Preprint, Institute of Mathematics, AS CR, Prague. 2007-12-20

Easton's theorem and large cardinals

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Abstract

The continuum function F on regular cardinals is known to have great freedom; if α, β are regular cardinals, then F needs only obey the following two restrictions: (1) $\mathrm{cf}(F(\alpha)) > \alpha$, (2) $\alpha < \beta \to F(\alpha) \le F(\beta)$. However, if we wish to preserve measurable cardinals in the generic extension, new restrictions must be put on F. We say that κ is $F(\kappa)$-*hypermeasurable* if there is an elementary embedding $j : V \to M$ with critical point κ such that $H(F(\kappa))^V \subseteq M$; j will be called the *witnessing embedding*. We will show that if κ, closed under F, is $F(\kappa)$-hypermeasurable in V and there is a witnessing embedding j such that $j(F)(\kappa) \ge F(\kappa)$, then κ will remain measurable in some generic extension realizing F.

Key words: Easton's theorem, hypermeasurable and strong cardinals, lifting.
AMS subject code classification: 03E35,03E55.

1 Introduction

By Easton's results [3] the function $\alpha \mapsto 2^\alpha$ on regular cardinals, which we will call the *continuum function*, has great freedom as regards the values of 2^α. In fact, apart from the obvious restrictions of monotonicity, i.e. $\alpha < \beta \to 2^\beta \le 2^\alpha$, and of Cantor's theorem $\alpha < 2^\alpha$, there is only one non-trivial condition, namely the König's inequality $\mathrm{cf}(2^\alpha) > \alpha$ (which obviously implies Cantor's theorem). Such freedom is however not compatible with large cardinals possessing some kind of reflection properties. We for instance know, by Scott's theorem, that if

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[1] Supported by FWF Grant P 19375-N18
[2] Supported by grants FWF P 19375-N18 and GA ČR no. 401/03/H047

κ is measurable, it cannot be the first cardinal where GCH fails. Acknowledging the importance of large cardinals in the universe of sets, is seems reasonable to inquire what kind of restrictions must be put on the continuum function if some large cardinal structure should be preserved.

In [9], Menas showed that if a continuum function F is "locally definable", then all supercompact cardinals are preserved in a generic extension realizing F. We show in this article how to extend this result to strong cardinals (see Section 3.2).

The main focus of this article lies with the preservation of measurability, however. We know by results of Gitik [5] that non-trivial, i.e. GCH-failing, values of the continuum function on measurable cardinals require consistency-wise larger cardinal strength than just measurability. The exact strength is captured by sequences of measures or extenders which compose together to create elementary embeddings which are "stronger" than the usual measure ultrapower embeddings; for instance $2^\kappa = \kappa^{++}$ with κ being measurable can be forced from the assumption that κ has Mitchell order $o(\kappa) = \kappa^{++}$. Such sequences of measures or extenders are easily obtained from the hypermeasurable embeddings, and we will use these slightly stronger assumptions in our proofs.

We will show that if κ is $F(\kappa)$-hypermeasurable, then κ remains measurable in a generic extension realizing the continuum function F provided the following single non-trivial condition is satisfied:

There is an embedding j witnessing the $F(\kappa)$-hypermeasurability of κ such that $j(F)(\kappa) \geq F(\kappa)$ (1)

By way of illustration, if F is defined as $F(\alpha) = \alpha^{++}$ for every regular α, then if κ is κ^{+2}-hypermeasurable and j is any witnessing embedding, it follows by elementarity that $j(F)(\kappa) = F(\kappa) = \kappa^{++}$ and consequently the conditions above are satisfied and the theorem will imply that κ will remain measurable.

2 Preliminaries

We will start with a definition of a hypermeasurable cardinal and some related concepts.

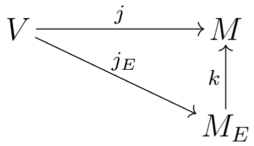
Definition 2.1 *A cardinal κ is λ-hypermeasurable (or λ-strong), where λ is a cardinal number, if there is an elementary embedding j with a critical point κ from V into a transitive class M such that $H(\lambda) \subseteq M$. As a part of the* *definition (although relevant only for singular λ), we also include the condition that $\lambda < j(\kappa)$.*

Definition 2.2 *An elementary embedding $j : V \to M$ is called an* extender embedding *if there are A and $B \subseteq j(A)$ such that $M = \{j(F)(a) \mid F : A \to V, a \in B\}$. In the context of this article, A will be the set of all finite subsets of the critical point of an embedding, i.e. $A = [\kappa]^{<\omega}$. It is equally possible to consider all subsets of size less than κ, in this case $A = H(\kappa) = V_\kappa$. We call B as above the* support *of the extender embedding.*

The following fact can be shown easily, for instance using the arguments in [8].

Fact 1 *If κ is $F(\kappa)$-hypermeasurable and $j : V \to M$ is a witnessing embedding, then j can be factored through some $j_E : V \to M_E$ and $k : M_E \to M$ such that j_E is an extender embedding with $A = [\kappa]^{<\omega}$ and $B = [F(\kappa)]^{<\omega}$ witnessing the $F(\kappa)$-hypermeasurability of κ. Moreover, if F is a continuum function and $j(F)(\kappa) \geq F(\kappa)$, then also $j_E(F)(\kappa) \geq F(\kappa)$.*

Proof. Consider the following commutative triangle:



By the construction of the extender, it follows that k is identity on $F(\kappa)$. The following holds: $k(j_E(F)(\kappa)) = k(j_E(F))(k(\kappa)) = k(j_E(F))(\kappa) = j(F)(\kappa)$. If $\mu = j_E(F)(\kappa) < F(\kappa)$ were true, then k would be identity at μ, implying that $j(F)(\kappa) = \mu$, which is a contradiction. $\square$

The above fact allows us to use only extender embeddings in our arguments and these will be used tacitly throughout.

Remark 2.3 Note that Definition 2.1 is slightly different from the definition found in [6] or [8], where κ is called $\kappa + \alpha$ or just α-strong if $V_{\kappa+\alpha}$ is included in M. The main difference is that we use the $H_\alpha = H(\alpha)$ hierarchy instead of the V_α hierarchy to measure the strength of the embedding j.[3] The definition using the V_α-hierarchy has the drawback of depending on the continuum function in the given universe; thus it may happen that the degree of hyper-

[3] Under GCH, there is a straightforward correspondence between the measurement of the strength of an embedding using the structures $H(\kappa^{+\alpha})$ and $V_{\kappa+\alpha}$ for an inaccessible κ and ordinal number α. It holds that if M is an inner model of ZFC then $V_{\kappa+\alpha} \subseteq M$ holds iff $H(\kappa^{+\alpha}) \subseteq M$ holds. This correspondence is however lost if GCH fails.

measurability of a given cardinal drops, although the witnessing embedding remains equally strong.

Example 2.4 In the article [4], one starts with GCH and κ being κ^{++}-hypermeasurable. Then one defines a forcing notion P of length $\kappa+1$ which iterates generalized Sacks forcing $\mathrm{Sacks}(\alpha, \alpha^{++})$ for $\alpha \leq \kappa$ inaccessible, making $2^\kappa = \kappa^{++}$ in the generic extension $V[G]$, where G is P-generic. It is shown that the original $j : V \to M$ witnessing the hypermeasurability of κ can be lifted to $j : V[G] \to M[j(G)]$. It is straightforward to verify that $H(\kappa^{++})^{V[G]}$ is included in $M[j(G)]$, so κ remains κ^{++}-hypermeasurable in $V[G]$ according to Definition 2.1. However, as 2^κ becomes κ^{++} in $V[G]$, κ may not stay $\kappa + 2$-strong in $V[G]$ according to the definition as in [6] since this would require that $\mathscr{P}(\kappa^{++})$ of $V[G]$ is included in $M[j(G)]$.

2.1 Product forcing

To avoid confusion, we explicitly state that we use the notion κ-distributive and κ-closed forcing notion for $<\kappa$-distributive or $<\kappa$-closed forcing notion (just as κ-cc is in fact used for antichains of size $<\kappa$).

The following lemma 2.5 is often called "Easton's lemma" as it first appeared in the proof by Easton in [3].

Lemma 2.5 *Assume $P, Q \in V$ are forcing notions, and P is κ-cc and Q is κ-closed. Then the following holds:*

(1) $1_P \Vdash \check{Q}$ *is κ-distributive;*
(2) $1_Q \Vdash \check{P}$ *is κ-cc;*
(3) As a corollary, if G is generic for P over V and H is generic for Q over V, then $G \times H$ is generic for $P \times Q$, i.e. G and H are mutually generic.

Proof. Ad (1). Assume $\dot{f}$ is forced by $\langle \tilde{p}, \tilde{q} \rangle$ to be a $P \times Q$-name for a function from $\mu < \kappa$ into ordinal numbers. We will find a condition $q_\infty \leq \tilde{q}$ such that for every G generic for P containing $\tilde{p}$ and H generic for Q over $V[G]$ containing q_∞, the condition q_∞ defines the function $\dot{f}^{G \times H}$ in $V[G]$.

The proof is an analogue of the proof of the fact that a κ-closed forcing notion doesn't add a new μ-sequence for $\mu < \kappa$, except that instead of determining a single value of $\dot{f}(\alpha)$, $<\kappa$ different values – corresponding to an antichain in P – will be found.

In preparation for the argument define for each $\alpha < \mu < \kappa$ and some $q_\alpha \leq \tilde{q} \in Q$ the following procedure. Simultaneously construct a decreasing sequence $\langle q_\alpha^\xi \mid \xi < \zeta_\alpha \rangle$ of conditions in Q and a sequence of pairwise incompatible conditions below $\tilde{p}$, $A_\alpha = \{p^\xi_\alpha \,|\, \xi < \zeta_\alpha\}$, and ordinals a^ξ_α for $\xi < \zeta_\alpha$ such that $\langle q^\xi_\alpha, p^\xi_\alpha\rangle \Vdash \dot{f}(\alpha) = a^\xi_\alpha$. As P is κ-cc, the construction will terminate – yielding a maximal antichain A_α – at some $\zeta_\alpha < \kappa$. Let $\tilde{q}_\alpha$ be the lower bound of q^ξ_α for $\xi < \zeta_\alpha$ (such bound exists as Q is κ-closed).

By induction carry out the above procedure for all $\alpha < \mu$, making sure that the resulting conditions $\tilde{q}_\alpha$ form a decreasing sequence $\langle \tilde{q}_\alpha \,|\, \alpha < \mu\rangle$. Let q_∞ be the lower bound of this sequence.

Let G be a generic for P containing $\tilde{p}$ and H generic for Q over $V[G]$ containing q_∞. It follows that $\dot{f}^{G\times H}(\alpha) = x$ iff x is identical to $a^{\xi_0}_\alpha$, where $p^{\xi_0}_\alpha$ is the unique element of A_α in G (A_α is a maximal antichain below $\tilde{p}$, and so is hit by all generics G containing $\tilde{p}$). This definition takes place in $V[G]$ and hence $\dot{f}^{G\times H} \in V[G]$.

