(u_0 + v))]v_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y = 0, \\ &\int_\Omega [\lambda s\Phi_{\alpha,\beta}(u_0 + v) + g(\lambda, s\Phi_{\alpha,\beta}(u_0 + v))]w_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y = 0 \end{aligned} \tag{3.7}$$

and $s + \|v\| + |\lambda - \lambda_0| + |\alpha - \alpha_0| + |\beta - \beta_0| < \zeta$, *then the couple* (λ, u) *with*

$$u = s\Phi_{\alpha,\beta}(u_0 + v) \tag{3.8}$$

satisfies (2.1), $\mathcal{A}(u) = [\alpha, \beta]$, $\|u\| + \|\frac{u}{\|u\|} - u_0\| < \eta$ *and* $u \in W^{2,p}(\Omega)$ *for all* $p > 1$.

(ii) *For any* $\varepsilon > 0$ *there exists* $\eta > 0$ *such that for any couple* $(\lambda, u) \in \mathbb{R}^+ \times (H \setminus \{0\})$ *satisfying (2.1),* $u \in W^{2,2}(\Omega)$ *and* $\|u\| + \|\frac{u}{\|u\|} - u_0\| + |\lambda - \lambda_0| < \eta$ *there exists* $(s, v, \alpha, \beta) \in \mathbb{R}^+ \times H_1 \times D$ *satisfying (3.6), (3.7), (3.8),* $s + \|v\| + |\alpha - \alpha_0| + |\beta - \beta_0| < \varepsilon$, $\mathcal{A}(u) = [\alpha, \beta]$.

For the proof we need additional notation and regularity results.

For any fixed $(\alpha, \beta) \in D$, let us define functions $X^{(1/2)}_\alpha$ and $Y^{(1/2)}_\beta$ by

$$\begin{aligned} X^{(1/2)}_\alpha(\alpha + r\cos\omega, r\sin\omega) &:= \chi(r)r^{1/2}\sin\tfrac{\omega}{2}, \\ Y^{(1/2)}_\beta(\beta + r\cos\omega, r\sin\omega) &:= \chi(r)r^{1/2}\sin\tfrac{\omega}{2}, \end{aligned}$$

where the function χ is from (3.23) and r, ω are as in the definition of $X^{(-1/2)}$, $Y^{(-1/2)}$ in (2.12).

Observation 3.2 *Let us consider an arbitrary given* $r > 1$. *We work in the two-dimensional case and therefore* $W^{1,2}(\Omega) \subset L_p(\Omega)$ *for all* $p > 1$. *Let us choose* $p = qr$ *with* q *from (1.5). Then the mapping* $(\lambda, u) \mapsto g(\lambda, u)$ *is continuous as a map of* $\mathbb{R}^+ \times L^p(\Omega)$ *into* $\in L^r(\Omega)$ *under the assumption (1.5) due to the theorems about Nemyckii operator (see e.g. [22, Proposition 26.6]). Consequently, it is continuous as a map of* $\mathbb{R}^+ \times H$ *into any* $L^r(\Omega)$, $r > 1$. *The same holds for the mapping* $(\lambda, u) \mapsto \partial_u g(\lambda, u)$, *and even for the mapping* $(\lambda, u) \mapsto \partial^{(j)}_u g(\lambda, u)$ $(j = 1, 2, ..., k)$ *if* g *is* C^k*-smooth and (2.21) is fulfilled with some* $k > 1$.

Remark 3.3 *Let* u *be a weak solution of the mixed boundary value problem*

$$-\Delta u = f \text{ in } \Omega \tag{3.9}$$

with (1.2), (2.3), where $f \in L_p(\Omega)$, $p > 1$ *is such that* $\frac{2}{p'} + \frac{1}{2}$ *(with* $\frac{1}{p} + \frac{1}{p'} = 1$*) is non integer (which is true if* $p \neq 4$*). It follows from [8, Theorem 2] that we can write* u *as*

$$u = \widetilde{u} + K^1_{\alpha,\beta}(f)X^{(1/2)}_\alpha + K^2_{\alpha,\beta}(f)Y^{(1/2)}_\beta, \tag{3.10}$$

where $\widetilde{u} \in W^{2,p}(\Omega)$ *and* $K^1_{\alpha,\beta}(f), K^2_{\alpha,\beta}(f) \in \mathbb{R}$*. The coefficients* $K^1_{\alpha,\beta}(f), K^2_{\alpha,\beta}(f)$ *are independent of the choice of* δ *and cut-off function* χ *because* $X^{(1/2)}_\alpha$ *and* $Y^{(1/2)}_\beta$ *are independent of this choice in* $B_{\delta/2}(\alpha)$ *and* $B_{\delta/2}(\beta)$*, respectively.*

Let us emphasize that the functions $X^{(1/2)}_\alpha$*,* $Y^{(1/2)}_\beta$ *belong neither to* $W^{2,2}(\Omega)$ *nor to* $C^1(\overline{\Omega})$*, because of the singularity in the first derivatives at* $(\alpha, 0)$ *or* $(\beta, 0)$*, respectively. In particular,*

$$\partial_x X^{(1/2)}_\alpha(\alpha-, 0) = -\infty, \quad \partial_x Y^{(1/2)}_\beta(\beta+, 0) = +\infty. \tag{3.11}$$

It follows that in the case $p > 2$ *we have* $u \in C^1(\overline{\Omega})$ *if and only if* $u \in W^{2,2}(\Omega)$*, and this is true if and only if* $K^1_{\alpha,\beta}(f) = K^2_{\alpha,\beta}(f) = 0$*. In this case even* $u \in W^{2,r}(\Omega)$ *for all* $r > 1$*.*

Lemma 3.4 *Assume that (1.5) is fulfilled with some* $q > 1$*. Let* $(\lambda, u, \alpha, \beta) \in \mathbb{R} \times H \times D$*, let* u *be a weak solution of (1.1), (1.2), (2.3). Then (3.10) holds with* $f = \lambda u + g(\lambda, u)$ *(i.e. with* $K^1_{\alpha,\beta}(\lambda u + g(\lambda, u)), K^2_{\alpha,\beta}(\lambda u + g(\lambda, u))$ *and some* $\widetilde{u}$ *satisfying* $\widetilde{u} \in W^{2,r}(\Omega)$ *for all* $r > 1$*.*

Proof. Consider a given $r > 1$. We have $f := \lambda u + g(\lambda, u) \in L^r(\Omega)$ (see Observation 3.2), and Remark 3.3 used for this f implies that (3.10) holds with $\tilde{u} \in W^{2,r}(\Omega)$. ■

Remark 3.5 *Let* $(\lambda, u, \alpha, \beta) \in \mathbb{R} \times H \times D$*, let* u *be a weak solution of (3.9), (1.2) and (2.3). It follows from [19, Theorem 4.2.3] that we have for* $r \to 0$

$$\begin{aligned} u(\alpha + r\cos\omega, r\sin\omega) &= K^1_{\alpha,\beta}(f) X^{(1/2)}_\alpha(\alpha + r\cos\omega, r\sin\omega) + O(r^{3/2}), \\ u(\beta + r\cos\omega, r\sin\omega) &= K^2_{\alpha,\beta}(f) Y^{(1/2)}_\beta(\beta + r\cos\omega, r\sin\omega) + O(r^{3/2}), \end{aligned} \tag{3.12}$$

where $K^1_{\alpha,\beta}(f), K^2_{\alpha,\beta}(f) \in \mathbb{R}$ *are the coefficients from Remark 3.3.*

Lemma 3.6 *For any* $(\alpha, \beta) \in D$ *we have*

$$\begin{gathered} \textstyle\int_\Omega \nabla v_{\alpha,\beta} \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega \nabla w_{\alpha,\beta} \cdot \nabla\varphi \, \mathrm{d}x \, \mathrm{d}y = 0 \\ \textit{for all } \varphi \in W^{2,2}(\Omega) \textit{ with } \varphi = 0 \textit{ on } \Gamma_D \cup I_{\alpha,\beta}, \end{gathered} \tag{3.13}$$

$$K^1_{\alpha,\beta}(f) = \frac{2}{\pi} \int_\Omega f v_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y, \ \ K^2_{\alpha,\beta}(f) = \frac{2}{\pi} \int_\Omega f w_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y \textit{ for all } f \in L^2(\Omega). \tag{3.14}$$

Proof. In order to prove (3.13), let us take such φ. Due to the definition of $\Phi_{\alpha,\beta}$, $v_{\alpha,\beta}$, $\nabla_{\alpha,\beta}$, Lemma 2.2(iii), (iv) and the imbedding $W^{2,2}(\Omega) \subset W^{1,q}(\Omega)$ for all $2 \le q < \infty$ we get

$$\begin{aligned} \textstyle\int_\Omega \nabla\varphi \cdot \nabla v_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y &= \textstyle\int_\Omega \nabla \Phi_{\alpha,\beta} \Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla \Phi_{\alpha,\beta}(X_{\alpha,\beta} + X^{(-1/2)}) \, \mathrm{d}x \, \mathrm{d}y \\ &= \textstyle\int_\Omega \nabla_{\alpha,\beta} \Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla_{\alpha,\beta} X_{\alpha,\beta} - \Phi^{-1}_{\alpha,\beta}\varphi \Delta X^{(-1/2)} \, \mathrm{d}x \, \mathrm{d}y \end{aligned}$$

which is zero due to (2.15) with $\Phi^{-1}_{\alpha,\beta}\varphi$ instead of φ there (let us observe that we have $\Phi^{-1}_{\alpha,\beta}\varphi \in W^{2,2}(\Omega)$ and $\Phi^{-1}_{\alpha,\beta}\varphi = 0$ on $\Gamma_D \cup I_{\alpha_0,\beta_0}$. Similarly one can show that $\int_\Omega \nabla\varphi \cdot \nabla w_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y = 0$ and the assertion (3.13) is proved.

In order to prove (3.14), let us realize that it follows from (2.9), (2.13), from Lemma 2.2(iv) and (2.15) that

$$\begin{aligned}&\int_\Omega \nabla\Phi_{\alpha,\beta}X_{\alpha,\beta}\cdot\nabla\Phi_{\alpha,\beta}\varphi - \Delta\Phi_{\alpha,\beta}X^{(-1/2)}\Phi_{\alpha,\beta}\varphi \,\mathrm{d}x\,\mathrm{d}y\\ &\quad= \int_\Omega \nabla_{\alpha,\beta}X_{\alpha,\beta}\cdot\nabla_{\alpha,\beta}\varphi - \Delta_{\alpha,\beta}X^{(-1/2)}\varphi \,\mathrm{d}x\,\mathrm{d}y\\ &\quad= \int_\Omega \nabla_{\alpha,\beta}X_{\alpha,\beta}\cdot\nabla_{\alpha,\beta}\varphi - \Delta X^{(-1/2)}\varphi \,\mathrm{d}x\,\mathrm{d}y = 0\\ &\qquad\text{for all } \varphi\in W^{1,2}(\Omega) \text{ with } \varphi=0 \text{ on } \Gamma_D\cup I_{\alpha_0,\beta_0}.\end{aligned}$$

It follows that $\Phi_{\alpha,\beta}X_{\alpha,\beta}$ is a weak solution to

$$\begin{aligned}&-\Delta\Phi_{\alpha,\beta}X_{\alpha,\beta} = \Delta\Phi_{\alpha,\beta}X^{(-1/2)} \quad \text{in } \Omega,\\ &\Phi_{\alpha,\beta}X_{\alpha,\beta} = 0 \quad \text{on } \Gamma_D\cup I_{\alpha,\beta},\\ &\partial_\nu\Phi_{\alpha,\beta}X_{\alpha,\beta} = 0 \quad \text{on } \Gamma_N\cup E_{\alpha,\beta}.\end{aligned} \tag{3.15}$$

Remark 3.5 implies that we have

$$\begin{aligned}(\Phi_{\alpha,\beta}X_{\alpha,\beta})(\alpha+r\cos\omega, r\sin\omega) &= \tilde K^1_{\alpha,\beta}X^{(1/2)}_\alpha(\alpha+r\cos\omega, r\sin\omega)+O(r^{3/2})\\ &= \tilde K^1_{\alpha,\beta}r^{1/2}\sin\tfrac{\omega}{2}+O(r^{3/2}) \quad \text{for } r\to 0_+, \omega\in[0,\pi]\end{aligned} \tag{3.16}$$

with some $\tilde K^1_{\alpha,\beta}\in\mathbb{R}$.

Furthermore, with help of the property (2.6) it is easy to see from (3.23) and (2.12) that

$$\begin{aligned}(\Phi_{\alpha,\beta}X^{(-1/2)})(\alpha+r\cos\omega, r\sin\omega) = X^{(-1/2)}(\alpha_0+r\cos\omega, r\sin\omega) = r^{-1/2}\sin\tfrac{\omega}{2}\\ \text{for } r\in(0,\delta/2], \omega\in[0,\pi].\end{aligned} \tag{3.17}$$

Adding (3.16) and (3.17) we obtain for $v_{\alpha,\beta} = \Phi_{\alpha,\beta}(X^{(-1/2)}+X_{\alpha,\beta})$ the expansion

$$\begin{aligned}v_{\alpha,\beta}(\alpha+r\cos\omega, r\sin\omega) = r^{-1/2}\sin\tfrac{\omega}{2}+\tilde K^1_{\alpha,\beta}r^{1/2}\sin\tfrac{\omega}{2}+O(r^{3/2})\\ \text{for } r\to 0_+, \omega\in[0,\pi].\end{aligned} \tag{3.18}$$

Moreover, (3.15) and Lemma 2.2(ii) and (iii) imply that $v_{\alpha,\beta}$ satisfies

$$\begin{aligned}&\Delta v_{\alpha,\beta} = 0 \quad \text{in } \Omega,\\ &v_{\alpha,\beta} = 0 \quad \text{on } \Gamma_D\cup I_{\alpha,\beta},\\ &\partial_\nu v_{\alpha,\beta} = 0 \quad \text{on } \Gamma_N\cup E_{\alpha,\beta}.\end{aligned} \tag{3.19}$$

Let us emphasize that we have only $v_{\alpha,\beta}\in W^{1,q}(\Omega)$ for $1\le q<4/3$ and $v_{\alpha,\beta}$ is neither classical nor weak solution of the boundary value problem (3.19) (which has in $W^{1,2}(\Omega)$ only the trivial solution). Cf. [10, Lemma 2.3.1] and the text after it. We have $\Delta u = -f$ in Ω, u satisfies the boundary conditions (1.2), (2.3) and the expansion (3.12) holds by Remark 3.5. This together with (3.18) imply that we have

$$\begin{aligned}\partial_r u &= \tfrac12 K^1_{\alpha,\beta}(f)r^{-1/2}\sin\tfrac{\omega}{2}+O(r^{1/2}),\\ \partial_r v_{\alpha,\beta} &= -\tfrac12 r^{-3/2}\sin\tfrac{\omega}{2}+\tfrac12\tilde K^1_{\alpha,\beta}r^{-1/2}\sin\tfrac{\omega}{2}+O(r^{1/2}).\end{aligned} \tag{3.20}$$

