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HOMOGENIZATION OF THE DIRICHLET PROBLEM FOR STEADY NAVIER-STOKES EQUATIONS IN DOMAINS WITH CHANNELS

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Abstract We consider a sequence of Dirichlet problems for steady Navier- Stokes equations in perforated domains $\Omega_s = \Omega \setminus \bigcup_{i=1}^{I(s)} \mathcal{F}_i^{(s)}$, where $\mathcal{F}_i^{(s)}$, $i = 1, \ldots, I(s)$, are nonintersecting closed subsets of Ω containing in small neighborhoods of some lines (perforated domains with channels). The number of this small sets tends to infinity as $s \to \infty$. We study the asymptotic behavior of solutions $u_s(x)$ of this problems in domains with channels as $s \to \infty$. Conditions under which solutions $u_s(x)$ converge to some limit function are established. The boundary value problem for the limit function is constructed. The result is based on the asymptotic expansion of the sequence $u_s(x)$ and on pointwise and integral estimates of auxiliary functions which are solutions of the model boundary value problems.

Keywords Homogenization · Navier-Stokes equation · Perforated domain

1 Introduction and formulation of the problem

Processes in locally inhomogeneous media, the local properties of which are subject to sharp small scale changes in the space, are of great interest in various fields of science. Various methods were applied to the investigation of such problems. We mention one of the most famous micro-macro approach which was used for homogenization of processes in porous periodic media, [17]. In particular, there were constructed the asymptotic expansions for flow in small channels of solid porous body and for moving of composite of solid elastic bodies and viscous fluids. It is known that it is possible to get **Darcy's**

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law in the limit. By the Darcy's law a slow fluid flows through a rigid medium can be modelled. Ene and Sanchez-Palencia seems to be first to give a derivation of it, from the Stokes system, using multiscale expansion [7]. This derivation was made rigorously by L.Tartar in [20]. This result was generalized by many authors and we mention one generalization, which was done by Allaire [1] and to random statistically porous medium by Beliaev and Kozlov [3]. Also we would like to mention the fundamental work of Jikov, Kozlov and Oleinik [10] and several fundamental works which were done by Jäger, Mikelic [9], Mikelic at el. [8,4]. If a rigid part of the porous medium Ω has the critical size, much smaller than $O(1)$, then it is possible to get **Brinkman's law**, (in detail, see [2]). In many applications the solid part is supposed to be elastic. In this case the effective filtration law is known as **Biot's law** [6].

Different problems in domains with nonperiodic structure, corresponding to the case of Brinkman's law, were considered in [14] - [18]. Firstly, elliptic systems of higher order in domains with fine-grained boundary were studied in [14] using potential theory and variational methods. This approach was applied, in particular, to Dirichlet problems for Navier-Stokes equations in domains with fine-grained boundary (see [11,12]). Another approach was developed by I.V.Skrypnik for nonlinear elliptic and parabolic equations (see, for example, [18,19] and reference therein). The study of this problem in nonlinear case is essentially different from the study of linear problem because for the construction of the limit boundary value problem there must be some strong convergence of gradients of solutions of the initial problems. The proof of such strong convergence is based on a special asymptotic expansion by which the solutions of nonlinear problem in perforated domains are approximated near sets $\mathcal{F}_i^{(s)}$ by solution of some model nonlinear problems. The main role in study of the asymptotic behavior of the solutions of nonlinear problems and in the construction of limit boundary value problems is played by pointwise estimates of the solution of the model problem. This approach, which gave us possibility to show the strong convergence of the gradients of the solutions, was applied to the homogenization of the Navier-Stokes equation with Dirichlet condition in domains with fine-grained boundary ([16]). In this paper we consider the problem of the homogenization of the Navier-Stokes equations with Dirichlet condition on the boundary in a sequence of domains with channels.

Notation. In what follows by C_j, $j = 0, 1, \ldots,$ we denote positive constants that depend on n, Ω and do not depend on i and s.

Now we formulate the problem. Let $\rho(x, G)$ be the distance from point x to the set $G \subset \mathbf{R}^n$, $n \geq 3$, and for every $\varepsilon > 0$ we define ε-neighborhood of the set G:

$$\mathcal{U}(G, \varepsilon) = \{x \in \mathbf{R}^n : \rho(x, G) < \varepsilon\}.$$

For any fixed $s \in \mathbf{N}$ we consider the finite number of lines $l_i^{(s)} \subset \overline{\Omega}$, positive numbers $d_i^{(s)}$, and closed domains $\mathcal{F}_i^{(s)}$ such that

$$\mathcal{F}_i^{(s)} \subset \mathcal{U}(l_i^{(s)}, d_i^{(s)}), \quad i = 1, \ldots, I(s).$$

The number of lines $I(s)$ tends to infinity as $s \to \infty$.

In the perforated domain with channels $\Omega_s = \Omega \setminus \bigcup\limits_{i=1}^{I(s)} \mathcal{F}_i^{(s)}$ we will study the following problem

$$\begin{aligned} \nu \Delta \mathbf{u}^{(s)} - (\mathbf{u}^{(s)} \cdot \nabla)\, \mathbf{u}^{(s)} = \nabla p^{(s)} + \mathbf{f}, \quad x \text{ in } \Omega_s, \\ \operatorname{div} \mathbf{u}^{(s)} = 0, \quad x \text{ in } \Omega_s, \\ \mathbf{u}^{(s)}|_{\partial\Omega_s} = 0, \end{aligned} \tag{1}$$

where $p^{(s)}(x)$ is pressure, $\mathbf{f}(x) \in L_2(\Omega)^n$.
In the study of this problem the following problems arise:

- to establish conditions under which the solutions of the problem (1) converge as $s \to \infty$
- to determine the boundary value problem for the limit function

We introduce the following spaces

$$H(\Omega_s) := \{\mathbf{u}_s(x) \in W_0^{1,2}(\Omega_s)^n : \ \operatorname{div} \mathbf{u}_s(x) = 0, \ x \in \Omega_s\},$$

$$H(\Omega) := \{\mathbf{u}(x) \in W_0^{1,2}(\Omega)^n : \ \operatorname{div} \mathbf{u}(x) = 0, \ x \in \Omega\}.$$

Definition 1 *We say that $\mathbf{u}^{(s)}(x) \in H(\Omega_s)$ is a weak solution of problem (1) if the following integral identity*

$$\int\limits_{\Omega_s} [\nu \nabla \mathbf{u}^{(s)} \nabla \boldsymbol{\varphi}^{(s)} - (\mathbf{u}^{(s)} \cdot \nabla) \boldsymbol{\varphi}^{(s)} \cdot \mathbf{u}^{(s)}] dx = - \int\limits_{\Omega_s} (\mathbf{f}, \boldsymbol{\varphi}^{(s)}) dx \tag{2}$$

is satisfied for every $\boldsymbol{\varphi}^{(s)} \in H(\Omega_s)$.