Ad (2). Assume $\tilde{q} \in Q$ forces that $\dot{A}$ is a sequence of length κ of elements of P in V^Q. Construct by induction a decreasing sequence $\langle q_\xi \,|\, \xi < \kappa\rangle$ of conditions below $\tilde{q}$ such that q_ξ decides the value of $\dot{A}$ at ξ, i.e. $q_\xi \Vdash \dot{A}(\xi) = a_\xi$. At limit stage of the construction, we can take a lower bound as Q is κ-closed. It follows that $\tilde{A} = \{a_\xi \,|\, \xi < \kappa\}$ is an antichain, if $\tilde{q}$ forced $\dot{A}$ to be an antichain. This is a contradiction.

Ad (3). It is enough to show that G hits all maximal antichains of P in $V[H]$. As by (2) P is still κ-cc in $V[H]$, and Q doesn't add new $<\kappa$ sequences, it follows that the maximal antichains in $V[H]$ coincide with the the maximal antichains in V.

Note that (1) was not needed to obtain the mutual generocity in (3). □

We say that a forcing notion P is κ-Knaster if every subset $X \subseteq P$ of size κ has a subfamily $Y \subseteq X$ of size κ such that all elements of Y are pairwise compatible. The property of being κ-Knaster is an obvious strengthening of the property of being κ-cc.

Lemma 2.6 *If P is κ-Knaster and Q is κ-cc, then $P \times Q$ is κ-cc.*

Proof. We will show that $1_P \Vdash Q$ is κ-cc. Assume $\tilde{p} \Vdash \dot{A}$ is an antichain in Q of size κ. Define a set $X = \{p_\alpha \,|\, \alpha < \kappa\}$ of conditions below $\tilde{p}$ such that $p_\alpha \Vdash \dot{A}(\alpha) = q_\alpha$ for some q_α. By κ-Knasterness of P, there is a subfamily $X' = \{p_{\alpha_\xi} \,|\, \xi < \kappa\}$ of X such that the conditions in X' are pairwise compatible. It follows that the set $A' = \{q_{\alpha_\xi} \,|\, p_{\alpha_\xi} \in X'\}$ is an antichain in V. This is a contradiction. □

As regards the preservation of measurability of a given cardinal κ in the generic extension, it turns out that the most suitable way to achieve this is to lift the original elementary embedding in the ground model.

Definition 2.7 *Let $P \in M$ be a forcing notion and $j : M \to N$ an elementary embedding from M to N, both transitive models of ZFC. Let G be an M-generic filter for P, and H an N-generic filter for $j(P)$. We say that j^* lifts the embedding j if j^* is an elementary embedding $j^* : M[G] \to N[H]$.*

There is a simple sufficient condition which guarantees the existence of a lifting j^*.

Lemma 2.8 (Lifting lemma) *Let $P \in M$ be a forcing notion and $j : M \to N$ an elementary embedding from M to N, both transitive models of ZFC. Let G be an M-generic filter for P, and H an N-generic filter for $j(P)$. If $j[G] \subseteq H$, i.e. if the pointwise image of G under j is included in H, then*

(1) j lifts to $j^* : M[G] \to N[H]$, and
(2) $j^*(G) = H$.

As we will be dealing with extender embeddings, it is useful to realize that by using names for the elements of $M[G]$, we can argue that if j is an extender embedding, then so will be the lift j^*:

Lemma 2.9 *Let the assumptions of Lemma 2.8 hold. Assume further that $j : M \to N$ is an extender embedding, i.e. $N = \{j(F)(a) \mid F \in M, F : A \to V, a \in B \subseteq j(A)\}$. Assume $j^* : M[G] \to N[j^*(G)]$ is a lift of j. Then j^* is also an extender embedding, and moreover the parameters A and B of the extender embedding j remain the same, i.e.*

$$N[j^*(G)] = \{j^*(F)(a) \mid F \in M[G], F : A \to M[G], a \in B\}.$$

Remark 2.10 If $j^* : M[G] \to N[j^*(G)]$ is a lift of an embedding $j : M \to N$ which witnessed the measurability of κ in M (i.e. j is definable in M), then κ is still measurable in $M[G]$, *providing* that j^* is definable in $M[G]$.

Remark 2.11 Assume $j^* : M[G] \to N[j^*(G)]$ is a lift of an embedding $j : M \to N$ which witnessed the measurability of κ in M. Assume further that j^* is definable in $M[G]$. Let $U_j = \{X \subseteq \kappa \mid X \in M, \kappa \in j(X)\}$ be the normal ultrafilter derived from j. Then U_{j^*} extends the ultrafilter U_j: $U_j \subseteq U_{j^*}$. Note however that the extension U_{j^*} is in general very difficult to find [4] unless some

[4] With the notable exception when the forcing notion P is of size $<\kappa$; in this case any normal ultrafilter U in M generates a normal ultrafilter in the generic extension.

powerful structural information such as the embedding j is available.

In view of the Lifting lemma 2.8, the crucial part of the arguments dealing with the preservation of measurability consists in finding an N-generic H containing the pointwise image of G. This may be rather difficult in some cases, but if the forcing notion P is sufficiently distributive and the embedding to be lifted is an extender embedding, the existence of such H is straightforward.

Lemma 2.12 *Assume $j : M \to N$ is an extender embedding as in Lemma 2.9 and $N = \{j(F)(a) \mid F \in M, F : A \to V, a \in B \subseteq j(A)\}$. Let G be M-generic for a forcing notion $P \in M$. If M satisfies that P is $|A|^+$-distributive, then*

$$H = \{q \in j(P) \mid \exists p \in G, j(p) \leq q\}$$

is N-generic for $j(P)$ and contains the pointwise image of G.

Proof. H is obviously a filter. We show it is a generic filter. Let $D = j(F)(d)$ be a dense open set. We may assume that the range of F consists of dense open sets in P. Let $\{a_\xi \mid \xi < |A|\}$ be the enumeration of A. By distributivity, $X = \bigcap_{\xi<|A|} F(a_\xi)$ is dense. Let $p \in X$ be in G; then $M \models \forall a \in A, p \in F(a)$, and by elementarity it follows that $N \models \forall a \in j(A), j(p) \in j(F)(a)$. In particular, $j(p) \in j(F)(d) = D$. □

However, it is not true conversely that if P fails to be $|A|^+$-distributive, then H cannot be in some sense generated from the generic filter G. In fact, the construction in [4] shows that in the context of Sacks forcing, distributivity can be replaced by the weaker property of $|A|^+$-fusion.

We close this preliminary section by a technical lemma which is useful in constructing generic filters and is tacitly used throughout the arguments.

Lemma 2.13 *Assume $N \subseteq M$ are inner models of ZFC and $M \models {}^\lambda N \subseteq N$, i.e. N is closed under λ-sequences in M. If $P \in N$ is λ^+-cc in M and G is P-generic over M, then $M[G] \models {}^\lambda N[G] \subseteq N[G]$, i.e. $N[G]$ is closed under λ-sequences in $M[G]$.*

2.3 Generalized Sacks forcing

The proof of the theorem is centered around the technique developed in [4] which uses the Sacks forcing instead of the Cohen forcing in the lifting arguments. We will give but a brief review here; for a more detailed account, the reader is referred to [4].

Though the concept of perfect tree can be formulated for an arbitrary regular cardinal, see also [7], we will use the forcing at inaccessible cardinals only and this introduces further simplifications.

Definition 2.14 *If α is an inaccessible cardinal, then $T \subseteq 2^{<\alpha}$ is a perfect α-tree if the following conditions hold:*

(1) If $s \in T, t \subseteq s$, then $t \in T$;
(2) Each $s \in T$ has a proper extension in T;
(3) If $s_0 \subseteq s_1 \cdots$ is a sequence in T of length less than α, then the union of s_i's belongs to T;
(4) For every $s \in T$ there is some $s \subseteq t$ such that t is a splitting node, i.e. both $t * 0$ and $t * 1$ belong to T;
(5) Let $\mathrm{Split}(T)$ denote the set of s in T such that both $s * 0$ and $s * 1$ belong to T. Then for some (unique) closed unbounded $C(T) \subseteq \alpha$, $\mathrm{Split}(T) = \{s \in T \,|\, \mathrm{length}(s) \in C(T)\}$.

A perfect α-tree is an obvious generalization of the perfect tree at ω ordered by inclusion; there is only one non-trivial condition, and this concerns the limit levels of the tree: if $s \in T$ is an element at a limit level and the splitting nodes $t \subseteq s$ are unbounded in s, then s must be a splitting node as well (continuous splitting). As α is inaccessible, and consequently every level of T is of size $<\alpha$, the trees obeying (5) above are dense in the trees having continuous splitting.

Generalized perfect trees can be used to define a natural forcing notion.

Definition 2.15 *The forcing notion* $\mathrm{Sacks}(\alpha, 1)$ *contains as conditions the perfect α-trees, the ordering is by inclusion (not the reverse inclusion), i.e. $S \leq T$ iff $S \subseteq T$. Or generally, the forcing notion* $\mathrm{Sacks}(\alpha, \lambda)$*, where $0 < \lambda$ is an ordinal number, is a product of length λ of the forcing* $\mathrm{Sacks}(\alpha, 1)$ *with support of size α.*

Let $\langle \alpha_i \,|\, i < \alpha \rangle$ be an enumeration of $C(T)$ and let $\mathrm{Split}_i(T)$ be the set of s in T of length α_i. Let us write $S \leq_\beta T$ iff $S \leq T$ and $\mathrm{Split}_i(S) = \mathrm{Split}_i(T)$ for $i < \beta$. The forcing $\mathrm{Sacks}(\alpha, 1)$ satisfies the following α-fusion property: suppose that $T_0 \geq T_1 \geq \cdots$ is a descending sequence in $\mathrm{Sacks}(\alpha, 1)$ of length α and suppose further that $T_{i+1} \leq_i T_i$ for each $i < \alpha$. Then the intersection of T_i, $i < \alpha$ is a perfect tree. The α-fusion property implies in particular that α^+ is preserved by $\mathrm{Sacks}(\alpha, 1)$. Little more calculation in fact shows that $\mathrm{Sacks}(\alpha, \lambda)$ preserves α^+ as well. Under GCH, the preservation of the other cardinals is straightforward, and it follows that $\mathrm{Sacks}(\alpha, \lambda)$ preserves all cardinals.

If G is generic for $\mathrm{Sacks}(\alpha, 1)$ then $g = \bigcap G = \bigcup_{T \in G} \mathrm{stem}(T)$ is a new subset of α such that $V[G] = V[g]$. It follows that $\mathrm{Sacks}(\alpha, \lambda)$ adds λ-many new subsets of α.

As discussed after Lemma 2.12, it is the α-fusion property which is strong enough to replace the restrictive condition of distributivity in Lemma 2.12. Assuming knowledge of [4], we will review only the important points. Let us for the purposes of this review fix a κ^{++}-hypermeasurable extender embedding $j : V \to M$ with critical point κ and let $\langle S_\alpha \,|\, \alpha \le \kappa+1\rangle$ be an Easton-supported forcing iteration of length $\kappa+1$ which at every inaccessible $\alpha \le \kappa$ adds α^{++} many new subsets of α using the forcing Sacks(α, α^{++}). Let us write the generic $G_{\kappa+1}$ for $\langle S_\alpha \,|\, \alpha \le \kappa+1\rangle$ as $G_\kappa * g$, where g is Sacks(κ, κ^{++})-generic over $V[G_\kappa]$. Assume that we have completed the generics on the M-side, partially lifting to $j : V[G_\kappa] \to M[G_\kappa * g * H]$. Now it remains to find a generic h for Sacks$(j(\kappa), j(\kappa)^{++})$ over $M[G_\kappa * g * H]$ containing the pointwise image of g. For simplicity, assume first that Sacks$(\kappa, 1)$ is used instead of Sacks(κ, κ^{++}) and let g be a generic for this forcing. A priori, it is not even obvious that $t = \bigcap_{p\in g} j(p)$ is nontrivial (i.e. containing something else besides $\emptyset$); however the next lemma shows that this intersection is a union of two branches in the trees $j(p)$ (in the terminology of [4], t is called $(\kappa, j(\kappa))$-*tuning fork*).