Let us set $\Omega_\rho(\alpha) := \Omega \setminus B_\rho(\alpha)$, $\Gamma_\rho(\alpha) := \partial B_\rho(\alpha) \cap \Omega$ for $\rho \in (0, \delta/2)$. The Green formula gives

$$\int_{\Omega_\rho(\alpha)} \Delta u v_{\alpha,\beta} - u \Delta v_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y = \int_{\partial\Omega_\rho(\alpha)} \partial_\nu u v_{\alpha,\beta} - u \partial_\nu v_{\alpha,\beta} \, \mathrm{d}\Gamma. \tag{3.21}$$

By using (1.1), (1.2), (2.3), (3.19) and (3.12), (3.18), (3.20) with $r = \rho$ we obtain from (3.21) that

$$\begin{aligned}
&\textstyle\int_{\Omega_\rho(\alpha)} -f v_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y \\
&\quad = \textstyle\int_{\Gamma_\rho(\alpha)} \partial_\nu u v_{\alpha,\beta} - u \partial_\nu v_{\alpha,\beta} \, \mathrm{d}\Gamma \\
&\quad = \textstyle\int_{\Gamma_\rho(\alpha)} \partial_r u v_{\alpha,\beta} - u \partial_r v_{\alpha,\beta} \, \mathrm{d}\Gamma \\
&\quad = \rho \textstyle\int_0^\pi (\frac{1}{2} K^1_{\alpha,\beta}(f) \rho^{-1/2} \sin\frac{\omega}{2} + O(\rho^{1/2}))(\rho^{-1/2} \sin\frac{\omega}{2} + \tilde{K}^1_{\alpha,\beta} \rho^{1/2} \sin\frac{\omega}{2} + O(\rho^{3/2})) \\
&\qquad\quad - (K^1_{\alpha,\beta}(f) \rho^{1/2} \sin\frac{\omega}{2} + O(\rho^{3/2}))(-\frac{1}{2}\rho^{-3/2} \sin\frac{\omega}{2} + \frac{1}{2} \tilde{K}^1_{\alpha,\beta} \rho^{-1/2} \sin\frac{\omega}{2} + O(\rho^{1/2})) \, \mathrm{d}\omega \\
&\quad = K^1_{\alpha,\beta}(f) \textstyle\int_0^\pi \sin^2\frac{\omega}{2} \, \mathrm{d}\omega + O(\rho) = K^1_{\alpha,\beta}(f) \frac{\pi}{2} + O(\rho).
\end{aligned}$$

The limiting process for $\rho \to 0_+$ then gives

$$\int_\Omega -f v_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y = K^1_{\alpha,\beta}(f) \frac{\pi}{2}$$

and our assertion follows. Similarly for $K^2_{\alpha,\beta}(f)$ and the proof is done. ■

Proposition 3.7 *A point $(s, \lambda, v, \alpha, \beta) \in \mathbb{R}^2 \times H_1 \times D$ satisfies (3.6) if and only if u from (3.8) is a weak solution of the boundary value problem (1.1), (1.2), (2.3). In this case, $(s, \lambda, v, \alpha, \beta)$ satisfies (3.7) if and only if $u \in W^{2,2}(\Omega)$. Then also $u \in W^{2,r}(\Omega)$ for all $r > 1$.*

Proof. The statement about the equivalence of (3.6) with a weak formulation of (1.1), (1.2), (2.3) follows from standard considerations and the fact that $\Phi_{\alpha,\beta}$ is a one-to-one mapping of H_{α_0,β_0} onto $H_{\alpha,\beta}$. The equivalence of the $C^1(\overline{\Omega})$ regularity with the condition (3.7) follows from Remark 3.3 and the form (3.14) of $K^j_{\alpha,\beta}(\lambda u + g(\lambda, u))$, $j = 1, 2$ (Lemma 3.6). ■

Let us set

$$E(u) := \{(x, 0) \in \Gamma_U;\ u(x, 0) \neq 0\},\ I(u) := \Gamma_U \setminus \overline{E(u)} = \operatorname{int} \mathcal{A}(u)) \times \{0\}.$$

(the interior in Γ_U).

Proposition 3.8 *Let $\Omega' \subset \Omega$ be a sub-domain such that $\{(\gamma_1, 0), (\gamma_2, 0)\} \notin \overline{\Omega'}$. Then for any $(\lambda, u) \in \mathbb{R}^+ \times H$ satisfying (2.1) we have $u \in W^{2,2}(\Omega')$, the equation (1.1) is fulfilled a.e. in Ω and the boundary conditions (1.2), (1.3) hold in the sense of traces. If, moreover, $\overline{I(u)} \subset \Gamma_U$ then $u \in W^{2,2}(\Omega)$.*

In fact we will have even $u \in W^{2,p}(\Omega)$ with all $p > 2$ for our solutions.

Proof. follows by a combination of known results formulated usually only for particular boundary value problems (pure Dirichlet, pure Neuman and Signorini-Dirichlet). For the completeness we will explain it in more details.

First, we will show that a solution of (2.1) satisfies $u \in W^{2,2}(\Omega'_U)$ for any sub-domain $\Omega'_U \subset \Omega$ such that $\overline{\Omega'_U} \cap \partial\Omega$ is a closed segment in Γ_U. It is known ([9, Theorem 3.2.3.1]) that if Ω_U is a convex domain with a smooth boundary $\partial\Omega_U$, Γ_U is an open part of $\partial\Omega_U$, then for any $f \in L^2(\Omega_U)$ there is unique weak solution $w \in W^{2,2}(\Omega_U)$ of the problem $-\Delta w + w = f$ in Ω_U with the Signorini condition (1.3) on Γ_U and $w = 0$ on $\partial\Omega \setminus \Gamma_U$. For any Ω'_U mentioned there exists a larger sub-domain $\Omega_U \subset \Omega$ such that $\overline{\Omega_U} \cap \partial\Omega$ is a closed segment in Γ_U containing $\overline{\Omega'_U} \cap \partial\Omega$ in the interior. There is a smooth cut-off function $\chi_U : \Omega \to [0,1]$ such that

$$\chi_U = 0 \text{ in } \Omega \setminus \Omega_U, \quad \chi_U = 1 \text{ in } \overline{\Omega'_U}, \quad \partial_y \chi_U = 0 \text{ on } \Gamma_U. \tag{3.22}$$

We will show below that the function $w_U := \chi_U u$ satisfies the variational inequality

$$\begin{aligned} &w \in K_U := \{\varphi \in W^{1,2}(\Omega_U) : \ \varphi \le 0 \text{ on } \Gamma_U, \ \varphi = 0 \text{ on } \partial\Omega_U \setminus \Gamma_U\}, \\ &\textstyle\int_{\Omega_U} \nabla w \nabla(\varphi - w) + w(\varphi - w)\, \mathrm{d}x\, \mathrm{d}y \ge \int_{\Omega_U} f(\varphi - w)\, \mathrm{d}x\, \mathrm{d}y \text{ for all } \varphi \in K_U \end{aligned} \tag{3.23}$$

with

$$f = (\lambda u + g(\lambda, u)) \cdot \chi_U - 2\nabla u \cdot \nabla \chi_U - \Delta \chi_U \cdot u + w_U \in L^2(\Omega_U).$$

This is a weak formulation of $-\Delta w + w = f$ in Ω_U with (1.3) on Γ_U, $w = 0$ on $\partial\Omega \setminus \Gamma_U$, and therefore $w_U \in W^{2,2}(\Omega_U)$, which implies immediately $u \in W^{2,2}(\Omega'_U)$.

Hence, for the proof of $u \in W^{2,2}(\Omega'_U)$ it is sufficient to show (3.23). Standard considerations imply that if u is a solution of (2.1) then $\Delta u \in L^2(\Omega)$, consequently the normal derivative is defined as a functional on the space of traces $W^{1/2,2}(\partial\Omega)$ (see e.g. [17]) or on the whole H (see [2, Remark 5.2] for a brief self-contained explanation), and that the equation (1.1) is fulfilled a.e. in Ω, the boundary conditions (1.3) are fulfilled in the sense of the functional mentioned (cf. [2, Observation 5.2] for details). In particular, we have

$$\int_{\partial\Omega} \partial_\nu u \varphi \, \mathrm{d}\Gamma = \int_{\Gamma_U} \partial_\nu u \varphi \, \mathrm{d}\Gamma \le 0 \text{ for all } \varphi \in K, \quad \int_{\partial\Omega} \partial_\nu u u \, \mathrm{d}\Gamma = \int_{\Gamma_U} \partial_\nu u u \, \mathrm{d}\Gamma = 0,$$

where the integrals are understood in the sense of the value of the functional $\partial_\nu u$ at φ and u, respectively. Using these facts we get for all $\varphi \in K_U$

$$\begin{aligned} \textstyle\int_{\Omega_U} \nabla w_U \nabla \varphi + w_U \varphi \, \mathrm{d}x\, \mathrm{d}y &= \textstyle\int_{\Omega_U} [\chi_U \nabla u + u \nabla \chi_U] \nabla \varphi + w_U \varphi \, \mathrm{d}x\, \mathrm{d}y \\ &= \textstyle\int_{\Omega_U} \nabla u \nabla(\chi_U \varphi) - \nabla u \nabla \chi_U \cdot \varphi + u \nabla \chi_U \nabla \varphi + w_U \varphi \, \mathrm{d}x\, \mathrm{d}y \\ &\ge \textstyle\int_{\Omega_U} [\lambda u + g(\lambda, u)] \chi_U \varphi - \nabla u \nabla \chi_U \cdot \varphi + u \nabla \chi_U \nabla \varphi + w_U \varphi \, \mathrm{d}x\, \mathrm{d}y \\ &= \textstyle\int_{\Omega_U} [\lambda u + g(\lambda, u)] \chi_U \varphi - 2\nabla u \nabla \chi_U \cdot \varphi + \nabla \chi_U (\nabla u \varphi + u \nabla \varphi) + w_U \varphi \, \mathrm{d}x\, \mathrm{d}y \\ &= \textstyle\int_{\Omega_U} [\lambda u + g(\lambda, u)] \chi_U \varphi - 2\nabla u \nabla \chi_U \cdot \varphi - \Delta \chi_U \cdot u \varphi + w_U \varphi \, \mathrm{d}x\, \mathrm{d}y, \end{aligned}$$

$$\int_{\Omega_U} \nabla w_U \nabla w_U + w_U w_U \,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega_U} -\Delta(\chi_U u)\chi_U u + w_U w_U \,\mathrm{d}x\,\mathrm{d}y + \int_{\partial\Omega} \partial_\nu(\chi_U u)\chi_U u \,\mathrm{d}\Gamma$$
$$= \int_{\Omega_U} (-\Delta u)\chi_U^2 u - 2\nabla u \nabla \chi_U \cdot \chi_U u - \Delta(\chi_U)\chi_U u^2 + w_U w_U \,\mathrm{d}x\,\mathrm{d}y + \int_{\partial\Omega} \partial_\nu(\chi_U u)\chi_U u \,\mathrm{d}\Gamma,$$

which implies (3.23).

Further, we will show that a solution of (2.1) satisfies $u \in W^{2,2}(\Omega'_D)$ for any sub-domain $\Omega'_D \subset \Omega$ such that $\overline{\Omega'_D} \cap \partial\Omega \subset \Gamma_D$. It is known ([9, Theorem 3.2.1.2]) that if Ω_D is a domain with a smooth boundary $\partial\Omega_D$, then for any $f \in L^2(\Omega_D)$ there is unique weak solution $w \in W^{2,2}(\Omega_D)$ of the problem $\Delta w = f$ in Ω_D, $w = 0$ on $\partial\Omega_D$. For any Ω'_D mentioned there exist a larger sub-domain $\Omega_D \subset \Omega$ such that $\overline{\Omega_D} \cap \partial\Omega \subset \Gamma_D$ and a cut-off function $\chi_D : \Omega \to [0,1]$ such that

$$\chi_D = 0 \text{ in } \Omega \setminus \Omega_D, \quad \chi_D = 1 \text{ in } \overline{\Omega'_D}.$$

If u satisfies (2.1) then $w = \chi_D u$ is a weak solution of the problem $\Delta w = f$ in Ω_D, $w = 0$ on $\partial\Omega_D$ with a certain $f \in L^2(\Omega)$ and therefore $w \in W^{2,2}(\Omega_D)$, which implies $u \in W^{2,2}(\Omega'_D)$.

Similarly we can derive $u \in W^{2,2}(\Omega'_N)$ for any sub-domain $\Omega'_N \subset \Omega$ such that $\overline{\Omega'_N} \cap \partial\Omega \subset \Gamma_N$ from the known fact that the week solution of the problem $-\Delta w + w = f$ in Ω_N, $\partial_\nu w = 0$ on $\partial\Omega_N$ is in $W^{2,2}(\Omega_N)$ (see [9, Theorem 3.2.1.3]). In this case we take a cut-off function satisfying

$$\chi_N = 0 \text{ in } \Omega \setminus \Omega_N, \quad \chi_N = 1 \text{ in } \overline{\Omega'_N}, \quad \partial_y \chi_N = 0 \text{ on } \Gamma_N \cap \partial\Omega_N.$$

Since our Ω has only right angles, the solution of (2.1) (which satisfies $\Delta u \in L^2(\Omega)$, (1.1) a.e. in Ω and (1.2)) is $W^{2,2}(\Omega_C)$ also in a neighbourhood Ω_C of the corners.

