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Asymptotic behavior of the motion of a viscous heat-conducting one-dimensional gas with radiation: the pure scattering case

SUMMARY

We study the large-time behaviour of the solution of an initial-boundary value problem for the equations of 1D motions of a compressible viscous heat-conducting gas coupled with radiation through a radiative transfer equation.

Assuming only scattering processes between matter and photons (neglecting absorption and emission) and suitable hypotheses on the transport coefficients, we prove that the unique weak solution of the problem converges toward the static state.

Keywords: compressible, viscous, heat conducting fluids, one-dimensional symmetry, radiative transfer. **AMS subject classification:** 35Q30, 76N10

1 Introduction

We consider the asymptotic behaviour of the compressible Navier-Stokes system when radiation is present with coupling terms between matter and radiation. These couplings depend on the radiative intensity I driven by the so called radiative transfert integro-differential equation introduced and discussed by Chandrasekhar in [4].

We consider (see [7] for a complete derivation) a radiative-hydrodynamics model (see [15] and [17] for a comprehensive study of such systems) given in Lagrangian mass coordinates by the coupled system

$$\begin{cases} \eta_t = v_x, \\ v_t = \sigma_x - \eta(S_F)_R, \\ \left(e + \frac{1}{2}\, v^2\right)_t = (\sigma v - q)_x\,, \\ I_t + \eta^{-1}(c\omega - v)I_x = cS, \end{cases} \tag{1}$$

in the domain $Q := \Omega \times \mathbf{R}^+$ with $\Omega := (0, M)$ (M is the total mass of matter), where the specific volume η (with $\eta := \frac{1}{\rho}$), the velocity v, the temperature θ and the radiative intensity I depends on the lagrangian mass coordinates (x, t) and also on two extra variables: the radiation frequency $\nu \in \mathbb{R}_+$ and the angular variable $\omega \in S^1 := [-1, 1]$. In $(1)_4$, c is the velocity of light.

We also denote by $\sigma := -p + \mu \, \frac{v_x}{\eta}$ the stress and by $q := -\kappa \, \frac{\theta_x}{\eta}$ the heat flux, and the source term in the last equation, including only scattering processes and neglecting absorption and emission phenomena (see [7] for the complete model), is given by the non-local expression

$$S(x, t, \nu, \omega) = \sigma_s(\nu; \eta, \theta) \, \left[\tilde{I}(x, t, \nu) - I(x, t, \nu, \omega)\right], \tag{2}$$

where $\tilde{I}$ is the averaged intensity $\tilde{I}(x, t, \nu) := \frac{1}{2} \int_{-1}^{1} I(x, t, \nu, \omega) \, d\omega$, and $\sigma_s > 0$ is the scattering coefficient (see below for hypotheses on the behaviour of this function).

Defining the radiative energy

$$E_R := \frac{1}{c} \int_{-1}^{1} \int_{0}^{\infty} I(x, t, \nu, \omega) \, d\nu \, d\omega, \tag{3}$$

the radiative flux

$$F_R := \int_{-1}^{1} \int_{0}^{\infty} \omega I(x, t, \nu, \omega) \, d\nu \, d\omega, \tag{4}$$

and the radiative pressure

$$P_R := \frac{1}{c} \int_{-1}^{1} \int_{0}^{\infty} \omega^2 I(x, t, \nu, \omega) \, d\nu \, d\omega, \tag{5}$$

we get the radiative force source in the right-and side of $(1)_2$

$$(S_F)_R := \frac{1}{c} \int_{-1}^{1} \int_{0}^{\infty} \omega S(x, t, \nu, \omega) \, d\nu \, d\omega. \tag{6}$$

Now, from $(1)_4$ and the definitions (3)-(6), one derives the equations

$$(\eta I)_t + ((c\omega - v) I)_x = c\eta S. \tag{7}$$

and after integrating in frequency and angular variables

$$\begin{cases} (\eta E_R)_t + (F_R - v E_R)_x = 0, \\ (\eta F_R)_t + (P_R - v F_R)_x = \eta \, (S_F)_R \, . \end{cases} \tag{8}$$

We consider Dirichlet-Neumann boundary conditions for the fluid unknowns

$$\begin{cases} v|_{x=0} = v|_{x=M} = 0, \\ q|_{x=0} = q|_{x=M} = 0, \end{cases} \tag{9}$$

and transparent boundary conditions for the radiative intensity (see [5])

$$\begin{cases} I|_{x=0} = 0 \quad \text{for } \omega \in (0,1) \\ I|_{x=M} = 0 \quad \text{for } \omega \in (-1,0), \end{cases} \tag{10}$$

for $t > 0$, and initial conditions

$$\eta|_{t=0} = \eta^0(x), \quad v|_{t=0} = v^0(x), \quad \theta|_{t=0} = \theta^0(x), \quad \text{on} \;\; \Omega. \tag{11}$$

and

$$I|_{t=0} = I^0(x,\nu,\omega) \;\; \text{on} \;\; \Omega \times \mathbb{R}_+ \times S^1. \tag{12}$$

Pressure and energy of the matter are related by the thermodynamical relation

$$e_\eta(\eta,\theta) = -p(\eta,\theta) + \theta p_\theta(\eta,\theta). \tag{13}$$

Finally we assume that state functions e, p and κ (resp. σ_s) are C^2 (resp C^0) functions of their arguments for $0 < \eta < \infty$ and $0 \leqslant \theta < \infty$, and , for any $\underline{\eta} \geqslant 0$ we suppose the following growth conditions for $\eta \geqslant \underline{\eta}$ and $\theta \geqslant 0$

$$\begin{cases} e(\eta,0) \geqslant 0, \quad c_1(1+\theta^r) \leqslant e_\theta(\eta,\theta) \leqslant C_1(\underline{\eta})(1+\theta^r), \\ -c_2\eta^{-2}(1+\theta^{1+r}) \leqslant p_\eta(\eta,\theta) \leqslant -C_2\eta^{-2}(1+\theta^{1+r}), \\ |p_\theta(\eta,\theta)| \leqslant C_3(\underline{\eta})\eta^{-1}(1+\theta^r), \\ \eta p(\eta,\theta) \leqslant C_4(1+\theta^{1+r}), \\ c_5(\underline{\eta})(1+\theta^{1+r}) \leqslant p(\eta,\theta) \leqslant C_5(\underline{\eta})(1+\theta^{1+r}), \\ c_6(1+\theta^q) \leqslant \kappa(\eta,\theta) \leqslant C_6(\underline{\eta})(1+\theta^q), \\ |\kappa_\eta(\eta,\theta)| + |\kappa_{\eta\eta}(\eta,\theta)| \leqslant C_7(\underline{\eta})(1+\theta^q), \\ \eta\sigma_s(\nu;\eta,\theta) \leqslant C_{11}k(\nu), \\ \left(\left|(\sigma_s)_\eta\right| + |(\sigma_s)_\theta|\right)(1+B+|B_\theta|) \leqslant C_{12}\ell(\nu), \end{cases} \tag{14}$$

where the numbers c_j, C_j, $j = 1, ..., 12$ are positive constants and the functions k, ℓ are such that

$$k, \ell \;\in L^{1+\gamma}(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+),$$

for an arbitrary small $\gamma > 0$.

For simplicity, we assume here that

$$r \in [0,1], \;\; q \geqslant r+1,$$

but one can check that our results also hold in more general general situations (see the book of Qin [18] for a general presentation).

Concerning the viscosity, we suppose that it does not depend on temperature and that $s \to \mu(s)$ satisfies, for any $s > 0$

$$0 < \mu_0 \leqslant \mu(s) \leqslant \mu_1 \quad \text{and } |\mu'(s)| \leqslant \mu_2, \tag{15}$$

for some positive constants μ_0, μ_1 and μ_2.

We consider weak solutions for the above problem with properties

$$\begin{cases} \eta \in L^\infty(Q_T), \quad \eta_t \in L^\infty([0,T], L^2(\Omega)), \\ v \in L^\infty([0,T], L^4(\Omega)), \quad v_t \in L^\infty([0,T], L^2(\Omega)), \quad v_x \in L^\infty([0,T], L^2(\Omega)), \\ \sigma_x \in L^\infty([0,T], L^2(\Omega)), \\ \theta \in L^\infty([0,T], L^q(\Omega)), \quad \theta_x \in L^\infty([0,T], L^2(\Omega)), \\ I \in L^\infty([0,T], L^1(\Omega \times \mathbb{R}_+ \times S^1)), \end{cases} \tag{16}$$

where $Q_T := \Omega \times (0,T)$ and for any $1 \leqslant q \leqslant r+1$.

We also assume the following conditions on the data:

$$\begin{cases} \eta^0 > 0 \text{ on } \Omega, \ \eta^0 \in L^1(\Omega), \\ v^0 \in L^2(\Omega), \quad v^0_x \in L^2(\Omega), \\ \theta^0 \in L^2(\Omega), \quad \inf_\Omega \theta^0 \geqslant 0, \\ I^0 \in L^1(\Omega \times \mathbb{R}_+ \times S^1). \end{cases} \tag{17}$$

Then our definition of a weak solution for the previous problem is

Definition 1.1. *We call (η, v, θ, I) a weak solution of (1) if it satisfies*

$$\eta(x,t) = \eta^0(x) + \int_0^t v_x \, ds, \tag{18}$$

for a.e. $x \in \Omega$ and any $t > 0$, and if, for any test function $\phi \in L^2([0,T], H^1(\Omega))$ with $\phi_t \in L^1([0,T], L^2(\Omega))$ such that $\phi(\cdot, T) = 0$, one has

$$\int_Q \phi_t v + \phi_x p - \frac{\mu \phi_x}{\eta} v_x \, dx \, dt = \int_\Omega \phi(0,x) \, v^0(x) \, dx, \tag{19}$$

with $\phi^0 = \phi(0,x)$,

$$\int_Q \phi_t (e + \frac{1}{2} \, v^2) + \phi_x(\sigma v - q) \, dx \, dt = \int_\Omega \phi^0 (e^0 + \frac{1}{2} \, {v^0}^2) \, dx, \tag{20}$$

and if, for any test function $\psi \in L^2([0,T], H^1(\Omega \times \mathbb{R}_+ \times S^1))$ *with* $\phi_t \in L^1([0,T], L^2(\Omega \times \mathbb{R}_+ \times S^1))$ *such that* $\phi(\cdot, T, \cdot, \cdot) = 0$*, one has*

$$\int_{Q\times\mathbb{R}_+\times S^1} [\phi_t \eta I + \phi_x (v - c\omega) I + \phi \eta S]\, d\nu\, d\omega\, dx\, dt = \int_{\Omega\times\mathbb{R}_+\times S^1} \phi^0 \eta^0 I^0\, d\nu\, d\omega\, dx. \tag{21}$$

In the following we use the notation $\mathcal{I}(x,t) := \int_0^\infty \int_{S^1} I(x,t;\omega,\nu)\, d\omega\, d\nu$ for the integrated radiative intensity.