Lemma 2.16 *t is a $(\kappa, j(\kappa))$-tuning fork, i.e. a subtree of $2^{<j(\kappa)}$ which is the union of two distinct cofinal branches which split at κ.*

Proof. Each $j(p)$, $p \in g$, obviously splits at κ. We will further show that for every $\kappa < \alpha < j(\kappa)$ there is a tree $p_\alpha \in g$ such that $j(p_\alpha)$ doesn't split between κ and α. Since by elementarity all $j(p)$ are compatible, this will imply that all $p \in g$ must have the unique two segments (as $j(p_\alpha)$ does split at κ) contained in p_α between κ and α. Using the property (5) in Definition 2.14 and the fact that it is dense in Sacks$(\kappa, 1)$ that there is a perfect tree below every closed unbounded set, it is enough to show that for every such α there is a closed unbounded set $C_\alpha \subseteq \kappa$ in $V[G_\kappa]$ such that $j(C_\alpha)$ doesn't contain α. Since the set of all limit cardinals C_L is closed unbounded in κ, we can work below C_L to find a suitable $C_\alpha \subseteq C_L$. Note that the least limit cardinal $\kappa^{+\omega}$ of M is greater than κ^{++} = the size of the support of the extender. Let α, a limit cardinal below $j(\kappa)$, be equal to $j(f)(a)$ for some $f : [\kappa]^{<\omega} \to \kappa$ and $a \in [\kappa^{++}]^{<\omega}$. Let C_α be the set of limit cardinals $\lambda < \kappa$ such that $f[[\lambda]^{<\omega}] \subseteq \lambda$; C_α is obviously closed unbounded. To reach contradiction, assume $\alpha = j(f)(a) \in C_\alpha$; then $[\alpha]^{<\omega}$ must be closed under $j(f)$. Since $\kappa^{++} < \alpha$, a is an element of $[\alpha]^{<\omega}$, implying that $j(f)(a) < \alpha$, which is a contradiction. The proof is finished by inductively arguing that not only a single point is avoided by a closed unbounded set, but the whole interval (κ, α) as well. □

A fusion-style argument is then applied to argue that either branch in the $(\kappa, j(\kappa))$-tuning fork will hit every dense set of Sacks$(j(\kappa), 1)$ in $M[G_\kappa * g * H]$. Further argumentation is needed to handle the product instead of the single Sacks forcing. In this setting p is a κ^{++}-sequence with support of size κ. In a nutshell, coordinates in $j(\kappa^{++})$ which are images of κ^{++} under j will obtain by taking the intersection of all the trees $j(p)(\alpha)$, $\alpha < \kappa^{++}$, a $(\kappa, j(\kappa))$-tuning fork, while the rest of the coordinates will obtain just a single cofinal branch. Taking for instance just the left branches in the tuning-forks, we can argue that these will constitute a $M[G_\kappa * g * H]$-generic over $\mathrm{Sacks}(j(\kappa), j(\kappa^{++}))$. The construction generalizes to any λ and $\mathrm{Sacks}(\kappa, \lambda)$ in place of κ^{++} satisfying the Easton criteria.

Thus it is possible to finish the lifting to $j : V[G_\kappa * g] \to M[G_\kappa * g * H * h]$, where h is the generic generated from $j[g]$ by the process described above. Amongst the main advantages of this approach, apart from the fact that we avoid the "modification" argument as in the Woodin-style approach (see for instance [1] or a slightly different argument in [2]) is that we don't have to enlarge the universe $V[G_\kappa * g]$ to complete the lifting. This adds a degree of uniformity which will be used later in this article.

3 Easton's theorem and large cardinals

3.1 Preservation of measurable cardinals

Definition 3.1 *A class function F defined on regular cardinals is called a* continuum function *if it satisfies the following two conditions which were shown by Easton to be the only conditions provable about the continuum function on regular cardinals in ZFC. Let κ, μ be arbitrary regular cardinals:*

(1) If $\kappa < \mu$, then $F(\kappa) \leq F(\mu)$;
(2) $\kappa < \mathrm{cf}(F(\kappa))$.

Note that Cantor's theorem $\kappa < 2^\kappa = F(\kappa)$ is implied by (2) above.

It is obvious, however, that if a given large cardinal κ should remain measurable in the generic extension realizing a given Easton function F, the properties of the cardinal κ and the properties of the function F need to combine in a suitable way which requires more than the conditions given in Definition 3.1.

Example 3.2 (1) If for some $\lambda < \kappa$, $\kappa \leq F(\lambda)$, then κ will not be even strongly inaccessible if F is realized.
(2) By the theorem of Scott, κ cannot be the least cardinal where GCH fails if it should remain measurable.
(3) Or more generally, F should not "jump" at κ. For instance if $F(\lambda) \leq \lambda^{++}$ for $\lambda < \kappa$ and $F(\kappa) = \kappa^{+3}$, then κ cannot remain measurable if F is realized.

We capture a sufficient condition for preservation of measurability in the following definition.

Definition 3.3 *We say that a cardinal* κ *is* good *for* F*, or shortly* F-good*, if the following properties hold:*

(1) $F[\kappa] \subseteq \kappa$*, i.e.* κ *is closed under* F*;*
(2) κ *is* $F(\kappa)$*-hypermeasurable and this is witnessed by an embedding* $j : V \to M$ *such that* $j(F)(\kappa) \geq F(\kappa)$*.*

As our aim is the preservation of large cardinals, we cannot use the standard Easton product-style forcing, but we need to use some kind of (reverse Easton) iteration. The iteration however needs some "space" as otherwise it would collapse cardinals, as the following observation shows.

Observation 3.4 *Assume* $\kappa < \lambda$ *are regular cardinals. If* κ^* *is a cardinal greater than* λ*, then forcing with* $\mathrm{Add}(\kappa, \kappa^*) * \mathrm{Add}(\lambda, 1)$ *collapses* κ^* *to* λ*.*

Proof. Let $\langle x_\xi \,|\, \xi < \kappa^* \rangle$ be the enumeration of subsets of κ in $V^{\mathrm{Add}(\kappa,\kappa^*)}$; for each $\xi < \kappa^*$, the set $D_\xi = \{p \in \mathrm{Add}(\lambda, 1) \,|\, \exists \alpha < \lambda, p \cap [\alpha, \alpha + \kappa) = x_\xi\}$ is dense. Consequently, there is a surjection from λ onto κ^* in the generic extension of V by $\mathrm{Add}(\kappa, \kappa^*) * \mathrm{Add}(\lambda, 1)$. □

The concept of the "space" mentioned above is technically captured by the closure points of the function F. We say that a cardinal κ is a closure point of F if $\mu < \kappa$ maps to $F(\mu) < \kappa$. We will denote the closed unbounded class of closure points of F as $I_F = I = \langle i_\alpha \,|\, \alpha < \mathrm{On} \rangle$. Note that every i_α must be a limit cardinal and $i_{\beta+1}$ has cofinality ω for every β. If $\kappa \in I$ is regular, then κ equals i_κ.

We will now give a full definition of a forcing notion to realize a continuum function F.

Definition 3.5 *Let a continuum function* F *satisfying the conditions (1)–(2) of 3.1 be given. Let* $I_F = I = \langle i_\alpha \,|\, \alpha < \mathrm{On} \rangle$ *be an enumeration of the closure points of* F*.*

We will define an iteration $\mathbb{P}^F = \Big\langle \langle \mathbb{P}_{i_\alpha} \,|\, \alpha < \mathrm{On} \rangle, \langle \dot{Q}_{i_\alpha} \,|\, \alpha < \mathrm{On} \rangle \Big\rangle$ *indexed by* $\langle i_\alpha \,|\, \alpha < \mathrm{On} \rangle$ *such that:*

- *If* i_α *is not an inaccessible cardinal, then*

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{Q}_{i_\alpha}, \tag{2}$$

  where $\dot{Q}_{i_\alpha} = \prod_{i_\alpha < \lambda < i_{\alpha+1}} \mathrm{Add}(\lambda, F(\lambda))$*, and where* λ *ranges over regular cardinals and the product has Easton support.*
- *If* i_α *is an inaccessible cardinal, then*

$$\mathbb{P}_{i_{\alpha+1}} = \mathbb{P}_{i_\alpha} * \dot{Q}_{i_\alpha}, \tag{3}$$

where $\dot{Q}_{i_\alpha} = \mathrm{Sacks}(i_\alpha, F(i_\alpha)) \times \prod_{i_\alpha<\lambda<i_{\alpha+1}} \mathrm{Add}(\lambda, F(\lambda))$, *and where* λ *ranges over regular cardinals and the product has Easton support.*

- *If* γ *is a limit ordinal, then* $\mathbb{P}_{i_\gamma}$ *is an inverse limit unless* γ *is a regular cardinal, in which case* $\mathbb{P}_{i_\gamma}$ *is a direct limit (the usual Easton support).*

Lemma 3.6 *Under GCH,* $\mathbb{P}^F$ *preserves all cofinalities.*

Proof. The only non-standard feature of $\mathbb{P}^F$ is the inclusion of the Sacks forcing. It is enough to show that $\mathrm{Sacks}(\kappa, F(\kappa)) \times \mathrm{Add}(\kappa^+, F(\kappa^+))$ preserves cofinalities. The product is κ-closed and κ^{++}-cc, hence just the cardinal κ^+ needs a special argument. It suffices to use the usual fusion-style argument for the Sacks forcing in $V^{\mathrm{Add}(\kappa^+, F(\kappa^+))}$. □

In fact, a more detailed analysis of the product $\mathrm{Sacks}(\kappa, \alpha)$ and $\mathrm{Add}(\kappa^+, \beta)$ for arbitrary ordinals α, β shows that the Cohen forcing $\mathrm{Add}(\kappa^+, \beta)$ remains κ^+-distributive after forcing with the Sacks forcing $\mathrm{Sacks}(\kappa, \alpha)$. This will be useful in the further arguments.

Lemma 3.7 *Let* κ *be an inaccessible cardinal, and* α, β *ordinal numbers.*

(1) Let $S = \mathrm{Sacks}(\kappa, 1)$ *and* $P = \mathrm{Add}(\kappa^+, \beta)$. *Then* $1_S \Vdash \check{P}$ *is* κ^+*-distributive.*
(2) Or more generally, if $S = \mathrm{Sacks}(\kappa, \alpha)$ *and* $P = \mathrm{Add}(\kappa^+, \beta)$, *then* $1_S \Vdash \check{P}$ *is* κ^+*-distributive.*

Proof. Ad (1). The proof is a generalization of the usual argument which shows that a κ^+-closed forcing notion doesn't add new κ-sequences. The difference is in the treatment of the Sacks coordinates in $S \times P$ which are obviously not κ^+-closed; however, they are closed under fusion limits of length κ, and this will suffice to argue that new κ sequences cannot appear between V^S and $V^{S \times P}$.