Any sub-domain Ω' from the assumptions can be covered by the domains Ω'_U, Ω'_D, Ω_N and Ω_C mentioned above, and it follows $W^{2,2}(\Omega')$. If we have in addition $\overline{I(u)} \subset \Gamma_U$ then $\partial_\nu u = 0$ in a neighborhoods (in $\overline{\Gamma}_U \cup \Gamma_N$) of the points $(\gamma_1, 0)$, $(\gamma_2, 0)$, and these neighbourhoods can be covered by extending of the set Ω_N and Ω'_N, respectively. Hence, we obtain $u \in W^{2,2}(\Omega)$. ■

Lemma 3.9 *Let $(\lambda_0, u_0) \in \mathbb{R}^+ \times H$ satisfy (2.2), let Ω' be a sub-domain of Ω, $\overline{\Omega'} \subset \Omega \cap \Gamma_D$. Then $u_0 \in W^{2,r}(\Omega')$ and $u \in W^{2,r}(\Omega')$ for any $r > 1$ and any $(\lambda, u) \in \mathbb{R}^+ \times H$ satisfying the variational inequality (2.1) or the mixed boundary value problem (1.1), (1.2), (2.3). For any $r > 1$, $R > 0$ there are $C > 0$ and $\eta > 0$ such that if $(\lambda, u) \in \mathbb{R}^+ \times H$ with $|\lambda| \le R$, $0 < \|u\| < \eta$ satisfies (2.1) or (1.1), (1.2), (2.3), then*

$$\|\frac{u}{\|u\|} - u_0\|_{W^{2,r}(\Omega')} \le C\left(|\lambda - \lambda_0| + \|\frac{u}{\|u\|} - u_0\| + \|u\|\right), \tag{3.24}$$

$$\|\frac{u}{\|u\|} - u_0\|_{C^2(\overline{\Omega'})} \le C\left(|\lambda - \lambda_0| + \|\frac{u}{\|u\|} - u_0\| + \|u\|\right). \tag{3.25}$$

Proof. Let us consider a smooth sub-domain Ω'' of Ω' such that $\overline{\Omega'} \subset \Omega'' \cup \Gamma_D$, $\overline{\Omega'} \cap \Gamma_D \subset$ int $(\overline{\Omega''} \cap \Gamma_D)$ (the interior in Γ_D). If $r > 1$ and $b_j \in L^\infty(\Omega'')$ $(j = 1, 2)$ then there is $C > 0$ such that for any weak solution w of the problem

$$\Delta w + \sum_{j=1,2} b_j(x)\partial_{x_j} w = f \text{ in } \Omega'',\ w = 0 \text{ on } \partial\Omega'' \tag{3.26}$$

with $f \in L^r(\Omega'')$ we have $w \in W^{2,r}(\Omega')$ and

$$\|w\|_{W^{2,r}(\Omega')} \leq C(\|w\|_{L^r(\Omega'')} + \|f\|_{L^r(\Omega'')}) \tag{3.27}$$

(see e.g. [7, Theorems 9.15, 9.13]). Let $\chi_D : \Omega \to [0,1]$ be a smooth cut-off function such that

$$\chi_D = 0 \text{ in } \Omega \setminus \Omega'', \quad \chi_D = 1 \text{ in } \overline{\Omega'}.$$

It is easy to see that $\chi_D u$, $\chi_D u_0$ satisfy in the weak sense

$$\Delta(\chi_D u) - 2\nabla\chi_D\nabla u = \chi_D\Delta u + u\Delta\chi_D = \chi_D(\lambda u + g(\lambda,u)) + u\Delta\chi_D \text{ in } \Omega'',\ \chi_D u = 0 \text{ on } \partial\Omega'',$$

$$\Delta(\chi_D u_0) - 2\nabla\chi_D\nabla u_0 = \chi_D\Delta u_0 + u_0\Delta\chi_D = \lambda_0\chi_D u_0 + u_0\Delta\chi_D \text{ in } \Omega'',\ \chi_D u_0 = 0 \text{ on } \partial\Omega''.$$

Dividing the first equation by $\|u\|$, subtracting and defining $w = \frac{\chi_D u}{\|u\|} - u_0$, we get (3.26) with $b_j = -2\partial_{x_j}\chi_D$ and

$$f = (\lambda - \lambda_0)\chi_D u_0 + (\lambda\chi_D + \Delta\chi_D)\left(\frac{u}{\|u\|} - u_0\right) + \chi_D\frac{g(\lambda,u)}{\|u\|}.$$

We have $f \in L^r(\Omega'')$ for any $r > 1$ by Observation 3.2 and therefore (3.27) holds. It follows from (1.4) and (1.5) that $\frac{\|g(\lambda,u)\|_{L^r(\Omega)}}{\|u\|} \to 0$ if $\|u\| \to 0$ (see Lemma 6.2 in Appendix for details). In particular, there is $\eta > 0$ such that $\frac{\|g(\lambda,u)\|_{L^r(\Omega)}}{\|u\|} \leq C\|u\|$ for all $(\lambda,u) \in \mathbb{R}^+ \times H$ satisfying $\|u\| < \eta$. Due to the continuity of the embedding $H \subset L_r(\Omega)$ and the boundedness of λ under consideration we obtain (3.24) from (3.27).

The proof of (3.25) will be done in a similar way. Because of the embedding of $W^{2,r}(\Omega')$ into $C^{1,\gamma}(\overline{\Omega'})$ with certain γ, we obtain from (3.24) the estimate

$$\|\frac{u}{\|u\|} - u_0\|_{C^{1,\gamma}(\overline{\Omega'})} \leq C\left(|\lambda - \lambda_0| + \|\frac{u}{\|u\|} - u_0\| + \|u\|\right). \tag{3.28}$$

the function w defined above satisfy due to (3.26) the equation

$$\Delta w = f \text{ in } \Omega'',\ w = 0 \text{ on } \partial\Omega'' \tag{3.29}$$

now with

$$f = (\lambda - \lambda_0)\chi_D u_0 + (\lambda\chi_D + \Delta\chi_D)\left(\frac{u}{\|u\|} - u_0\right) + 2\nabla\chi_D\left(\frac{\nabla u}{\|u\|} - \nabla u_0\right) + \chi_D\frac{g(\lambda,u)}{\|u\|}$$

(i.e. the gradients of u and u_0 are now included into f). In order to use the estimate

$$\|w\|_{C^2(\overline{\Omega'})} \leq C(\|w\|_{C(\overline{\Omega'})} + \|f\|_{C^{0,\gamma}(\Omega')}) \tag{3.30}$$

from [7, Theorem 4.12] it remains because of (3.28) to realize that

$$\frac{\|g(\lambda,u)\|_{C^{0,\gamma}(\overline{\Omega'})}}{\|u\|} \le C\|u\|. \tag{3.31}$$

This becomes obvious by having in mind again the embedding of $W^{2,r}(\Omega')$ into $C^{1,\gamma}(\overline{\Omega'})$, Lemma 6.2 and Hölder inequality by which we obtain

$$\frac{\|g(\lambda,u)\|_{C^{0,\gamma}(\overline{\Omega'})}}{\|u\|} \le C\frac{\|g(\lambda,u)\|_{W^{2,r}(\Omega')}}{\|u\|} \le C\frac{\|\partial_u g(\lambda,u)\partial_x u + \partial_u g(\lambda,u)\partial_y u\|_{L^r(\Omega')}}{\|u\|} \le C\|u\|.$$

Now, (3.30) together with (3.28) give (3.25). ■

Remark 3.10 *For any $p > 2$ sufficiently close to 2 there is $R > 0$ such that all weak solutions of the mixed boundary value problem (3.9), (1.2), (2.3) with $f \in L^2(\Omega)$, $(\alpha,\beta) \in D$, satisfy*

$$\|u\|_{W^{1,p}(\Omega)} \le R\|f\|_{L^2(\Omega)}. \tag{3.32}$$

See e.g. [11, Theorem 1]. The problem (3.9), (1.2), (2.3) can be transformed by the transformation $\Phi_{\alpha,\beta}$ to a boundary value problem with fixed $\alpha = \alpha_0$, $\beta = \beta_0$ in (2.3) and with parameters α, β in the coefficient of the equation. (Cf. [3], proof of Lemma 3.7 for a concrete weak form of such equations.) The ellipticity coefficients of these equations are independent of α, β and it follows from the result mentioned that the constant C in (3.32) is independent of $(\alpha,\beta) \in D$.

The embedding theorems and the continuity of the Nemyckii operator (see Observation 3.2) imply that if u is small in H and λ close to λ_0 then $f = \lambda u + g(\lambda,u)$ is small in $L^2(\Omega)$. Hence, the estimate (3.32) guarantees that if $(\lambda,u) \in \mathbb{R}^+ \times H$ is a solution of (1.1), (1.2), (2.3) with $\|u\|$ small enough, λ close to λ_0 and $(\alpha,\beta) \in D$ then $\|u\|_{C(\overline{\Omega})}$ is small. In particular, $|\partial_u g(\lambda,u(x,y))| < \lambda$ for a. a. $(x,y) \in \Omega$, λ close to λ_0 and $\|u\|$ small, and therefore $\operatorname{sign}(\lambda v(x,y) + \partial_u g(\lambda,u(x,y))v(x,y)) = \operatorname{sign} v(x,y)$ *for all such (λ,u), any $v \in W^{1,2}(\Omega)$ and a.a. $(x,y) \in \Omega$. Furthermore, due to (1.4) and (3.32), for all $(\lambda,u) \in \mathbb{R}^+ \times H$ satisfying (1.1), (1.2), (2.3) with $\|u\|$ small enough we have $|g(\lambda,u(x,y))| < \lambda|u(x,y)|$, and consequently* $\operatorname{sign}(\lambda u(x,y) + g(\lambda,u(x,y))) = \operatorname{sign} u(x,y)$ *for a.a. $(x,y) \in \Omega$ with $u(x,y) \neq 0$.*

Observation 3.11 *Let Γ_0 be an open subset of Γ_U, let (λ_n,u_n) satisfy in a weak sense the equation (1.1) and the boundary condition*

$$\partial_\nu u = 0 \text{ on } \Gamma_0, \tag{3.33}$$

that means

$$\int_\Omega \nabla u_n \nabla\varphi - (\lambda u + g(\lambda,u))\varphi \, \mathrm{d}x\, \mathrm{d}y = 0 \textit{ for all } \varphi \in H \textit{ satisfying } \varphi = 0 \textit{ on } \partial\Omega \setminus \Gamma_0.$$

If $\|u_n\| \to 0$, $\|\frac{u_n}{\|u_n\|} - u_0\| \to 0$, $\lambda_n \to \lambda_0$ *then* $g(\lambda_n, u_n)/\|u_n\| \to 0$ *in* $L^2(\Omega)$ *(see Lemma 6.2) and we obtain by the limiting process (by using (1.4)) that*

$$\int_\Omega \nabla u_0 \nabla \varphi - \lambda_0 u_0 \varphi \, \mathrm{d}x \, \mathrm{d}y = 0 \text{ for all } \varphi \in H \text{ satisfying } \varphi = 0 \text{ on } \partial\Omega \setminus \Gamma_0,$$

i.e. (λ_0, u_0) *satisfies in the weak sense the equation (1.6) and the condition (3.33).*

In the following lemmas we consider automatically the assumptions of Theorem 3.1.

Lemma 3.12 *There exist* $\varepsilon_0 > 0$ *and* $\eta > 0$ *such that if* $(\lambda, u) \in \mathbb{R}^+ \times H$ *satisfies (2.1),* $u \in W^{2,2}(\Omega)$ *and*

$$\|u\| + \|\frac{u}{\|u\|} - u_0\| + |\lambda - \lambda_0| < \eta \tag{3.34}$$

then

$$\partial_y u \geq 0 \quad \text{in } \Omega_{\varepsilon_0}. \tag{3.35}$$

Lemma 3.13 *There exist* $\varepsilon_0 > 0$ *and* $\eta > 0$ *such that if* $(\lambda, u) \in \mathbb{R}^+ \times H$ *satisfies (1.1), (1.2), (2.3) with* $u \in W^{2,2}(\Omega)$, $(\alpha, \beta) \in D$ *and (3.34) then (3.35) holds.*

Lemma 3.14 *For any* $\varepsilon \in (0, \varepsilon_0)$ *with* ε_0 *from Lemma 3.12 there is* $\eta > 0$ *with the following property: if* $(\lambda, u) \in \mathbb{R}^+ \times H$ *satisfies (2.1), (3.34) and* $u \in W^{2,2}(\Omega)$ *then there exists* $(\alpha, \beta) \in D$, $|\alpha - \alpha_0| + |\beta - \beta_0| < \varepsilon$ *such that* $\mathcal{A}(u) = [\alpha, \beta]$, $u \in W^{2,p}(\Omega)$ *for all* $p > 1$ *and*

$$\partial_y u > 0 \quad \text{in } I_{\alpha,\beta} \cup \Omega_{\varepsilon_0}, \tag{3.36}$$

$$u < 0 \qquad \text{on } E_{\alpha,\beta}. \tag{3.37}$$

Lemma 3.15 *For any* $\varepsilon \in (0, \varepsilon_0)$ *with* ε_0 *from Lemma 3.12 there is* $\eta > 0$ *such that if* $(\lambda, u) \in \mathbb{R}^+ \times H$ *satisfies (1.1), (1.2), (2.3) with* $(\alpha, \beta) \in D$, *(3.34) and* $u \in W^{2,2}(\Omega)$ *then* $|\alpha - \alpha_0| + |\beta - \beta_0| < \varepsilon$, $\mathcal{A}(u) = [\alpha, \beta]$, $u \in W^{2,p}(\Omega)$ *for all* $p > 1$ *and (3.36), (3.37) are fulfilled.*

Proof of Lemma 3.12. Let us consider an arbitrary (λ, u) satisfying (2.1). Let us set $v := \partial_y u$. Due to our assumptions we have $v \in W^{1,2}(\Omega_d)$, consequently also $v^- \in W^{1,2}(\Omega_d)$, and (1.2), (1.3) imply

$$v = v^- = 0 \text{ on } \Gamma_N \cup E(u), \; u = \partial_x u = 0 \text{ on } I(u) \tag{3.38}$$

in the sense of traces. We will show below that there is $C > 0$ and $\tilde{\varepsilon} \in (0, d)$ with the following property. For any $\varepsilon \in (0, \tilde{\varepsilon})$ there is $\eta > 0$ such that if $(\lambda, u) \in \mathbb{R}^+ \times H$ satisfies (2.1) and (3.34), then

$$\|\nabla v^-\|_{L^2(\Omega_\varepsilon)} \leq C \|v^-\|_{L^2(\Omega_\varepsilon)} \tag{3.39}$$

(where C is independent of $\varepsilon \in (0, \tilde{\varepsilon})$). Let us assume for a moment that this is true. The Poincare inequality implies

$$\|v^-\|_{L^2(\Omega_\varepsilon)} \leq C(\Omega_\varepsilon)\|\nabla v^-\|_{L^2(\Omega_\varepsilon)} \text{ with } C(\Omega_\varepsilon) \to 0 \text{ for } \operatorname{meas} \Omega_\varepsilon \to 0$$

(see e.g. [7]). Clearly there is $\varepsilon_0 \in (0, \tilde{\varepsilon})$ such that the last two estimates can be fulfilled simultaneously for $\varepsilon \in (0, \varepsilon_0)$ only if v^- is identically zero (cf. [13, Theorem III.6.1] for the basic idea of the last trick). Hence, it must be $\partial_y u \geq 0$ in Ω_{ε_0}. Thus, for the proof of (3.35) it is sufficient to show the existence of C, $\tilde{\varepsilon}$ with the properties mentioned above.