Analogously to [13] the existence of the solution to problem (1) and the a priory estimate are proved

$$\|\mathbf{u}^{(s)}\|_{H(\Omega_s)} \leq C_0 \|\mathbf{f}(x)\|_{L_2(\Omega_s)^n}.$$

Extending the function $\mathbf{u}^{(s)}(x) \in H(\Omega_s)$ into $\bigcup_{i=1}^{I(s)} \mathcal{F}_i^{(s)}$ by zero and keeping the same notation, we obtain the function $\mathbf{u}^{(s)}(x) \in H(\Omega)$ which satisfies the following estimate

$$\|\mathbf{u}^{(s)}\|_{H(\Omega)} \leq C_0 \|\mathbf{f}\|_{L_2(\Omega)^n}.$$

Then, there exists a subsequence of the sequence $\{\mathbf{u}^{(s)}(x)\}_{s=1}^{\infty}$ converging weakly in $H(\Omega)$ and strongly in $L_p(\Omega)^n$ $(p < 6)$ as $s \to \infty$. We denote this weak limit by $\mathbf{u}^{(0)}(x) \in H(\Omega)$.

Remark 1 We suppose that $n \geq 4$. For the case $n = 3$ the same result can be obtained using analogous methods.

2 Assumptions on sequence of perforated domains $\{\Omega_s\}_{s=1}^{\infty}$

We denote a ball of radius r with center at x by $B(x,r)$. Let for every $i = 1,\ldots,I(s)$, $r_i^{(s)}$ be numbers such that the following conditions with constant λ independent of s and i are satisfied:

- (**i**) $(2+C_1)d_i^{(s)} \le r_i^{(s)}$, $\lim_{s\to\infty} r^{(s)} = 0$, *where* $r^{(s)} = \max\limits_{1\le i\le I(s)} r_i^{(s)}$, *and*

$$\sum_{i=1}^{I(s)} \frac{(d_i^{(s)})^{2(n-3)}}{(r_i^{(s)})^{(n-1)}} \le C_2;$$

- (**ii**) *for any choice of s and i there exists finite number of points* $z_{i,p}^{(s)}$, $l = 1,\ldots,P(i,s)$, *such that for every* $t_i^{(s)} \in [d_i^{(s)}, r_i^{(s)}]$ *the inclusion*

$$\mathcal{U}\left(T_i^{(s)}(\{t_i^{(s)}\}), t_i^{(s)}\right) \subset \bigcup_{p=1}^{P(i,s)} B(z_{i,p}^{(s)}, \lambda t_i^{(s)})$$

holds, where

$$T_i^{(s)}(\{t_i^{(s)}\}) = \mathcal{U}(l_i^{(s)}, t_i^{(s)}) \bigcap \Big\{ \bigcup_{j\ne i} \mathcal{U}(l_j^{(s)}, t_j^{(s)}) \bigcup \partial\Omega \Big\},$$

$$\mathcal{U}(l_i^{(s)}, r_i^{(s)}) \subset \Omega, \quad B(z_{i,p}^{(s)}, \lambda r_i^{(s)}) \subset \Omega;$$

- (**iii**) *for every* $s = 1,2,\ldots$ *the order of families of sets*

$$\{\mathcal{U}(l_i^{(s)}, r_i^{(s)}), \quad i = 1,\ldots,I(s)\},$$

$$\{B(z_{i,p}^{(s)}, \lambda r_i^{(s)}), \quad i = 1,\ldots,I(s), \quad p = 1,\ldots,P(i,s)\}$$

do not exceed the number p_0. *That is,* p_0 *is the largest positive number for which* (p_0+1) *sets from these families with the common points exist;*

We suppose the following conditions on regularity of the lines $l_i^{(s)}$:

- (**iv**) *for every* $s = 1,2,\ldots$, $i = 1,\ldots,I(s)$, *there exists a diffeomorphisms* $g_i^{(s)}$ *from class* C^1, $g_i^{(s)} : \mathcal{U}(l_i^{(s)},1) \to g_i^{(s)}(\mathcal{U}(l_i^{(s)},1)) \subset \mathbf{R}^n$ *such that the following inclusion holds*

$$g_i^{(s)}(l_i^{(s)}) \subset \{y \in \mathbf{R}^n : \quad y_1 = \ldots = y_{n-1} = 0\};$$

- (**v**) *there exists a constant* $\kappa > 0$ *independent of i and s such that the inequalities*

$$\|\frac{\partial g_i^{(s)}(x)}{\partial x}\| \le \kappa, \quad det\frac{\mathcal{D}g_i^{(s)}(x)}{\mathcal{D}x} \ge \kappa^{-1},$$

are satisfied for $x \in \mathcal{U}(l_i^{(s)},1)$, *where we denote the Jacobi matrix of* $g_i^{(s)}(x)$ *in point* x *by* $\frac{\mathcal{D}g_i^{(s)}(x)}{\mathcal{D}x}$.

Remark 2 The meaning of the previous condition is the following: there exists a constant C_3 depending on n and κ only such that for every $x_1, x_2 \in \mathcal{U}(l_i^{(s)},1)$ the inequality holds

$$\frac{1}{C_3}|x_1 - x_2| \le |g_i^{(s)}(x_1) - g_i^{(s)}(x_2)| \le C_3|x_1 - x_2|. \tag{3}$$

3 Auxiliary functions

To formulate additional conditions for sets $\mathcal{F}_i^{(s)}$ that guarantee possibility of construction of an averaged problem, we define auxiliary functions which are solutions of the appropriate *"model"* problems.

Let $a > 0$ and

$$Q_a = \{x \in R^n;\ x = (x', x_n),\ x' = (x_1, \ldots, x_{n-1}),\ |x'| \leq a,\ |x_n| \leq H\}.$$

Let $\mathcal{F}$ be a closed set, $l = \operatorname{diam}\mathcal{F}$, and $0 < d < H < 1/4$. By Q_d we denote the smallest cylinder such that

$$\mathcal{F} \subset Q_d.$$

We consider the main *"model"* problem in $B(0,1) \setminus \mathcal{F}$

$$\begin{gathered} \Delta \mathbf{v}^k(x) = \nabla p^k(x), \\ \operatorname{div} \mathbf{v}^k(x) = 0,\ x \in B(0,1) \setminus \mathcal{F} \\ \mathbf{v}^k(x)|_{\partial\mathcal{F}} = \mathbf{e}^k, \quad \mathbf{v}^k(x)|_{\partial B(0,1)} = 0, \end{gathered} \tag{4}$$

where $\mathbf{e}_k$ is the unit vector of axis $0X_k$, $k = 1, \ldots, n$.

The existence and uniqueness of the function $\mathbf{v}^k(x,t)$ follows from [13] for any fixed $s \in \mathbf{N}$ and $i = 1, \ldots, I(s)$. The solution $\mathbf{v}^k(x,t)$ can be find as a minimum of the following functional

$$J(\mathbf{v}^k) = \int\limits_{B(0,1)} |\nabla \mathbf{v}^k|^2 dx = \int\limits_{B(0,1)} \sum_{l,p=1}^{n} \left| \frac{\partial v_l^k}{\partial x_p} \right|^2 dx$$

in the class of functions

$$H_k(\mathcal{F}) := \{\mathbf{v}^k(x) \in W_0^{1,2}(B(0,1)) :\ \operatorname{div}\ \mathbf{v}^k = 0,\ x \in B(0,1),\ \mathbf{v}^k = \mathbf{e}^k,\ x \in \mathcal{F}\}.$$

In the study of homogenization problem for a sequence of domains with channels the key role is played by the pointwise estimate. In particular, the following statement is proved using methods of [11], [12].