Let $\langle s, p\rangle$ force that $\dot{f} : \kappa \to \mathrm{On}$. It is enough to find a condition $\langle \tilde{s}, \tilde{p}\rangle \leq \langle s, p\rangle$ such that if $\langle \tilde{s}, \tilde{p}\rangle \in G \times F$, for arbitrary generic $G \times F$ for $S \times P$, then $\dot{f}^{G\times F} = f$ can be defined in $V[G]$.

We will define a decreasing sequence of conditions $\langle\langle s_\alpha, p_\alpha\rangle \,|\, \alpha < \kappa\rangle$ deciding the elements of $\dot{f}$ where $\tilde{s}$ will be a fusion limit of $\langle s_\alpha \,|\, \alpha < \kappa\rangle$ and $\tilde{p}$ will be the lower bound of $\langle p_\alpha \,|\, \alpha < \kappa\rangle$.

Set $\langle s_0, p_0\rangle = \langle s, p\rangle$. Assume $\langle s_\alpha, p_\alpha\rangle$ is constructed and we will construct $\langle s_{\alpha+1}, p_{\alpha+1}\rangle$. Let S_α denote the set of splitting nodes of rank α in s_α (the first splitting node has rank 0). Pick some $t \in S_\alpha$, and considering its immediate continuations $t*0$ and $t*1$, find conditions $\langle r_{t*0}, p_{t*0}\rangle$, $\langle r_{t*1}, p_{t*1}\rangle$ and ordinals α_{t*0}, α_{t*1} such that the following conditions hold:

(1) $p_\alpha \geq p_{t*0} \geq p_{t*1}$;

(2) $r_{t*0} \leq s_\alpha \restriction t*0$, $r_{t*1} \leq s_\alpha \restriction t*1$;
(3) $\langle r_{t*0}, p_{t*0}\rangle \Vdash \dot{f}(\alpha) = \alpha_{t*0}$ and $\langle r_{t*1}, p_{t*1}\rangle \Vdash \dot{f}(\alpha) = \alpha_{t*1}$.

Continue in this fashion considering successively all $t \in S_\alpha$, taking care to form a decreasing chain $p_\alpha \geq p_{t*0} \geq p_{t*1} \ldots \geq p_{t'*0} \geq p_{t'*1} \geq \ldots$, for $t, t' \in S_\alpha$, in the Cohen forcing. We define:

(1) $p_{\alpha+1} =$ the lower bound of $p_\alpha \geq p_{t*0} \geq p_{t*1} \ldots \geq p_{t'*0} \geq p_{t'*1} \geq \ldots$;
(2) $s_{\alpha+1} =$ the amalgamation of the subtrees r_{t*0}, r_{t*1} for all $t \in S_\alpha$.

Finally, define $\langle \tilde{s}, \tilde{p}\rangle$ as the fusion limit of s_α at the first coordinate and as the lower bound at the second coordinate.

Let $G \times F$ be a generic for $S \times P$ containing $\langle \tilde{s}, \tilde{p}\rangle$. In $V[G]$ define a function $f' : \kappa \to \mathrm{On}$ as follows: $f'(\alpha) = \beta$ iff $\beta = \alpha_{t*i}$, for $i \in \{0,1\}$, where t is a splitting node of rank α in $\tilde{s}$ and $t*i \subseteq \bigcup_{s\in G} \mathrm{stem}(s)$.

It is straightforward to verify that $f' = f = \dot{f}^{G\times F}$.

Ad (2). The proof proceeds in exactly the same way as (1) except that a generalized fusion is used for the Sacks(κ, α) forcing (it is essential here that the conditions in the Sacks forcing can have support of size κ). □

We can now state the main theorem.

Theorem 3.8 *Assume GCH and let F be a continuum function according to Definition 3.1. Then the generic extension by $\mathbb{P}^F$ realizes F, i.e. $2^\kappa = F(\kappa)$ for every regular cardinal κ, and moreover if a cardinal κ is good for F, then it will remain measurable.*

The proof will be given in a sequence of lemmas.

It is obvious that the continuum function F is realized in $V^{\mathbb{P}^F}$. It remains to prove that each F-good cardinal κ remains measurable in the generic extension. Let an F-good cardinal κ be fixed. Fix also a $j : V \to M$ an $F(\kappa)$-hypermeasurable extender embedding witnessing the F-goodness of κ.

$$V \xrightarrow{\quad j \quad} M$$

The properties of the continuum function F imply that $\mathrm{cf}(F(\kappa)) > \kappa$, so in particular M is closed under κ-sequences in V. It also holds that $F(\kappa) < j(\kappa) < F(\kappa)^+$, $j(F)(\kappa) \geq F(\kappa)$ (by goodness), $M = \{j(f)(a) \mid f : [\kappa]^{<\omega} \to V, a \in [F(\kappa)]^{<\omega}\}$, and $H(F(\kappa))^V = H(F(\kappa))^M$. Note that M is generally not closed even under κ^+-sequences in V, but the correct capturing of $H(F(\kappa))$ implies that ${}^{<\mathrm{cf}(F(\kappa))}H(F(\kappa)) \subseteq M$, so M is closed under $< \mathrm{cf}(F(\kappa))$-sequences providing that they refer to objects in $H(F(\kappa))$.

We fix some notation first. Let G be a generic for $\mathbb{P}^F$. As usual, we will write G_α for the generic G restricted to $\mathbb{P}_\alpha$. For ease of notation, the generic for $\dot{Q}_\alpha$ taken in $V[G_\alpha]$ will be denoted as g_α; it follows that $G_{\alpha+1} = G_\alpha * g_\alpha$.

Again for reasons of notational simplicity, we write $\mathbb{P}^M$ for $j(\mathbb{P}^F)$. Recall that $\mathbb{P}^F$ is defined as an iteration along the closure points $\langle i_\alpha \,|\, \alpha < \mathrm{On}\rangle$ of F; by elementarity, $\mathbb{P}^M$ is defined using the closure points of $j(F)$, which we will denote as $\langle i^M_\alpha \,|\, \alpha < \mathrm{On}\rangle$. Since j is the identity on $H(\kappa)$, the closure points of F and $j(F)$ coincide up to an including κ, i.e. $\langle i_\alpha \,|\, \alpha \leq \kappa\rangle = \langle i^M_\alpha \,|\, \alpha \leq \kappa\rangle$. Because κ is regular, we also have that $i_\kappa = \kappa$. By elementarity, $j(\kappa)$ is closed under $j(F)$, and as $j(\kappa)$ is regular in M, it follows that $j(\kappa) = i^M_{j(\kappa)}$ and so in particular $F(\kappa) \leq j(F)(\kappa) < i^M_{\kappa+1} < j(\kappa) < F(\kappa)^+ < i_{\kappa+1}$.

The general strategy of the proof is to lift the embedding j to $V[G]$. This amounts to finding a suitable generic for $\mathbb{P}^M$. As the cardinal structure between V and M is the same up to an including $F(\kappa)$, it follows that the generics for the V-regular cardinals $\leq F(\kappa)$ must be "copied" from the $V[G]$-side. The forcing $\mathbb{P}^M$ at the M-cardinals in the interval $(F(\kappa), j(\kappa))$ (and at $F(\kappa)$ if $F(\kappa)$ is singular in V but regular in M) will be shown to be sufficiently well-behaved so that the corresponding generics can be constructed in $V[G]$. The next step is the forcing $\mathbb{P}^M$ at $j(\kappa)$ where the task is twofold: not only do we need to find a generic, but we need to find one which contains the pointwise image under j of g_κ. Precisely to resolve this difficult point, we have included the Sacks forcing $\mathrm{Sacks}(\kappa, F(\kappa))$ at stage κ because by [4], the point-wise image of the generic g_κ (or rather of its Sacks part) will (almost) generate the correct generic for $j(\kappa)$. Finally, we lift to all of $V[G]$ using Lemma 2.12.

We will first lift the embedding j to $V[G_\kappa]$. As $H(\kappa)^V = H(\kappa)^M$, $\mathbb{P}_\kappa = \mathbb{P}^M_\kappa$ and it follows we can copy the generic G_κ.

Note: In order to keep track of where we are, we will use the following dotted arrow convenction to indicate that we are in the process of lifting the embedding j to $V[G_\kappa]$, but we have not yet completed the lifting. Once we lift the embedding, the arrow will be printed in solid line.

$$V[G_\kappa] \overset{j}{\dashrightarrow} M[G_\kappa]$$

Recall by the definition of $\mathbb{P}^F$ that the next step of iteration Q_κ of $V[G_\kappa]$ is the product $\mathrm{Sacks}(\kappa, F(\kappa)) \times \prod_{\kappa<\lambda<i_{\kappa+1}} \mathrm{Add}(\lambda, F(\lambda))$, where λ ranges over regular cardinals and the product has Easton support; the corresponding forcing in $M[G_\kappa]$, to be denoted Q^M_κ, is $\mathrm{Sacks}(\kappa, j(F)(\kappa)) \times \prod_{\kappa<\lambda<i^M_{\kappa+1}} \mathrm{Add}(\lambda, j(F)(\lambda))$. As mentioned above, $F(\kappa) < i^M_{\kappa+1} < i_{\kappa+1}$, so both Q_κ and Q^M_κ go past the cardinal $F(\kappa)$. Let us denote the product Q^M_κ with indices restricted up to $F(\kappa)$, i.e. $\mathrm{Sacks}(\kappa, j(F)(\kappa)) \times \prod_{\kappa<\lambda\leq F(\kappa)} \mathrm{Add}(\lambda, j(F)(\lambda))$ by $Q^M_{\leq F(\kappa)}$, and similarly on the $V[G_\kappa]$-side let $Q_{\leq F(\kappa)}$ stand for $\mathrm{Sacks}(\kappa, F(\kappa)) \times \prod_{\kappa<\lambda\leq F(\kappa)} \mathrm{Add}(\lambda, F(\lambda))$.

Let the rest of the product in $M[G_\kappa]$ up to $i^M_{\kappa+1}$ be denoted as $Q^M_{>F(\kappa)}$, obtaining $Q^M_\kappa \cong Q^M_{\leq F(\kappa)} \times Q^M_{>F(\kappa)}$. Further, let $g_{\leq F(\kappa)}$ be the $Q_{\leq F(\kappa)}$-generic obtained by truncating the generic g_κ for Q_κ at $F(\kappa)$. We will show that $g_{\leq F(\kappa)}$ can be used to find an $M[G_\kappa]$-generic for Q^M_κ in $V[G_\kappa]$.

