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# AN INEQUALITY CONCERNING THE RADEMACHER FUNCTIONS

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Abstract. Let $r_j$, $j = 0, 1, \dots$ be the Rademacher functions on $[0, 1]$. We prove that for every measurable subset $E$ of $[0, 1]$ with $|E| > 0$ and for each $\lambda > 1$ there exists a positive integer $N$ such that for all real-valued sequences $\{a_j\}$ there exists a subset $J$ of $[0, 1]$ such that $\sum_{j=N}^{\infty} |a_j|^2 \le \frac{\lambda}{|J|} \int_{J\cap E} \left|\sum_{j=0}^{\infty} a_j r_j(t)\right|^2 dt$.

## 1. Introduction

The $j^{\text{th}}$ Rademacher function $r_j$ on $[0, 1)$, $j = 0, 1, 2, \dots$ is defined as follows: $r_0 = 1$, $r_1 = 1$ on $[0, 1/2)$ and $r_1 = -1$ on $[1/2, 1)$, $r_2 = 1$ on $[0, 1/4) \cup [1/2, 3/4)$ and $r_2 = -1$ on $[1/4, 1/2) \cup [3/4, 1)$, etc.

In this article we investigate a local property of the Rademacher functions related to Khintchine's inequality. The following is a classical result that can be found in Zygmund [9] (page 213): For every subset $E$ of $[0, 1]$ and every $\lambda > 1$, there is a positive integer $N$ such that for all complex-valued square-summable sequences $\{a_j\}$ we have

$$\sum_{j\ge N} |a_j|^2 \le \lambda \sup_{t\in E} \Big|\sum_{j\ge N} a_j r_j(t)\Big|^2 . \tag{1}$$

The next statement is contained in Lemma 2 of Stein [8] (page 147): For every subset $E$ of $[0, 1]$ there is a positive integer $N_E$ and a constant $C_E$ such that for all complex-valued square-summable sequences $\{a_j\}$ we have

$$\sum_{j\ge N_E} |a_j|^2 \le C_E \sup_{t\in E} \Big|\sum_{j\ge 0} a_j r_j(t)\Big|^2 . \tag{2}$$

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Estimate (2) has been referred to in the literature as Stein's lemma and has been found to be a useful tool in applications concerning almost everywhere convergence, see for instance [1], [8], [6]. Unpublished versions of Stein's lemma have been independently obtained by several authors, including D. Burkholder A. M. Garsia, R. F. Gundy, P. A. Meyer, S. Sawyer, and G. Weiss (c.f. [2], [3]). A version of this lemma in the context of independent sequences of random variables with very good control of the constants has been published by Burkholder [2]. Other authors have published related results. Sagher and Zhou [4] published a version of inequality (1) in which the supremum is replaced by the $L^p$ average over $E$. In [5] the same authors proved analogous inequalities for lacunary series. Carefoot and Flett [3] have obtained a version of inequality (2) in which the $\ell^2$ norm on the left is replaced by a supremum of truncated $\ell^1$ norms. Recently, Slavin and Volberg [7] have obtained a profound local version of the Chang-Wilson-Wolff inequality which may be thought as analogous to the aforementioned local versions of Khintchine's inequality.

In this note we discuss yet another generalization of Stein's lemma. The inequality we prove is of $L^2$ nature and presents certain quantitative advantages: as in (1) there is a near-optimal constant $\lambda > 1$ but the supremum in (1) and (2) is replaced by the sharper quadratic average over a certain set $J$ of the truncated Rademacher series localized on $E$. This set $J$ has small measure and is contained in a dyadic interval with twice its measure, hence it precisely pinpoints the part of $E$ on which the Rademacher series is largest. Moreover, the truncated Rademacher series may start at any point less than $N$. The exact formulation of the result is as follows.

**Theorem 1.1.** *For every measurable subset $E$ of $[0,1]$ with $|E| > 0$ and each $\lambda > 1$ there exists a positive integer $N = N(E,\lambda)$ such that for any $n \in \{0,1,\dots,N\}$ and for any real-valued square-summable sequence $a_j$ there is a subset $J = J(n,\{a_j\})$ of $[0,1]$ of measure $2^{-N-1}$ such that*

$$\sum_{j\geq N} |a_j|^2 \leq \frac{\lambda}{|J|}\int_{J\cap E}\Big|\sum_{j\geq n} a_j r_j(t)\Big|^2 dt\,. \tag{3}$$

Naturally, estimate (3) implies both estimates (1) and (2) for real-valued sequences. It also yields (1) with an additional factor of 2 on the right for complex-valued sequences and it implies (2) with a constant $C_E$ independent of the set $E$; in fact, it follows from (3) that the constant $C_E$ in (2) can be taken to be $1+\delta$ for real-valued sequences and $C_E = 2+2\delta$ for complex-valued sequences, for any $\delta > 0$.