Theorem 1 *There exists positive constant C_4 such that for $|\alpha| = 0, 1, 2,\ k = 1, \ldots, n,\ n \geq 4$ the following estimates for the solution $v^k(x,t)$ of problem (4) are satisfied*

$$|D^\alpha v_i^k(x)| \leq C_4 \frac{l^{n-3}}{|x'|^{n-3+|\alpha|}} \tag{5}$$

for every $x \in B(0,1) \setminus Q_a$, where $a \geq C_1 l$, for every $i = 1, \ldots, n$, and

$$\int\limits_{B(0,1)} (D^\alpha \mathbf{v}^k(x), D^\alpha \mathbf{v}^k(x)) \leq C_4 (l^{n-3} + l^{n-1-2|\alpha|}) \tag{6}$$

where $D^\alpha v_i^k = \frac{\partial^{|\alpha|} v_i^k}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n}$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Proof Let denote by $\mathbf{G}(x,y)=\|G_{ij}(x,y)\|_{i,j=1}^n$ Green's function to the Dirichlet problem for the Stokes system:

$$\nu\Delta_x G_{ij}(x,y)-\frac{\partial}{\partial x_i}q_j(x,y)=\delta(x,y)\delta_{ij},\ (x,y)\in B(0,1)\times B(0,1),$$

$$\sum_{i=1}^n\frac{\partial}{\partial x_i}G_{ij}(x,y)=0,\ (x,y)\in B(0,1)\times B(0,1),$$

$$G_{ij}(x,y)|_{x\in\partial B(0,1)}=0,\ y\in B(0,1).$$

If $x\neq y,\ y\in B(0,b),\ a<b<1$ then $D_x^\alpha D_y^\beta G_{ij}(x,y)$ for $(x,y)\in B(0,1)\times B(0,b)$ exists and are continuous. They look like

$$\mathcal{A}(x,y)|x-y|^{-n+2-|\alpha|-|\beta|}$$

where functions $\mathcal{A}(x,y)$ are bounded in $B(0,1)\times B(0,b)$. There exist $\tilde{\mathbf{G}}_{\mathbf{j}}(x,y)$ such that

$$\mathbf{G}_j(x,y)=\mathrm{curl}_x\tilde{\mathbf{G}}_{\mathbf{j}}(x,y)$$

with $D_x^\alpha D_y^\beta\tilde{G}_{ij}(x,y)=\mathcal{B}(x,y)|x-y|^{-n+3-|\alpha|-|\beta|}$ for $(x,y)\in B(0,1)\times B(0,b), x\neq y$, where $\mathcal{B}(x,y)$ are bounded.

We will use Green's formulas in the next consideration:

$$\int\limits_{B(0,1)\setminus\mathcal{F}}\Big(\Delta\mathbf{v}-\nabla p,\mathbf{u}\Big)dx=\int\limits_{B(0,1)\setminus\mathcal{F}}(\nabla\mathbf{v},\nabla\mathbf{u})dx+\int\limits_{\partial B(0,1)\bigcup\partial\mathcal{F}}\Big(\frac{\partial\mathbf{v}}{\partial n}-p\mathbf{n},\mathbf{u}\Big)\,ds,\tag{7}$$

$$\int\limits_{B(0,1)\setminus\mathcal{F}}\Big\{(\Delta\mathbf{v}-\nabla p,\mathbf{u})-(\Delta\mathbf{u}-\nabla q,\mathbf{v})\Big\}dx=$$
$$\int\limits_{\partial B(0,1)\bigcup\partial\mathcal{F}}\Big\{(\tfrac{\partial\mathbf{v}}{\partial n}-p\mathbf{n},\mathbf{u})-(\tfrac{\partial\mathbf{u}}{\partial n}-q\mathbf{n},\mathbf{v})\Big\}ds,\tag{8}$$

where $\mathbf{n}$ is an outer normal vector to $B(0,1)\setminus\mathcal{F}$. Let in (8) $\mathbf{v}(x)$ be solution of (4) and $\mathbf{u}(x)=\mathbf{G}_j(x,y)$, then

$$\int\limits_{B(0,1)\setminus\mathcal{F}}\Big\{(\Delta\mathbf{v}^k-\nabla p^k,\mathbf{G}_j(x,y))-(\Delta_x\mathbf{G}_j(x,y)-\nabla q,\mathbf{v}^k)\Big\}dx=$$
$$\int\limits_{\partial B(0,1)\bigcup\partial\mathcal{F}}\Big\{(\tfrac{\partial\mathbf{v}^k}{\partial n}-p^k\mathbf{n},\mathbf{G}_j(x,y))-(\tfrac{\partial_x\mathbf{G}_j(x,y)}{\partial n}-q_j\mathbf{n},\mathbf{v}^k)\Big\}d_xs,\tag{9}$$

We use that

$$D^\alpha\mathbf{v}^k(x)=D_x\mathbf{G}_j(x,y)=0\ \text{ if } x\in\partial B(0,1),\ |\alpha|=0,1,$$

$$\mathbf{v}^k(x)=\mathbf{e}^k,\ \frac{\partial\mathbf{v}^k}{\partial n}\ \text{ if } x\in\partial\mathcal{F},$$

$$(\Delta_x\mathbf{G}_j(x,y)-\nabla q_j,\mathbf{v}^k)=\delta(x,y)v_j^k(x)=v_j^k(y)\text{ if }(x,y)\in B(0,1)\times B(0,b).$$

Then from (9) we get

$$v_j^k(y) = \int\limits_{\partial\mathcal{F}} \Big((\frac{\partial \mathbf{v}^k}{\partial n} - p^k \mathbf{n}, \mathbf{G}_j(x,y)) - (\frac{\partial_x \mathbf{G}_j(x,y)}{\partial n} - q_j \mathbf{n}, \mathbf{v}^k) \Big) d_x s \ \forall y \in B(0,b). \quad (10)$$

To prove that

$$\int\limits_{\partial\mathcal{F}} \Big(\frac{\partial_x \mathbf{G}_j(x,y)}{\partial n} - q_j \mathbf{n}, \mathbf{v}^k \Big) d_x s = 0, \quad (11)$$

we use formula (7) where $\mathbf{v} = \mathbf{G}_j(x,y),\ \mathbf{u} = \mathbf{e}_k,\ y \in \mathcal{F}$, and domain of integration is $\mathcal{F} \setminus \partial\mathcal{F}$. Then we obtain

$$\int\limits_{\mathcal{F}\setminus\partial\mathcal{F}} \Big(\Delta_x \mathbf{G}_j(x,y) - \nabla q_j, \mathbf{e}_k \Big) dx =$$

$$\int\limits_{\mathcal{F}\setminus\partial\mathcal{F}} (\nabla_x \mathbf{G}_j(x,y), \nabla \mathbf{e}_k) dx + \int\limits_{\partial\mathcal{F}} (\tfrac{\partial \mathbf{G}_j(x,y)}{\partial n} - q_j(x)\mathbf{n}, \mathbf{e}_k) d_x s. \quad (12)$$

From this it follows (11). Using (10) and (11) for all $y \in Q_1,\ |\alpha| = 0,1,2$, we have

$$D^\alpha v_j^k(y) = \int\limits_{\partial\mathcal{F}} \Big(\frac{\partial \mathbf{v}^k}{\partial n} - p^k \mathbf{n}, D_y^\alpha \mathbf{G}_j(x,y)) \Big) d_x s \ \forall y \in B(0,1). \quad (13)$$