Let $\varepsilon \in (0, d)$ be arbitrary. It is easy to see that the function $v = \partial_y u_0$ is a weak solution of the equation $\Delta v + \lambda_0 v = 0$. It follows from (2.18) that v attains its minimum over Ω_ε in all points $(0, y)$, $(1, y)$, $y \in (0, \varepsilon)$. The strong maximum principle implies that $\partial_x v(0, y) = \partial_{x,y} u_0(0, y) > 0 > \partial_x v(1, y) = \partial_{x,y} u_0(1, y)$ for all $y \in (0, \varepsilon)$. It follows by using (2.18) and Lemma 3.9 that if (λ, u) satisfies (2.1) and (3.34) with $\eta > 0$ small enough then

$$\partial_y u > 0 \text{ in } [0, 1] \times [\frac{\varepsilon}{2}, \varepsilon]. \tag{3.40}$$

Due to (3.40), (3.38) there exist smooth functions φ_n on Ω_ε such that

$$\begin{aligned} &\varphi_n \to v^- \text{ in } W^{1,2}(\Omega_\varepsilon), \\ &\varphi_n \text{ have compact supports in } [(0, 1) \times (0, \varepsilon/2)] \cup I(u), \\ &\partial_y \varphi_n = 0 \text{ on } I(u). \end{aligned} \tag{3.41}$$

(We can take first $\tilde{\varphi}_n$ with a compact support in $[(\gamma_1 + \theta, \gamma_2 - \theta) \times (0, \varepsilon/2 - 1/n)] \cup I(u)$ and define $\varphi_n(x, y) = \tilde{\varphi}_n(x, y - 1/n)$ for $y \in (1/n, \varepsilon/2)$, $\varphi_n(x, y) = \tilde{\varphi}_n(x, 0)$ for $y \in (0, 1/n)$. Then also the last condition in (3.41) is fulfilled.)

Let us show that

$$\int_{\Omega_\varepsilon} |\nabla(v^-)|^2 \, dx \, dy = -\int_{\Omega_\varepsilon} \nabla(v)\nabla(v^-) \, dx \, dy. \tag{3.42}$$

If $v \in H$ is smooth then the set $\{(x, y) \in \Omega_\varepsilon : v(x, y) < 0\}$ is open and v^- coincides with $-v$ on a neighbourhood of any its point, i.e. also derivatives of v^- coincide with those of $-v$. Both integrands are zero on $\{(x, y) \in \Omega_\varepsilon : v(x, y) > 0\}$. Further, $\nabla(v^-) = 0$ in the points where $\nabla(v) = 0$. Finally, the set $\{(x, y) \in \Omega_\varepsilon^0 : v(x, y) = 0, \nabla v \neq 0\}$ is of measure zero because in a neigbourhood of any its point it forms a smooth curve due to the implicit function theorem. Hence, (3.42) holds. For general $v \in H$ we get (3.42) via approximation by smooth functions.

Using the equality (3.42), integration by parts with respect to y, the Green formula and the equation (1.1) (see also Proposition 3.8) we get

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |\nabla(v^-)|^2 \,\mathrm{d}x\,\mathrm{d}y = -\lim_{n\to+\infty} \int_{\Omega_\varepsilon} \nabla(\partial_y u)\nabla\varphi_n \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \lim_{n\to+\infty} \left(\int_{\Omega_\varepsilon} \nabla u \nabla(\partial_y \varphi_n) \,\mathrm{d}x\,\mathrm{d}y - \int_0^1 \nabla u \nabla\varphi_n \,\mathrm{d}x \Big|_{y=0}^{y=\varepsilon} \right) \\
&\quad = \lim_{n\to+\infty} \left(-\int_{\Omega_\varepsilon} \Delta u \cdot \partial_y \varphi_n \,\mathrm{d}x\,\mathrm{d}y + \int_{\partial\Omega_\varepsilon} \partial_\nu u \partial_y \varphi_n \,\mathrm{d}\Gamma - \int_0^1 \nabla u \nabla\varphi_n \,\mathrm{d}x \Big|_{y=0}^{y=\varepsilon} \right) \\
&\quad = \lim_{n\to+\infty} \left(-\int_{\Omega_\varepsilon} \Delta u \partial_y \varphi_n \,\mathrm{d}x\,\mathrm{d}y \qquad + \int_{\partial\Omega_\varepsilon} \partial_\nu u \partial_y \varphi_n \,\mathrm{d}\Gamma - \int_0^1 \nabla u \nabla\varphi_n \,\mathrm{d}x \Big|_{y=0}^{y=\varepsilon} \right) \\
&\quad = \lim_{n\to+\infty} \left(\int_{\Omega_\varepsilon} (\lambda u + g(\lambda,u)) \partial_y \varphi_n \,\mathrm{d}x\,\mathrm{d}y \qquad + \int_{\partial\Omega_\varepsilon} \partial_\nu u \partial_y \varphi_n \,\mathrm{d}\Gamma - \int_0^1 \nabla u \nabla\varphi_n \,\mathrm{d}x \Big|_{y=0}^{y=\varepsilon} \right) \\
&\quad = \lim_{n\to+\infty} \left(-\int_{\Omega_\varepsilon} (\lambda v + \partial_u g(\lambda,u) v)\varphi_n \,\mathrm{d}x\,\mathrm{d}y) \right) \\
&\qquad + \int_0^1 (\lambda u + g(\lambda,u))\varphi_n \,\mathrm{d}x \Big|_{y=0}^{y=\varepsilon} \qquad + \int_{\partial\Omega_\varepsilon} \partial_\nu u \partial_y \varphi_n \,\mathrm{d}\Gamma - \int_0^1 \nabla u \nabla\varphi_n \,\mathrm{d}x \Big|_{y=0}^{y=\varepsilon} \Bigg).
\end{aligned}$$

Both integrals over the interval $(0,1)$ and that over $\partial\Omega_\varepsilon$ vanish for all n because of (3.41), (3.38) and (2.3). Hence, the limiting process gives

$$\int_{\Omega_\varepsilon} |\nabla(v^-)|^2 \,\mathrm{d}x\,\mathrm{d}y \le \int_{\Omega_\varepsilon} \left|(\lambda v + \partial_u g(\lambda,u)v)\cdot v^-\right| \,\mathrm{d}x\,\mathrm{d}y. \tag{3.43}$$

Due to Observation 3.2 there is $C > 0$ such that $\|\partial_u g(\lambda,u)\|_{L^4(\Omega_\varepsilon)} \le C$ for all (λ,u) satisfying (3.34) and

$$\int_{\Omega_\varepsilon} \left|\partial_u g(\lambda,u)v\cdot v^-\right| \,\mathrm{d}x\,\mathrm{d}y \le \|\partial_u g(\lambda,u)\|_{L^4(\Omega_\varepsilon)}\|v\|_{L^4(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)} \le C\|\nabla v\|_{L^2(\Omega_\varepsilon)}\|v\|_{L^2(\Omega_\varepsilon)}.$$

Dividing (3.43) by $\|\nabla v\|_{L^2(\Omega_\varepsilon)}$ we obtain (3.39).

As we explained on the begining, the proof of (3.35) is finished. ■

Proof of Lemma 3.13. is the same as that of Lemma 3.12.

Proof of Lemma 3.14. Let ε_0 and the corresponding η be from Lemma 3.12, that means (3.35) holds for all (λ,u) satisfying (2.1) with $(\alpha,\beta)\in D$ and (3.34).

First, let us show that if η is small enough and (λ,u) satisfies (2.1) and (3.34) then there are no $x_1,x_2\in(0,1)$ such that

$$u(x_1,0) = u(x_2,0) = 0,\ u(x,0) < 0 \text{ for all } x\in(x_1,x_2). \tag{3.44}$$

Let us assume by way of contradiction that there are such x_1,x_2. Hence, due to (3.35) we have

$$u(x_1,y)\ge 0,\ u(x_2,y)\ge 0 \text{ for all } y\in(0,\varepsilon_0). \tag{3.45}$$

Due to (2.18) and Lemma 3.9 we could take η simultaneously such that (3.40) holds for all solutions under consideration. Let $\varepsilon\in(0,\varepsilon_0)$ be arbitrary and let us consider the rectangle $\Omega_\varepsilon^x = (x_1,x_2)\times(0,\varepsilon)$. We have $u\in W^{2,2}(\Omega_\varepsilon^x)$ by the assumption. Hence, we can multiply (1.1) (holding a.e. in Ω) by u^-, integrate over Ω_ε^x and use the Green Formula. The boundary terms vanish because of $u^-(x_1,y) = u^-(x_2,y) = 0$ for all $y \in (0,\varepsilon)$ by (3.45), $u^-(x,\varepsilon) = 0$ for all $x \in (x_1,x_2)$ by (3.40) and $\partial_y u \cdot u^- = 0$ on Γ_U (see Proposition 3.8). Therefore we get

$$\int_{\Omega_\varepsilon^x} |\nabla u^-|^2 \,\mathrm{d}x\,\mathrm{d}y = -\int_{\Omega_\varepsilon^x} \nabla u \nabla u^- \,\mathrm{d}x\,\mathrm{d}y = -\int_{\Omega_\varepsilon^x} (\lambda u + g(\lambda,u))u^- \,\mathrm{d}x\,\mathrm{d}y.$$

Due to the assumption (1.4) there is $C>0$ such that $\|g(\lambda,u)\|_{L^2(\Omega_\varepsilon^x)} \le C\|u\|_{L^2(\Omega_\varepsilon^x)}$ if (3.34) is fulfilled with η small (see Lemma 6.2 in Appendix), and it follows that

$$\int_{\Omega_\varepsilon^x} |\nabla u^-|^2 \,\mathrm{d}x\,\mathrm{d}y \le C\|u^-\|^2_{L_2(\Omega_\varepsilon^x)}$$

with C independent of $\varepsilon \in (0,\varepsilon_0)$. Simultaneously

$$\int_{\Omega_\varepsilon^x} (u^-)^2 \,\mathrm{d}x\,\mathrm{d}y \le C(\Omega_\varepsilon^x)\int_{\Omega_\varepsilon^x} |\nabla(u^-)|^2 \,\mathrm{d}x\,\mathrm{d}y \text{ with } C(\Omega_\varepsilon^x) \to 0 \text{ for } meas(\Omega_\varepsilon^x) \to 0$$

(see e.g. [7]). If (3.44) holds then u^- is nontrivial and therefore the last two inequalities cannot be fulfilled for ε small. This is a contradiction and (3.44) is excluded for $x_1, x_2 \in (\gamma_1,\gamma_2)$ and all (λ,u) satisfying (2.1) and (3.34) with η from our considerations.

Let us show that if $\mu \in (0,\delta_0)$ then our $\eta > 0$ can be chosen so small that for all (λ,u) satisfying (2.1) and (3.34) we have

$$u(x,0) < 0 \text{ for all } x \in [\gamma_1+\mu, \alpha_0-\mu] \cup [\beta_0+\mu, \gamma_2-\mu]. \tag{3.46}$$

Under our assumptions about $\mathcal{A}(u_0)$ we have

$$u_0 < 0 \quad \text{ on } E_{\alpha_0,\beta_0} \cup \{(\gamma_1,0),(\gamma_2,0)\} \tag{3.47}$$

and therefore there exists $\varepsilon' \in (0,\varepsilon_0]$ such that

$$u_0(x,y) < 0 \text{ for all } (x,y) \in ([\gamma_1+\mu,\alpha_0-\mu] \cup [\beta_0+\mu,\gamma_2-\mu]) \times [0,\varepsilon'].$$

It follows from Lemma 3.9 that if (λ,u) satisfies (2.1) and (3.34) with η small enough then

$$u(x,\varepsilon') < 0 \text{ for all } x \in [\gamma_1+\mu,\alpha_0-\mu] \cup [\beta_0+\mu,\gamma_2-\mu].$$

If $u(x_0,0) \ge 0$ for some $x_0 \in [\gamma_1+\mu,\alpha_0-\mu] \cup [\beta_0+\mu,\gamma_2-\mu]$ then (3.35) implies $u(x_0,y) \ge 0$ for all $y \in (0,\varepsilon']$, which is the contradiction. Hence, (3.46) must hold.

Now we will prove that if $\eta > 0$ is small enough then for any (λ,u) under consideration

$$\text{there are } x_1,x_2 \in [\alpha_0,\beta_0] \text{ such that } x_1 < x_2,\ u(x,0) = 0 \text{ for all } x \in (x_1,x_2). \tag{3.48}$$

Indeed, otherwise due to the fact that (3.44) is already excluded, we see that we would have a sequence (λ_n,u_n) such that $\lambda_n \to \lambda_0$, $\|u_n\| \to 0$, $\|\frac{u_n}{\|u_n\|} - u_0\| \to 0$ with $u_n < 0$ on Γ_U maybe with the exception of one point $(x_n,0) \in \Gamma_U$, where $u_n(x_n,0)=0$. We would have $\partial_\nu u_n(x,0)=0$ on Γ_U (see also Proposition 3.8), and Observation 3.11 would imply that also $\partial_y u_0 = 0$ on Γ_0, which contradicts the assumption (2.18).

Since (3.48), (3.46) are proved and (3.44) is excluded for all solutions (λ, u) under consideration with η small, it is easy to see that for any such (λ, u) there are α, β with $\mathcal{A}(u) = [\alpha,\beta]$, $\alpha_0 - \mu < \alpha < \beta < \beta_0 + \mu$. We could choose $\mu < \varepsilon/2$. It remains to prove that $\alpha < \alpha_0 + \varepsilon/2$, $\beta > \beta_0 - \varepsilon/2$ if η is small enough. In the opposite case we would have (λ_n, u_n) such that $\lambda_n \to \lambda_0$, $\|u_n\| \to 0$, $\|\frac{u_n}{\|u_n\|} - u_0\| \to 0$ and satisfying (2.1) and the zero Neumann condition at least on $(\gamma_1, \alpha_0 + \varepsilon/2)$ or on $(\beta_0 - \varepsilon/2, \gamma_2)$. It would follow by using Observation 3.11 that also u_0 should satisfy zero Neumann boundary condition on $(\alpha_0, \alpha_0+\varepsilon/2)$ or on $(\beta_0 - \varepsilon/2, \beta_0)$, which is a contradiction with (2.18). Hence, it must be $|\alpha - \alpha_0| + |\beta - \beta_0| < \varepsilon$ if η is small enough.