In [7] (see also [6] for a simplified model) we have proved the following existence result

Theorem 1. *Suppose that the initial data satisfy (17) and that* T *is an arbitrary positive number.*

Then the problem (1)(9)(10)(11) (12) possesses a unique global weak solution satisfying (16) together with properties (18), (19) and (20).

Let us consider the static problem corresponding to (1). The following result is easily checked

Proposition 1. *The unique static solution* $(\eta_\infty, v_\infty \equiv 0, \theta_\infty, I_\infty)$ *, of the problem (1) satisfies the system*

$$\begin{cases} P_x = 0, \\ Q_x = 0, \\ I_\infty = 0, \end{cases} \tag{22}$$

where $P = p(\eta_\infty, \theta_\infty)$*,* $Q = q(\eta_\infty, \theta_\infty)$*, with boundary condition*

$$\begin{cases} I_\infty|_{x=0} = 0 \quad \text{for } \omega \in (0,1), \\ I_\infty|_{x=M} = 0 \quad \text{for } \omega \in (-1,0), \end{cases} \tag{23}$$

for $t > 0$*, and is given by the formulas*

$$\begin{cases} \eta_\infty = \frac{1}{M}\int_\Omega \eta^0(x)\, dx, \\ e(\eta_\infty, \theta_\infty) = \frac{1}{M}\int_\Omega \left(e^0 + \frac{1}{2}\, {v^0}^2\right)\, dx, \\ I_\infty = 0, \end{cases} \tag{24}$$

Then our main result is

Theorem 2. *Suppose that the initial data satisfy (17) and that* T *is an arbitrary positive number.*

The unique solution $(\eta, v, \theta, \mathcal{I})$ *of the problem (1)(9)(23)(11) (12) satisfying (16) together with properties (18), (19) and (20) decays to the constant state* $(\eta_\infty, v_\infty = 0, \theta_\infty, \mathcal{I}_\infty = 0)$ *given by Proposition 1.*

The convergence holds in $H^1(\Omega)$ for (η, v, θ) and in $L^2(\Omega)$ for $\mathcal{I}$. Moreover there exist two positive numbers T_∞ and Γ such that

$$\|\eta - \eta_\infty\|_{L^2(\Omega)} + \|\theta - \theta_0\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} + \|\mathcal{I} - \mathcal{I}_\infty\|_{L^2(\Omega)} \leqslant K e^{-\Gamma t}, \quad (25)$$

for $t \geqslant T_\infty$.

The proof of this result (Section 3) exploits ideas of Okada-Kawashima [16] and Jiang [12]. It relies on suitable time-independent estimates given in Section 2, which constitute the main part of the paper.

Let us recall that the investigation of 1D viscous flows for compressible media goes back to the pioneer work of Antonsev-Kazhikov-Monakov [1] for the perfect gas. Subsequent works [13], [10], [11] deal with “real gas” with much more general equations of state (see [8] [9] and [18] for more recent presentations in the heat-conductive case)

2 Time-independent a priori estimates

Let T be an arbitrary positive number and let us denote by K, K_j $j = 1, 2, ...$ various positive constants which do not depend on T, but only on the physical constants of the problem.

We first get mass-energy estimates

Lemma 1. *Under the following condition on the data*

$$\left\|v^0\right\|_{L^2(\Omega)} + \left\|\eta^0\right\|_{L^1(\Omega)} + \left\|\theta^0\right\|_{L^1 \cap L^{r+1}(\Omega)} + \left\|I^0\right\|_{L^1(\Omega \times \mathbb{R}_+ \times S^1)} \leqslant N, \quad (26)$$

there exist a positive constant $K = K(N)$ such that

1. *the mass conservation*

$$\int_\Omega \eta \, dx = \int_\Omega \eta^0 \, dx, \quad (27)$$

2. *the energy equality*

$$\int_\Omega \left[e + \frac{1}{2} v^2 \right] \, dx = \int_\Omega \left[e^0 + \frac{1}{2} (v^0)^2 \right] \, dx, \quad (28)$$

3. *the estimate*

$$\|\eta\|_{L^\infty(0,T;L^1(\Omega))} + \|v\|_{L^\infty(0,T;L^2(\Omega))} + \|\theta\|_{L^\infty(0,T;L^\delta(\Omega))} \leqslant K, \quad (29)$$

   for any $1 \leqslant \delta \leqslant r + 1$,

4. *the condition*

$$\theta(x, t) > 0 \quad \text{for any } (x, t) \in Q_T, \quad (30)$$

hold.

Proof. 1. Integrating the first two equations (1) and using boundary conditions give (27) and (28).

2. Estimate (29) follows directly from (27) and (28).

4. Using (14), the positivity of $\theta(x,t)$ follows from that of $\theta^0(x)$ after the maximum principle applied to the third equation (1) $\square$

Lemma 2. *Any solution of the integro-differential problem*

$$\begin{cases} \dfrac{\partial}{\partial t}\ [\eta I(x,t;\nu,\omega)] + \dfrac{\partial}{\partial x}\ [(c\omega - v)\, I(x,t;\nu,\omega)] \\ = c\eta\sigma_s(\nu,\eta;\theta)\left[\tilde{I}(x,t;\nu) - I(x,t;\nu,\omega)\right] \qquad on\ \Omega\times[0,T]\times\mathbb{R}_+\times S^1, \\ I(0;\nu,\omega) = 0 \qquad for\ \omega\in(0,1), \\ I(M;\nu,\omega) = 0 \qquad for\ \omega\in(-1,0), \\ I(x,0;\nu,\omega) = I^0(x;\nu,\omega) \qquad on\ \Omega\times\mathbb{R}_+\times S^1 \end{cases} \tag{31}$$

satisfies the following bounds

$$\max_{[0,T]}\int_\Omega\int_0^\infty\int_{S^1}\eta I(x,t;\nu,\omega)\ d\omega\ d\nu\ dx \leqslant K, \tag{32}$$

$$\int_0^T\int_0^\infty\int_{S^1} I(M,t;\nu,\omega)\ d\omega\ d\nu\ dt \leqslant K, \tag{33}$$

$$\int_0^T\int_0^\infty\int_{S^1} I(0,t;\nu,\omega)\ d\omega\ d\nu\ dt \leqslant K, \tag{34}$$

$$\max_{[0,T]}\int_\Omega\int_0^\infty\int_{S^1}\eta I^2(x,t;\nu,\omega)\ d\omega\ d\nu\ dx \leqslant K, \tag{35}$$

$$\int_{Q_T}\int_0^\infty\int_{S^1}\eta\sigma_s(\eta,\theta;\nu)\left(\tilde{I}(x,t;\nu) - I(x,t;\nu,\omega)\right)^2\ d\omega\ d\nu\ dx\ dt \leqslant K. \tag{36}$$

$$\int_0^T\int_0^\infty\int_{S^1} I^2(M,t;\nu,\omega)\ d\omega\ d\nu\ dt \leqslant K, \tag{37}$$

$$\int_0^T\int_0^\infty\int_{S^1} I^2(0,t;\nu,\omega)\ d\omega\ d\nu\ dt \leqslant K, \tag{38}$$

$$\left|\int_{Q_T}\eta\,(S_F)_R\ dx\ dt\right| \leqslant K. \tag{39}$$

Proof. 1. Integrating the last equation (1) on $Q_T\times\mathbb{R}_+\times S^1$ and using boundary conditions, we get

$$\int_\Omega\int_0^\infty\int_{S^1}\eta I(x,t;\nu,\omega)\ d\omega\ d\nu\ dx - \int_\Omega\int_0^\infty\int_{S^1}\eta^0 I^0(x;\nu,\omega)\ d\omega\ d\nu\ dx$$

$$+c\int_0^T\int_0^\infty\int_0^1 \omega I(M,t;\nu,\omega)\,d\omega\,d\nu\,dt - c\int_0^T\int_0^\infty\int_{-1}^0 \omega I(0,t;\nu,\omega)\,d\omega\,d\nu\,dt = 0.$$

We get then (32), (33) and (34).

2. Multiplying $(31)_1$ by I, integrating on $\Omega\times S^1$ and using boundary conditions, we get

$$\frac{1}{2}\,\frac{d}{dt}\int_\Omega\int_{S^1}\eta I^2\;dx\;d\omega + \frac{1}{2}\;c\int_{S^1}\omega I^2(M,t;\nu,\omega)\;d\omega - \frac{1}{2}\;c\int_{S^1}\omega I^2(0,t;\nu,\omega)\;d\omega$$
$$+\int_\Omega\int_{S^1}\eta\sigma_s\left(\tilde{I}-I\right)^2\;dx\;d\omega = 0.$$

Integrating on time and frequency, we get (35), (36), (37) and (38).