Now we introduce cut-off functions

– **(1)** $\varphi(t) \in C^\infty(\mathbf{R}^1),\ \varphi(t) = 1,$ if $t \le 0,\ \ \varphi(t) = 0,$ if $t \ge \frac{C_1}{2}$,

$$\varphi(\frac{|x'| - d/2}{l}) = 1 \ \text{ if } |x'| \le \frac{d}{2}, \quad \varphi(\frac{|x'| - d/2}{l}) = 0 \ \text{ if } |x'| \ge \frac{d}{2} + \frac{c_1 l}{2},$$

$$|D^\alpha \varphi(\frac{|x'| - d/2}{l})| \le \frac{\tilde{C}}{l^{|\alpha|}};$$

– **(2)** $\psi(t) \in C^\infty(\mathbf{R}^1)$,

$$\psi(\frac{y_n - x_n}{H}) = 1 \text{ if } |y_n - x_n| \le \frac{H}{2}, \quad \psi(\frac{y_n - x_n}{H}) = 0 \text{ if } |y_n - x_n| \ge H.$$

We define the function

$$\mathbf{u}_j^\alpha(x,y) = \mathrm{curl}_x[D_y^\alpha \tilde{\mathbf{G}}_j(x,y)\varphi(\frac{|x'| - d/2}{l})\psi(\frac{y_n - x_n}{H})].$$

If $x \in \partial\mathcal{F} \subset Q_a$ then $\mathbf{u}_j^\alpha(x,y) = D_y^\alpha \mathbf{G}_j(x,y)$. Moreover, $\mathbf{u}_j^\alpha(x,y) \in C^\infty(Q_a)$ if $|y'| \ge \frac{d}{2} + \frac{c_1 l}{2}$ and $\mathbf{u}_j^\alpha(x,y) = 0$ if $|x'| \ge \frac{d}{2} + \frac{c_1 l}{2}$. Let in (7) $\mathbf{v}$ be a solution of (4), $\mathbf{u} = \mathbf{u}_j^\alpha(x,y)$

$$\int\limits_{B(0,1)\setminus\mathcal{F}} \Big(\Delta \mathbf{v}^k - \nabla p^k, \mathbf{u}_j^\alpha(x,y) \Big) dx =$$

$$\int\limits_{B(0,1)\setminus\mathcal{F}} (\nabla \mathbf{v}^k, \nabla_x \mathbf{u}_j^\alpha(x,y)) dx + \int\limits_{\partial B(0,1) \bigcup \partial\mathcal{F}} (\tfrac{\partial \mathbf{v}^k}{\partial n} - p^k \mathbf{n}, \mathbf{u}_j^\alpha(x,y)) d_x s.$$

Then we obtain

$$\int\limits_{\partial\mathcal{F}} (\frac{\partial \mathbf{v}^k}{\partial n} - p^k \mathbf{n}, \mathbf{D}_y^\alpha \mathbf{G}_j(x,y)) d_x s = - \int\limits_{B(0,1)\setminus\mathcal{F}} (\nabla \mathbf{v}^k, \nabla_x \mathbf{u}_j^\alpha(x,y)) dx. \quad (14)$$

From (13) and (14) we get

$$\mathbf{D}_y^\alpha v_j^k(y) = \int\limits_{B(0,1)\setminus\mathcal{F}} (\nabla \mathbf{v}^k, \nabla_x \mathbf{u}_j^\alpha(x,y))dx,$$

$$|\mathbf{D}_y^\alpha v_j^k(y)| \leq \left(\int\limits_{B(0,1)\setminus\mathcal{F}} |\nabla \mathbf{v}^k|^2 dx \right)^{\frac{1}{2}} \left(\int\limits_{B(0,1)\setminus\mathcal{F}} |\nabla_x \mathbf{u}_j^\alpha(x,y)|^2 dx \right)^{\frac{1}{2}} \qquad (15)$$

Function $\boldsymbol{\omega}^k(x) = \mathrm{curl}\{\tilde{\mathbf{e}}_k \varphi(\frac{|x'|-d/2}{l})\psi(\frac{y_n-x_n}{H})\} \in H_k(\mathcal{F})$. Since $d < l$ we have

$$\int\limits_{B(0,1)} |\nabla \mathbf{v}^k|^2 dx \leq \sum_{i,j=1}^n \int\limits_{B(0,1)} |\frac{\partial \omega_i^k(x)}{\partial x_j}|^2 dx \leq \int\limits_{B(0,1)} |\nabla(\varphi(\frac{|x'|-d/2}{l})\psi(\frac{y_n-x_n}{H}))|^2 dx \leq$$

$$C\ meas\{Q_{\frac{d}{2}+\frac{C_1 l}{2}} \setminus Q_{\frac{d}{2}}\} \leq \frac{C}{l^2}(\frac{d}{2}+\frac{C_1 l}{2})^{n-1} \leq C l^{n-3}, \qquad (16)$$

hereafter C are generic constants independent of d, l.

We consider the second integral of (15):

$$\int\limits_{B(0,1)\setminus\mathcal{F}} |D_x D_y^\alpha(\tilde{\mathbf{G}}_j(x,y)\varphi(\frac{|x'|-d/2}{l})\psi(\frac{y_n-x_n}{H}))|^2 dx.$$

Now,

$$|D_x D_y^\alpha(\tilde{\mathbf{G}}_j(x,y)\varphi(\frac{|x'|-d/2}{l})\psi(\frac{y_n-x_n}{H}))|^2 \leq |D_x[\varphi(\frac{|x'|-d/2}{l})\times$$

$$(D_y^\alpha \tilde{\mathbf{G}}_j(x,y)\psi(\frac{y_n-x_n}{H}) + D_y^\alpha \psi(\frac{y_n-x_n}{H})\tilde{\mathbf{G}}_j(x,y))]|^2 = |D_x\varphi(\frac{|x'|-d/2}{l})\times$$

$$(D_y^\alpha \tilde{\mathbf{G}}_j(x,y)\psi(\frac{y_n-x_n}{H}) + D_y^\alpha \psi(\frac{y_n-x_n}{H})\tilde{\mathbf{G}}_j(x,y)) + \varphi(\frac{|x'|-d/2}{l})\times$$

$$(D_x D_y^\alpha \tilde{\mathbf{G}}_j(x,y)\psi(\frac{y_n-x_n}{H}) + D_x D_y^\alpha \psi(\frac{y_n-x_n}{H})\tilde{\mathbf{G}}_j(x,y))|^2 \leq$$

$$\leq C[|D_x\varphi(\frac{|x'|-d/2}{l})|^2 |D_y^\alpha \tilde{\mathbf{G}}_j(x,y)|^2 + |D_x\varphi(\frac{|x'|-d/2}{l})|^2 |\tilde{\mathbf{G}}_j(x,y)|^2 +$$

$$|D_x D_y^\alpha \tilde{\mathbf{G}}_j(x,y)|^2 + |D_y^\alpha \tilde{\mathbf{G}}_j(x,y)|^2] \leq$$

$$C\left(\frac{1}{l^2|x-y|^{2(n-3+|\alpha|)}} + \frac{1}{l^2|x-y|^{2(n-3)}} + \frac{1}{|x-y|^{2(n-2+|\alpha|)}} + \frac{1}{|x-y|^{2(n-3+|\alpha|)}}\right) \leq$$

$$\frac{C}{l^2||y'|-|x'||^{2(n-3+|\alpha|)}}.$$

Since $|y'| \geq C_1 l + d$ and $|x'| \leq \frac{C_1 l}{2} + \frac{d}{2}$, then $|y'| - |x'| \geq |y'| - \frac{C_1 l}{2} - \frac{d}{2} \geq \frac{|y'|}{2}$ and we obtain

$$\int\limits_{B(0,1)\setminus\mathcal{F}} |D_x D_y^\alpha(\mathbf{G}_j(x,y)\varphi(\frac{|x'|-d/2}{l})\psi(\frac{y_n-x_n}{H}))|^2 dx \leq C\frac{l^{n-3}}{|y'|^{2(n-3+|\alpha|)}}. \qquad (17)$$

From (15)-(17) we deduce (5).