In particular, we have $I(u) = I_{\alpha,\beta}$ and therefore u is a solution of the mixed boundary value problem (1.1), (1.2), (2.3). We assume that $u \in W^{2,2}(\Omega)$ and therefore we have even $u \in W^{2,p}(\Omega)$ for all $p > 1$ by Lemma 3.4 and Remark 3.3.

It remains to prove the inequality (3.36). It is easy to see that the function v is a weak solution of the equation

$$\Delta v = -\lambda v - \partial_u g(\lambda, u) v.$$

It follows from (3.35) and Remark 3.10 that the right hand side of the last equation is non-positive in Ω_{ε_0} for all (λ, u) under consideration with λ close to λ_0 if η is small enough. The function v is nontrivial by (3.40) and the Maximum Principle for weak solutions (see e.g. [7, Theorem 8.19]) implies that it cannot attain its minimum over $\overline{\Omega_{\varepsilon_0}}$ in the interior of Ω_{ε_0}. This minimum is zero because of (3.35) and $v = \partial_y u = 0$ on $E_{\alpha,\beta}$. Hence, $\partial_y u > 0$ in Ω_{ε_0}. The second derivatives are continuous up to the Dirichlet part of the boundary (see e.g. [7]) and we have $\partial_y v = \partial_{yy} u = -\partial_{xx} u - \lambda u - g(\lambda, u) = 0$ on $I_{\alpha,\beta}$. The Strong Maximum Principle implies that v can attain its minimum at no point $(x_0, 0) \in I_{\alpha,\beta}$ because it would follow that $\partial_y v(x_0,0) > 0$ (see e.g. [7, Lemma 3.4]), which would be a contradiction. Hence, (3.36) must hold.

The condition (3.37) follows from $u \in K$, $\mathcal{A}(u) = [\alpha, \beta]$, and the definitions of $E_{\alpha,\beta}$, $\mathcal{A}(u)$. ■

Proof of Lemma 3.15. Let ε_0 and $\eta > 0$ be from Lemma 3.13, that means (3.35) holds for any (λ, u) satisfying (1.1), (1.2), (2.3) and (3.34).

Let $\varepsilon \in (0, \varepsilon_0)$, $\mu \in (0, \delta_0)$ be given. Let as choose $\mu \in (0, \theta)$. In the same way as in the proof of Lemma 3.14 we can show that η can be chosen such that (3.46) and (3.40) are valid for all solutions under consideration. Due to the boundary conditions (2.3) we have $[\alpha, \beta] \subset \mathcal{A}(u)$. We can show in the same way as in the proof of Lemma 3.14 that (3.44) cannot occur. In contrast to Lemma 3.14, u need not be non-positive. Therefore in order to verify $\mathcal{A}(u) = [\alpha, \beta]$, besides excluding (3.44) we need also to show that there is no interval $[x_1, x_2] \subset [\alpha - \mu, \alpha] \cup [\beta, \beta + \mu]$ such that

$$u(x_1, 0) = u(x_2, 0) = 0,\ u(x, 0) \geq 0 \text{ for all } x \in (x_1, x_2). \tag{3.49}$$

Let us assume by contradiction that there are x_1, x_2 satisfying (3.49). First, let $x_2 \leq \alpha$. Since (3.44) is excluded and we have $u(x, 0) = 0$ for all $x \in [\alpha, \beta]$, it must be

$$u(x, 0) \geq 0 \text{ for all } x \in (x_2, x_2 + \tilde{\mu}) \tag{3.50}$$

with some $\tilde{\mu} > 0$. Due to (3.35), u is nonnegative on $\Omega_\varepsilon^x := [x_1, x_2 + \tilde{\mu}] \times [0, \varepsilon]$ and attains at $(x_2, 0)$ its zero minimum over this rectangle. The function u is not identically zero because of (3.40) and $\lambda u + g(\lambda, u) \geq 0$ in Ω_ε^x by Remark 3.10. We have $u \in C^1(\overline{\Omega})$ by our assumptions and the embedding theorem, and therefore the normal derivative at $(x_2, 0)$ exists. Hence, the Strong Maximum Principle implies that it should be $\partial_\nu u(x_1, 0) = -\partial_y u(x_1, 0) < 0$. However, $(x_1, 0) \in \overline{E_{\alpha,\beta}}$ and therefore $\partial_y u(x_1, 0) = 0$ by (2.3), which is a contradiction. The case $\beta \leq x_1$ can be treated symmetrically and (3.49) for x_1, x_2 discussed is excluded. Hence, $\mathcal{A}(u) = [\alpha, \beta]$ is proved.

It follows from (3.35), (3.46) that (λ, u) under consideration satisfies also (2.1) if η is small enough. Thus, the conditions (3.36), (3.37) and $|\alpha - \alpha_0| + |\beta - \beta_0| < \varepsilon$ follow from Lemma 3.14. ■

Proof of Theorem 3.1. To prove the assertion (i), let $\eta > 0$ be given. It follows from Lemma 2.1 that there exists $\zeta > 0$ such that for all $(s, \lambda, v, \alpha, \beta) \in \mathbb{R}^+ \times \mathbb{R}^+ \times H_1 \times D$ satisfying $s + \|v\| + |\lambda - \lambda_0| + |\alpha - \alpha_0| + |\beta - \beta_0| < \zeta$ the corresponding u defined by (3.8) satisfies $\|u\| + \|\frac{u}{\|u\|} - u_0\| < \eta$. If, moreover, $(s, \lambda, v, \alpha, \beta)$ satisfies (3.6), (3.7) then Proposition 3.7 implies that $u \in W^{2,p}(\Omega)$ for all $p > 1$ and that (λ, u) satisfies (1.1), (1.2), (2.3). Further, Lemma 3.15 implies that if ζ is chosen sufficiently small then we have $\mathcal{A}(u) = [\alpha, \beta]$ and (3.37) holds, hence $u \in K$. Finally, standard considerations (multiplication (1.1) by $\varphi - u$, integration over Ω and using Green's formula and boundary conditions (1.2), (2.3)) give that (λ, u) satisfies (2.1) and (i) is proved.

To prove (ii), let $\varepsilon > 0$ be given. Because of Lemma 3.14, for any $\varepsilon' \in (0, \varepsilon)$ there exists $\eta > 0$ such that for any $(\lambda, u) \in \mathbb{R}^+ \times W^{2,2}(\Omega)$ satisfying (2.1), $\|u\| \neq 0$ and (3.34) there exists $(\alpha, \beta) \in D$ such that $\mathcal{A}(u) = [\alpha, \beta]$, $|\alpha - \alpha_0| + |\beta - \beta_0| < \varepsilon'$ and (3.37) holds. In particular $\overline{I(u)} = [\alpha, \beta] \subset \Gamma_U$ and therefore (λ, u) satisfy(1.1), (1.2), (2.3). Define

$$s := \langle \Phi_{\alpha,\beta}^{-1} u, u_0 \rangle, \quad v := \frac{\Phi_{\alpha,\beta}^{-1} u}{s} - u_0. \tag{3.51}$$

Then (3.8) holds. Due to (2.5) and Lemma 2.1, if ε' and η are small then $\frac{s}{\|u\|} = \left\langle \Phi_{\alpha,\beta}^{-1}\left(\frac{u}{\|u\|}\right), u_0 \right\rangle$ is close to $\langle u_0, u_0 \rangle = 1$, $\Phi_{\alpha,\beta}^{-1}\left(\frac{u}{s}\right)$ is close to u_0. Hence, if ε_0' and η are sufficiently small then

$s+|\lambda-\lambda_0|+\|v\|+|\alpha-\alpha_0|+|\beta-\beta_0|<\varepsilon$. Remark 3.3 and Lemma 3.4 imply $u\in W^{2,p}(\Omega)$ for all $p>2$. Hence, Proposition 3.7 ensures that (3.6) and (3.7) are fulfilled. ■

4 Application of the Implicit Function Theorem: Continuation for the Operator Equation

In this section we will use the notation from Sections 2, 3. In particular, the transformations $\Phi_{\alpha,\beta}$ are from (2.7), the subspaces H_0 and H_1 are from (3.2) and (3.3), respectively, and the functions $v_{\alpha,\beta}$, $w_{\alpha,\beta}$ are from (3.5). Our goal is to describe the set of solutions to (3.6), (3.7) in a neighbourhood of $(\lambda_0,0,\alpha_0,\beta_0)$, which is done in the following theorem.

Theorem 4.1 *Let $(\lambda_0,u_0,\alpha_0,\beta_0)$ satisfy (2.2), (1.7), (2.19), and (2.20). Let g be C^1-smooth and let (1.4), (1.5) hold.*

Then there exist $s_0>0$, neighbourhoods $\Lambda_0\subset\mathbb{R}^+$ of λ_0, $W_0\subset H_0$ of the origin and $W_\alpha,W_\beta\subset\mathbb{R}$ of α_0 and β_0, respectively, and C^1-mappings $\hat\lambda:(-s_0,s_0)\to\Lambda_0$, $\hat v:(-s_0,s_0)\to W_0$, $\hat\alpha:(-s_0,s_0)\to W_\alpha$, $\hat\beta:(-s_0,s_0)\to W_\beta$ such that $\hat\lambda(0)=\lambda_0$, $\hat v(0)=0$, $\hat\alpha(0)=\alpha_0$, $\hat\beta(0)=\beta_0$ and that $(s,\lambda,v,\alpha,\beta)\in(-s_0,s_0)\times\Lambda_0\times W_0\times W_\alpha\times W_\beta$ satisfies (3.6), (3.7) if and only if $\lambda=\hat\lambda(s)$, $v=\hat v(s)$, $\alpha=\hat\alpha(s)$, $\beta=\hat\beta(s)$.

If g is C^k-smooth with some $k\geq 2$ and (2.21) holds then $\hat\lambda$, $\hat\alpha$, $\hat\beta$ and $\hat v$ are C^k-smooth.

Theorem 4.1 will be proved by means of the Implicit Function Theorem. Hence, we need C^1-smoothness of the operators involved.

Lemma 4.2 *The mapping F defined in (3.1) is C^1-smooth. If, moreover, the function g is C^k-smooth and (2.21) holds with some $k>1$ then F is even C^k-smooth.*

Proof follows by standard considerations using Observation 3.2. ■

Let us introduce the map $G:\mathbb{R}\times\mathbb{R}^+\times H_0\times D\to H_0\times\mathbb{R}^2$, $G=(G_1,G_2,G_3)$ by

$$\begin{aligned}
&G_1(s,\lambda,v,\alpha,\beta):=\tfrac{1}{s}\Phi^*_{\alpha,\beta}F(\lambda,s\Phi_{\alpha,\beta}(u_0+v))\text{ for }s\neq 0,\\
&G_1(0,\lambda,v,\alpha,\beta):=\Phi^*_{\alpha,\beta}\partial_uF(\lambda,0)\Phi_{\alpha,\beta}(u_0+v),\\
&G_2(s,\lambda,v,\alpha,\beta):=\textstyle\int_\Omega[\lambda\Phi_{\alpha,\beta}(u_0+v)+g(\lambda,s\Phi_{\alpha,\beta}(u_0+v))/s]v_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\text{ for }s\neq 0,\\
&G_2(0,\lambda,v,\alpha,\beta):=\textstyle\int_\Omega\lambda\Phi_{\alpha,\beta}(u_0+v)v_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y,\\
&G_3(s,\lambda,v,\alpha,\beta):=\textstyle\int_\Omega[\lambda\Phi_{\alpha,\beta}(u_0+v)+g(\lambda,s\Phi_{\alpha,\beta}(u_0+v))/s]w_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\text{ for }s\neq 0,\\
&G_3(0,\lambda,v,\alpha,\beta):=\textstyle\int_\Omega\lambda\Phi_{\alpha,\beta}(u_0+v)w_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{4.1}$$

Lemma 4.3 *The mapping G is C^1-smooth. If (2.20) holds then the partial derivative $\partial_{(\lambda,v,\alpha,\beta)}G(0,\lambda_0,0,\alpha_0,\beta_0)$ is an isomorphism from $\mathbb{R}\times H_1\times D$ onto $H_0\times D$ where H_0, H_1 are given by (3.2), (3.3). If g is C^k-smooth with $k>1$ and (2.21) holds then G is C^k-smooth.*

Proof. Because of (1.4) there is a continuous function $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ such that

$$h(\lambda, 0) = 0, \quad g(\lambda, u) = h(\lambda, u)u \quad \text{for all } (\lambda, u) \in \mathbb{R}^+ \times \mathbb{R}.$$

Introducing a new integration variable $\bar{x} = \xi_{\alpha,\beta}(x)$, renaming $\bar{x}$ again by x, and using (2.9), for any $\varphi \in H_0$ we get

$$\begin{aligned}
&\langle G_1(s, \lambda, v, \alpha, \beta), \varphi\rangle = \\
&\quad \textstyle\int_\Omega \nabla\Phi_{\alpha,\beta}(u_0 + v)\nabla\Phi_{\alpha,\beta}\varphi - [\lambda\Phi_{\alpha,\beta}(u_0 + v) + h(\lambda, s\Phi_{\alpha,\beta}(u_0 + v))\Phi_{\alpha,\beta}(u_0 + v)]\Phi_{\alpha,\beta}\varphi \, \mathrm{d}x \, \mathrm{d}y \\
&= \textstyle\int_\Omega \partial_x [u_0(x, y) + v(x, y)] \, \partial_x\varphi(x, y) \cdot \xi'_{\alpha,\beta}\left(\xi^{-1}_{\alpha,\beta}(x)\right) \\
&\quad +\partial_y [u_0(x, y) + v(x, y)] \, \partial_y\varphi(x, y) \tfrac{1}{\xi'_{\alpha,\beta}\left(\xi^{-1}_{\alpha,\beta}(x)\right)} \\
&\quad -[\lambda \left(u_0(x, y) + v(x, y)\right) + h(\lambda, s(u_0 + v))(u_0 + v)]\varphi(x, y) \tfrac{1}{\xi'_{\alpha,\beta}\left(\xi^{-1}_{\alpha,\beta}(x)\right)} \, \mathrm{d}x \, \mathrm{d}y
\end{aligned}$$

Hence, we have

$$\begin{aligned}
G_1(s, \lambda, v, \alpha, \beta) = M_1 &\left(\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))\partial_x(u_0 + v), \tfrac{\partial_y(u_0+v)}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))}\right) \\
&- M_2 \left((\lambda(u_0 + v) + h(\lambda, s(u_0 + v))(u_0 + v)) \tfrac{1}{\xi'_{\alpha,\beta}\left(\xi^{-1}_{\alpha,\beta}(x)\right)}\right),
\end{aligned}$$

where the linear bounded operators $M_1 : (L^2(\Omega))^2 \to H_0$ and $M_2 : L^2(\Omega) \to H_0$ are defined by $\langle M_1(v_1, v_2), \varphi\rangle := \int_\Omega(v_1\partial_x\varphi + v_2\partial_y\varphi) \, \mathrm{d}x \, \mathrm{d}y$ and $\langle M_2 v, \varphi\rangle := \int_\Omega v\varphi \, \mathrm{d}x \, \mathrm{d}y$ for all $\varphi \in H_0$. Using the C^∞-smoothness of the map

$$(\alpha, \beta, u) \in D \times W^{1,2}(\Omega) \mapsto \left(\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta})\partial_x u, \frac{\partial_y u}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta})}\right) \in \left(L^2(\Omega)\right)^2,$$

we get the C^∞-smoothness of the part with M_1.