## 2. Two lemmata

It is an easy fact to check that the system $\{r_k\, r_\ell\}_{k\neq\ell}$ indexed by all nonnegative integers $k$ and $\ell$ satisfying $k\neq\ell$ is orthonormal in $L^2[0,1]$. In particular we have the inequality

$$\sum_{\substack{k,\ell\geq 0\\ k\neq\ell}} |\langle f, r_k r_\ell\rangle|^2 \leq \|f\|_{L^2}^2 \tag{4}$$

for all $f\in L^2[0,1]$.

We will need the following two auxiliary results:

**Lemma 2.1.** *For every square-summable complex sequence $\{a_j\}_{j=0}^\infty$ and every measurable subset $E\subseteq[0,1]$ with positive measure, we have:*

$$\int_E \Big|\sum_{j\geq 0} a_j r_j\Big|^2 \leq \Big(|E|+\sqrt{|E|}\Big)\int_0^1 \Big|\sum_{j\geq 0} a_j r_j\Big|^2 .$$

*Proof.* Expanding out the square on the left we obtain

$$\begin{aligned}
\int_E \Big|\sum_{j\geq 0} a_j r_j\Big|^2 &\leq |E|\sum_{j=0}^\infty |a_j|^2 + \sum_{j\neq k} a_j\overline{a_k}\int_E r_j r_k\,dt \\
&\leq |E|\sum_{j=0}^\infty |a_j|^2 + \big(\sum_{j\neq k}|a_j a_k|^2\big)^{\frac12}\Big(\sum_{j\neq k}\Big|\int_E r_j r_k\,dt\Big|^2\Big)^{\frac12} \\
&\leq |E|\sum_{j=0}^\infty |a_j|^2 + \big(\sum_{j=0}^\infty |a_j|^2\big)\Big(\sum_{j\neq k}\Big|\int_E r_j r_k\,dt\Big|^2\Big)^{\frac12} \\
&\leq (|E|+|E|^{1/2})\sum_{j=0}^\infty |a_j|^2 ,
\end{aligned}$$

making use of (4). This completes the proof since $\int_0^1 \big|\sum_{j\geq 0} a_j r_j\big|^2 = \sum_{j\geq 0}|a_j|^2$. $\square$

For a dyadic subinterval $I_N = [m2^{-N},(m+1)2^{-N})$ of $[0,1)$ and a real sequence $\{a_j\}_{j\in\mathbb{N}}$ define sets depending on $\{a_j\}$

$$\begin{aligned}
I_N^{++} &= \Big\{t\in I_N : \sum_{j\geq N} a_j r_j(t) > 0\Big\}, \\
I_N^{--} &= \Big\{t\in I_N : \sum_{j\geq N} a_j r_j(t) < 0\Big\}, \\
I_N^{0} &= \Big\{t\in I_N : \sum_{j\geq N} a_j r_j(t) = 0\Big\}.
\end{aligned}$$

It is easy to see that the sets $I_N^{++}$ and $I_N^{--}$ have equal measure. Next, we find disjoint subsets $I_N^{0,+}$ and $I_N^{0,-}$ of $I_N^0$ of equal measure whose union is $I_N^0$ and we define $I_N^+ = I_N^{++} \cup I_N^{0,+}$ and $I_N^- = I_N^{--} \cup I_N^{0,-}$ Then we have $I_N^+ \cup I_N^- = I_N$ and by construction we have $|I_N^+| = |I_N^-| = |I_N|/2$. Moreover we have that $\sum_{j\geq N} a_j r_j \geq 0$ on $I_N^+$ and $\sum_{j\geq N} a_j r_j \leq 0$ on $I_N^-$. Next we have the following:

**Lemma 2.2.** *For any real-valued square-summable sequence $\{a_j\}$, for any positive integer $N$, for every dyadic interval $I_N \subseteq [0,1)$ with $|I_N| = 2^{-N}$, and any measurable subset $E \subseteq [0,1]$ satisfying*

$$\frac{|E^c \cap I_N|}{|I_N|} + \sqrt{\frac{|E^c \cap I_N|}{|I_N|}} < \frac{1}{2},$$

*we have*

$$\int_{I_N} \Big|\sum_{j\geq N} a_j r_j\Big|^2 \leq \frac{1}{\left(\frac{1}{2} - \frac{|E^c\cap I_N|}{|I_N|} - \sqrt{\frac{|E^c\cap I_N|}{|I_N|}}\right)} \int_{I'_N\cap E} \Big|\sum_{j\geq N} a_j r_j\Big|^2$$