3. Multiplying $(31)_1$ by ω, integrating on $\Omega\times S^1$ and using boundary conditions, we get

$$\frac{d}{dt}\int_\Omega\int_{S^1}\eta\omega I\;dx\;d\omega + \;c\int_{S^1}\omega^2 I(M,t;\nu,\omega)\;d\omega - \;c\int_{S^1}\omega^2 I(0,t;\nu,\omega)\;d\omega$$
$$= c\int_\Omega\int_{S^1}\eta(S_F)_R\;dx.$$

Integrating on time and frequency, we get

$$\left|\int_{Q_T}\eta\,(S_F)_R\;\;dx\;dt\right| \leqslant c^{-1}\max_{[0,T]}\int_\Omega\int_0^\infty\int_{S^1}\eta I\;d\omega\;d\nu\;dx$$
$$+\int_0^T\int_0^\infty\int_{S^1}\omega^2 I(M,t;\nu,\omega)\;d\omega\;d\nu\;dt + \int_0^T\int_0^\infty\int_{S^1}\omega^2 I(0,t;\nu,\omega)\;d\omega\;d\nu\;dt.$$

Using (32), (33) and (34) in the right-hand gives (39) $\square$

Lemma 3. *Under conditions (26) on the data, the following entropy inequality holds*

$$\int_{Q_t}\left(\frac{\kappa(\eta,\theta)}{\eta\theta^2}\;\theta_x^2 + \frac{\mu(\eta)}{\eta\theta}\;v_x^2\right)\;dx\;ds \leqslant K, \tag{40}$$

Proof. Total entropy $s = s_m + s_R$ is the sum of the entropy of matter s_m and entropy of radiation s_R, and the second principle of thermodynamics tells us that $\theta(s_m)_t = e_t + p\eta_t$, so using (1), and the isotropy of scattering in the lagrangian coordinates (see [3] for a general derivation, and also [6]), one gets

$$(s_m)_t = -\left(\frac{\kappa\theta_x}{\eta\theta}\right)_x + \frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2}. \tag{41}$$

¿From statistical mechanics mechanics, the entropy per mode of a boson gas is $k_B[(n+1)\log(n+1) - n\log n]$, where n is the occupation number related to I by

$$n = n(I) := \frac{c^2}{2h}\;\frac{I}{\nu^3}.$$

Multiplying by the number of modes, we find the entropy per mass unit

$$s_R = \eta \int_0^\infty \int_{S^1} \frac{2k_B\nu^2}{c^3} \; [(n+1)\log(n+1) - n\log n] \; d\nu \; d\omega.$$

Using the last equation (1), observing that for any regular function $n \to \chi(n)$ one has the identity

$$(\eta\chi)_t + [(c\omega - v)\chi]_x = \frac{c^3}{2h\nu^3} \; \chi' \eta S,$$

and choosing $\chi(n) = (n+1)\log(n+1) - n\log n$, we get after a direct computation

$$(s_R)_t + \left[\int_0^\infty \int_{S^1} \frac{2k_B\nu^2}{c^3}(c\omega - v) \; [(n+1)\log(n+1) - n\log n] \; d\nu \; d\omega\right]_x$$
$$= \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \; \log\frac{n+1}{n} \; S \; d\nu \; d\omega =: Q_R. \tag{42}$$

The right-hand side of (42) reads

$$Q_R = -\frac{\eta}{\theta} \, v \, (S_F)_R + \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde{I})+1}{n(\tilde{I})}\right] \; \sigma_s(\tilde{I} - I) \; d\nu \; d\omega.$$

As $u \to \log\frac{u+1}{u}$ is decreasing for $u > 0$, the last terms is positive, and we get finally

$$(s_R)_t + \left[\int_0^\infty \int_{S^1} \frac{2k_B\nu^2}{c^3}(c\omega - v) \; [(n+1)\log(n+1) - n\log n] \; d\nu \; d\omega\right]_x$$
$$= \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \left[\log\frac{n(I)+1}{n(I)} - \log\frac{n(\tilde{I})+1}{n(\tilde{I})}\right] \; \sigma_s(\tilde{I} - I) \; d\nu \; d\omega, \tag{43}$$

Where we used the fact that σ_s does not depend on ω.

Using the technique of [10] and defining the free energy $\psi := e - \theta s_m$ of the fluid, with $\psi_\theta = -s_m$ and $\psi_\eta = -p$, let us introduce the auxiliary function

$$\mathcal{E}(\eta,\theta) := \psi(\eta,\theta) - \psi(1,\theta_0) - (\eta - 1)\psi_\eta(1,\theta_0) - (\theta - \theta_0)\psi_\theta(\eta,\theta) - \theta_0 s_R.$$

A direct computation gives

$$\left(\mathcal{E} + \frac{1}{2}\, v^2\right)_t = [\sigma v + p(1,\theta_0)v - q]_x - \theta_0 s_t.$$

Plugging (41) and (43) in the right-hand side, we get finally

$$\left(\mathcal{E} + \frac{1}{2}\, v^2\right)_t + \theta_0 \left(\frac{\mu v_x^2}{\eta\theta} + \frac{\kappa\theta_x^2}{\eta\theta^2}\right)$$

$$+\theta_0\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)}-\log\frac{n(\tilde I)+1}{n(\tilde I)}\right]\ \sigma_s(I-\tilde I)\ d\nu\ d\omega$$

$$=[\sigma v+p(1,\theta_0)v-q]_x$$

$$+\left[\theta_0\int_0^\infty\int_{S^1}\frac{2k_B\nu^2}{c^3}(c\omega-v)\ [(n+1)\log(n+1)-n\log n]\ d\nu\ d\omega\right]_x.$$

Integrating on Q_t and using (28) and (9) the contribution of the first three boundary term is zero. Moreover using (10) to compute the contribution of the radiative terms boundary terms we have the final equality

$$\int_\Omega\left(\mathcal{E}+\frac{1}{2}\ v^2\right)\ dx+\theta_0\int_{Q_t}\left(\frac{\mu v_x^2}{\eta\theta}+\frac{\kappa\theta_x^2}{\eta\theta^2}\right)\ dx\ ds$$

$$+\theta_0\int_{Q_t}\eta\int_0^\infty\int_{S^1}\frac{k_B}{h\nu}\left[\log\frac{n(I)+1}{n(I)}-\log\frac{n(\tilde I)+1}{n(\tilde I)}\right]\ \sigma_s(\tilde I-I)\ d\nu\ d\omega\ dx\ ds$$

$$+\int_0^t\int_0^\infty\int_0^1\omega I(M,s;\omega,\nu)\ d\nu\ d\omega\ ds$$

$$-\int_0^t\int_0^\infty\int_{-1}^0\omega I(0,s;\omega,\nu)\ d\nu\ d\omega\ ds$$

$$+\theta_0\int_0^t\int_0^\infty\int_0^1\frac{2k_B\nu^2}{c^2}\ \omega\ [(n+1)\log(n+1)-n\log n](M,s;\omega,\nu)\ d\nu\ d\omega\ ds$$

$$-\theta_0\int_0^t\int_0^\infty\int_{-1}^0\frac{2k_B\nu^2}{c^2}\ \omega\ [(n+1)\log(n+1)-n\log n](0,s;\omega,\nu)\ d\nu\ d\omega\ ds$$

$$=\int_\Omega\left(\mathcal{E}^0+\frac{1}{2}\ {v^0}^2\right)\ dx. \tag{44}$$

Now we argue in the same way as [10] noting that, by using Taylor formula, for any $\eta>0$

$$\mathcal{E}(\eta,\theta)-\psi(\eta,\theta)+\psi(\eta,\theta_0)+(\theta-\theta_0)\psi_\theta(\eta,\theta)-\theta_0 s_R$$
$$=\psi(\eta,\theta_0)-\psi(1,\theta_0)-(\eta-1)\psi_\eta(1,\theta_0)\geqslant 0,$$

and that

$$\psi(\eta,\theta)-\psi(\eta,\theta_0)-(\theta-\theta_0)\psi_\theta(\eta,\theta)=-(\theta-\theta_0)^2\int_0^1(1-\alpha)\psi_{\theta\theta}(\eta,\theta+\alpha(\theta_0-\theta))\ d\alpha.$$

Using $\psi_{\theta\theta}=-\theta^{-1}e_\theta$ and estimates (14), we find

$$\psi(\eta,\theta)-\psi(\eta,\theta_0)-(\theta-\theta_0)\psi_\theta(\eta,\theta)\geqslant\frac{1}{4}\ K\left(\theta+\theta^{1+r}\right)-K.$$

Now one checks by elementary computations that $\eta E_R-\theta_0 s_R\geqslant K$, so we deduce that

$$\mathcal{E}(\eta,\theta)+\eta E_R\geqslant\frac{1}{4}\ K\left(\theta+\theta^{1+r}\right)-K,$$

and by plugging this into (44) we conclude, after (28), that (40) holds $\square$

Lemma 4. *Under the previous condition on the data (26), there exists positive constants* $\underline{\eta}$ *and* $\overline{\eta}$ *independent of* T *such that*

$$\underline{\eta} \leqslant \eta(x,t) \leqslant \overline{\eta} \quad \textit{for } (t,x) \in Q_T. \tag{45}$$

Proof.

1. Introducing the strictly increasing function $s \to \mathcal{M}(s) := \int_1^s \frac{\mu(\xi)}{\xi}\, d\xi$, one observes that $\mathcal{M}$ maps $(0, \inf_\Omega \eta^0]$ onto $(-\infty, 0)$.