For $|\alpha| = 1$ the estimate (6) follows from (16). Let us prove (6) for $|\alpha| = 0$. We divide the domain of integration $B(0,1)$ into two parts

$$B(0,1) = (B(0,1) \setminus Q_{2l}) \bigcup Q_{2l}.$$

The integral over $B(0,1) \setminus Q_{2l}$ we estimate using pointwise estimate (5):

$$\int\limits_{B(0,1)\setminus Q_{2l}} |v_i^k(x)|^2 dx \le A \int\limits_{B(0,1)\setminus Q_{2l}} \left(\frac{l}{|x'|}\right)^2 dx \le Bl^{n-3} \tag{18}$$

For $x \in Q_{2l} = \{x \in \mathbf{R}^n : \ |x'| \le 2l, \ |x_n| \le H\}$ we denote by $x'_t = x' \frac{t}{|x'|}$, then

$$v_i^k(x) = v_i^k(x'_{2l}, x_n) - \int\limits_{|x'|}^{2l} \frac{dv_i^k(x'_t, x_n)}{dt} dt.$$

Using Hölder's inequality we get:

$$|v_i^k(x)|^2 = 2|v_i^k(x'_{2l}, x_n)|^2 + 2\left(\int\limits_{|x'|}^{2l} t|\frac{dv_i^k(x'_t, x_n)}{dt}|\frac{1}{t}dt\right)^2$$

$$\le 2|v_i^k(x^{2l})|^2 + 2\int\limits_{|x'|}^{2l} \frac{dt}{t^2} \int\limits_0^{2l} |\nabla v_i^k|^2 t^2 dt.$$

Integrating the last inequality over cylinder Q_{2l} we obtain:

$$\int\limits_{Q_{2l}} |v_i^k(x)|^2 dx \le 2\int\limits_{Q_{2l}} |v_i^k(x'_{2l}, x_n)|^2 dx + \int\limits_{|x'|}^{2l} \frac{dt}{t^2} \int\limits_{Q_{2l}} \int\limits_0^{2l} |\nabla v_i^k|^2 t^2 dt.$$

The first integral in the right-hand side we estimate using (5), the second analogously to (16). Then we get (6). The proof is completed.

Remark 3 The estimates in the case $n = 3$ we obtain analogously using logarithmic cut-off functions. As a result we obtain that there exist positive constants C_5 such that for $|\alpha| = 0, 1, 2, \ k = 1, 2, 3$, the following estimates for the solution $\mathbf{v}^k(x,t)$ of problem (4) are satisfied

$$|D^\alpha v_i^k(x)| \le C_5 \frac{1}{|x'|^{|\alpha|}} \ln^{-1} \frac{1}{l}$$

for every $x \in B(0,1) \setminus Q_a$, where $a \ge C_1 l$, for every $i = 1, 2, 3$, and

$$\int\limits_{B(0,1)} (D^\alpha \mathbf{v}^k(x), D^\alpha \mathbf{v}^k(x)) \le C_5 \ln^{-1} \frac{1}{l}.$$

To construct the limit boundary value problem we define $\mathbf{v}^k(x; G, \mathcal{F})$ as the solution of the problem

$$\begin{aligned} \Delta \mathbf{v}^k(x) = \nabla p^k(x),\ x \text{ in } \mathcal{U}(G, 1/2), \\ \operatorname{div} \mathbf{v}^k(x) = 0,\ x \text{ in } \mathcal{U}(G, 1/2) \setminus \mathcal{F}, \\ \mathbf{v}^k(x)|_{\partial\mathcal{F}} = \mathbf{e}^k,\ \mathbf{v}^k(x)|_{\partial\mathcal{U}(G,1/2)} = 0, \end{aligned} \tag{19}$$

for any open set $G \subset \mathbf{R}^n$ and a closed set $\mathcal{F} \subset G$.

We suppose that for every s the inequality $r^{(s)} \leq 1/2^{n-2}$ is valid. We define sequences of numbers $\{\lambda_s\},\ \{\mu_s\}$:

$$\lambda_s^2 = \max\{[\ln \frac{1}{r^{(s)}}]^{-1}, \|u^{(0)}\|^2_{W^{1,2}(\mathcal{U}^{(s)})^n} + \|f\|^2_{W^{1,2}(\mathcal{U}^{(s)})^n}\},\quad \mu_s = \lambda_s^{\frac{1}{n-3}}, \tag{20}$$

where $\mathcal{U}^{(s)} = \bigcup\limits_{i=1}^{I(s)} \mathcal{U}(l_i^{(s)}, \bar{\rho}_i^{(s)}),\ \bar{\rho}_i^{(s)} = \frac{1}{2} d_i^{(s)} + [r_i^{(s)}]^{\frac{n-1}{n-2}}$.

For $s = 1, 2, \ldots,\ i = 1, \ldots, I(s)$, we introduce subsets of indices

$$I'(s) = \{i : i = 1, \ldots, I(s),\ d_i^{(s)} \geq [r_i^{(s)}]^{\frac{n-1}{n-3}} \mu_s\},$$

$$I''(s) = \{i : i = 1, \ldots, I(s), d_i^{(s)} < [r_i^{(s)}]^{\frac{n-1}{n-3}} \mu_s\},$$

and a sequence of numbers

$$\rho_i^{(s)} = d_i^{(s)} \text{ for } i \in I'(s),\qquad \rho_i^{(s)} = 2C_3 [r_i^{(s)}]^{\frac{n-1}{n-3}} \mu_s \text{ if } i \in I''(s), \tag{21}$$

where C_3 is the constant from (3). We assume that $2C_3 \rho_i^{(s)} \leq r_i^{(s)}$ for $i \in I''(s)$. For every $s = 1, 2, \ldots,\ i \in I''(s)$, we consider sets

$$L_i^{(s)} = \{x \in l_i^{(s)} :\ \rho(x, T_i^{(s)}(\{2\rho_i^{(s)}\})) \geq 2\rho_i^{(s)}\} = L_{i,1}^{(s)} \bigcup L_{i,2}^{(s)}.$$

We denote $L_{i,1}^{(s)}$ is the union of all connected sets from $L_i^{(s)}$ which length are not less than $\lambda_s^{-1}\rho_i^{(s)}$. The closed set $L_{i,2}^{(s)}$ consists of all curve segments of $l_i^{(s)}$ such that their lengths are less or equal $\lambda_s^{-1}\rho_i^{(s)}$,

$$dist\{\partial L_{i,2}^{(s)}, T_i^{(s)}(\{2\rho_i^{(s)}\})\} = 2\rho_i^{(s)}.$$

We divide every curvilinear segment from the set $L_{i,1}^{(s)}$ on the finite number segments of equal length such that

$$L_{i,1}^{(s)} = \bigcup_{m=1}^{M(i,s)} L_i^{(s)}(m),\quad \frac{\rho_i^{(s)}}{2\lambda_s} \leq |L_i^{(s)}(m)| \leq \frac{\rho_i^{(s)}}{\lambda_s},$$

where $|L_i^{(s)}(m)|$ is the length of curvilinear segment $L_i^{(s)}(m)$.