For ease of notation let $P(\lambda, F(\lambda))$ for $\kappa \leq \lambda \leq F(\kappa)$ denote either the Sacks forcing or the Cohen forcing according to the index λ. We will first correct the possible discrepancy between the values of $F(\lambda)$ and $j(F)(\lambda)$ for regular λ in the interval $[\kappa, F(\kappa)]$. By the F-goodness of κ, $F(\kappa) \leq j(F)(\kappa)$. Let λ_0 be least such that $F(\kappa) < F(\lambda_0)$. Then $\lambda_0 \leq F(\kappa)$ and as $F(\kappa) \leq j(F)(\kappa)$, we have: $\kappa \leq \mu < \lambda_0$ maps to $F(\mu) = F(\kappa) \leq j(F)(\kappa) \leq j(F)(\mu) < j(\kappa) < F(\kappa)^+$ and $\lambda_0 \leq \tilde{\mu} \leq F(\kappa)$ maps to $j(F)(\tilde{\mu}) < j(\kappa) < F(\kappa)^+ \leq F(\tilde{\mu})$. For μ's fix 1-1 functions k_μ from $F(\mu) \to j(F)(\mu)$ and let k'_μ be the derived isomorphism between $P(\mu, F(\mu))$ and $P^*(\mu) = P(\mu, j(F)(\mu))$. These k'_μ generate an obvious isomorphism between $Q_{\leq F(\kappa)}$ and $Q'_{\leq F(\kappa)} = \prod_{\kappa \leq \mu < \lambda_0} P^*(\mu) \times \prod_{\lambda_0 \leq \tilde{\mu} \leq F(\kappa)} P(\tilde{\mu}, F(\tilde{\mu}))$. Let $g'_{\leq F(\kappa)}$ be the derived generic from $g_{\leq F(\kappa)}$ for $Q'_{\leq F(\kappa)}$. For $\tilde{\mu} \geq \lambda_0$, the forcing on the $V[G_\kappa]$-side is longer than on the $M[G_\kappa]$-side, so we will truncate the forcing $P(\tilde{\mu}, F(\tilde{\mu}))$ at the ordinal $j(F)(\tilde{\mu})$, denoting this by $P^*(\tilde{\mu})$. Let $g^*_{\leq F(\kappa)}$ be the resulting generic for $Q^*_{\leq F(\kappa)} = \prod_{\kappa \leq \lambda \leq F(\kappa)} P^*(\lambda)$.

Notice that regular cardinals in M and V coincide up to $F(\kappa)$, with $F(\kappa)$ possibly being singular in V while being regular or singular in M. Also note that even when $F(\kappa)$ is regular, $Q^*_{\leq F(\kappa)}$ is not identical to $Q^M_{\leq F(\kappa)}$ because the supports are not identical. (Taking a specific example, if $F(\kappa) = \kappa^{++}$, then the forcing $\mathrm{Add}(\kappa^{++}, j(F)(\kappa^{++}))$ extends past κ^{++} in $M[G_\kappa]$, so it fails to capture all supports of size κ^+ available in $V[G_\kappa]$.) Accordingly, we have (even when $F(\kappa)$ is regular) only $Q^M_{\leq F(\kappa)} \subseteq Q^*_{\leq F(\kappa)}$.

We will deal separately with the two cases: $F(\kappa)$ regular in V, and $F(\kappa)$ singular in V.

Lemma 3.9 *Assume $F(\kappa)$ is* regular *in V. There is in $V[G_\kappa * g_{\leq F(\kappa)}]$ a $M[G_\kappa]$-generic for Q^M_κ, which we will denote as $\tilde{g}_\kappa$.*

Proof. As $F(\kappa)$ is regular in V, it is also regular in M. Consequently, the forcing $Q^M_{\leq F(\kappa)}$ is $F(\kappa)^+$-cc in $M[G_\kappa]$ and as $Q^M_{>F(\kappa)}$ is $F(\kappa)^+$-closed, the forcings are mutually generic in the sense of Lemma 2.5. It follows that we can deal with $Q^M_{\leq F(\kappa)}$ and $Q^M_{>F(\kappa)}$ separately.

A) The product $Q^M_{\leq F(\kappa)}$.

We will use $g^*_{\leq F(\kappa)}$ to obtain the required generic; in fact we will show that the intersection $\tilde{g}_{\leq F(\kappa)} = g^*_{\leq F(\kappa)} \cap Q^M_{\leq F(\kappa)}$ is $M[G_\kappa]$-generic for $Q^M_{\leq F(\kappa)}$.

We will argue that a maximal antichain $A \in M[G_\kappa]$ for $Q^M_{\leq F(\kappa)}$ will stay maximal in $Q^*_{\leq F(\kappa)}$, and so will be hit by $\tilde{g}_{\leq F(\kappa)}$. For $p \in Q^*_{\leq F(\kappa)}$ write

$$\mathrm{supp}(p) = \{\langle \lambda, \alpha \rangle \,|\, p(\lambda)(\alpha) \neq 1\}, \tag{4}$$

and analogously for $A \subseteq Q^*_{\leq F(\kappa)}$,

$$\mathrm{supp}(A) = \{\langle \lambda, \alpha \rangle \,|\, \exists p \in A, \langle \lambda, \alpha \rangle \in \mathrm{supp}(p)\}. \tag{5}$$

We will show that if $A \in M[G_\kappa]$ is a maximal antichain in $Q^M_{\leq F(\kappa)}$ and $p \in Q^*_{\leq F(\kappa)}$ is arbitrary, then

$$X = \mathrm{supp}(p) \cap \mathrm{supp}(A) \in M[G_\kappa] \text{ and } p \restriction X \in M[G_\kappa]. \tag{6}$$

Providing we know (6), $p \restriction X$ must be compatible with some $a \in A$, and because p and a are compatible on the supports, they must be compatible everywhere. It follows that A stays maximal in $V[G_\kappa]$. To argue for (6), the $F(\kappa)^+$-cc of $Q^M_{\leq F(\kappa)}$ in $M[G_\kappa]$ implies that the size of $\mathrm{supp}(A)$ in $M[G_\kappa]$ is at most $F(\kappa)$. Since the size of $\mathrm{supp}(p)$ is strictly less than $F(\kappa)$, (6) will follow from the following property (7).

$$\text{If } x \subseteq \mathrm{On} \text{ has size } F(\kappa) \text{ in } M[G_\kappa], \text{ then } {}^{<F(\kappa)}x \subseteq M[G_\kappa]. \tag{7}$$

Let $f : x \to F(\kappa)$ be a 1-1 function, $f \in M[G_\kappa]$, and let $\vec{s} \in {}^{<F(\kappa)}x$ be given. Working in $V[G_\kappa]$, it is obvious that $f[\vec{s}] \in H(F(\kappa))$. Since $H(F(\kappa))$ is the same in $V[G_\kappa]$ and $M[G_\kappa]$, $f[\vec{s}] \in M[G_\kappa]$. But as f is in $M[G_\kappa]$, so is $f^{-1}[f[\vec{s}]] = \vec{s}$.

B) The product $Q^M_{>F(\kappa)}$.

Notice that every dense open set of $Q^M_{>F(\kappa)}$ in $M[G_\kappa]$ is of the form $(j(f)(a))^{G_\kappa}$, $a \in [F(\kappa)]^{<\omega}$, where $j(f)(a)$ is a $\mathbb{P}^M_\kappa$-name, for some $f : [\kappa]^{<\omega} \to H(\kappa^+)$. Without loss of generality, we may assume that the range of all such f contains just names for dense open sets.[5] For each such f, the set $\{\langle j(f)(a), 1\rangle \,|\, a \in [F(\kappa)]^{<\omega}\}$ is a $\mathbb{P}^M_\kappa$-name in M, which interprets as a family $\{(j(f)(a))^{G_\kappa} \,|\, a \in [F(\kappa)]^{<\omega}\}$ of $F(\kappa)$ many dense open sets in $M[G_\kappa]$ – it follows the intersection $\mathscr{D}_f = \bigcap_{a \in [F(\kappa)]^{<\omega}} (j(f)(a))^{G_\kappa}$ is dense in $Q^M_{>F(\kappa)}$, since the forcing notion $Q^M_{>F(\kappa)}$ is $F(\kappa)^+$-distributive in $M[G_\kappa]$. As there are only $(\kappa^+)^\kappa = \kappa^+$ such f, and $M[G_\kappa]$ is closed under κ-sequences in $V[G_\kappa]$, we can construct a generic in $V[G_\kappa]$ meeting all the dense sets $\mathscr{D}_f$ for all suitable f. Let us denote this generic as $\tilde{g}_{>F(\kappa)}$.

[5] Formally, $f(s)$ will be a name for a dense open set in the forcing $\mathbb{P}_\kappa$, and so $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ will be a name for a dense open set in $\mathbb{P}^M_{j(\kappa)}$. We will abuse the notation and identify every $j(f)(a)$ with a $\mathbb{P}_\kappa$-name for a dense open set in $Q^M_{>F(\kappa)}$.

We finish the proof of Lemma 3.9 by setting $\tilde{g}_\kappa = \tilde{g}_{\leq F(\kappa)} \times \tilde{g}_{>F(\kappa)}$. □

Lemma 3.10 *Assume $F(\kappa)$ is* singular *in V with cofinality δ, where $\kappa^+ \leq \delta < F(\kappa)$. There is in $V[G_\kappa * g_{<F(\kappa)}]$ a $M[G_\kappa]$-generic for Q^M_κ, which we will denote as $\tilde{g}_\kappa$.*

Proof. The singularity of $F(\kappa)$ implies that $M[G_\kappa]$ may not be closed under $<F(\kappa)$-sequences of elements of $F(\kappa)$, but just under $<\delta$-sequences. It follows that the argument given in Lemma 3.9 cannot be used as it stands. However, we will argue that the desired generic can be constructed via "approximations" by induction along some sequence of regular cardinals cofinal in $F(\kappa)$.

In preparation for the argument, we will define a certain procedure which will be used in the argument. Let $\langle \gamma_i \,|\, i < \delta\rangle$ be a sequence of regular cardinals cofinal in $F(\kappa)$, with $\delta < \gamma_0$ (we may assume that this sequence belongs to M if $F(\kappa)$ is singular in M, as in that case, $F(\kappa)$ has the same cofinality in M as it has in M). Generalizing the above notation, if γ_i is in the sequence $\langle \gamma_i \,|\, i < \delta\rangle$, let $Q^M_{\leq\gamma_i}$ denote the relevant part of the product Q^M_κ with the indices restricted to the closed interval $[\kappa, \gamma_i]$. Similarly, let $Q^M_{>\gamma_i}$ denote the remaining part of Q^M_κ after $Q^M_{\leq\gamma_i}$, i.e. $Q^M_\kappa \cong Q^M_{\leq\gamma_i} \times Q^M_{>\gamma_i}$. Sometimes we will work with $Q^M_{\leq F(\kappa)}$ instead of Q^M_κ; if this is obvious from the context, $Q^M_{>\gamma_i}$ will denote the remaining part of the forcing $Q^M_{\leq F(\kappa)}$, i.e. $Q^M_{\leq F(\kappa)} \cong Q^M_{\leq\gamma_i} \times Q^M_{>\gamma_i}$. Also, if $p \in Q^M_\kappa$ is a condition, let p_{γ_i} denote p restricted to $Q^M_{\leq\gamma_i}$ (the "lower part of p") and p^{γ_i} denote p restricted to $Q^M_{>\gamma_i}$ (the "upper part of p"). Note that for each γ_i, $Q^M_{>\gamma_i}$ is γ_i^+-closed and $Q^M_{\leq\gamma_i}$ is γ_i^+-cc in $M[G_\kappa]$.