If $\phi(x,t) := \int_0^t \sigma\ ds + \int_0^x v^0\ dy - \int_0^t \int_0^x \eta(S_F)_R\ dy\ ds$, then ϕ satisfies the equations $\phi_x = v$ and $\phi_t = \frac{\mu(\eta)}{\eta}\ v_x - p - \int_0^x \eta(S_F)_R\ dy$. Multiplying the last equation by η we find that

$$(\eta\phi)_t = (v\phi)_x + \mu\phi_{xx} - p\eta - v^2 - \eta \int_0^x \eta(S_F)_R\ dy.$$

Integrating on Q_t, and using boundary conditions we find

$$\int_\Omega \phi\eta\ dx = \int_\Omega \phi^0\eta^0\ dx$$
$$- \int_{Q_t} \left(p\eta + v^2\right)\ dx\ ds - \int_0^t \int_\Omega \eta \int_0^x \eta(S_F)_R\ dy\ dx\ ds. \tag{46}$$

Using (27) and a standard argument of [1], there exists a point $X(t) \in \Omega$ such that $\phi(X(t),t) = \frac{1}{R} \int_\Omega \phi\eta\ dx$ with $R := M \int_\Omega \eta^0\ dx$. Then after the definition of ϕ and (46), we find

$$\int_0^t \sigma(X(t),t)\ ds + \int_0^{X(t)} v^0 dy - \int_0^t \int_0^{X(t)} \eta(S_F)_R\ dy\ ds$$
$$= \frac{1}{R} \left\{ \int_\Omega \eta^0(x) \int_0^x v^0(y) dy\ dx - \int_{Q_t} \left(p\eta + v^2\right)\ dx\ ds \right.$$
$$\left. - \int_0^t \int_\Omega \eta \int_0^x \eta(S_F)_R\ dy\ dx\ ds \right\}. \tag{47}$$

Now rewriting the second equation (1) as $\mathcal{M}_{xt} = v_t + p_x + \eta\,(S_F)_R$ and integrating it first on $[0,t]$ then on $[X(t),x]$, we find

$$\mathcal{M}(x,t) - \mathcal{M}(X(t),t) - \mathcal{M}^0(x) + \mathcal{M}^0(X(t))$$
$$= \int_{X(t)}^x \left(v(y,t) - v^0(y)\right)\ dy + \int_0^t p(x,s)\ ds - \int_0^t p(X(t),s)\ ds$$
$$+ \int_{X(t)}^x \int_0^t \eta(S_F)_R\ ds\ dy.$$

As the definition of $\mathcal{M}$ gives $\int_0^t \sigma(X(t),t)\ ds = -\int_0^t p(X(t),s)\ ds + \mathcal{M}(X(t),t) - \mathcal{M}^0(X(t))$, we get

$$\mathcal{M}(\eta(x,t)) = \mathcal{M}(\eta^0(x)) + \int_0^t p\ ds + \int_{X(t)}^x \big(v(y,t) - v^0(y)\big)\ dy$$

$$+\int_0^t \sigma(X(t),t)\ ds + \int_{X(t)}^x \int_0^t \eta\,(S_F)_R\ dt\ dx, \tag{48}$$

and using (47), we obtain

$$\mathcal{M}(\eta(x,t)) = \int_0^t p\ ds + \Psi(x,t), \tag{49}$$

where

$$\Psi(x,t) := \mathcal{M}(\eta^0(x)) + \int_{X(t)}^x \big(v(y,t) - v^0(y)\big)\ dy$$

$$+\int_{X(t)}^x \int_0^t \eta(S_F)_R\ dy\ ds + \int_0^t \int_0^{X(t)} \eta(S_F)_R\ dy\ ds - \int_0^{X(t)} v^0 dy$$

$$+\frac{1}{R}\int_\Omega \eta^0(x) \int_0^x v^0(y) dy\ dx$$

$$-\frac{1}{R}\int_0^t \int_\Omega \eta \int_0^x \eta\,(S_F)_R\ dy\ dx\ ds - \frac{1}{R}\int_0^t \int_\Omega (v^2 + p\eta)\ dx\ ds = \sum_{j=1}^{8} \Psi_j(x,t).$$

Integrating (15) we find $\mu_0 \log\eta \leqslant \mathcal{M}(\eta) \leqslant \mu_1 \log\eta$, then after a standard computation we get from (49) the inequalities

$$\eta(x,t) \leqslant \left\{ \exp\left(\frac{1}{\mu_1}\ \Psi(x,t)\right) \left[1 + \frac{1}{\mu_1}\int_0^t (p\eta)(x,s) \exp\left(-\frac{1}{\mu_1}\ \Psi(x,s)\right) ds\right]\right\}^{\frac{\mu_1}{\mu_0}}, \tag{50}$$

and

$$\eta(x,t) \geqslant \left\{ \exp\left(\frac{1}{\mu_0}\ \Psi(x,t)\right) \left[1 + \frac{1}{\mu_0}\int_0^t (p\eta)(x,s) \exp\left(-\frac{1}{\mu_0}\ \Psi(x,s)\right) ds\right]\right\}^{\frac{\mu_0}{\mu_1}}, \tag{51}$$

so we are led to bound the right (resp. left)-hand side in (50) (resp. (51)).

One first easily check by using conditions on initial data, (28) and Lemma 2 that

$$K^{-1} \leqslant \exp\left(\frac{1}{\mu_0}\sum_{j=1}^{6} \Psi_j(x,t)\right) \leqslant K.$$

Integrating by parts in Ψ_7, we get

$$\int_0^t \int_\Omega \eta \int_0^x \eta\,(S_F)_R\ dy\ dx\ ds = \int_\Omega \eta\ dx \times \int_0^t \int_\Omega \eta\,(S_F)_R\ dx\ ds$$

$$-\int_0^t\int_\Omega\left(\int_0^x \eta\ dy\right)\eta\left(S_F\right)_R\ dx\ ds,$$

and using Lemmas 1 and 2, we get

$$K^{-1}\leqslant\exp\left(\frac{1}{\mu_0}\ \Psi_7(x,t)\right)\leqslant K.$$

Then we get

$$\eta(x,t)\leqslant K_1\left[1+\frac{1}{\mu_1}\int_0^t(p\eta)(x,s)\exp\left(-\frac{1}{\mu_1}\ \int_s^t\int_\Omega(v^2+p\eta)\ dy\ d\tau\right)ds\right]^{\frac{\mu_1}{\mu_0}},\tag{52}$$

and

$$\eta(x,t)\geqslant K_1^{-1}\left[\int_0^t(p\eta)(x,s)\exp\left(-\frac{1}{\mu_0}\ \int_s^t\int_\Omega(v^2+p\eta)\ dy\ d\tau\right)ds\right]^{\frac{\mu_0}{\mu_1}}.\tag{53}$$

In (52) we have, after (14)

$$\exp\left(-\frac{1}{\mu_1}\ \int_s^t\int_\Omega(v^2+p\eta)\ dy\ d\tau\right)\leqslant e^{-Mc_4(t-s)},$$

then

$$\eta(x,t)\leqslant K_1\left[1+\frac{C_4}{\mu_1}\int_0^t(1+\theta^{1+r})\ e^{-K(t-s)}ds\right]^{\frac{\mu_1}{\mu_0}}.\tag{54}$$

Now, for any $t>0$, there is a number $a(t)\in\Omega$ such that $\theta(a(t),t)=\frac{1}{M}\int_\Omega\theta\ dx$, so from the inequality

$$\theta^{\frac{r+1}{2}}(x,t)\leqslant\theta^{\frac{r+1}{2}}(a(t),t)+\frac{r-1}{2}\int_\Omega\theta^{\frac{r-1}{2}}|\theta_x|\ dx,$$

and using Cauchy-Schwarz inequality and Lemma 1, we get that $\theta^{1+r}\leqslant K(1+V(t))$, where $V\in L^1(0,T)$. Putting this into (54), we get clearly that $\eta(x,t)\leqslant\overline{\eta}$, for a positive constant $\overline{\eta}$ independent of T.

In the same stroke for (53) we see, after Lemma 1 and (14) that $\int_\Omega(v^2+p\eta)\ dx\leqslant K$, so

$$\exp\left(-\frac{1}{\mu_0}\ \int_s^t\int_\Omega(v^2+p\eta)\ dy\ d\tau\right)\geqslant e^{-K(t-s)},$$

then it is sufficient to show that a lower bound exists, for any $t\geqslant T_0$ and some $T_0\geqslant 0$.

We have

$$\eta(x,t)\geqslant K_1^{-1}\left[\int_0^t\theta^{1+r}e^{-K(t-s)}ds\right]^{\frac{\mu_0}{\mu_1}}\geqslant K_1^{-1}\left[\int_0^t(1-V(s))e^{-K(t-s)}ds\right]^{\frac{\mu_0}{\mu_1}}$$

$$\geqslant K_2 \left[1 - e^{-Kt} - \int_0^t V(s) e^{-K(t-s)} ds\right]^{\frac{\mu_0}{\mu_1}} \geqslant K_2^{-1},$$

which implies that $\eta(x,t) \geqslant \underline{\eta}$, for a positive constant $\underline{\eta}$ independent of T. □
As the first and third equation (1) are similar to those studied by Jiang in [12], we will use some of his estimates without proof (see [12] for details).

Lemma 5.

$$K(1 - V(t)) \leqslant \theta^{2\lambda}(x,t) \leqslant K(1 + V(t)), \tag{55}$$

where $V(t) := \int_\Omega \frac{1+\theta^q}{\theta^2}\, \theta_x^2\, dx$*, for any* $\lambda \leqslant \frac{q+r+1}{2}$.

Proof. Just use the inequality $\theta^\lambda(x,t) \leqslant K + K \int_\Omega \theta^{\lambda-1}|\theta_x|\, dx$ together with (40) and Lemma 4 □

Lemma 6.

$$\int_{Q_T} (S_F)_R^2\ dx\ dt \leqslant K. \tag{56}$$

Proof. After the definitions of $(S_F)_R$ and (45), we get

$$\int_{Q_T} (S_F)_R^2\ dx\ dt \leqslant \int_{Q_T} \left(\int_0^\infty \int_{S^1} \eta\sigma_s(\tilde{I} - I)\ d\omega\ d\nu\right)^2 dx\ ds$$

$$\leqslant K \int_{Q_T} \int_0^\infty \int_{S^1} \eta\sigma_s(\tilde{I} - I)^2\ d\omega\ d\nu\ dx\ ds,$$

which implies (56) □

Lemma 7.

$$\int_\Omega v^2\ dx + \int_\Omega \eta_x^2\ dx + \int_{Q_t} v_x^2\ dx\ ds + \int_{Q_t} (1 + \theta^{1+r})\eta_x^2\ dx\ ds \leqslant K. \tag{57}$$