*where $I'_N = I_N^+$ or $I'_N = I_N^-$.*

*Proof.* First take $I'_N = I_N^+$. We write

$$\begin{aligned} &\int_{I_N} \Big|\sum_{j\geq N} a_j r_j\Big|^2 = \\ &\int_{I_N^+\cap E} \Big|\sum_{j\geq N} a_j r_j\Big|^2 + \int_{I_N^-\cap E} \Big|\sum_{j\geq N} a_j r_j\Big|^2 + \int_{I_N\cap E^c} \Big|\sum_{j\geq N} a_j r_j\Big|^2 \end{aligned} \tag{5}$$

but obviously

$$\int_{I_N^-\cap E} \Big|\sum_{j\geq N} a_j r_j\Big|^2 \leq \int_{I_N^-} \Big|\sum_{j\geq N} a_j r_j\Big|^2 \tag{6}$$

and, since the system of Rademacher functions $\{r_j\}_{j\geq N}$ on $I_N$ has the same properties as the system $\{r_j\}_{j\geq 0}$ on $[0,1]$, we have that

$$\int_{I_N^-} \Big|\sum_{j\geq N} a_j r_j\Big|^2 = \frac{1}{2}\int_{I_N} \Big|\sum_{j\geq N} a_j r_j\Big|^2. \tag{7}$$

On the other hand, by a simple change of variables we get

$$\int_{I_N\cap E^c} \Big|\sum_{j\geq N} a_j r_j\Big|^2 = |I_N| \int_F \Big|\sum_{j\geq 0} r_j a_{j+N}\Big|^2 \tag{8}$$

for some $F \subseteq [0,1]$ with measure $|F| = \dfrac{|I_N \cap E^c|}{|I_N|}$. By Lemma 2.1 we obtain

$$\begin{aligned} |I_N| \int_F \left| \sum_{j\geq 0} r_j a_{j+N} \right|^2 &\leq |I_N|(|F| + \sqrt{|F|}) \int_0^1 \Big| \sum_{j\geq 0} r_j a_{j+N} \Big|^2 \\ &= |I_N|(|F| + \sqrt{|F|}) \frac{1}{|I_N|} \int_{I_N} \Big| \sum_{j\geq N} r_j a_j \Big|^2 . \end{aligned}$$

Combining (5), (6), (7), and (8) yields

$$\int_{I_N} \Big| \sum_{j\geq N} a_j r_j \Big|^2 \leq$$

$$\int_{I_N^+ \cap E} \Big| \sum_{j\geq N} a_j r_j \Big|^2 + \frac{1}{2} \int_{I_N} \Big| \sum_{j\geq N} a_j r_j \Big|^2 + (|F| + \sqrt{|F|}) \int_{I_N} \Big| \sum_{j\geq N} a_j r_j \Big|^2$$

which implies

$$\left( \frac{1}{2} - \frac{|I_N \cap E^c|}{|I_N|} - \sqrt{\frac{|I_N \cap E^c|}{|I_N|}} \right) \int_{I_N} \Big| \sum_{j\geq N} a_j r_j \Big|^2 \leq \int_{I_N^+ \cap E} \Big| \sum_{j\geq N} a_j r_j \Big|^2 .$$

Obviously we can repeat the proof replacing $I_N^-$ with $I_N^+$ and the claimed result follows.

$\square$

## 3. Proof of the Theorem

*Proof.* Given $\lambda > 1$, pick an $\epsilon > 0$ small enough such that

$$0 < \frac{1}{1/2 - \epsilon - \sqrt{\epsilon}} < 2\lambda .$$

By standard measure theory, we have that for every measurable subset $E \subseteq [0,1]$ there exists a dyadic subinterval $I_N$ of $[0,1]$ of size $2^{-N}$ such that

$$\frac{|I_N \cap E^c|}{|I_N|} < \epsilon .$$

Since $\{r_j\}_{j\in\mathbb{N}}$ is an orthogonal system in $L^2([0,1])$, by a change of variables we obtain

$$\sum_{j\geq N} |a_j|^2 = \frac{1}{|I_N|} \int_{I_N} |\sum_{j\geq N} a_j r_j|^2$$

and an application of Lemma 2.2 gives

$$\sum_{j\geq N}|a_j|^2 \leq \frac{1}{|I_N|}\frac{1}{(1/2-\epsilon-\sqrt{\epsilon})}\int_{I'_N\cap E}|\sum_{j\geq N}a_jr_j|^2 \tag{9}$$

where $I'_N = I^+_N$ or $I'_N = I^-_N$.