Let $\alpha_{i,m}^{(s)},\ \beta_{i,m}^{(s)},\ s = 1, 2, \ldots,\ i \in I''(s),\ m = 1, \ldots, M(i, s)$ are endings of curvilinear piece $L_i^{(s)}(m)$. We denote for $s = 1, 2, \ldots,\ i = 1, \ldots, I(s),\ m = 1, \ldots, M(i, s)$ and some sufficiently large constant γ

$$G_{i,m}^{(s)}(\gamma) = \mathcal{U}(L_i^{(s)}(m), 2\rho_i^{(s)}) \setminus \overline{\{B(\alpha_{i,m}^{(s)}, \gamma\rho_i^{(s)}) \bigcup B(\beta_{i,m}^{(s)}, \gamma\rho_i^{(s)})\}}, \tag{22}$$

$$\mathbf{v}_{k,i,m}^{(s)}(x) := \mathbf{v}^k(x; G_{i,m}^{(s)}(\gamma), \mathcal{F}_i^{(s)} \bigcap G_{i,m}^{(s)}(\gamma)),$$

$$C_m^{k\ell}(\mathcal{F}_i^{(s)}) = \int\limits_{\mathcal{U}(G_{i,m}^{(s)}(\gamma),1/2)} (\nabla \mathbf{v}_{k,i,m}^{(s)}, \nabla \mathbf{v}_{\ell,i,m}^{(s)})dx, \ k, \ell = 1, \dots, n.$$

We assume that the following additional condition is satisfied:

– **(vi)** *There exists a continuous nonnegative matrix* $||c(x)||$ *such that for every ball* $B \subset \Omega$ *we have*

$$\lim_{s\to\infty} \sum_{(i,m)\in I_s(B)} C_m^{k\ell}(\mathcal{F}_i^{(s)}) = \int\limits_B c^{k\ell}(x)dx.$$

By $I_s(B)$ *we denote the set of all pairs* $(i, m),\ i \in I''(s),\ m = 1, \dots, M(i, s)$ *such that* $x_{i,m}^{(s)} \in B$*, where* $x_{i,m}^{(s)}$ *is a middle of the curvilinear piece* $L_i^{(s)}(m)$*.*

4 Auxiliary statements and cut-off functions

To specify some geometrical properties of such perforation, we formulate additional statements (see [18]). In particular, it can be shown that

$$\lim_{s\to\infty} \lambda_s = 0, \quad \lim_{s\to\infty} \mu_s = 0, \quad \lim_{s\to\infty} \lambda_s \mu_s^{2(n-3)} = 0.$$

The following inclusion is also valid

$$\mathcal{U}(l_i^{(s)}, 2\rho_i^{(s)}) \subset \mathcal{U}(L_{i,1}^{(s)}, 2\rho_i^{(s)}) \bigcup \mathcal{U}(T_i^{(s)}(\{2\rho_i^{(s)}\}), 2\rho_i^{(s)}(2 + \frac{1}{\lambda_s})).$$

It can be proved the following results ([18])

Lemma 1 *There exists* γ *large enough independent of* s *and* i *such that*
(I) *for any fixed* s, i *sets* $G_{i,m}^{(s)}(\gamma),\ m = 1, \dots, M(i, s),$ *are disjoint;*
(II) *the following inclusion holds*

$$\mathcal{U}(l_i^{(s)}, 2\rho_i^{(s)}) \subset \bigcup_{m=1}^{M(i,s)} \{G_{i,m}^{(s)}(\gamma + 1/3) \bigcup B(\alpha_{i,m}^{(s)}, (\gamma + 2/3)\rho_i^{(s)}) \bigcup \tag{23}$$

$$B(\beta_{i,m}^{(s)}, (\gamma + 2/3)\rho_i^{(s)})\} \bigcup \Big\{ \bigcup_{p=1}^{P(i,s)} B(z_{i,p}^{(s)}, 2\lambda\rho_i^{(s)}(2 + 1/\lambda_s)) \Big\}. \tag{24}$$

Now we define the cut-off functions in a different way for indices $i \in I'(s), i \in I''(s)$. If $i \in I'(s)$ we divide the curve onto the finite number of curvilinear segments with equal length such that it belongs to the segment $[\frac{d_i^{(s)}}{2}, d_i^{(s)}]$. Then we have

$$l_i^{(s)} = \bigcup_{r=1}^{R(i,s)} l_i^{(s)}(r), \qquad \frac{d_i^{(s)}}{2} \le |l_i^{(s)}(r)| \le d_i^{(s)},$$

$$\mathcal{U}(l_i^{(s)}, 2d_i^{(s)}) = \bigcup_{r=1}^{R(i,s)} \mathcal{U}(l_i^{(s)}(r), 2d_i^{(s)}), \quad i \in I'(s).$$

Let us define the cut-off functions for $i \in I''(s)$. Let $m = 1, \dots, M(i,s)$, $p = 1, \dots, P(i,s)$, $j \in I'(s)$, $r = 1, \dots, R(j,s)$, then we denote:

$$B_{i,m}^{(s,1)} = B(\alpha_{i,m}, (\gamma+1)\rho_i^{(s)}), \quad B_{i,m}^{(s,2)} = B(\beta_{i,m}^{(s)}, (\gamma+1)\rho_i^{(s)}),$$

$$B_{i,m}^{(s,3)} = B\Big(x_{i,m}^{(s)}, \Big(2 + \frac{1}{\lambda_s}\Big)\rho_i^{(s)}\Big),$$

$$\hat{B}_{i,m}^{(s)} = B\Big(z_{i,m}^{(s)}, 2\lambda\rho_i^{(s)}\Big(3 + \frac{1}{\lambda_s}\Big)\Big),$$

$$D_{j,r}^{(s)} = \mathcal{U}(l_j^{(s)}(r), 2d_j^{(s)}), \qquad G_{i,m}^{(s)} = G_{i,m}^{(s)}(\gamma),$$

where $x_{i,m}^{(s)}$ is a middle of curve segment $L_i^{(s)}(m)$. Numbers λ_s and $\rho_i^{(s)}$ are defined in (20) and (21) accordingly, and $\alpha_{i,m}^{(s)}, \beta_{i,m}^{(s)}, \gamma$ take the same values as (22).