Proof. 1. Multiplying the second equation (1) by v and integrating by parts on Q_t for any $t \in [0,T]$, we get

$$\int_\Omega v^2\ dx + \int_{Q_t} \frac{\mu}{\eta}\ v_x^2\ dx\ ds = \int_\Omega (v^0)^2\ dx + \int_{Q_t} p_x v\ dx\ ds$$

$$- \int_{Q_t} \eta v (S_F)_R\ dx\ ds. \tag{58}$$

In the right-hand side, the last term in the right-hand side is bounded as follows, using Lemmas 1 and 4 and (14)

$$\left|\int_{Q_t} \eta v (S_F)_R\ dx\ ds\right| \leqslant \int_{Q_t} |v| \int_0^\infty \int_{S^1} \eta\sigma_s(\tilde{I} - I)\ d\omega\ d\nu\ dx\ ds$$

$$\leqslant K\int_{Q_t}|v|\left(\int_0^\infty\int_{S^1}\eta\sigma_s(\tilde I-I)^2\ d\omega\ d\nu\right)^{1/2}\ dx\ ds$$

$$\leqslant K\left(\int_{Q_t}v^2\ dx\ ds\right)^{1/2}\left(\int_{Q_t}\int_0^\infty\int_{S^1}\eta\sigma_s(\tilde I-I)^2\ d\omega\ d\nu\ dx\ ds\right)^{1/2}.$$

The last integral is bounded after Lemma 6 and, using

$$\int_{Q_t}v^2\ dx\ ds\leqslant K\int_0^t\max_\Omega v^2\ ds\leqslant K\int_0^t\left(\int_\Omega|v_x|\ dx\right)^2\ ds$$

$$\leqslant K\int_0^t\int_\Omega\frac{v_x^2}{\eta\theta}\ dx\ ds\leqslant K,$$

the first one is also bounded, so (58) rewrites

$$\int_\Omega v^2\ dx+\int_{Q_t}\frac{\mu}{\eta}\ v_x^2\ dx\ ds\leqslant K+\int_{Q_t}|p_xv|\ dx\ ds$$

$$\leqslant K+\int_{Q_t}(1+\theta^{1+r})|\eta_xv|\ dx\ ds+\int_{Q_t}(1+\theta^r)|\theta_xv|\ dx\ ds$$

$$\leqslant K_\varepsilon+\varepsilon\int_{Q_t}(1+\theta^{1+r})\eta_x^2\ dx\ ds+\int_0^t\max_\Omega v^2\int_\Omega(1+\theta^{1+r})\ dx\ ds$$

$$+K\int_{Q_t}\frac{(1+\theta^r)\theta_x^2}{\eta\theta^2}\ dx\ ds.$$

So using (45), we get

$$\int_\Omega v^2\ dx+\int_{Q_t}v_x^2\ dx\ ds\leqslant K_\varepsilon+\varepsilon\int_{Q_t}(1+\theta^{1+r})\eta_x^2\ dx\ ds. \tag{59}$$

2. Multiplying the second equation (1) by $\mathcal{M}_x$ and integrating by parts on Q_t for any $t\in[0,T]$, we get

$$\frac12\int_\Omega\mathcal{M}_x^2\ dx-\frac12\int_\Omega{\mathcal{M}^0}_x^2\ dx\leqslant\int_{Q_t}v_t\mathcal{M}_x\ dx-K\int_{Q_t}(1+\theta^{1+r})\eta_x^2\ dx\ ds$$

$$+K\int_{Q_t}(1+\theta^r)|\eta_x\theta_x|\ dx\ ds+\int_{Q_t}\eta(S_F)_R\mathcal{M}_x\ dx\ ds. \tag{60}$$

After integrating by parts, the first term in the right-hand side reads

$$\int_{Q_t}v_t\mathcal{M}_x\ dx\ ds=\int_\Omega v\mathcal{M}_x\ dx-\int_\Omega v^0{\mathcal{M}^0}_x\ dx+\int_{Q_t}v_x\mathcal{M}_t\ dx\ ds.$$

So

$$\int_{Q_t}v_t\mathcal{M}_x\ dx\ ds\leqslant K+\varepsilon\int_\Omega\mathcal{M}_x^2\ dx+K\int_\Omega v_x^2\ dx\ ds.$$

Using Cauchy-Schwarz inequality and Lemma 6, each of the two last terms is bounded by

$$K_\varepsilon + \varepsilon \int_{Q_t} (1 + \theta^{1+r})\eta_x^2 \, dx \, ds.$$

Plugging all of these estimates into (60) and using (45), we have

$$\int_\Omega \eta_x^2 \, dx + \int_{Q_t} (1 + \theta^{1+r})\eta_x^2 \, dx \, ds \leqslant K_\varepsilon + K \int_{Q_t} v_x^2 \, dx \, ds. \tag{61}$$

Finally, multiplying (61) by 2ε, adding to(59) and choosing $\varepsilon < \frac{1}{4K}$, we recover the estimate (57) $\square$

Lemma 8.

$$\int_\Omega v_x^2 \, dx + \int_0^t \max_\Omega v_x^2(\cdot, s) \, ds + \int_{Q_t} v_{xx}^2 \, dx \, ds \leqslant K. \tag{62}$$

Proof. 1. Multiplying the third equation (1) by e and integrating by parts on Ω, one get first (see [12])

$$\int_\Omega (\theta^2 + \theta^{2r+2}) \, dx + \int_{Q_t} (1 + \theta^{q+r})\theta_x^2 \, dx \, ds$$

$$\leqslant K_\varepsilon + \varepsilon \int_{Q_t} v_{xx}^2 \, dx \, ds + K \int_0^t V(s) \int_\Omega \theta^{2r+2} \, dx. \tag{63}$$

2. Multiplying the second equation (1) by $-v_{xx}$ and integrating by parts on Ω, we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_x v_{tx} \, dx + \int_\Omega \frac{\mu}{\eta} v_{xx}^2 \, dx = \int_\Omega v_x p_x \, dx - \int_\Omega \left(\frac{\mu}{\eta}\right)_x v_x v_{xx} \, dx + \int_\Omega v_{xx} \eta (S_F)_R \, dx.$$

Integrating on $[0,t]$ and using (15), we find

$$\frac{1}{2}\int_\Omega v_x v_{tx} \, dx + \frac{\mu_1}{\overline{\eta}} \int_{Q_t} v_{xx}^2 \, dx \, ds \leqslant \int_{Q_t} |v_x(p_\eta \eta_x + p_\theta \theta_x)| \, dx \, ds$$

$$+ \left(\frac{\mu_1}{\overline{\eta}^2} + \frac{\mu_2}{\overline{\eta}}\right) \int_{Q_t} |\eta_x v_x v_{xx}| \, dx \, ds + \int_{Q_t} |v_{xx}\eta(S_F)_R| \, dx \, ds$$

$$\leqslant \varepsilon \int_{Q_t} v_{xx}^2 \, dx \, ds + K_\varepsilon \int_{Q_t} v_x^2 \eta_x^2 \, dx \, ds$$

$$+ K_\varepsilon \int_{Q_t} (1 + \theta^{q+r})\theta_x^2 \, dx \, ds + K_\varepsilon \int_{Q_t} (1 + \theta^{2+2r})\eta_x^2 \, dx \, ds.$$

In order to bound the second term in the right-hand side we remark that, using Lemma 7

$$\max_\Omega v_x^2(\cdot, t) \leqslant K \int_\Omega v_x^2 \, dx + \varepsilon \int_\Omega v_{xx}^2 \, dx \leqslant K + \varepsilon \int_\Omega v_{xx}^2 \, dx.$$

To bound the third term in the right-hand side, we observe that, after (28) and (40) (see [12]), for each $t > 0$ there exists a $a(t) \in \Omega$ such that $\theta(a(t), t) \leqslant \theta^*$, for a positive θ^* independent of time. Then for any $\lambda > 0$

$$\left(\theta^\lambda(x,t) - \theta^{*\lambda}\right)^2 \leqslant KV(t) \int_\Omega \theta^{\lambda-q+1}\ dx. \tag{64}$$

In particular

$$\int_0^t \left(\theta^\lambda(x,t) - \theta^{*\lambda}\right)^2\ ds \leqslant K,$$

for any $\lambda \leqslant r + q$.

So we have, using Lemma 7

$$\int_{Q_t} (1 + \theta^{2+2r})\eta_x^2\ dx\ ds \leqslant K + \int_{Q_t} (\theta^{1+r} - \theta^*)^2 \eta_x^2\ dx\ ds \leqslant K.$$

Finally, we get for ε small enough

$$\int_\Omega v_x^2\ dx + \int_{Q_t} v_{xx}^2\ dx\ ds \leqslant K + K \int_{Q_t} (1 + \theta^{q+r})\theta_x^2\ dx\ ds. \tag{65}$$

Now adding (65) to (63) and applying Gronwall's Lemma gives (62) $\square$

Lemma 9. *Let us introduce the two quantities*

$$Y(t) := \int_\Omega \left(1 + \theta^{2q}\right) \theta_x^2\ dx, \qquad X(t) := \int_{Q_t} \left(1 + \theta^{q+r}\right) \theta_t^2\ dx\ ds.$$

The following estimates hold

$$X(t) + Y(t) \leqslant K, \tag{66}$$

and

$$\max_{Q_t} \theta \leqslant K. \tag{67}$$

Proof. From the previous Lemma 8, we have

$$\theta^{q+r+2} - \theta^{*\,q+r+2} \leqslant KY^{1/2}(t).$$

Then

$$\max_{Q_t} \theta \leqslant KY^{\frac{1}{2q+2r+4}}. \tag{68}$$

¿From (1), the equation for the internal energy reads

$$e_\theta \theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\ v_x^2 = \left(\frac{\kappa \theta_x}{\eta}\right)_x.$$

Defining the auxiliary function $K(\eta,\theta) := \int_0^\theta \frac{\kappa(\eta,u)}{u}\ du$, multiplying the previous equation by K_t and integrating by parts, we get

$$\int_{Q_t} \left(e_\theta \theta_t + \theta p_\theta v_x - \frac{\mu}{\eta}\ v_x^2 \right) K_s\ dx\ ds + \int_{Q_t} \left(\frac{\kappa \theta_x}{\eta} \right) K_{sx}\ dx\ ds = 0. \qquad (69)$$

Observing that $K_t = K_\eta v_x + \frac{\kappa}{\eta}\ \theta_t$, $K_{xt} = \left(\frac{\kappa\theta_x}{\eta}\right)_t + K_{\eta\eta} v_x \eta_x + \left(\frac{\kappa}{\eta}\right)_\eta \eta_x \theta_t$ and that after (14) $|K_\eta| + |K_{\eta\eta}| \leqslant C(1+\theta^{q+1})$, we can estimate all the contributions in (69).