It remains to show the C^1-smoothness of the Nemyckii operator

$$(s, \lambda, v) \mapsto h(\lambda, s(u_0 + v))(u_0 + v) \tag{4.2}$$

from $\mathbb{R} \times \mathbb{R}^+ \times H_1$ into $L^2(\Omega)$. We show this by showing that all partial derivatives exist and are continuous.

The partial derivative of (4.2) with respect to s is

$$(s, \lambda, v) \mapsto \partial_u h(\lambda, s(u_0 + v))(u_0 + v)^2 = \frac{1}{s}\partial_u g(\lambda, s(u_0 + v))(u_0 + v) - \frac{1}{s^2}g(\lambda, s(u_0 + v)).$$

This map is continuous from $\mathbb{R} \times \mathbb{R}^+ \times H_1$ to $\in L^2(\Omega)$ because of (1.4), (1.5) and Theorem about Nemyckii operator (see e.g. [22, Proposition 26.6]). The partial derivative of (4.2) with respect to λ is

$$(s, \lambda, v) \mapsto \partial_\lambda h(\lambda, s(u_0 + v))(u_0 + v) = \partial_\lambda g(\lambda, s(u_0 + v)).$$

This map is again continuous from $\mathbb{R} \times \mathbb{R}^+ \times H_1$ to $\in L^2(\Omega)$ because of (1.5) and [22, Proposition 26.6]. The partial derivative of (4.2) with respect to v in the point (s, λ, v) is the linear bounded operator

$$w \in H_1 \mapsto \partial_u h(\lambda, s(u_0 + v))(u_0 + v)sw + h(\lambda, s(u_0 + v))w \in L^2(\Omega).$$

Using (1.5) and [22, Proposition 26.6], it is easy to show that it depends continuously in the uniform operator norm in $\mathcal{L}(H^1; L^2(\Omega))$ on (s, λ, v).

Concerning G_2 and G_3 we have

$$\begin{aligned} G_2(s, \lambda, v, \alpha, \beta) &= \textstyle\int_\Omega [\lambda \Phi_{\alpha,\beta}(u_0 + v) + h(\lambda, s\Phi_{\alpha,\beta}(u_0 + v))\Phi_{\alpha,\beta}(u_0 + v)] v_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y \\ &= \textstyle\int_\Omega [\lambda(u_0 + v) + h(\lambda, s(u_0 + v))(u_0 + v)](X_{\alpha,\beta} + X^{(-1/2)}) \frac{1}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))} \, \mathrm{d}x \, \mathrm{d}y, \\ G_3(s, \lambda, v, \alpha, \beta) &= \textstyle\int_\Omega [\lambda \Phi_{\alpha,\beta}(u_0 + v) + h(\lambda, s\Phi_{\alpha,\beta}(u_0 + v))\Phi_{\alpha,\beta}(u_0 + v)] w_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y \\ &= \textstyle\int_\Omega [\lambda(u_0 + v) + h(\lambda, s(u_0 + v))(u_0 + v)](Y_{\alpha,\beta} + Y^{(-1/2)}) \frac{1}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))} \, \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

It follows from the form (3.14) of $K^1_{\alpha,\beta}(f)$, $K^2_{\alpha,\beta}(f)$ with $f = \lambda \Phi_{\alpha,\beta}(u_0 + v) + h(\lambda, s\Phi_{\alpha,\beta}(u_0 + v))\Phi_{\alpha,\beta}(u_0 + v)$ and their independence of δ, χ in (3.23) that also G_2, G_3 are independent of δ, χ (let us remark that δ, χ in G_1 are not even involved).

We will show the C^1-smoothness of G_2 again by showing that all its partial derivatives exist and are continuous. The map

$$(\alpha, \beta) \in D \mapsto G_2(s, \lambda, v, \alpha, \beta) \in \mathbb{R}$$

is C^∞-smooth, and the map $(s, \lambda, v, \alpha, \beta) \mapsto \partial_\alpha G_2(s, \lambda, v, \alpha, \beta)$ from $\mathbb{R} \times \mathbb{R}^+ \times H_0 \times D$ to $\mathbb{R}$ is continuous with respect to all variables. Similarly for the derivative with respect to β.

We have $X_{\alpha,\beta}, Y_{\alpha,\beta} \in W^{1,2}(\Omega)$, $X^{(-1/2)}, Y^{(-1/2)} \in L^q(\Omega)$ for all $1 \le q < 4$, $X^{(-1/2)}, Y^{(-1/2)} \in W^{1,q}(\Omega)$ for $1 \le q < 4/3$ (see Lemma 2.2). The partial derivative of G_2 with respect to v equals

$$\partial_v G_2(s, \lambda, v, \alpha, \beta) w = \textstyle\int_\Omega [\lambda + \partial_u h(\lambda, s(u_0 + v))(u_0 + v) + h(\lambda, s(u_0 + v))] w \frac{X_{\alpha,\beta} + X^{(-1/2)}}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))} \, \mathrm{d}x \, \mathrm{d}y,$$

and the map $(s, \lambda, v, \alpha, \beta) \mapsto \partial_v G_2(s, \lambda, v, \alpha, \beta)$ from $\mathbb{R} \times \mathbb{R}^+ \times H_0 \times D$ to the space of linear functionals on H_0 is continuous with respect to all variables.

The smooth differentiability with respect to s and λ is clear thanks to the form of the expressions considered.

The C^1-smoothness of G_3 can be proved in a similar way.

In the case of the C^k-smoothness of g, the C^k-smoothness of G_1, G_2, G_3 can be proved similarly under the assumption (2.21).

Now we are going to prove that the partial derivative $\partial_{(\lambda,v,\alpha,\beta)} G(0, \lambda_0, 0, \alpha_0, \beta_0)$ is an isomorphism from $\mathbb{R}^+ \times H_1 \times D$ to $H_0 \times D$.

Let us define an operator $F_0 : \mathbb{R}^+ \to \mathcal{L}(H_0; H_0)$ by

$$\langle F_0(\lambda)u, \varphi\rangle = \int_\Omega \nabla u \nabla\varphi - \lambda u\varphi \,\mathrm{d}x\,\mathrm{d}y \qquad \text{for all } \varphi \in H_0.$$

Hence, $F_0(\lambda)u = 0$ is a weak formulation of the problem (1.6), (1.2), (2.2). Because of (2.19) and the obvious Fredholmness of the operator $F_0(\lambda_0)$ we have

$$H_1 = \operatorname{im} F_0(\lambda_0), \tag{4.3}$$

$$\ker F_0(\lambda_0) = \operatorname{span}\{u_0\}, \tag{4.4}$$

$$H_0 = H_1 \oplus \operatorname{span}\{u_0\}, \tag{4.5}$$

$$F_0(\lambda_0)|_{\operatorname{im} F_0(\lambda_0)} \text{ is an isomorphism from } \operatorname{im} F_0(\lambda_0) \text{ onto itself.} \tag{4.6}$$

We denote $L_j := \partial_{(\lambda,v,\alpha,\beta)} G_j(0, \lambda_0, 0, \alpha_0, \beta_0)$, $j = 1, 2, 3$.

Because of $\xi''_{\alpha_0,\beta_0}(x) = 0$ we get

$$\begin{aligned}
\partial_\alpha \left[\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))\right]_{\alpha=\alpha_0,\beta=\beta_0} &= \partial_\alpha \xi'_{\alpha_0,\beta_0}(x),\\
\partial_\alpha \left[\left(\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))\right)^{-1}\right]_{\alpha=\alpha_0,\beta=\beta_0} &= -\partial_\alpha \xi'_{\alpha_0,\beta_0}(x).
\end{aligned}$$

Furthermore, it holds

$$\left[\partial_\alpha \Phi_{\alpha_0,\beta_0}\varphi\right](x, y) = \partial_x\varphi(x, y)\partial_\alpha\xi_{\alpha_0,\beta_0}(x).$$

Therefore, if $\varphi \in W^{2,2}(\Omega)$ satisfies $\varphi = 0$ on $\Gamma_D \cup I_{\alpha_0,\beta_0}$ then $\partial_\alpha\Phi_{\alpha_0,\beta_0}\varphi \in W^{1,2}(\Omega)$ also satisfies homogeneous Dirichlet boundary conditions on $\Gamma_D \cup I_{\alpha_0,\beta_0}$ (because $\partial_\alpha\xi_{\alpha_0,\beta_0}(0) = \partial_\alpha\xi_{\alpha_0,\beta_0}(1) = 0$ and $\partial_x\varphi(x, 0) = 0$ for $x \in [\alpha_0, \beta_0]$). Similarly for $\partial_\beta\Phi_{\alpha_0,\beta_0}\varphi$. Therefore, $\partial_\alpha\Phi_{\alpha_0,\beta_0}\varphi, \partial_\beta\Phi_{\alpha_0,\beta_0}\varphi \in H_0$ for all $\varphi \in H_0 \cap W^{2,2}(\Omega)$. In particular, we have by (3.4) that

$$\begin{aligned}
\textstyle\int_\Omega \nabla u_0\nabla\partial_\alpha\Phi_{\alpha_0,\beta_0}\varphi - \lambda_0 u_0 \partial_\alpha\Phi_{\alpha_0,\beta_0}\varphi \,\mathrm{d}x\,\mathrm{d}y = 0,\\
\textstyle\int_\Omega \nabla u_0\nabla\partial_\beta\Phi_{\alpha_0,\beta_0}\varphi - \lambda_0 u_0 \partial_\beta\Phi_{\alpha_0,\beta_0}\varphi \,\mathrm{d}x\,\mathrm{d}y = 0
\end{aligned} \qquad \text{for all } \varphi \in H_0 \cap W^{2,2}(\Omega). \tag{4.7}$$

Direct calculation using (2.5) yields

$$\begin{aligned}
&\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))\big|_{\alpha=\alpha_0,\beta=\beta_0} = 1, \quad \xi''_{\alpha_0,\beta_0}(x) = 0,\\
&\partial_\alpha(\xi_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x)))\big|_{\alpha=\alpha_0,\beta=\beta_0} = \partial_\alpha(\xi'_{\alpha,\beta}(x))\big|_{\alpha=\alpha_0,\beta=\beta_0},\\
&\partial_\alpha(\xi^{-1}_{\alpha,\beta}(x))\big|_{\alpha=\alpha_0,\beta=\beta_0} = -\,\partial_\alpha(\xi_{\alpha,\beta}(x))\big|_{\alpha=\alpha_0,\beta=\beta_0},\\
&\partial_\alpha(1/(\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))))\big|_{\alpha=\alpha_0,\beta=\beta_0} = -\,\partial_\alpha(\xi'_{\alpha,\beta}(x))\big|_{\alpha=\alpha_0,\beta=\beta_0}
\end{aligned}$$

and similar expressions for ∂_β. Realizing (4.7) we obtain that

$$\begin{aligned}
\langle L_1(\bar\alpha, \bar\beta, \bar\lambda, \bar v), \varphi\rangle = \textstyle\int_\Omega &\nabla\left(\bar v + \partial_x u_0\left(\bar\alpha\partial_\alpha\xi_{\alpha_0,\beta_0} + \bar\beta\partial_\beta\xi_{\alpha_0,\beta_0}\right)\right)\nabla\varphi\\
&- \lambda_0\left(\bar v + \partial_x u_0\left(\bar\alpha\partial_\alpha\xi_{\alpha_0,\beta_0} + \bar\beta\partial_\beta\xi_{\alpha_0,\beta_0}\right)\right)\varphi - \bar\lambda u_0\varphi \,\mathrm{d}x\,\mathrm{d}y \text{ for all } \varphi \in H_0.
\end{aligned} \tag{4.8}$$

Similarly we obtain

$$\begin{aligned}L_2(\bar\alpha,\bar\beta,\bar\lambda,\bar v) = \lambda_0 \textstyle\int_\Omega \bar v(X_{\alpha_0,\beta_0}+X^{(-1/2)}) + u_0[\bar\alpha\left(\partial_\alpha X_{\alpha_0,\beta_0} + (X_{\alpha_0,\beta_0}+X^{(-1/2)})\partial_\alpha \xi'_{\alpha_0,\beta_0}\right)\\ +\bar\beta\left(\partial_\beta X_{\alpha_0,\beta_0} + (X_{\alpha_0,\beta_0}+X^{(-1/2)})\partial_\beta \xi'_{\alpha_0,\beta_0}\right)]\,\mathrm{d}x\,\mathrm{d}y,\end{aligned} \tag{4.9}$$

$$\begin{aligned}L_3(\bar\alpha,\bar\beta,\bar\lambda,\bar v) = \lambda_0 \textstyle\int_\Omega \bar v(Y_{\alpha_0,\beta_0}+Y^{(-1/2)}) + u_0[\bar\alpha\left(\partial_\alpha Y_{\alpha_0,\beta_0} + (Y_{\alpha_0,\beta_0}+Y^{(-1/2)})\partial_\alpha \xi'_{\alpha_0,\beta_0}\right)\\ +\bar\beta\left(\partial_\beta Y_{\alpha_0,\beta_0} + (Y_{\alpha_0,\beta_0}+Y^{(-1/2)})\partial_\beta \xi'_{\alpha_0,\beta_0}\right)]\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.10}$$

It remains to show that the map $\partial_{(\lambda,v,\alpha,\beta)}G(0,\lambda_0,0,\alpha_0,\beta_0) = (L_1,L_2,L_3)$ is bijective.