We define the sequences of cut-off functions from $C^\infty(\mathbf{R}^n)$ with values in $[0,1]$

$$\{\varphi_{i,m}^{(s)}(x)\}, \ \ \{\psi_{i,m}^{(s,1)}(x)\}, \ \ \{\psi_{i,m}^{(s,2)}(x)\}, \ \ \{\chi_{i,p}^{(s)}(x)\}, \ \ \{\omega_{j,r}^{(s)}(x)\}$$

for $i \in I''(s),\ m = 1, \dots, M(i,s),\ p = 1, \dots, P(i,s),\ j \in I'(s),\ r = 1, \dots, R(j,s)$, with the following properties:

(a) $\operatorname{supp} \varphi_{i,m}^{(s)}(x) \subset G_{i,m}^{(s)}$, $\operatorname{supp} \psi_{i,m}^{(s,1)}(x) \subset B_{i,m}^{(s,1)}$, $\operatorname{supp} \psi_{i,m}^{(s,2)}(x) \subset B_{i,m}^{(s,2)}$,

$$\operatorname{supp} \chi_{i,p}^{(s)}(x) \subset \hat{B}_{i,p}^{(s)}, \ \ \operatorname{supp} \omega_{j,r}^{(s)}(x) \subset D_{j,r}^{(s)};$$

(b) $\sum_{j\in I'(s)} \omega_j^{(s)}(x) + \sum_{j\in I''(s)} \sigma_i^{(s)}(x) = 1, \quad$ for $\ x \in \bigcup_{i=1}^{I(s)} \mathcal{U}(l_i^{(s)}, \rho_i),$

where $\quad \omega_j^{(s)}(x) = \sum_{r=1}^{R(j,s)} \omega_{j,r}^{(s)}(x)$ and

$$\sigma_i^{(s)} = \sum_{m=1}^{M(i,s)} \big[\varphi_{i,m}^{(s)}(x) + \psi_{i,m}^{(s,1)}(x) + \psi_{i,m}^{(s,2)}(x)\big] + \sum_{p=1}^{P(i,s)} \chi_{i,p}^{(s)}(x);$$

(c) $\varphi_{i,m}^{(s)}(x) = 1$ for $x \in \mathcal{U}(l_i^{(s)}, \rho_i^{(s)}) \cap G_{i,m}^{(s)}(\gamma+1)$;

(d) there exists the constant C_6 independing of i, s such that

$$|\nabla\varphi_{i,m}^{(s)}(x)| + |\nabla\psi_{i,m}^{(s,1)}(x)| + |\nabla\psi_{i,m}^{(s,2)}(x)||\nabla\chi_{i,p}^{(s)}(x)| \le \frac{C_6}{\rho_i^{(s)}}, \quad \text{for } i \in I''_s,$$

$$|\nabla\omega_{i,r}^{(s)}(x)| \le \frac{C_6}{d_i^{(s)}}, \quad \text{if } i \in I'_s,$$

(e) order of families of sets

$$\{\operatorname{supp}\varphi_{i,m}^{(s)},\ \operatorname{supp}\psi_{i,m}^{(s,1)},\ \operatorname{supp}\psi_{i,m}^{(s,2)},\ \operatorname{supp}\chi_{i,p}^{(s)},\ \operatorname{supp}\omega_{j,r}^{(s)}\},$$

where $i\in I''(s),\ m=1,\dots,M(i,s),\ p=1,\dots,P(i,s),\ j\in I'(s), r=1,\dots,R(j,s)$, is less or equal than constant independing of s.

These functions were constructed in [18]. It was shown that

$$\{\operatorname{supp}\varphi_{i',m'}^{(s)}(x)\}\cap\{\operatorname{supp}\varphi_{i'',m''}^{(s)}(x)\}=\emptyset \quad \text{if} \quad (i',m')\neq(i'',m'')$$

for a fixed s.

5 Asymptotic expansion of solutions and formulation of the main result

The construction of an asymptotic expansion is connected with the separation of leading terms which are constructed by means of solutions of local boundary value problems. To construct the asymptotic expansion of the solution to the problem (1) we need the following theorem about the representation of solenoidal vectors from the space $L_2(\Omega)^n$ in the form of rotors [5].

Theorem 2 *Let G be a domain of $\mathbf{R}^n$ which is the diffeomorphic image of a ball. Let $J(G)$ be the closer of lineal of smooth solenoidal functions from $L_2(G)^n$. Then for every $u(x)\in J(G)$ the following representations are valid*

$$\mathbf{u}(x)=curl\widetilde{\mathbf{u}}(x),$$

$$\tilde{\mathbf{u}}(x)\in W^{1,2}(G)^n,\ div\,\tilde{\mathbf{u}}(x)=0,\ \frac{\partial\tilde{\mathbf{u}}}{\partial\mathbf{n}}|_{\partial G}=0,$$

$$\|\tilde{\mathbf{u}}\|_{W^{1,2}(G)^n}\le C_7\|\mathbf{u}\|_{L_2(G)^n},$$

where $C_7=C_7(G)$. The vector-function $\widetilde{\mathbf{u}}$ is defined by these conditions in a unique way.

Let $G=B(x_0,R)$, then from this theorem and reasons of the similarity the following estimates can be obtained:

$$\|\tilde{\mathbf{u}}\|_{L_2(G)^n}\le C_8R\|\mathbf{u}\|_{L_2(G)^n},\ \|D\tilde{\mathbf{u}}\|_{L_2(G)^n}\le C_8\|\mathbf{u}\|_{L_2(G)^n},$$

where the constant C_8 does not depend on R and $\mathbf{u}(x)$.

For an arbitrary function $\mathbf{g}(x)=(g_1(x),\dots,g_n(x))\in L_1(\Omega)^n$ we define:

$$M_{i,m}^{(s)}[g_k]=\frac{1}{measB_{i,m}^{(s,3)}}\int\limits_{B_{i,m}^{(s,3)}}g_k(x)\,dx,\quad M_{i,m}^{(s,t)}[g_k]=\frac{1}{measB_{i,m}^{(s,t)}}\int\limits_{B_{i,m}^{(s,t)}}g_k(x)\,dx,$$

$$\hat{M}_{i,p}^{(s)}[g_k]=\frac{1}{mes\hat{B}_{i,p}^{(s)}}\int\limits_{\hat{B}_{i,p}^{(s)}}g_k(x)\,dx,\quad \bar{M}_{j,r}^{(s)}[g_k]=\frac{1}{measD_{j,r}^{(s)}}\int\limits_{D_{j,r}^{(s)}}g_k(x)\,dx,$$

where $i\in I''(s),\ m=1,\dots,M(i,s),\ t=1,2,\ p=1,\dots,P(i,s),\ j\in I'(s), r=1,\dots,R(i,s),\ k=1,\dots,n.$

Let us denote by

$$\mathbf{u}_{i,m}^{(s)} = \big(M_{i,m}^{(s)}[u_1^{(0)}],\dots,M_{i,m}^{(s)}[u_n^{(0)}]\big), \quad \mathbf{u}_{i,m}^{(s,t)} = \big(M_{i,m}^{(s,t)}[u_1^{(0)}],\dots,M_{i,m}^{(s,t)}[u_n^{(0)}]\big),$$

$$\hat{\mathbf{u}}_{i,p}^{(s)} = \big(\hat{M}_{i,p}^{(s)}[u_1^{(0)}],\dots,\hat{M}_{i,p}^{(s)}[u_n^{(0)}]\big), \quad \bar{\mathbf{u}}_{j,r}^{(s)} = \big(\bar{M}_{j,r}^{(s)}[u_1^{(0)}],\dots,\bar{M}_{j,r}^{(s)}[u_n^{(0)}]\big),$$

$$\mathbf{f}_{i,m}^{(s)} = \big(M_{i,m}^{(s)}[f_1],\dots,M_{i,m}^{(s)}[f_n]\big), \quad \mathbf{f}_{i,m}^{(s,t)} = \big(M_{i,m}^{(s,t)}[f_1],\dots,M_{i,m}^{(s,t)}[f_n]\big),$$

$$\hat{\mathbf{f}}_{i,p}^{(s)} = \big(\hat{M}_{i,p}^{(s)}[f_1],\dots,\hat{M}_{i,p}^{(s)}[f_n]\big), \quad \bar{\mathbf{f}}_{j,r}^{(s)} = \big(\bar{M}_{j,r}^{(s)}[f_1],\dots,\bar{M}_{j,r}^{(s)}[f_n]\big).$$