After (14) we have the lower bound

$$\int_{Q_t} \kappa e_\theta \theta_s^2\ dx\ ds \geqslant \frac{c_6 c_1}{\overline{\eta}}\ X(t),$$

Using (14) and Lemma 4

$$\left| \int_{Q_t} e_\theta \theta_s K_\eta v_x\ dx\ ds \right| \leqslant K \int_{Q_t} (1+\theta)^{q+r+1} |\theta_s v_x|\ dx\ ds$$

$$\leqslant \frac{c_6 c_1}{8\overline{\eta}}\ X(t) + K(1 + \max_{Q_t} \theta^{q+r+2}).$$

In the same stroke

$$\left| \int_{Q_t} \left(\theta p_\theta v_x - \frac{\mu}{\eta}\ v_x^2 \right) K_s\ dx\ ds \right|$$

$$\leqslant K \int_{Q_t} (1+\theta)^{q+r+2} v_x^2\ dx\ ds + K \int_{Q_t} (1+\theta)^{q+1} |v_x|^3\ dx\ ds$$

$$+K \int_{Q_t} (1+\theta)^{q+1} |v_x \theta_s|\ dx\ ds + K \int_{Q_t} (1+\theta)^q v_x^2 |\theta_s|\ dx\ ds$$

$$\leqslant \frac{c_6 c_1}{8\overline{\eta}}\ X + K(1 + \max_{Q_t} \theta^{q+r+2}).$$

Using (14) we have

$$\left| \int_{Q_t} \frac{\kappa\theta_x}{\eta} \left(\frac{\kappa\theta_x}{\eta} \right)_s\ dx\ ds \right| \geqslant \frac{c_6^2}{2\overline{\eta}^2}\ Y(t) - K.$$

$$\left| \int_{Q_t} \frac{\kappa\theta_x}{\eta}\ (K_\eta v_{xx} + K_{\eta\eta} v_x \eta_x)\ dx\ ds \right| \leqslant K \int_{Q_t} (1+\theta)^{2q+1} |\theta_x| (|v_{xx}| + |v_x \eta_x|)\ dx\ ds$$

$$\leqslant K \left(\int_{Q_t} (1+\theta)^{4q+2} \theta_x^2\ dx\ ds \right)^{1/2} \leqslant K \left(1 + \max_{Q_t} \theta^{1+\frac{3q}{2}} \right).$$

Using (14) we have also

$$\left| \int_{Q_T} \frac{\kappa\theta_x}{\eta}\ \left(\frac{\kappa}{\eta} \right)_\eta\ \eta_x \theta_t\ dx\ ds \right| \leqslant \frac{c_6 c_1}{8\overline{\eta}}\ X(t) + K \int_{Q_t} \left[\frac{\kappa\theta_x}{\eta} \right]^2 (1 + \theta^{q-r}) \eta_x^2\ dx\ ds$$

$$\leqslant K + \frac{c_6 c_1}{8\overline{\eta}}\ X(t) + K\left(1 + \max_{Q_t} \theta^{2q-2r}\right)$$

$$+K\left(1 + \max_{Q_t} \theta^{q-r}\right) \times \int_{Q_t} \left|\frac{\kappa\theta_x}{\eta}\right| \left|\left[\frac{\kappa\theta_x}{\eta}\right]_x\right|\ dx\ ds.$$

But the last integral is estimated by

$$\int_{Q_t} \left|\frac{\kappa\theta_x}{\eta}\right| \left|\left[\frac{\kappa\theta_x}{\eta}\right]_x\right|\ dx\ ds.$$

$$\leqslant \left(\int_{Q_t} (1+\theta^{q-r}) \left[\frac{\kappa\theta_x}{\eta}\right]_x^2\ dx\ ds\right)^{1/2}$$

$$\leqslant K\left(\int_{Q_t} (1+\theta^{q-r})\theta_s^2 + (1+\theta^{q+r+2})v_x^2 + (1+\theta^{q-r})v_x^4\ dx\ ds\right)^{1/2}$$

$$\leqslant KX(t) + K\left(1 + \max_{Q_t} \theta^{\frac{q+r+2}{2}}\right),$$

so finally

$$\left|\int_{Q_t} \frac{\kappa\theta_x}{\eta}\ \left(\frac{\kappa}{\eta}\right)_\eta\ \eta_x\theta_t\ dx\ ds\right| \leqslant \frac{c_6 c_1}{4\overline{\eta}}\ X(t) + K\left(1 + \max_{Q_t} \theta^{2q+1}\right),$$

Plugging all the previous estimates into (69), we get

$$\frac{c_6 c_1}{2\overline{\eta}}\ X(t) + \frac{c_6^2}{2\overline{\eta}^2}\ Y(t) \leqslant K\left(1 + \max_{Q_t} \theta^{2q+1}\right).$$

Using (68), we end with

$$\frac{c_6 c_1}{2\overline{\eta}}\ X(t) + \frac{c_6^2}{2\overline{\eta}^2}\ Y(t) \leqslant K\left(1 + Y^{\frac{2q+1}{2q+2r+4}}\right),$$

which ends the proof $\square$

Corollary 1. *The quantities*

$$\int_{Q_t} \theta_{xx}^2\ dx\ ds, \quad \int_{Q_t} v_s^2\ dx\ ds, \quad \int_{Q_t} \theta_t^2\ dx\ dt, \tag{70}$$

are bounded independently of time.

Proof. The first bound is a consequence of the following inequality (itself following from the third equation (1))

$$\theta_{xx}^2 \leqslant K[\theta_t^2 + v_x^2 + v_x^4 + \eta_x^2\theta_x^2 + \theta_x^4].$$

We know from Lemmas that $v_x \in L^2(Q_T)$ and $\theta_t \in L^2(Q_T)$. Moreover as

$$\int_{Q_t} v_x^4 \, dx \, ds \leqslant \int_0^t \max_\Omega v_x^2 \int_\Omega v_x^2 \, dx \, ds \leqslant K,$$

after Lemma 8,

$$\int_{Q_t} \theta_x^4 \, dx \, ds \leqslant \int_0^t \max_\Omega \theta_x^2 \int_\Omega \theta_x^2 \, dx \, ds \leqslant K,$$

after Lemma 9 and

$$\int_{Q_t} \eta_x^2 \theta_x^2 \, dx \, ds \leqslant \int_0^t \max_\Omega \theta_x^2 \int_\Omega \eta_x^2 \, dx \, ds \leqslant K,$$

after Lemmas 7, the bound (70 follows □

Lemma 10.

$$\max_{[0,T]} \int_\Omega \int_0^\infty \int_{S^1} I_t^2 \, d\omega \, d\nu \, dx \leqslant K, \tag{71}$$

$$\max_{[0,T]} \int_\Omega \int_0^\infty \int_{S^1} I_x^2 \, d\omega \, d\nu \, dx \leqslant K. \tag{72}$$

Proof. Going back to Eulerian coordinates, it is sufficient to prove that $I_\tau \in L^2(\mathcal{O} \times [0,T] \times \mathbb{R}_+ \times S^1)$ and $I_y \in L^2(\mathcal{O} \times [0,T] \times \mathbb{R}_+ \times S^1)$, where $I(y,\tau;\nu,\omega)$ solves the problem

$$\begin{cases} \dfrac{\partial}{\partial \tau} I(y,\tau;\nu,\omega) + c\omega \dfrac{\partial}{\partial y} I(y,\tau;\nu,\omega) = c\sigma_s(\nu,\eta,\theta) \left[\tilde{I}(y,\tau;\nu) - I(y,\tau;\nu,\omega)\right] \\ \qquad =: S(I;y,\tau;\nu,\omega) \quad \text{on } \mathcal{O} \times [0,T] \times \mathbb{R}_+ \times S^1, \\ I(0;\nu,\omega) = 0 \quad \text{for } \omega \in (0,1), \\ I(L;\nu,\omega) = 0 \quad \text{for } \omega \in (-1,0), \\ I(y,0;\nu,\omega) = I^0(y;\nu,\omega) \quad \text{on } \mathcal{O} \times \mathbb{R}_+ \times S^1 \end{cases} \tag{73}$$

We can use a bootstrap method. Derivating the equation with respect to τ and putting $J := I_\tau$, one checks that J solves the problem

$$\begin{cases} J_\tau + c\omega J_y = S_\tau \quad \text{on } \mathcal{O} \times [0,T] \times \mathbb{R}_+ \times S^1, \\ J(0;\nu,\omega) = 0 \quad \text{for } \omega \in (0,1), \\ J(L;\nu,\omega) = 0 \quad \text{for } \omega \in (-1,0), \\ J(y,0;\nu,\omega) = J^0(y,0;\nu,\omega) \\ = -\omega I_y^0(y;\nu,\omega) + S(I^0(y;\nu,\omega)) \quad \text{on } \mathcal{O} \times \mathbb{R}_+ \times S^1, \end{cases} \tag{74}$$

with the right-hand side

$$S_\tau = S(J; y, \tau; \nu, \omega) + \Phi(I; y, \tau; \nu, \omega),$$

where $\Phi \in L^2\left(\mathcal{O} \times [0,T] \times \mathbb{R}_+ \times S^1\right)$, after the Eulerian counterparts of Lemmas 1-10. Note that we have used the equation to derive the initial condition.