Let γ_i, $f : [\kappa]^{<\omega} \to H(\kappa^+)$, and $a \in [\gamma_i]^{<\omega}$ be given and assume that $j(f)(a)$ is a $\mathbb{P}^M_\kappa$-name for a dense open set in $Q^M_{<\mu}$, where μ is either $F(\kappa)+1$ or $i^M_{\kappa+1}$. Let us denote $(j(f)(a))^{G_\kappa}$ as D. Assume further that $p \in Q^M_{<\mu}$ is also given.

Definition 3.11 *$\overline{q} \in Q^M_{>\gamma_i}$ is said to γ_i-*reduce *D* below *p if the following holds:*

(1) $\overline{q}$ extends the upper part of p, i.e. $\overline{q} \leq p^{\gamma_i}$ in $Q^M_{>\gamma_i}$;
(2) The set $\overline{D} = \{q \leq p_{\gamma_i} \in Q^M_{\leq\gamma_i} \,|\, q \cup \overline{q} \in D\}$ is dense open in $Q^M_{\leq\gamma_i}$ below the lower part of p.

We will show how to construct such a reduction $\overline{q}$ (as $p \in M[G_\kappa]$ by assumption, the construction can be carried out in $M[G_\kappa]$ and consequently $\overline{q}$ will also be in $M[G_\kappa]$). Set $(r_0, s_0) = (p_{\gamma_i}, p^{\gamma_i})$. At stage ξ of the construction, let r'_ξ be any condition which is incompatible with the set of all previous conditions $\{r_\zeta \,|\, \zeta < \xi\}$ (if there such) and let $r_\xi \leq r'_\xi$, $s_\xi \leq s_\zeta$ for all $\zeta < \xi$ be such that $r_\xi \cup s_\xi \in D$. The construction is well-defined since $Q^M_{\leq\gamma_i}$ is γ_i^+-cc and consequently the process will stop at some $\rho < \gamma_i^+$. Set $\overline{q}$ to be the lower bound of all s_ζ for $\zeta < \rho$. We will show that $\overline{q}$ indeed γ_i-reduces D below p

according to Definition 3.11. We only need to check the condition (2) as (1) is obvious. Let $q \leq p_{\gamma_i}$ be given. It follows from the construction that there is some r_ζ such that q and r_ζ are compatible with some lower bound $\tilde{r}$. Also, $r_\zeta \cup s_\zeta \in D$ and consequently $\tilde{r} \cup \overline{q} \in D$ by openness.

***Case (1)**: Cofinality of $F(\kappa)$ in V is κ^+; $F(\kappa)$ can be either regular or singular in M.*

Fix the two following sequences:

(1) Sequence $\langle \gamma_i \,|\, i < \kappa^+ \rangle$ of regular cardinals cofinal in $F(\kappa)$;
(2) Sequence $\langle j(f_\alpha) \,|\, \alpha < \kappa^+ \rangle$, where $\langle f_\alpha \,|\, \alpha < \kappa^+ \rangle$ enumerates all $f : [\kappa]^{<\omega} \to H(\kappa^+)$ such that $f(s)$ is a name for a dense open set in $\mathbb{P}_\kappa$ for every $s \in [\kappa]^{<\omega}$; $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ will thus range over names for dense open sets in $\mathbb{P}^M_{j(\kappa)}$ but we will abuse the notation and identify every $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ with a name restricted to Q^M_κ in $M[G_\kappa]$.

By induction on $i < \kappa^+$, we will construct reductions $p_i \in M[G_\kappa]$ and a "master condition" p_∞ (possibly outside $M[G_\kappa]$) which will reduce all dense open sets in Q^M_κ according to Definition 3.11.

Assume that p_i have been constructed for all $i < j$ and we need to construct p_j. First let r_j be a lower bound of p_i for $i < j$ and work below this condition. Let $\langle a_\zeta \,|\, \zeta < \gamma_j \rangle$ be some enumeration of finite subsets of γ_j and $\langle j(f_\xi) \,|\, \xi < j \rangle$ be the first j-many functions $j(f)$ from the sequence $\langle j(f_\alpha) \,|\, \alpha < \kappa^+ \rangle$ fixed above. Carry out the following construction in $M[G_\kappa]$. By induction on pairs $(j(f_\xi), a_\zeta)$ construct a decreasing chain of conditions $\overline{q}_{(\xi,\zeta)}$ in $Q^M_{>\gamma_j}$ as follows. At stage (ξ, ζ), let first $r_{(\xi,\zeta)}$ be the lower bound of $\overline{q}_{(\xi',\zeta')}$ for $(\xi', \zeta') < (\xi, \zeta)$. Using the argument below Definition 3.11, set $\overline{q}_{(\xi,\zeta)}$ to be a condition which γ_j-reduces the dense open set with the name $j(f_\xi)(a_\zeta)$ below $(r_j)_{\gamma_j} \cup r_{(\xi,\zeta)}$. Since the induction has length γ_j, the lower bound of all $\overline{q}_{(\xi,\zeta)}$ exists in $M[G_\kappa]$. Denoting this lower bound $\overline{q}$, we set p_j to be equal to the union of the lower part of r_j and $\overline{q}$, i.e. $p_j = (r_j)_{\gamma_j} \cup \overline{q}$.

Notice that for all $i < \kappa^+$, p_i will lie in $M[G_\kappa]$ since only $\leq\kappa$-many functions $j(f)$ are considered at each step. Set p_∞ to be the lower bound of $\langle p_i \,|\, i < \kappa^+ \rangle$ (p_∞ may exist only in $V[G_\kappa]$).

Define the desired generic $\tilde{g}_\kappa$ as follows. Let us write $p^{\leftarrow}_\infty$ for p_∞ restricted to the interval $[\kappa, F(\kappa))$ and $p^{\rightarrow}_\infty$ for the rest of p_∞ defined at the interval $[F(\kappa), i^M_{\kappa+1})$. Assume that h is a $Q^*_{<F(\kappa)}$-generic containing $p^{\leftarrow}_\infty$, and set $h' = \{p^{\rightarrow}_\infty\} \cup \{q \in Q^M_\kappa \restriction [F(\kappa), \to) \,|\, p^{\rightarrow}_\infty \leq q\}$. We claim that $\tilde{g}_\kappa = (h \times h') \cap M[G_\kappa]$ is $M[G_\kappa]$-generic for Q^M_κ.

Let $D = (j(f)(a))^{G_\kappa}$ dense open be given, where $a \in [\gamma_{j'}]^{<\omega}$ for some $j' < \kappa^+$. We will show that $\tilde{g}_\kappa$ meets D. Assume that the set D was dealt with at substage (ξ, ζ) of the inductive construction of p_∞ at stage $j \geq j'$, where $j(f)$ is considered. Under this notation, recall that the set $\overline{D} = \{q \leq (r_j)_{\gamma_j} \,|\, q \cup \overline{q}_{(\xi,\zeta)} \in D\}$ is dense in $M[G_\kappa]$ below $(r_j)_{\gamma_j} \cup r_{(\xi,\zeta)}$ in $Q^M_{\leq\gamma_j}$. If A is a maximal antichain contained in $\overline{D}$, then by the reasoning as in Lemma 3.9, A will stay maximal in $V[G_\kappa]$ and consequently must be hit by h restricted to $Q^*_{\leq\gamma_i}$; let a be an element of h such that $a_{\gamma_j} \in \overline{D}$. It follows that $a_{\gamma_j} \cup \overline{q}_{(\xi,\zeta)}$ meets D. As both a and $p^{\leftarrow}_\infty$ are in h, there is some $a' \in h$ below both of them. But then $a' \cup p^{\rightarrow}_\infty \in h \times h'$ and $a' \cup p^{\rightarrow}_\infty \leq a_{\gamma_j} \cup \overline{q}_{\xi,\zeta}$, and so $a_{\gamma_j} \cup \overline{q}_{\xi,\zeta}$ is in $\tilde{g}_\kappa$ and meets D.

We finish the proof by arguing that $g^*_{<F(\kappa)}$, the restriction of g^*_κ to $[\kappa, F(\kappa))$, can be used to find in $V[G_\kappa * g_{\leq F(\kappa)}]$ some such generic h containing $p^{\leftarrow}_\infty$. By the homogeneity of the forcing $Q^*_{<F(\kappa)}$ [6] there is $r \in g^*_{<F(\kappa)}$ and an automorphism $\pi : Q^*_{<F(\kappa)} \cong Q^*_{<F(\kappa)}$ such that $\pi(r) = p^{\leftarrow}_\infty$; it follows that $h = \pi[g^*_{<F(\kappa)}]$ is $Q^*_{<F(\kappa)}$-generic containing $p^{\leftarrow}_\infty$.

Case (2)*: Cofinality of $F(\kappa)$, which we denote δ, is greater than κ^+ in V.*

We will need to distinguish two subcases.

Case (2a)*: $F(\kappa)$ is regular in M.*

Recall the sequences $\langle \gamma_i \,|\, i < \delta \rangle$ and $\langle j(f_\alpha) \,|\, \alpha < \kappa^+ \rangle$ which we used in the inductive construction in Case (1). Unlike Case (1), we do not make the assumption that $\delta = \kappa^+$. Thus the two inductions cannot be merged together as in Case (1) and a more complicated argument is called for. We will construct the desired generic for Q^M_κ in two steps.

A) The forcing $Q^M_{\leq F(\kappa)}$.

Intuitively, we need to define a generic for $Q^M_{\leq F(\kappa)}$ by building a decreasing list of conditions using induction along $\langle \gamma_i \,|\, i < \delta \rangle$ and simultaneously along $\langle j(f_\alpha) \,|\, \alpha < \kappa^+ \rangle$. [7] As both inductions can lead the construction outside the model $M[G_\kappa]$, we need to find a way to compatibly extend conditions "locally" without leaving the class $M[G_\kappa]$. We shall do this by dividing the available supports of the conditions into segments corresponding to some elementary substructures existing in $M[G_\kappa]$.

Let m_α for $\alpha < \kappa^+$ denote the following elementary substructure of some large

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[6] In fact, we need homogeneity only for the Cohen forcing part of the forcing $Q^*_{<F(\kappa)}$ above γ_0 as the master condition $p^{\leftarrow}_\infty$ is trivial below γ_0. This implies that we can disregard the Sacks forcing here.

[7] This time, $j(f)(a)$ for $a \in [F(\kappa)]^{<\omega}$ will be identified with $\mathbb{P}_\kappa$-names for dense open sets in $Q^M_{\leq F(\kappa)}$.

enough $H(\theta)^{M[G_\kappa]}$ which is closed under $<F(\kappa)$-sequences existing in $M[G_\kappa]$:

$$m_\alpha = \mathrm{SkolemHull}^{H(\theta)^{M[G_\kappa]}}\Big(\{Q^M_{\le F(\kappa)}\} \cup F(\kappa)+1 \cup \{j(f_\xi)\,|\,\xi \le \alpha\}\Big). \tag{8}$$

Notice that each m_α has size $F(\kappa)$ in $M[G_\kappa]$ and contains as elements all dense open sets of the form $j(f_\xi)(a)$ for $a \in [F(\kappa)]^{<\omega}$ and $\xi \le \alpha$.