The important observation is that the functions $r_j$, $j = 0, 1, \ldots, N$ are constant on $I_N$. This implies that for all $n \in \{0, 1, \ldots, N\}$, the sum $\sum_{j=n}^N a_jr_j$ is a real-valued constant on $I_N$. We may first assume that $\sum_{j=n}^N a_jr_j > 0$ on $I_N$. Then we have

$$|\sum_{j=N}^\infty a_jr_j| = \sum_{j=N}^\infty a_jr_j \leq \sum_{j=n}^\infty a_jr_j = |\sum_{j=n}^\infty a_jr_j| \qquad \text{on } I^+_N.$$

Choosing $I'_N = I^+_N$ in (9) we write

$$\begin{aligned}\sum_{j\geq N}|a_j|^2 &\leq \frac{1}{|I_N|}\frac{1}{(1/2-\epsilon-\sqrt{\epsilon})}\int_{I^+_N\cap E}|\sum_{j\geq N}a_jr_j|^2\\ &\leq \frac{1}{|I_N|}\frac{1}{(1/2-\epsilon-\sqrt{\epsilon})}\int_{I^+_N\cap E}|\sum_{j\geq n}a_jr_j|^2\\ &\leq \frac{2\lambda}{|I_N|}\int_{I^+_N\cap E}\Big|\sum_{j\geq n}a_jr_j\Big|^2\\ &= \frac{\lambda}{|J|}\int_{J\cap E}\Big|\sum_{j\geq n}a_jr_j\Big|^2\end{aligned}$$

where $J = I^+_N$. We argue likewise when $\sum_{j=n}^N a_jr_j$ is a negative constant on $I_N$, in which case we pick $J = I^-_N$. The theorem is proved. □

**Corollary 3.1.** *For every measurable subset $E \subseteq [0, 1]$ with positive measure and for every $\lambda > 1$ there exists a positive integer $N$ such that for all $n \in \{0, 1, \ldots, N\}$ and all $1 \leq p \leq \infty$ we have*

$$\sum_{j\geq N}|a_j|^2 \leq \lambda\, 2^{(N+1)/p}\Big\|\sum_{j\geq n}r_ja_j\Big\|^2_{L^{2p}(E)} \tag{10}$$

*for every square-summable real-valued sequence $\{a_j\}_{j\geq 0}$.*

*Proof.* We write the right hand side of (3) as:

$$\frac{\lambda}{|J|}\int_E\Big|\sum_{j\geq n}a_jr_j\Big|^2\chi_J$$

and we apply Hölder's inequality to the integral with exponents $p$ and $p' = p/(p-1)$. □

## 4. A Remark

We know that the constant on the right hand side of inequality (10) must depend on $N$ and hence on the set $E$. Indeed, let us illustrate this in the case $p = 1$.

**Remark 4.1.** *For all positive constants $C$ there exists a measurable subset $E \subset [0,1]$ such that for every natural number $N$ and any $n \in \{0, 1, ..., N\}$ we can find a real sequence $\{a_j\}_{j\in\mathbb{N}} \in \ell^2$ such that the following holds:*

$$\sum_{j\geq N} |a_j|^2 > C \| \sum_{j\geq n} r_j a_j \|^2_{L^2(E)}. \tag{11}$$

*Proof.* Let $C$ be an arbitrary positive constant. Set

$$C' = [\log_2(C+2)] + 1,$$

where $[\ ]$ denotes the integer part; so $C' \in \mathbb{N}$. Then we just need to choose $E$ to be a dyadic subinterval of $[0,1]$ with $|E| = 2^{-C'}$. Also let $t_0$ be an interior point of $E$. For this choice of $E$, the following holds: for every real sequence $\{a_j\}_{j\in\mathbb{N}}$ and every natural number $n$ we have

$$\begin{aligned} \left\| \sum_{j\geq n} r_j a_j \right\|^2_{L^2(E)} &= \int_E \left| \sum_{j\geq n} a_j r_j(t) \right|^2 dt \\ &= \left( \left| \sum_{n\leq j\leq \mathbb{C}'} a_j r_j(t_0) \right|^2 + \sum_{j>\max\{C',n\}} |a_j|^2 \right) |E| \end{aligned} \tag{12}$$

where the sum $\sum_{n\leq j\leq C'} a_j r_j(t_0)$ is understood to be equal to zero when $n > C'$.

Now, fix $N \in \mathbb{N}$ and define $a_j = 0$ when $j \neq \max\{N, C'\}$ and $a_{\max\{N,C'\}} = 1$. Then for this choice of $\{a_j\}_{j\in\mathbb{N}}$ and for $n \in \{0, 1, ..., N\}$, (11) is a consequence of (12) since

$$\begin{aligned} C \| \sum_{j\geq n} a_j r_j \|^2_{L^2(E)} &= C \cdot 1 \cdot |E| \\ &< (C+2) 2^{-C'} \\ &= 2^{\log_2(C+2)} \cdot 2^{-C'} \\ &< 2^{[\log_2(C+2)]+1} \cdot 2^{-C'} \\ &= 2^{C'} \cdot 2^{-C'} \\ &= |a_{\max\{N,C'\}}|^2 \\ &= \sum_{j\geq N} |a_j|^2 . \end{aligned}$$

□

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