Now we proceed as in Lemma 2. Multiplying equation (74) by J integrating by parts on $[0,\tau] \times \mathcal{O} \times \mathbb{R}_+ \times S^1$ and using Cauchy-Schwarz, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{O}}\int_0^\infty\int_{S^1} J^2 \, d\omega\, d\nu\, dy - \frac{1}{2}\int_{\mathcal{O}}\int_0^\infty\int_{S^1} (J^0)^2 \, d\omega\, d\nu\, dy \\
&+c\int_0^\infty\int_0^1 \omega J^2(L,\tau;\nu,\omega)\, d\omega\, d\nu - c\int_0^\infty\int_{-1}^0 \omega J^2(0,\tau;\nu,\omega)\, d\omega\, d\nu \\
&+\int_0^\tau\int_{\mathcal{O}}\int_0^\infty\int_{S^1} \eta\sigma_s\left(\tilde{J} - J\right)^2 \, d\omega\, d\nu\, dy\, dt \\
&\leqslant \frac{1}{2}\int_0^\tau\int_{\mathcal{O}}\int_0^\infty\int_{S^1} \Phi^2 \, d\omega\, d\nu\, dy\, dt.
\end{aligned}$$

After (14) the right-hand side is bounded, so this last inequality clearly implies (71).

In the same stroke, derivating the equation with respect to y and putting $K := I_y$, one checks that K solves the problem

$$\begin{cases}
K_\tau + c\omega K_y = S_\tau \quad \text{on } \mathcal{O} \times [0,T] \times \mathbb{R}_+ \times S^1, \\
K(0;\nu,\omega) = 0 \quad \text{for } \omega \in (0,1), \\
K(L;\nu,\omega) = \quad \text{for } \omega \in (-1,0), \\
K(y,0;\nu,\omega) = K^0(y,0;\nu,\omega) = I^0_y(y;\nu,\omega) \quad \text{on } \mathcal{O} \times \mathbb{R}_+ \times S^1,
\end{cases} \tag{75}$$

with the right-hand side

$$S_y = S(K; y, \tau; \nu, \omega) + \Psi(K; y, \tau; \nu, \omega),$$

where $\Psi \in L^2\left(\mathcal{O} \times [0,T] \times \mathbb{R}_+ \times S^1\right)$, after the Eulerian counterparts of Lemmas 1-10.

As previously, multiplying equation (75) by K and integrating by parts on $[0,\tau] \times \mathcal{O} \times \mathbb{R}_+ \times S^1$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\mathcal{O}}\int_0^\infty\int_{S^1} K^2 \, d\omega\, d\nu\, dy - \frac{1}{2}\int_{\mathcal{O}}\int_0^\infty\int_{S^1} (K^0)^2 \, d\omega\, d\nu\, dy \\
&+c\int_0^\infty\int_0^1 \omega K^2(L,\tau;\nu,\omega)\, d\omega\, d\nu - c\int_0^\infty\int_{-1}^0 \omega K^2(0,\tau;\nu,\omega)\, d\omega\, d\nu
\end{aligned}$$

$$+\int_0^\tau \int_{\mathcal{O}} \int_0^\infty \int_{S^1} \eta\sigma_s \left(\tilde{K}-K\right)^2 \, d\omega\, d\nu\, dy\, dt$$
$$\leqslant \frac{1}{2} \int_0^\tau \int_{\mathcal{O}} \int_0^\infty \int_{S^1} \Phi^2 \, d\omega\, d\nu\, dy\, dt.$$

This inequality clearly implies (72) □

3 Proof of Theorem 2

1. Applying the following elementary result (see Brézis [2]) " if, for a $1 \leqslant p < \infty$, the function u is in $W^{1,p}(\mathbb{R}_+)$ then $\lim_{t\to\infty} u(t) = 0$" to the quantities $\|\eta - \eta_\infty\|_{H^1(\Omega)}$, $\|v\|_{H^1(\Omega)}$,, $\|\theta - \theta_\infty\|_{H^1(\Omega)}$ and $\|\mathcal{I}\|_{L^2(\Omega)}$, one has first to check that

$$\int_0^\infty \left[\left| \frac{d}{dt} \int_\Omega \eta_x^2 \, dx \right| + \left| \frac{d}{dt} \int_\Omega v_x^2 \, dx \right| + \left| \frac{d}{dt} \int_\Omega \theta_x^2 \, dx \right| \right] dt \leqslant K,$$

which follows from the fact that $\eta_t, v_t, \theta_t, \eta_{xx}, v_{xx}$ and θ_{xx} are in $L^2(\Omega)$ after the results of Section 2, and

$$\int_0^\infty \left| \frac{d}{dt} \int_\Omega \mathcal{I}^2 \, dx \right| \, dt \leqslant K,$$

which follows from Lemmas 2 and 10.

Finally it remains to prove that

$$\int_0^\infty \left[\int_\Omega |\eta - \eta_\infty|^2 \, dx + \int_\Omega v^2 \, dx + \int_\Omega |\theta - \theta_\infty|^2 \, dx + \int_\Omega \mathcal{I}^2 \, dx \right] dt \leqslant K.$$

After (27), the only thing to check is the convergence of temperature, for which we can reproduce verbatim the argument of Jiang in [12].

2. The exponential decay is finally obtained by applying the method of [16].

Let us define the modified energy of the matter

$$E(\eta, v, \theta) := \frac{1}{2} v^2 + \psi(\eta, \theta) - \psi(\eta_0, \theta_0) - (\eta - \eta_0)\psi_\eta(\eta_0, \theta_0) - (\theta - \theta_0)\psi_\theta(\eta, \theta),$$

where ψ is the free energy.

Introduce the set

$$\mathcal{O}_{k_1,k_2} := \left\{ \eta, \theta \ : \ \log\left|\frac{\eta}{\eta_0}\right| < k_1, \ \log\left|\frac{\theta}{\theta_0}\right| < k_2 \right\}.$$

We have the following two-sided inequalities for the energy and "reduced" production of radiative entropy

Lemma 11. *1. There exist $a > 0$ such that $\forall(\eta, \theta) \in \mathcal{O}(k_1, k_2)$,*

$$\frac{1}{2} v^2 + a^{-1} \left(|\eta - \eta_0|^2 + |\theta - \theta_0|^2\right) \leqslant E \leqslant \frac{1}{2} v^2 + a \left(|\eta - \eta_0|^2 + |\theta - \theta_0|^2\right) \tag{76}$$

where the parameter a depends on k_1 and k_2.

2. There exist $d > 0$ such that $\forall I > 0, \theta > 0$

$$d^{-1}\int_0^\infty \int_{S^1} |\tilde{I} - I|^2 \ d\omega \ d\nu \leqslant \mathcal{Q} \leqslant d \int_0^\infty \int_{S^1} |\tilde{I} - I|^2 \ d\omega \ d\nu \tag{77}$$

with

$$\mathcal{Q} := \theta_0 \eta \int_0^\infty \int_{S^1} \frac{k_B}{h\nu} \left[\log \frac{n(I)+1}{n(I)} - \log \frac{n(\tilde{I})+1}{n(\tilde{I})} \right] \ \sigma_s(\tilde{I} - I) \ d\nu \ d\omega.$$

Proof. The first inequality (76) is a slight modification of the result of Okada and Kawashima (see Lemma 3.1 of [16]) and the second (77) follows after an elementary analysis of the integrand in $\mathcal{Q}$ $\square$.

Now we rewrite equation (44) as

$$E_t + \frac{\theta_0}{\theta} \left(\frac{\mu v_x^2}{\eta} + \frac{\kappa \theta_x^2}{\eta\theta} \right) + \mathcal{Q}$$

$$= \Bigg[(p(1,\theta_0) - p(\eta,\theta))\, v + \frac{\mu}{\eta} \ v v_x + \left(1 - \frac{\theta_0}{\theta}\right) \ \frac{\kappa}{\eta} \ \theta_x$$

$$- \int_0^\infty \int_{S^1} \frac{2k_B \nu^2}{c^3} (c\omega - v) \ [(n+1)\log(n+1) - n\log n] \ d\nu \ d\omega \Bigg]_x . \tag{78}$$

In the same stroke, multiplying the second equation (1) by $\mathcal{M}_x$, we get

$$\left(\frac{1}{2} \mathcal{M}_x^2 - \mathcal{M}_x v \right)_t - \frac{\mu}{\eta} \ p_\eta \eta_x^2$$

$$= \frac{\mu}{\eta} \ v_x^2 - \mathcal{M}_x p_\theta \theta_x - \left(\frac{\mu}{\eta} \ v v_x \right)_x + \mathcal{M}_x \eta (S_F)_R. \tag{79}$$

Multiplying (78) by $e^{\beta_1 t}$ then (79) by $\beta_2 e^{\beta_1 t}$ with $\beta_1 > 0$, $\beta_2 > 0$ and adding the resulting identities, we get

$$\frac{\partial}{\partial t} \ e^{\beta_1 t} \left\{ E + \beta_2 \left(\frac{1}{2} \mathcal{M}_x^2 - \mathcal{M}_x v \right) \right\}$$

$$+ e^{\beta_1 t} \left\{ \frac{\theta_0}{\theta} \ \left(\frac{\mu v_x^2}{\eta} + \frac{\kappa \theta_x^2}{\eta\theta} \right) + \mathcal{Q} + \beta_2 \left(-\frac{\mu}{\eta} \ p_\eta \eta_x^2 - \frac{\mu}{\eta} \ v_x^2 + \mathcal{M}_x p_\theta \theta_x - \mathcal{M}_x \eta (S_F)_R \right) \right\}$$

$$= \beta_1 e^{\beta_1 t} \left\{ E + \beta_2 \left(\frac{1}{2} \mathcal{M}_x^2 - \mathcal{M}_x v \right) \right\}$$

$$+ e^{\beta_1 t} \Bigg[(p(\eta_0,\theta_0) - p(\eta,\theta))\, v + (1 - \beta_2) \frac{\mu}{\eta} \ v v_x + \left(1 - \frac{\theta_0}{\theta}\right) \ \frac{\kappa}{\eta} \ \theta_x$$

$$- \int_0^\infty \int_{S^1} \frac{2k_B \nu^2}{c^3} (c\omega - v) \ [(n+1)\log(n+1) - n\log n] \ d\nu \ d\omega \Bigg]_x . \tag{80}$$

Now, after the proof of Lemma 2 we have

$$\frac{1}{2}\,\frac{d}{dt}\int_\Omega\int_0^\infty\int_{S^1}\eta I^2\;d\omega\;d\nu\;dx+\frac{c}{2}\int_0^\infty\int_{S^1}\omega I^2(M,t;\nu,\omega)\;d\omega\;d\nu$$

$$-\frac{c}{2}\int_0^\infty\int_{S^1}\omega I^2(0,t;\nu,\omega)\;d\omega\;d\nu+\int_\Omega\int_0^\infty\int_{S^1}\eta\sigma_s(\tilde I-I)^2\;d\omega\;d\nu\;dx\leqslant 0.$$