We will build a matrix of conditions $\{p_{i,\alpha}\,|\,i<\delta, \alpha<\kappa^+\}$ in $Q^M_{\le F(\kappa)}$ with δ-many rows each of length κ^+ such that the conditions will be decreasing both in the rows and the columns. Moreover, for every $i<\delta$ and every $\alpha<\kappa^+$, the sequence of conditions in the α-th column up to i, i.e. $\langle p_{j,\alpha}\,|\,j<i\rangle$, will exist in m_α. We will construct the matrix in δ-many steps, each of length κ^+ (i.e. we will be completing rows first).

The first "square" of the matrix $p_{0,0}$ will be filled in as follows. By definition of m_0, all dense open sets in $Q^M_{\le F(\kappa)}$ of the form $(j(f_0)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$ are in m_0; by elementarity, they are dense open in m_0. Working inside m_0, carry out the reduction argument described in Case (1) of the present lemma. In particular, $p_{0,0}$ will γ_0-reduce all dense open sets $(j(f_0)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$ (below the trivial condition 1 as we are filling in the first square). The square $p_{0,1}$ will be filled in in exactly the same way, but working below the condition $p_{0,0}$ which is present in m_1. In particular $p_{0,1}$ will γ_0-reduce below $p_{0,0}$ all dense open sets of the form $(j(f_1)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$. Proceed this way at every successor ordinal, obtaining $p_{0,\alpha+1}$. At a limit ordinal $\lambda<\kappa^+$, first take a lower bound q of $\langle p_{0,\alpha}\,|\,\alpha<\lambda\rangle$ which by the closure properties of m_λ exists in m_λ, and then work below this lower bound; the resulting $p_{0,\lambda}$ will γ_0-reduce below q all dense open sets of the form $(j(f_\lambda)(a))^{G_\kappa}$ for $a \in [\gamma_0]^{<\omega}$. After κ^+ steps we have completed the 0-th row of the matrix. Note that the lower bound of $\langle p_{0,\alpha}\,|\,\alpha<\kappa^+\rangle$ may not exist in $M[G_\kappa]$.

We now need to complete the row 1. In order to complete the first square in row 1, we need to find $p_{1,0}$ compatible with all conditions in the 0-th row of the matrix. Though the lower bound of these conditions may not exist in $M[G_\kappa]$, we will argue that an intersection of the union of the conditions in the 0-th row with m_0 is in $M[G_\kappa]$, and even in m_0, i.e.

$$m_0 \cap \textstyle\bigcup_{\alpha<\kappa^+} p_{0,\alpha} \in m_0 \tag{9}$$

To see that (9) is true, we argue similarly as in Lemma 3.9. Each $p_{0,\alpha}$ is obviously in $M[G_\kappa]$, and consequently $p_{0,\alpha}\cap m_0$ is in $M[G_\kappa]$ and in particular in m_0. The intersection (9) can thus be viewed as the union of a κ^+-sequence of elements in m_0. But as m_0 has size $F(\kappa)$ in $M[G_\kappa]$, such a sequence exists in $M[G_\kappa]$ due to the following closure property

$$^{\kappa^+}F(\kappa) \in M[G_\kappa], \tag{10}$$

which is implied by the $F(\kappa)$-hypermeasurability of κ and the fact that κ^+ is smaller than the cofinality of $F(\kappa)$.

It follows there is $p_{1,0}$ which γ_1-reduces all dense open sets $(j(f_0)(a))^{G_\kappa}$ for $a \in [\gamma_1]^{<\omega}$ below the condition $m_0 \cap \bigcup_{\alpha<\kappa^+} p_{0,\alpha}$. In general for $\alpha < \kappa^+$, the condition $p_{1,\alpha}$ will reduce the relevant dense open sets below the common lower bound of $m_\alpha \cap \bigcup_{\alpha<\kappa^+} p_{0,\alpha}$ and the union of previous $p_{1,\beta}$ for $\beta < \alpha$.

It is immediate that the above construction can be repeated for any successor ordinal $i+1$ below δ, i.e. if the matrix has been completed up to the stage i, we can fill in the $i+1$-th row by the above argument.

Assume now that $i < \delta$ is a limit ordinal. First consider the sequence $\langle p_{j,\alpha} \,|\, j < i\rangle$ for a single $\alpha < \kappa^+$. As the sequence is of length less then cofinality $F(\kappa)$ in $M[G_\kappa]$ and contains elements from m_α, which has size $F(\kappa)$ in $M[G_\kappa]$, we can infer from

$$ {}^{i}F(\kappa) \in M[G_\kappa] \tag{11} $$

that the sequence exists in $M[G_\kappa]$, and in particular in m_α. Let $q_{i,\alpha} \in m_\alpha$ denote the lower bound of the sequence $\langle p_{j,\alpha} \,|\, j < i\rangle$ for each $\alpha < \kappa^+$. Now repeat the above argument for the successor step considering the intersections of m_α and $\bigcup_{\alpha<\kappa^+} q_{i,\alpha}$.

We finish the construction by taking the limit of the whole matrix $\{p_{i,\alpha} \,|\, i < \delta, \alpha < \kappa^+\}$, obtaining some condition p_∞ existing in $V[G_\kappa]$. Let $p_\infty^{\leftarrow}$ denote the condition p_∞ restricted to the interval $[\kappa, F(\kappa))$ and $p_\infty^{\rightarrow}$ the condition p_∞ restricted to $\{F(\kappa)\}$. Arguing as at the end of Case (1) of the present lemma, we find a $Q^*_{<F(\kappa)}$-generic h, where $p_\infty^{\leftarrow}$ is in h, and define h' to be generated by $p_\infty^{\rightarrow}$, such that $h \times h' \cap M[G_\kappa]$ is $Q^M_{\leq F(\kappa)}$-generic over $M[G_\kappa]$. Let us denote this generic as $\tilde{g}_{\leq F(\kappa)}$.

B) The forcing $Q^M_{>F(\kappa)}$ in $M[G_\kappa]$.

The regularity of $F(\kappa)$ in M implies that $Q^M_{\leq F(\kappa)}$ and $Q^M_{>F(\kappa)}$ are mutually generic. Working in $V[G_\kappa]$, we construct the generic $\tilde{g}_{>F(\kappa)}$ for $Q^M_{>F(\kappa)}$ exactly as in case B) of Lemma 3.9.

It follows that $\tilde{g}_\kappa = \tilde{g}_{\leq F(\kappa)} \times \tilde{g}_{>F(\kappa)}$ is the desired $M[G_\kappa]$-generic for Q^M_κ.

Case (2b)*: $F(\kappa)$ is singular in M.*

Recall once again the sequences $\langle \gamma_i \,|\, i < \delta\rangle$ and $\langle j(f_\alpha) \,|\, \alpha < \kappa^+\rangle$ which we used in the inductive construction in Case (1) and Case (2a).

The singularity of $F(\kappa)$ in $M[G_\kappa]$ introduces important simplification into the construction: $\langle \gamma_i \,|\, i < \delta\rangle$ can be picked in $M[G_\kappa]$ this time. Just run the argument for Case (1) with the following modification: Start with $j(f_0)$ and run the argument for Case (1) using just this one function $j(f_0)$, obtaining some master condition $p_\infty^{f_0}$. Since the sequence $\langle \gamma_i \,|\, i < \delta\rangle$ is in $M[G_\kappa]$, so is $p_\infty^{f_0}$. Now deal with $j(f_1)$ and so on by induction on $\alpha < \kappa^+$. At each $\alpha < \kappa^+$ we can take the lower bound of the conditions $p_\infty^{f_\beta}$ for $\beta < \alpha$ as we have closure under κ-sequences. Denote the constructed generic as $\tilde{g}_\kappa$.

This ends the proof of Lemma 3.10. $\square$

It follows we have completed one more step in finding suitable generics for $\mathbb{P}^M$.

$$V[G_\kappa] \overset{j}{\dashrightarrow} M[G_\kappa * \tilde{g}_\kappa]$$

In order to construct another generic, we need to verify that we have preserved closure under κ-sequences of $M[G_\kappa * \tilde{g}_\kappa]$ in $V[G_\kappa * g_{\leq F(\kappa)}]$.

Lemma 3.12 *$M[G_\kappa * \tilde{g}_\kappa]$ is closed under κ-sequences in $V[G_\kappa * g_{\leq F(\kappa)}]$.*

Proof. As mentioned above, $M[G_\kappa]$ remains closed under κ-sequences in $V[G_\kappa]$ as the forcing $\mathbb{P}_\kappa$ is κ-cc. Let us denote as g_S the projection of g_κ to the Sacks forcing. By Lemma 3.7, the forcing $\mathrm{Add}(\kappa^+, F(\kappa^+))$ is κ^+-distributive after the Sacks forcing $\mathrm{Sacks}(\kappa, F(\kappa))$, and consequently it is enough to show closure just in $V[G_\kappa * g_S]$. Recall the modification argument just before Lemma 3.9 which removes the discrepancy between the values $F(\kappa)$ and $j(F)(\kappa)$; the modification of $\mathrm{Sacks}(\kappa, F(\kappa))$ changes this forcing to $S^* = \mathrm{Sacks}(\kappa, j(F)(\kappa))$. Due to closure of $M[G_\kappa]$ under κ-sequences, S^* is the same in $V[G_\kappa]$ and in $M[G_\kappa]$ and is the first step of the product Q_κ^M in the iteration $\mathbb{P}^M$ at stage κ. We are going to work in $V[G_\kappa * g_S^*] = V[G_\kappa * g_S]$, where g_S^* is the generic for S^*, and as such is present in $M[G_\kappa * \tilde{g}_\kappa]$.

Let X be a κ-sequence of ordinal numbers in $V[G_\kappa * g_S^*]$, and let this be forced by some $p_0 \in g_S^*$. By the fusion argument (carried out in $V[G_\kappa]$), there is for every $r \leq p_0$ some $p_X \leq r$ such that if p_X is in g_S^*, then X can be uniquely determined from p_X and g_S^* restricted to the support of p_X. Since such p_X are dense below p_0, some such p_X is in g_S^*, and as p_X and g_S^* are present in $M[G_\kappa * \tilde{g}_\kappa]$, so is X. $\square$

The preservation of closure allows us to prove:

Lemma 3.13 *We can construct an $M[G_\kappa * \tilde{g}_\kappa]$-generic for the stage $\mathbb{P}^M_{[i^M_{\kappa+1}, j(\kappa))}$ in $V[G_\kappa * g_{\leq F(\kappa)}]$.*

Proof. As in Lemma 3.9, case B), work in $V[G_\kappa * g_{\leq F(\kappa)}]$ and construct a generic H hitting all dense sets. $\square$

It follows we can lift partially to $V[G_\kappa]$:

$$V[G_\kappa] \xrightarrow{\quad j \quad} M[G_\kappa * \tilde{g}_\kappa * H]$$

The next step is to lift to the Q_κ-generic g_κ, where $Q_\kappa = \mathrm{Sacks}(\kappa, F(\kappa)) \times \prod_{\kappa<\lambda<i_{\kappa+1}} \mathrm{Add}(\lambda, F(\lambda))$ taken in $V[G_\kappa]$.