Let us have arbitrary $(w,a,b)\in H_0\times\mathbb{R}^2$ such that

$$\begin{aligned}L_1(\bar\alpha,\bar\beta,\bar\lambda,\bar v) &= w,\\ L_2(\bar\alpha,\bar\beta,\bar\lambda,\bar v) &= a,\\ L_3(\bar\alpha,\bar\beta,\bar\lambda,\bar v) &= b.\end{aligned} \tag{4.11}$$

Putting $\varphi := u_0$ into (4.8) and observing (by similar arguments used to derive (4.7)) that

$$\bar w := \bar v + \partial_x u_0\left(\bar\alpha\partial_\alpha\xi_{\alpha_0,\beta_0} + \bar\beta\partial_\beta\xi_{\alpha_0,\beta_0}\right)\in H_0 \tag{4.12}$$

we get by (3.4) that

$$\int_\Omega -\bar\lambda|u_0|^2\,\mathrm{d}x\,\mathrm{d}y = \langle w,u_0\rangle$$

leading immediatelly to

$$\bar\lambda = -\langle w,u_0\rangle/\int_\Omega |u_0|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Simultaneously, (4.8) being zero becomes

$$\begin{aligned}&\textstyle\int_\Omega \nabla\left(\bar v + \partial_x u_0\left(\bar\alpha\partial_\alpha\xi_{\alpha_0,\beta_0}+\bar\beta\partial_\beta\xi_{\alpha_0,\beta_0}\right)\right)\nabla\varphi\\ &\quad -\lambda_0\left(\bar v + \partial_x u_0\left(\bar\alpha\partial_\alpha\xi_{\alpha_0,\beta_0}+\bar\beta\partial_\beta\xi_{\alpha_0,\beta_0}\right)\right)\varphi - \bar\lambda u_0\varphi\,\mathrm{d}x\,\mathrm{d}y\\ &\quad = \textstyle\int_\Omega \nabla w\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \varphi\in H_0\end{aligned} \tag{4.13}$$

or, in other words,

$$\int_\Omega \nabla\bar w\nabla\varphi - \lambda_0\bar w\varphi\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \nabla w\nabla\varphi + \bar\lambda u_0\varphi\,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \varphi\in H_0,$$

where $\bar w$ is from (4.12). Hence,

$$\langle F_0(\lambda_0)\bar w,\varphi\rangle = \int_\Omega \nabla w\nabla\varphi + \bar\lambda u_0\varphi\,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \varphi\in H_0$$

which reads as

$$F_0(\lambda_0)\bar w = w + z$$

where $z\in H_0$ is such that

$$\langle z,\varphi\rangle = \int_\Omega \bar\lambda u_0\varphi\,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \varphi\in H_0.$$

Writting $w+z=c_0u_0+w_1$ with $c_0:=\langle w+z,u_0\rangle$ and $w_1\in H_1$ we get

$$\bar{v}+\partial_x u_0\left(\bar{\alpha}\partial_\alpha\xi_{\alpha_0,\beta_0}+\bar{\beta}\partial_\beta\xi_{\alpha_0,\beta_0}\right)=\bar{w}=(F_0(\lambda_0))^{-1}w_1=:w_2,$$

with w_2 already known. In particular,

$$\bar{w}\in H_1$$

and

$$\bar{v}=-\partial_x u_0\left(\bar{\alpha}\partial_\alpha\xi_{\alpha_0,\beta_0}+\bar{\beta}\partial_\beta\xi_{\alpha_0,\beta_0}\right)+w_2. \tag{4.14}$$

Inserting this in (4.11) divided by λ_0 we get a linear homogeneous 2×2-system of equations with unknowns $\bar{\alpha}$, $\bar{\beta}$ of the form

$$\begin{aligned} a_{11}\bar{\alpha}+a_{12}\bar{\beta}&=b_1,\\ a_{21}\bar{\alpha}+a_{22}\bar{\beta}&=b_2\end{aligned} \tag{4.15}$$

where a_{ij} are from (2.17) and

$$\begin{aligned} b_1&:=a/\lambda_0-\textstyle\int_\Omega w_2(X_{\alpha_0,\beta_0}+X^{(-1/2)})\,\mathrm{d}x\,\mathrm{d}y,\\ b_2&:=b/\lambda_0-\textstyle\int_\Omega w_2(Y_{\alpha_0,\beta_0}+Y^{(-1/2)})\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{4.16}$$

To see it, let us realize that in more detail we have

$$\begin{aligned}&\textstyle\int_\Omega -\partial_x u_0\left(\bar{\alpha}\partial_\alpha\xi_{\alpha_0,\beta_0}+\bar{\beta}\partial_\beta\xi_{\alpha_0,\beta_0}\right)(X_{\alpha_0,\beta_0}+X^{(-1/2)})\\ &\qquad+u_0[\bar{\alpha}\left(\partial_\alpha X_{\alpha_0,\beta_0}+(X_{\alpha_0,\beta_0}+X^{(-1/2)})\partial_\alpha\xi'_{\alpha_0,\beta_0}\right)+\bar{\beta}\left(\partial_\beta X_{\alpha_0,\beta_0}+(X_{\alpha_0,\beta_0}+X^{(-1/2)})\partial_\beta\xi'_{\alpha_0,\beta_0}\right)]\,\mathrm{d}x\,\mathrm{d}y\\ &=a/\lambda_0-\textstyle\int_\Omega w_2(X_{\alpha_0,\beta_0}+X^{(-1/2)})\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

$$\begin{aligned}&\textstyle\int_\Omega -\partial_x u_0\left(\bar{\alpha}\partial_\alpha\xi_{\alpha_0,\beta_0}+\bar{\beta}\partial_\beta\xi_{\alpha_0,\beta_0}\right)(Y_{\alpha_0,\beta_0}+Y^{(-1/2)})\\ &\qquad+u_0[\bar{\alpha}\left(\partial_\alpha Y_{\alpha_0,\beta_0}+(Y_{\alpha_0,\beta_0}+Y^{(-1/2)})\partial_\alpha\xi'_{\alpha_0,\beta_0}\right)+\bar{\beta}\left(\partial_\beta Y_{\alpha_0,\beta_0}+(Y_{\alpha_0,\beta_0}+Y^{(-1/2)})\partial_\beta\xi'_{\alpha_0,\beta_0}\right)]\,\mathrm{d}x\,\mathrm{d}y\\ &=b/\lambda_0-\textstyle\int_\Omega w_2(Y_{\alpha_0,\beta_0}+Y^{(-1/2)})\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Now, putting appropriate terms together we see that the coefficients a_{ij} in (4.15) are really those from (2.17). Now, because of (2.20), and (4.14) gives also a unique $\bar{v}$. The system (4.15) is uniquelly solvable and (4.14) gives also a unique $\bar{v}$. Hence, the bijectivity of $\partial_{(\lambda,v,\alpha,\beta)}G(0,\lambda_0,0,\alpha_0,\beta_0)$ is proved. Thus, $\partial_{(\lambda,v,\alpha,\beta)}G(0,\lambda_0,0,\alpha_0,\beta_0)$ is an isomorphism. ■

Proof of Theorem 4.1. The problem (3.6), (3.7) is equivalent to

$$G(s,\lambda,v,\alpha,\beta)=0, \tag{4.17}$$

where G is defined by (4.1). Lemma 4.3 ensures that this mapping is C^1-smooth and that under the condition (2.20), $\partial_{(\lambda,v,\alpha,\beta)}G(0,\lambda_0,0,\alpha_0,\beta_0)$ is an isomorphism from $\mathbb{R}^+\times H_1\times D$ to $H_0\times D$. Hence, it follows from the Implicit Function Theorem that there exist $s_0>0$, neighbourhoods $\Lambda_0\subset\mathbb{R}^+$ of λ_0, $W_0\subset H_0$ of the origin and $W_\alpha, W_\beta\subset\mathbb{R}$ of α_0 and β_0, respectively, and C^1-mappings $\hat{\lambda}:(-s_0,s_0)\to\Lambda_0$, $\hat{v}:(-s_0,s_0)\to W_0$, $\hat{\alpha}:(-s_0,s_0)\to W_\alpha$, $\hat{\beta}:(-s_0,s_0)\to W_\beta$ such that $\hat{\lambda}(0)=\lambda_0$, $\hat{v}(0)=0$, $\hat{\alpha}(0)=\alpha_0$, $\hat{\beta}(0)=\beta_0$ and that $(\lambda,v,\alpha,\beta)\in\Lambda_0\times W_0\times W_\alpha\times W_\beta$ satisfies (3.6), (3.7) if and only if $\lambda=\hat{\lambda}(s)$, $v=\hat{v}(s)$, $\alpha=\hat{\alpha}(s)$, $\beta=\hat{\beta}(s)$. ■

5 Proof of the Main Results

Proof of Theorem 2.3 Theorem 4.1 implies that there are neighbourhoods $\Lambda_0 \subset \mathbb{R}$, $W_0 \subset H_0$, $W_\alpha, W_\beta \subset \mathbb{R}$ of λ_0, 0, α_0, β_0, respectively, and C^1-mappings $\hat{\lambda} : (-s_0, s_0) \to \Lambda_0$, $\hat{v} : (-s_0, s_0) \to W_0$, $\hat{\alpha} : (-s_0, s_0) \to W_\alpha$, $\hat{\beta} : (-s_0, s_0) \to W_\beta$ such that (i) in Theorem 2.3 holds and $(\lambda, v, \alpha, \beta) = (\hat{\lambda}(s), \hat{v}(s), \hat{\alpha}(s), \hat{\beta}(s))$, $s \in (-s_0, 0) \cup (0, s_0)$, are exactly all nontrivial solutions of the problem (3.6), (3.7) lying in a neighbourhood of $(\lambda_0, 0, \alpha_0, \beta_0)$. That means due to Proposition 3.7 that $(\lambda, u) = (\hat{\lambda}(s), \hat{u}(s))$, $s \in (-s_0, 0) \cup (0, s_0)$, with $u = \hat{u}(s)$ defined in the assertion (iv) of Theorem 2.3 are all weak nontrivial solutions of the boundary value problems (1.1), (1.2), (2.3) with α, β close to α_0, β_0, lying in a neighbourhood of $(\lambda_0, 0)$ and satisfying $u \in W^{2,2}(\Omega)$. Theorem 3.1(i) guarantees that those $(\lambda, u) = (\hat{\lambda}(s), \hat{u}(s))$ with positive s simultaneously satisfy (ii) in Theorem 2.3, and also (iv) follows. The assertion (iii) as well as the very last assertion of Theorem 2.3 follows from the smoothness given by Theorem 4.1, the form of $\hat{u}(s)$ in (iv) and from Lemma 2.1. Finally, 3.1(ii) implies that also the assertion (v) in Theorem 2.3 is true.

■

Proof of Theorem 2.4. Let us consider the case when the nonlinearity g is identically zero in Theorem 2.3. Then the problem (2.1) coincides with (2.2). The couples (λ, u) with $\lambda = \lambda_0$, $u = su_0$, $s > 0$ satisfy (2.2) and it follows from the assertion (v) of Theorem 2.3 that all (λ, u) satisfying (2.2) with $u \in W^{2,2}(\Omega)$, $|\lambda - \lambda_0| + \|u\| + \|\frac{u}{\|u\|} - u_0\|$ small enough should be of this form. If the assertion of Theorem 2.4 were not true then we would have (λ_n, u_n) satisfying (2.2), $\|u_n\| \neq 0$, $0 < |\lambda_n - \lambda_0| + \|\frac{u_n}{\|u_n\|} - u_0\| \to 0$. Due to the fact that the problem (2.2) is positively homogeneous, we could choose u_n such that $\|u_n\| \to 0$, which would be a contradiction. ■

6 Appendix

For the proof of Lemma 2.1 we need the following modification of the continuity in the mean property in $L^q(\Omega)$. In this lemma we can consider an arbitrary open bounded set Ω in $\mathbb{R}^n$.

Lemma 6.1 *Let M be a precompact set in $L^q(\Omega)$ with $q > 1$. Then for any $\varepsilon > 0$ and $r > 0$ there is $\delta > 0$ such that the following holds: If $\varphi_j : \overline{\Omega} \to \overline{\Omega}$ $(j = 1, 2)$ are bijective, $\varphi_j \in C^1(\overline{\Omega})$ and $\|\varphi_1 - \varphi_2\|_{C(\overline{\Omega})} < \delta$, and if the Jacobians J_{φ_j} satisfy $|J_{\varphi_j}(x)| \geq r$ for all $x \in \overline{\Omega}$, then*

$$\left(\int_\Omega |h(\varphi_1(z)) - h(\varphi_2(z))|^q dz \right)^{1/q} < \varepsilon \text{ for all } h \in M. \tag{6.1}$$

Proof. Because M is precompact in $L^q(\Omega)$ and because $C(\overline{\Omega})$ is continuously and densely embedded into $L^q(\Omega)$, for any $\tilde{\varepsilon} > 0$ there exists a finite set $\tilde{M} \subset C(\overline{\Omega})$ such that for any

$h \in M$ there is a $\tilde{h} \in \tilde{M}$ with

$$\left(\int_\Omega |h(z) - \tilde{h}(z)|^q dz\right)^{1/q} < \tilde{\varepsilon}.$$

Hence

$$\left(\int_\Omega |h(\varphi_j(z)) - \tilde{h}(\varphi_j(z))|^q dz\right)^{1/q} = \left(\int_\Omega |h(z) - \tilde{h}(z)|^q |J_{\varphi_j}(\varphi_j^{-1}(z)|^{-1} dz\right)^{1/q} < \frac{\tilde{\varepsilon}}{r},$$

and the integral from (6.1) can be estimated by

$$\left(\int_\Omega |h(\varphi_1(z)) - \tilde{h}(\varphi_1(z))|^q dz\right)^{1/q} + \left(\int_\Omega |\tilde{h}(\varphi_1(z)) - \tilde{h}(\varphi_2(z))|^q dz\right)^{1/q} +$$

$$+ \left(\int_\Omega |\tilde{h}(\varphi_2(z) - h(\varphi_2(z)|^q dz\right)^{1/q} < \frac{2\tilde{\varepsilon}}{r} + \left(\int_\Omega |\tilde{h}(\varphi_1(z)) - \tilde{h}(\varphi_2(z))|^q dz\right)^{1/q}.$$

Moreover, because $\tilde{M}$ is finite and all $\tilde{h} \in \tilde{M}$ are uniformly continuous, there exists $\tilde{\delta} > 0$ such that

$$|\tilde{h}(z_1) - \tilde{h}(z_2)| < \tilde{\varepsilon} \text{ for all } \tilde{h} \in \tilde{M} \text{ and } z_1, z_2 \in \Omega \text{ with } |z_1 - z_2| < \tilde{\delta}.$$

Hence, if $\|\varphi_1 - \varphi_2\|_{C(\overline{\Omega})} < \tilde{\delta}$ then the integral from (6.1) can be estimated by $\left(\frac{2}{r} + (\text{mes } \Omega)^{1/q}\right) \tilde{\varepsilon}$. ■

Proof of Lemma 2.1. Define $\mathcal{F} : D \times W^{1,q}(\Omega) \to W^{1,q}(\Omega)$ by $\mathcal{F}(\alpha, \beta, f) := \Phi_{\alpha,\beta} f$. First we will prove the C^1-smoothness of $\mathcal{F}$ as a map from $D \times W^{1,q}(\Omega)$ into $L^q(\Omega)$.