Multiplying by $\beta_3 e^{\beta_1 t}$ with $\beta_3>0$, integrating on $(0,t)$ and using (14) we get

$$e^{\beta_1 t}\int_\Omega\int_0^\infty\int_{S^1}\frac{1}{2}\,\beta_3\eta I^2\;d\nu\;d\omega\;dx+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\beta_3\eta\sigma_s(\tilde I-I)^2\;d\nu\;d\omega\;dx\;ds$$

$$\leqslant K+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\frac{1}{2}\,\beta_1\beta_3\eta I^2\;d\nu\;d\omega\;dx\;ds. \tag{81}$$

Integrating now (80) on $(0,t)$, adding to (81) and using Lemma 11, we obtain finally

$$\mathbf{Q}:=e^{\beta_1 t}\int_\Omega\left\{a^{-1}\left(|\eta-\eta_0|^2+|\theta-\theta_0|^2\right)+\frac{1}{2}\,v^2+\frac{1}{2}\,\beta_2\mathcal{M}_x^2+\int_0^\infty\int_{S^1}\frac{1}{2}\,\beta_3\eta I^2\;d\nu\;d\omega\right\}\;dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega d^{-1}\left\{\int_0^\infty\int_{S^1}|\tilde I-I|^2\;d\nu\;d\omega+\right\}\;dx\;ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\left\{a_1v_x^2+a_2\theta_x^2+a_3\mathcal{M}_x^2\right\}\;dx\;ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\beta_3\eta\sigma_s(\tilde I-I)^2\;d\nu\;d\omega\;dx\;ds$$

$$\leqslant K+e^{\beta_1 t}\int_\Omega\beta_2\mathcal{M}_x v\;dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\left\{\beta_1\left[a\left(|\eta-\eta_0|^2+|\theta-\theta_0|^2\right)+\frac{1}{2}\,v^2+d\int_0^\infty\int_{S^1}|\tilde I-I|^2\;d\nu\;d\omega\right]\right.$$

$$+\beta_2\left[\mathcal{M}_x\eta(S_F)_R-\mathcal{M}_x p_\theta\theta_x\right]+\beta_1\beta_2\left[\frac{1}{2}\,\mathcal{M}_x^2-\mathcal{M}_x v\right]+\beta_2\frac{\mu}{\eta}\,v_x^2$$

$$\left.+\int_0^\infty\int_{S^1}\frac{1}{2}\,\beta_1\beta_3\eta I^2\;d\nu\;d\omega\right\}\;dx\;ds=:\mathbf{R}, \tag{82}$$

with the constants $a_1=\frac{\theta_0\mu_0}{\bar\theta\bar\eta}$, $a_2=\frac{c_6\theta_0(1+\underline{\theta}^{1+r})\mu_0}{\bar\theta\bar\eta}$ and $a_3=\frac{C_2\underline{\eta}(1+\underline{\theta}^{1+r})\mu_0}{\bar\eta^2}$.

Bounding the left-hand side from below we get

$$\mathbf{Q}\geqslant e^{\beta_1 t}\int_\Omega\left\{a^{-1}\left(|\eta-\eta_0|^2+|\theta-\theta_0|^2\right)+\frac{1}{2}\,v^2+\frac{1}{2}\,\beta_2\mathcal{M}_x^2+\int_0^\infty\int_{S^1}\frac{1}{2}\,\beta_3\eta I^2\;d\nu\;d\omega\right\}\;dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega d^{-1}\int_0^\infty\int_{S^1}\tilde I^2\;d\nu\;d\omega\;dx\;ds+\int_0^t e^{\beta_1 s}\int_\Omega d^{-1}\int_0^\infty\int_{S^1}I^2\;d\nu\;d\omega\;dx\;ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega \left\{a_1 v_x^2 + a_2\theta_x^2 + a_3\mathcal{M}_x^2\right\}\ dx\ ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\beta_3\eta\sigma_s(\tilde{I}-I)^2\ d\nu\ d\omega\ dx\ ds$$

The right-hand side is estimated by using Cauchy-Schwarz inequality.

$$|\mathbf{R}| \leqslant K + e^{\beta_1 t}\int_\Omega \frac{1}{2}\ \beta_2\left(\varepsilon_1\mathcal{M}_x^2 + \frac{1}{\varepsilon_1}\ v^2\right)\ dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\left\{a\beta_1\left(|\eta-\eta_0|^2+|\theta-\theta_0|^2\right)+\frac{1}{2}\ \beta_1 v^2+\beta_1 d\int_0^\infty\int_{S^1} I^2\ d\nu\ d\omega\right\}\ dx\ ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega \frac{1}{2}\ \beta_2\left(\varepsilon_2\mathcal{M}_x^2+\frac{1}{\varepsilon_2}\ [\eta(S_F)_R]^2\right)\ dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega \frac{1}{2}\ \beta_2\left(\varepsilon_3\mathcal{M}_x^2+\frac{1}{\varepsilon_3}\ p_\theta^2\theta_x^2\right)\ dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega \frac{1}{2}\ \beta_1\beta_2\left(\varepsilon_4\mathcal{M}_x^2+\frac{1}{\varepsilon_4}\ v^2\right)\ dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\left[\beta_2\frac{\mu}{\eta}\ v_x^2+\int_0^\infty\int_{S^1}\frac{1}{2}\ \beta_1\beta_3\eta I^2\ d\nu\ d\omega\right]dx\ ds,$$

for $0\leqslant\varepsilon_{1,2,3,4}\leqslant 1$.

Exploiting the structure of $\eta(S_F)_R$ and using (14), we see that

$$[\eta(S_F)_R]^2\leqslant\gamma\int_\Omega\int_0^\infty\int_{S^1}\beta_3\eta\sigma_s(\tilde{I}-I)^2\ d\nu\ d\omega,$$

where $\gamma=\overline{\eta}C_{11}\|k\|_{L^1(\mathbb{R}_+)}$, moreover using the inequalities

$$(\eta-\eta_0)^2\leqslant\delta\int_\Omega\mathcal{M}_x^2\ dx,\quad (\theta-\theta_0)^2\leqslant M\int_\Omega\theta_x^2\ dx\quad\text{and}\quad v^2\leqslant M\int_\Omega v_x^2\ dx,$$

where $\delta=\mu_0^{-1}\|k\|_{L^1(\Omega)}$, we get

$$|\mathbf{R}|\leqslant K+e^{\beta_1 t}\int_\Omega\frac{1}{2}\ \beta_2\left(\varepsilon_1\mathcal{M}_x^2+\frac{1}{\varepsilon_1}\ v^2\right)\ dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\beta_1\beta_3\overline{\eta}I^2\ d\nu\ d\omega\ dx\ ds,$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\left(\beta_2 a_4+\frac{1}{2}\ M\beta_1+\frac{1}{2\varepsilon_4}\ M\beta_1\beta_2\right)v_x^2\ dx\ ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\left(aM\beta_1+\frac{1}{2\varepsilon_3}\ \beta_2 a_5\right)\theta_x^2\ dx\ ds$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega \frac{1}{2}\ \left(2aM\delta\beta_1+\varepsilon_2\beta_2+\varepsilon_3\beta_2+\beta_1\beta_2\varepsilon_4\right)\mathcal{M}_x^2$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1}\gamma\frac{\beta_2}{2\varepsilon_2}\ \eta\sigma_s(\tilde{I}-I)^2\ dx\ d\nu\ d\omega$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\int_0^\infty\int_{S^1} d\beta_1\ (\tilde{I}-I)^2\ d\nu\ d\omega\ dx\ ds,$$

for any $\varepsilon_{1,2,3}>0$, for the positive constants $a_4=\frac{\mu_1}{\underline{\eta}}$, and $a_5=\frac{C_3(1+\overline{\theta}^r)}{2\underline{\eta}}$.

One checks that, in order to absorb the terms in the right-hand side, parameters $\varepsilon_{1,2,3,4}$ and $\beta_{1,2,3}$ have to satisfy the constraints

$$\begin{cases} \beta_2<\varepsilon_1<1, \\ \beta_1\beta_3<\frac{1}{2d\overline{\eta}}, \\ \beta_2 a_4+\frac{1}{2}\ M\beta_1+\frac{M\beta_1\beta_2}{2\varepsilon_4}<a_1, \\ aM\beta_1+\frac{\beta_2}{2\varepsilon_3}\ a_6<a_2, \\ 2aM\delta\beta_1+\varepsilon_2\beta_2+\varepsilon_3\beta_2+\beta_1\beta_2\varepsilon_4<2a_3, \\ \frac{\beta_2}{2\varepsilon_2}\ \gamma<\beta_3, \\ \beta_1<\frac{1}{2d^2} \end{cases}$$

where the a_j $j=1,...5$ are positive numbers depending only on the physical constants $(M,c,h,\theta_0,\overline{\eta},\underline{\eta},\overline{\theta},\underline{\theta})$ and those appearing in (14).

An elementary analysis of this system shows that, taking for example $\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4=1/2$ and $\beta_1=\beta_2$ it admits non-trivial solutions $(\beta_1,\beta_2,\beta_3)$ in a neighbourhood of $(\beta_1,\beta_2,\beta_3)=(0,0,0)$.

Then we end with the estimate

$$e^{\beta_1 t}\int_\Omega\left\{\left(|\eta-\eta_0|^2+|\theta-\theta_0|^2\right)+\frac{1}{2}\ v^2+\eta_x^2+\int_0^\infty\int_{S^1} I^2\ d\nu\ d\omega\right\}\ dx$$

$$+\int_0^t e^{\beta_1 s}\int_\Omega\left\{v_x^2+\theta_x^2+\eta_x^2\right\}\ dx\ ds\leqslant K, \tag{83}$$

which gives the exponential decay of Theorem 2, for $\Gamma=\beta_1$.

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