Again due to Lemma 3.7, coupled with Lemma 2.12, the only non-trivial part of this step is to lift to the generic filter g_S for $\mathrm{Sacks}(\kappa, F(\kappa))$. This follows from the technique in [4], which is briefly reviewed in section 2.3. There is only one point which requires a slight generalization; in the terminology of the present article, [4] uses some continuum function F_0 which is defined by $F_0(\alpha) = \alpha^{++}$ for $\alpha \le \kappa$ inaccessible, and trivial elsewhere. The set of closure points of F_0 below κ thus coincides with the set of limit cardinals C_L, and this is used in the proof of Lemma 2.16. However, instead of C_L, one may use in the argument the set $C_F = \langle i_\alpha \,|\, \alpha < \kappa \rangle$ of closure points of F.

Let us denote the generic generated by $j[g_S]$ as h_0. It follows we can lift as follows:

$$V[G_\kappa * g_S] \xrightarrow{\quad j \quad} M[G_\kappa * \tilde{g}_\kappa * H * h_0]$$

We finish the lifting by an application of Lemma 2.12 in two stages, first to the rest of the product Q_κ and then to the rest of the iteration $\mathbb{P}^F$:

$$V[G] \xrightarrow{\quad j \quad} M[j(G)]$$

This proves Theorem 3.8 and shows that κ remains measurable in the generic extension by $\mathbb{P}^F$.

3.2 Preservation of strong cardinals

In [9], Menas showed using a "master condition" argument that *locally definable* (see Definition 3.14 below) continuum functions F can be realized while preserving supercompact cardinals. We will show how to extend his result to strong cardinals using the above arguments.

Definition 3.14 *A continuum function F, see Definition 3.1, is said to be* locally definable *if the following condition hold:*

There is a sentence ψ and a formula $\varphi(x, y)$ with two free variables such that ψ is true in V and for all cardinals γ, if $H(\gamma) \models \psi$, then $F[\gamma] \subseteq \gamma$ and

$$\forall \alpha, \beta \in \gamma (F(\alpha) = \beta \Leftrightarrow H(\gamma) \models \varphi(\alpha, \beta)). \tag{12}$$

Theorem 3.15 *(GCH) Assume F is locally definable in the sense of Definition 3.14. If $\mathbb{P}^F$ is the forcing notion as in Definition 3.5, then $V^{\mathbb{P}^F}$ realizes F and preserves all strong cardinals.*

Proof. First note that since ψ is true in V, there exists a closed unbounded class of cardinals C_ψ such that if $\beta \in C_\psi$, then $H(\beta) \models \psi$.

Assume κ is a strong cardinal. We first show that κ is closed under F. Let $\beta > \kappa$ be a cardinal such that $H(\beta) \models \psi$ and let $j : V \to M$ be a β^+-hypermeasurable extender embedding; in particular $\beta^+ < j(\kappa) < \beta^{++}$ and $M = \{j(f)(a) \,|\, f : [\kappa]^{<\omega} \to V, a \in [\beta^+]^{<\omega}\}$. As $H(\beta)^V = H(\beta)^M$, $H(\beta)^M$ satisfies ψ. Let M_β be the ultrapower of V by $U_\beta = \{X \subseteq \kappa \,|\, \beta \in j(X)\}$ and $i_\beta : V \to M_\beta$ and $k_\beta : M_\beta \to M$ the elementary embeddings forming the commutative triangle $j = k_\beta \circ i_\beta$; in particular $k_\beta : M_\beta \to M$ is defined by $k_\beta([f]) = j(f)(\beta)$, where $[f]$ is an equivalence class in M_β and where we identify β and $\{\beta\}$ for simplicity. If h takes each infinite $\xi < \kappa$ to $H(|\xi|)$, then $k_\beta([h]) = j(h)(\beta) = H(\beta)^M = H(\beta)$; it follows by elementarity that $M_\beta \models ([h] \models \psi)$ and consequently the set $A = \{\xi < \kappa \,|\, H(|\xi|) \models \psi\}$ is in U_β. In particular, A is an unbounded subset of κ such that if $\xi \in A$, then $|\xi| < \kappa$ is closed under F. From this we can infer that κ is closed under F as well (although it may not be true that $H(\kappa)$ satifies ψ).

Let G be a generic filter for $\mathbb{P}^F$. Assume that α is given and let $\beta > \alpha$ be a singular cardinal such that $H(\beta)$ satisfies ψ, and assume for technical convenience that β is indexed by a successor ordinal in the enumeration of closure points of F. We claim that every extender embedding $j : V \to M$ witnessing the β^+-hypermeasurability of κ can be lifted to a $j^* : V[G] \to M[j(G)]$ with $H(\beta^+)$ of $V[G]$ included in $M[j(G)]$, thereby witnessing that κ is still β^+-hypermeasurable in $V[G]$. As α can be arbitrarily large, this implies that κ is still strong in $V[G]$.

Let $\beta > \alpha$ be a singular cardinal with $H(\beta) \models \psi$ such that β is indexed by a succesor ordinal in the enumeration of closure points of F, and let the previous closure point be denoted as $\overline{\beta} < \beta$. Let $j : V \to M$ be a β^+-hypermeasurable witnessing embedding; as above, we can assume $\beta^+ < j(\kappa) < \beta^{++}$ and $M = \{j(f)(a) \,|\, f : [\kappa]^{<\omega} \to V, a \in [\beta^+]^{<\omega}\}$. Since κ is closed under F, $j(\kappa)$ is closed under $j(F)$. Moreover, since $j(F)$ is locally definable in M via the formulas ψ and $\varphi(x, y)$ and $H(\beta)^M = H(\beta)^V$, it follows that $H(\beta)^M \models \psi$ and consequently F and $j(F)$ are identical on the interval $[\omega, \beta)$; in particular β is closed under $j(F)$. The fact that $H(\beta^+)$ is correctly captured by M implies that $\mathbb{P}^F$ and $j(\mathbb{P}^F)$ coincide up to stage β, i.e. $\mathbb{P}^F_\beta = j(\mathbb{P}^F)_\beta$, and thus we may "copy" the generic G_β, i.e. G restricted to β, and use it as a generic for $j(\mathbb{P}^F)_\beta$.

$M[G_\beta]$ is easily seen to be still closed under κ-sequences in $V[G_\beta]$. However, we also need to make sure that $H(\beta^+)$ of $V[G_\beta]$ is included in $M[G_\beta]$, and this requires a little argument. Recall that $\overline{\beta}$ is the immediate previous closure point of F below β. Considering nice names, it follows immediately that $H(\beta^+)$ of $V[G_{\overline{\beta}}]$ is included in $M[G_{\overline{\beta}}]$. The argument for the product $Q_{\overline{\beta}}$ (an Easton-supported product of $\mathrm{Add}(\lambda, F(\lambda))$ for regular cardinals between $\overline{\beta}$ and β, preceded with the Sacks forcing $\mathrm{Sacks}(\overline{\beta}, F(\overline{\beta}))$ in case $\overline{\beta}$ is regular) in $V[G_{\overline{\beta}}]$ is more delicate as the nice names for subsets of β in $Q_{\overline{\beta}}$ can have size β^+ since β is singular. However, using the reduction argument set forth in Definition 3.11, we can show that for any nice $Q_{\overline{\beta}}$-name σ for a subset of β and every $q \in Q_{\overline{\beta}}$ there is some $p \leq q$ and a name σ^* of size β such that

$$p \Vdash \sigma^* = \sigma \tag{13}$$

To obtain (13), we first fix a sequence of regular cardinals $\langle \gamma_n \,|\, n \in \omega \rangle$ cofinal in β and for technical convenience require that $\gamma_0 > \overline{\beta}^{++}$. (Cofinality of β is ω as it is a closure point indexed by a successor ordinal.) Mimicking the argument below Definition 3.11, work below q and find at stage γ_i some $q_i \leq q_{i-1}$ which γ_i-reduces γ_i-many dense open sets of conditions in $Q_{\overline{\beta}}$ deciding the membership of the elements of γ_i in σ. Note that if D_ξ is such a dense open set deciding the membership of $\xi < \gamma_i$, then its reduction $\overline{D}_\xi$ has size $|(Q_{\overline{\beta}})_{\leq \gamma_i}|$, and hence the dense set of conditions deciding $\xi \in \sigma$ is reduced by q_i to size less than β. The final p is defined to be the greatest lower bound of all q_i for $i \in \omega$ and σ^* is defined to contain for every $\alpha < \beta$ all pairs $\langle \alpha, r \rangle$ such that $r \leq p$ is in $Q_{\overline{\beta}}$, an initial part of r lies in a reduction $\overline{D}_\alpha$ at some stage $\gamma_i > \alpha$, and r forces $\alpha \in \sigma$.

Let us denote as $g_{\overline{\beta}}$ the generic for $Q_{\overline{\beta}}$ such that $G_\beta = G_{\overline{\beta}} * g_{\overline{\beta}}$. Let $X \subseteq \beta$ in $V[G_\beta]$ be given and let σ_X be a nice name for X; we will show that X is also in $M[G_\beta]$. Since the conditions p with names σ^*_X satisfying (13) are dense in $Q_{\overline{\beta}}$, there is some $p \in g_{\overline{\beta}}$ and a name σ^*_X such that p forces that $\sigma_X = \sigma^*_X$. It follows it is enough to show that σ^*_X is in $M[G_{\overline{\beta}}]$. But this is immediate from the fact that σ^*_X is an element of $H(\beta^+)$ of $V[G_{\overline{\beta}}]$ which is included in $M[G_{\overline{\beta}}]$.

We now argue as in Lemma 3.9 that there is in $V[G_\beta * g_\beta]$ a $M[G_\beta]$-generic for Q_β. Since β is singular, the forcing at β is trivial. The first non-trivial forcing in Q_β is $\mathrm{Add}(\beta^+, j(F)(\beta^+))$, where $j(F)(\beta^+) < j(\kappa) < \beta^{++}$. We find an $M[G_\beta]$-generic for this forcing by arguing as in Lemma 3.9, Case A), making use of the fact that $H(\beta^+)$ of $V[G_\beta]$ is included in $M[G_\beta]$. The rest of the forcing Q_β above β^+ is sufficiently distributive, and hence the argument as in Lemma 3.9, Case B) can be used. The same argument can actually be applied to the rest of the iteration $j(\mathbb{P}^F)$ up to $j(\kappa)$, see Lemma 3.13. We finish the proof by first lifting to the Sacks forcing at κ, using [4], and then to the rest of the forcing above κ (see the end of the proof for Theorem 3.8, just before this Section 3.2). □

References

[1] James Cummings. Iterated forcing and elementary embeddings. To be included in the Handbook of Set Theory.

[2] James Cummings. A model where GCH holds at successors but fails at limits. *Transactions of the American Mathematical Society*, 329(1), 1992.

[3] William B. Easton. Powers of regular cardinals. *Annals of Mathematical Logic*, 1, 1970.

[4] Sy D. Friedman and Kathrine Thompson. Perfect trees and generic modification. Preprint.

[5] Moti Gitik. The negation of singular cardinal hypothesis from $o(\kappa) = \kappa^{++}$. *Annals of Pure and Applied Logic*, 43(3), 1989.

[6] Tomáš Jech. *Set Theory.* Springer, 2003.

[7] Akihiro Kanamori. Perfect-set forcing for uncountable cardinals. *Annals of Mathematical Logic*, 19, 1980.

[8] Akihiro Kanamori. *Higher Infinite.* Springer, 2003.

[9] Telis K. Menas. Consistency results concerning supercompactness. *Transactions of the American Mathematical Society*, 223, 1976.