If $\mathcal{F}$ is differentiable then the partial derivatives can be calculated pointwise:

$$\left.\begin{aligned} \partial_\alpha \mathcal{F}(\alpha, \beta, f)(x, y) &= \partial_x f(\xi_{\alpha,\beta}(x), y) \partial_\alpha \xi_{\alpha,\beta}(x), \\ \partial_\beta \mathcal{F}(\alpha, \beta, f)(x, y) &= \partial_x f(\xi_{\alpha,\beta}(x), y) \partial_\beta \xi_{\alpha,\beta}(x), \\ (\partial_f \mathcal{F}(\alpha, \beta, f) g)(x, y) &= g(\xi_{\alpha,\beta}(x), y). \end{aligned}\right\} \tag{6.2}$$

In other words, the right hand sides of (6.2) are candidates for being the partial derivatives of $\mathcal{F}$. In order to show that the candidate for $\partial_\alpha \mathcal{F}$ is really the partial derivative of $\mathcal{F}$ with respect to α, we have to prove that

$$\left(\int_\Omega \left| \frac{f(\xi_{\tilde{\alpha},\beta}(x), y) - f(\xi_{\alpha,\beta}(x), y)}{\tilde{\alpha} - \alpha} - \partial_x f(\xi_{\alpha,\beta}(x), y) \partial_\alpha \xi_{\alpha,\beta}(x) \right|^q \mathrm{d}x\, \mathrm{d}y\right)^{1/q} \to 0 \tag{6.3}$$

for $\tilde{\alpha} \to \alpha$.

Let $f_1, f_2, \ldots \in C^1(\overline{\Omega})$ be a sequence converging to f in $W^{1,q}(\Omega)$. For (6.3) we have to show that

$$\left(\int_\Omega \left| \frac{f_n(\xi_{\tilde{\alpha},\beta}(x), y) - f_n(\xi_{\alpha,\beta}(x), y)}{\tilde{\alpha} - \alpha} - \partial_x f_n(\xi_{\alpha,\beta}(x), y) \partial_\alpha \xi_{\alpha,\beta}(x) \right|^q \mathrm{d}x\, \mathrm{d}y\right)^{1/q} \to 0 \tag{6.4}$$

for $\tilde{\alpha} \to \alpha$ uniformly with respect to n. The expression in the absolute value in the integral in (6.4) can be written as

$$\int_0^1 \frac{\partial_x f_n(s\xi_{\tilde{\alpha},\beta}(x) + (1-s)\xi_{\alpha,\beta}(x), y)(\xi_{\tilde{\alpha},\beta}(x), y) - \xi_{\alpha,\beta}(x), y)}{\tilde{\alpha} - \alpha} \,\mathrm{d}s - \partial_x f_n(\xi_{\alpha,\beta}(x), y)\partial_\alpha \xi_{\alpha,\beta}(x) =$$

$$= \int_0^1 \partial_x f_n(s\xi_{\tilde{\alpha},\beta}(x) + (1-s)\xi_{\alpha,\beta}(x), y)\,\mathrm{d}s \int_0^1 \partial_\alpha \xi_{t\tilde{\alpha}+(1-t)\alpha,\beta}(x)\,\mathrm{d}t - \partial_x f_n(\xi_{\alpha,\beta}(x), y)\partial_\alpha \xi_{\alpha,\beta}(x).$$

There exists $c > 0$ such that $|\partial_\alpha \xi_{\alpha,\beta}(x)| \le c$ for all $(\alpha, \beta) \in D$ and $x \in [0, 1]$. Therefore the whole integral in (6.4) can be estimated by

$$c\left(\int_\Omega \left|\int_0^1 (\partial_x f_n(s\xi_{\tilde{\alpha},\beta}(x) + (1-s)\xi_{\alpha,\beta}(x), y) - \partial_x f_n(\xi_{\alpha,\beta}(x), y))\, ds\right|^q \mathrm{d}x\,\mathrm{d}y\right)^{1/q} +$$

$$+\left(\int_\Omega \left|\partial_x f_n(\xi_{\alpha,\beta}(x), y) \int_0^1 \left(\partial_\alpha \xi_{t\tilde{\alpha}+(1-t)\alpha,\beta}(x) - \partial_\alpha \xi_{\alpha,\beta}(x)\right) dt\right|^q \mathrm{d}x\,\mathrm{d}y\right)^{1/q}. \qquad (6.5)$$

Using

$$\left|\int_0^1 (\partial_x f_n(s\xi_{\tilde{\alpha},\beta}(x) + (1-s)\xi_{\alpha,\beta}(x), y) - \partial_x f_n(\xi_{\alpha,\beta}(x), y))\,\mathrm{d}s\right|^q \le$$

$$\le \int_0^1 |\partial_x f_n(s\xi_{\tilde{\alpha},\beta}(x) + (1-s)\xi_{\alpha,\beta}(x), y) - \partial_x f_n(\xi_{\alpha,\beta}(x), y)|^q\,\mathrm{d}s,$$

the first term in (6.5) can be estimated by

$$c\left(\int_0^1 \int_\Omega |\partial_x f_n(s\xi_{\tilde{\alpha},\beta}(x) + (1-s)\xi_{\alpha,\beta}(x), y) - \partial_x f_n(\xi_{\alpha,\beta}(x), y)|^q\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\right)^{1/q}$$

Due to Lemma 6.1 (with $M = \{f_1, f_2, \ldots\}$, $\varphi_1(x, y) = (\xi_{\alpha,\beta}(x), y)$ and $\varphi_2(x, y) = (s\xi_{\tilde{\alpha},\beta}(x) + (1-s)\xi_{\alpha,\beta}(x), y)$ and, hence, $(\varphi_1 - \varphi_2)(x, y) = (-s(\xi_{\alpha,\beta}(x) - \xi_{\tilde{\alpha},\beta}), 0)$), the interior integral over Ω tends to zero for $\tilde{\alpha} \to \alpha$ uniformly with respect to n and s, hence the first term in (6.5) tends to zero for $\tilde{\alpha} \to \alpha$ uniformly with respect to n. The second term in (6.5) can be estimated by

$$\max_{0\le t\le 1} \left|\partial_\alpha \xi_{t\tilde{\alpha}+(1-t)\alpha,\beta}(x) - \partial_\alpha \xi_{\alpha,\beta}(x)\right| \left(\int_\Omega |\partial_x f_n(\xi_{\alpha,\beta}(x), y)|^q\,\mathrm{d}x\,\mathrm{d}y\right)^{1/q},$$

which obviously tends to zero for $\tilde{\alpha} \to \alpha$ uniformly with respect to n.

Analogously one can show that $\mathcal{F}$ is partially differentiable with respect to β.

For any fixed $(\alpha, \beta) \in D$, the map $f \in W^{1,q}(\Omega) \mapsto \mathcal{F}(\alpha, \beta, f) = \Phi_{\alpha,\beta} f \in L^q(\Omega)$ is linear and continuous, hence $\mathcal{F}$ is partially differentiable with respect to f.

In order to get the C^1-smoothness of $\mathcal{F}$ it remains to show that the maps $(\alpha, \beta, f) \in D \times W^{1,q}(\Omega) \mapsto \partial_\alpha \mathcal{F}(\alpha, \beta, f) \in L^q(\Omega)$, $(\alpha, \beta, f) \in D \times W^{1,q}(\Omega) \mapsto \partial_\beta \mathcal{F}(\alpha, \beta, f) \in L^q(\Omega)$ and $(\alpha, \beta, f) \in D \times W^{1,q}(\Omega) \mapsto \partial_f \mathcal{F}(\alpha, \beta, f) \in \mathcal{L}\left(W^{1,q}(\Omega), L^q(\Omega)\right)$ are continuous.

First continuity of $\partial_\alpha \mathcal{F}$: We have

$$\begin{aligned}
&\|\partial_\alpha \mathcal{F}(\tilde{\alpha},\tilde{\beta},\tilde{f}) - \partial_\alpha \mathcal{F}(\alpha,\beta,f)\|_{L^q(\Omega)} = \\
&\quad = \left(\int_\Omega \left|\partial_x \tilde{f}(\xi_{\tilde{\alpha},\tilde{\beta}}(x),y)\partial_\alpha \xi_{\tilde{\alpha},\tilde{\beta}}(x) - \partial_x f(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x)\right|^q \mathrm{d}x\,\mathrm{d}y\right)^{1/q} = \\
&\quad \le \left(\int_\Omega \left|\left(\partial_x \tilde{f}(\xi_{\tilde{\alpha},\tilde{\beta}}(x),y) - \partial_x \tilde{f}(\xi_{\alpha,\beta}(x),y)\right)\partial_\alpha \xi_{\tilde{\alpha},\tilde{\beta}}(x)\right|^q \mathrm{d}x\,\mathrm{d}y\right)^{1/q} + \\
&\qquad + \left(\int_\Omega \left(\partial_x \tilde{f}(\xi_{\alpha,\beta}(x),y) - \partial_x f(\xi_{\alpha,\beta}(x),y)\right)\partial_\alpha \xi_{\tilde{\alpha},\tilde{\beta}}(x)^q \,\mathrm{d}x\,\mathrm{d}y\right)^{1/q} + \\
&\qquad + \left(\int_\Omega \partial_x \tilde{f}(\xi_{\alpha,\beta}(x),y)\left(\partial_\alpha \xi_{\tilde{\alpha},\tilde{\beta}}(x) - \partial_\alpha \xi_{\alpha,\beta}(x)\right)^q \mathrm{d}x\,\mathrm{d}y\right)^{1/q}.
\end{aligned}$$

Due to Lemma 6.1, all these integrals tend to zero if $\tilde{f} \to f$ in $W^{1,q}(\Omega)$, $(\tilde{\alpha},\tilde{\beta}) \to (\alpha,\beta)$, and the continuity of $\partial_\alpha \mathcal{F}$ is proved.

Similarly for $\partial_\beta \mathcal{F}$.

Finally, let us show the continuity of $\partial_f \mathcal{F}$. We have

$$\|\partial_f \mathcal{F}(\tilde{\alpha},\tilde{\beta},\tilde{f})g - \partial_f \mathcal{F}(\alpha,\beta,f)g\|_{L^q(\Omega)} = \left(\int_\Omega \left|g(\xi_{\tilde{\alpha},\tilde{\beta}}(x),y) - g(\xi_{\alpha,\beta}(x),y)\right|^q \mathrm{d}x\,\mathrm{d}y\right)^{1/q},$$

which converges to zero for $(\tilde{\alpha},\tilde{\beta}) \to (\alpha,\beta)$ uniformly with respect to $\|g\|_{W^{1,q}(\Omega)} \le 1$ because of the compact embedding of $W^{1,q}(\Omega)$ into $L^q(\Omega)$ and Lemma 6.1.

In order to show the continuity of $\mathcal{F}$ as a map from $W^{1,q}(\Omega)$ into $W^{1,q}(\Omega)$ we have to show that $\mathcal{F}$ is continuous as a map from $W^{1,q}(\Omega)$ into m $L^q(\Omega)$ (what follows from its differentiability shown above) and that the map $\mathcal{F}_1 : W^{1,q}(\Omega) \to L^q(\Omega)$, which is defined by $\mathcal{F}_1(\alpha,\beta,f)(x,y) := \partial_x f(\xi_{\alpha,\beta}(x),y)\xi'_{\alpha,\beta}(x)$, is continuous. The continuity of $\mathcal{F}_1$ can be shown as that of $\partial_\alpha \mathcal{F}$.

Analogously we can prove the assertions for $\Phi^{-1}_{\alpha,\beta}$ and $\Phi^*_{\alpha,\beta}$ by using the formula (2.8). ■

Lemma 6.2 *Let (1.4), (1.5) hold, let $p > 1$ be arbitrary. There are $\delta > 0$, $C > 0$ such that*

$$\|g(\lambda,u)\|_{L^p(\Omega)} \le C\|u\|^2,\ \|\partial_u g(\lambda,u)\|_{L^p(\Omega)} \le C\|u\| \text{ for all } (\lambda,u) \in \mathbb{R}^+ \times H,\ \|u\| < \delta. \tag{6.6}$$

Proof. It follows from (1.4) that there exist $\delta > 0$ and $C > 0$ such that

$$\text{if } |s|^q < \delta \text{ then } |g(\lambda,s)| < Cs^2. \tag{6.7}$$

For the proof of the first part of our assertion it is sufficient to show that if $(\lambda_n,u_n) \in \mathbb{R}^+ \times H$, $\|u_n\| \to 0$ then there is $C > 0$ such that

$$\|g(\lambda_n,u_n)\|_{L^p(\Omega)} \le C\|u_n\|^2. \tag{6.8}$$

Let u_n be such a sequence. Let us introduce the sets

$$E^n_\delta := \{x \in \Omega : |u_n(x)|^q \ge \delta\}.$$

We obtain by (6.7) and the embedding theorems (see Observation 3.2) that

$$\left(\int_{\Omega\setminus E_\delta^n} |g(\lambda_n,u_n)|^p \,\mathrm{d}x\right)^{1/p} \le C\left(\int_{\Omega\setminus E_\delta^n} (|u_n|^2)^p \,\mathrm{d}x\right)^{1/p} \le C\|u_n\|^2_{L^{2p}(\Omega)} \le C\|u_n\|^2. \tag{6.9}$$

We have $1 \le \delta^{-1}|u_n(x)|^q$ for all $x \in E_\delta^n$ and we obtain by using (1.5) that

$$\begin{aligned}\left(\int_{E_\delta^n} |g(\lambda,u_n)|^p \,\mathrm{d}x\right)^{1/p} &\le \left(\int_\Omega (c(1+|u_n|^q))^p \,\mathrm{d}x\right)^{1/p}\\ &\le \left(\int_\Omega (c(\delta^{-1}+1)|u_n|^q))^p \,\mathrm{d}x\right)^{1/p} \le C\|u_n\|^q_{L^{pq}(\Omega)} \le C\|u_n\|^q.\end{aligned} \tag{6.10}$$

Since $q > 2$, the last expression is not larger then $C\|u_n\|^2$ for small $\|u_n\|$, and the first inequality from our assertion follows.

The second estimate can be shown analogously, replacing (6.7) by the inequality $|\partial_u g(\lambda,s)| < C|s|$ which is fulfilled if $|s|^q < \delta$. ∎

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