Using Theorem 2, we denote functions

$$\mathbf{v}_{k,i,m}^{(s)} = \operatorname{curl}\tilde{\mathbf{v}}_{k,i,m}^{(s)}(x),\ \mathbf{u}_{i,m}^{(s)} - \mathbf{u}^{(0)}(x) = \operatorname{curl}\tilde{\mathbf{u}}_{i,m}^{(s)}(x),\ \mathbf{u}_{i,m}^{(s,t)} - \mathbf{u}^{(0)}(x) = \operatorname{curl}\tilde{\mathbf{u}}_{i,m}^{(s,t)}(x),$$

$$\bar{\mathbf{u}}_{i,r}^{(s)} - \mathbf{u}^{(0)}(x) = \operatorname{curl}\tilde{\bar{\mathbf{u}}}_{i,r}^{(s)}(x),\ \hat{\mathbf{u}}_{i,p}^{(s)} - \mathbf{u}^{(0)}(x) = \operatorname{curl}\tilde{\hat{\mathbf{u}}}_{i,p}^{(s)}(x),\ \mathbf{f}_{i,m}^{(s)} - \mathbf{f}(x) = \operatorname{curl}\tilde{\mathbf{f}}_{i,m}^{(s)}(x),$$

$$\mathbf{f}_{i,m}^{(s,t)} - \mathbf{f}(x) = \operatorname{curl}\tilde{\mathbf{f}}_{i,m}^{(s,t)}(x),\ \bar{\mathbf{f}}_{i,r}^{(s)} - \mathbf{f}(x) = \operatorname{curl}\tilde{\bar{\mathbf{f}}}_{i,r}^{(s)}(x),\ \hat{\mathbf{f}}_{i,p}^{(s)} - \mathbf{f}(x) = \operatorname{curl}\tilde{\hat{\mathbf{f}}}_{i,p}^{(s)}(x),$$

with the mentioned properties.

We define the asymptotic expansion of the solution of problem (1) by equality:

$$\mathbf{u}^{(s)} = \mathbf{u}^{(0)}(x) + \mathbf{r}^{(s)}(x) + \sum_{j=1}^{4}\mathbf{r}_j^{(s)}(x) + \mathbf{w}_s(x), \tag{25}$$

where

$$\mathbf{r}^{(s)}(x) = \sum_{i\in I_s''}\sum_{m=1}^{M(i,s)}\operatorname{curl}\Big\{\sum_{k=1}^{n}\mathbf{v}_{k,i,m}^{(s)}(x)\mathbf{u}_{i,m,k}^{(s)}\varphi_{i,m}^{(s)}(x)\Big\},$$

$$\mathbf{r}_1^{(s)}(x) = \sum_{i\in I_s'}\sum_{r=1}^{R(i,s)}\operatorname{curl}\Big\{\big([\bar{\mathbf{u}}_{i,r}^{(s)} - \mathbf{u}^{(0)}(x)] + [\mathbf{f}(x) - \bar{\mathbf{f}}_{i,r}]\big)\omega_{i,r}^{(s)}(x)\Big\}$$

$$+\sum_{i\in I_s''}\sum_{m=1}^{M(i,s)}\Bigg(\operatorname{curl}\Big\{\big([\mathbf{u}_{i,m}^{(s)} - \mathbf{u}^{(0)}(x)] + [\mathbf{f}(x) - \mathbf{f}_{i,m}^{(s)}]\big)\varphi_{i,m}^{(s)}(x)\Big\}$$

$$+\sum_{t=1}^{2}\operatorname{curl}\Big\{\big([\mathbf{u}_{i,m}^{(s,t)} - \mathbf{u}^{(0)}(x)] + [\mathbf{f}(x) - \mathbf{f}_{i,m}^{(s,t)}]\big)\psi_{i,m}^{(s,t)}(x)\Big\}\Bigg)$$

$$+\sum_{i\in I_s''}\sum_{p=1}^{P(i,s)}\operatorname{curl}\Big\{\big([\hat{\mathbf{u}}_{i,p}^{(s)} - \mathbf{u}^{(0)}(x)] + [\mathbf{f}(x) - \hat{\mathbf{f}}_{i,p}^{(s)}]\big)\chi_{i,p}^{(s)}(x)\Big\},$$

$$\mathbf{r}_2^{(s)}(x) = \sum_{i\in I_s'}\sum_{r=1}^{R(i,s)}\operatorname{curl}\Big\{\sum_{k=1}^{n}\bar{\mathbf{v}}_{k,i,r}^{(s)}(x)[\bar{\mathbf{f}}_{i,r,k}^{(s)} - \mathbf{u}_{i,r,k}^{(s)}]\omega_{i,r}^{(s)}(x)\Big\},$$

$$\mathbf{r}_3^{(s)}(x) = \sum_{i\in I_s''}\sum_{m=1}^{M(i,s)}\sum_{t=1}^{2}\operatorname{curl}\Big\{\sum_{k=1}^{n}\mathbf{v}_{k,i,m}^{(s,t)}(x)[\mathbf{f}_{i,m,k}^{(s,t)} - \mathbf{u}_{i,m,k}^{(s,t)}]\psi_{i,m}^{(s,t)}(x)\Big\},$$

$$\mathbf{r}_4^{(s)}(x) = \sum_{i\in I''(s)} \sum_{p=1}^{P(i,s)} \mathrm{curl}\big\{ \sum_{k=1}^{n} \hat{\mathbf{v}}_{k,i,p}^{(s)}(x)[\hat{\mathbf{f}}_{i,p,k}^{(s)} - \tilde{\mathbf{u}}_{i,p,k}^{(s)}]\chi_{i,p}^{(s)}(x)\big\},$$

and

$$\bar{\mathbf{v}}_{k,i,r}^{(s)}(x) = \mathbf{v}^k(x; D_{i,r}^{(s)}, D_{i,r}^{(s)} \cap [F^{(s)} \cap \partial\Omega]),$$

$$\mathbf{v}_{k,i,m}^{(s,t)}(x) = \mathbf{v}^k(x; B_{i,m}^{(s,t)}, B_{i,m}^{(s,t)} \cap [F^{(s)} \cap \partial\Omega]),$$

$$\mathbf{v}_{k,i,p}^{(s)}(x) = \mathbf{v}^k(x; \hat{B}_{i,p}^{(s)}, \hat{B}_{i,p}^{(s)} \cap [F^{(s)} \cap \partial\Omega]).$$

By $\mathbf{w}_s(x) \in H(\Omega_s)$ we denote the remainder term of asymptotic expansion.

We recall asymptotic properties which were proved in [18]:

$$\lim_{s\to\infty} \sum_{i\in I''(s)} M(i,s)\frac{[\rho_i^{(s)}]^n}{\lambda_s} = 0, \ \lim_{s\to\infty} \sum_{i\in I''(s)} M(i,s)[\rho_i^{(s)}]^{n-2} = 0, \tag{26}$$

$$\lim_{s\to\infty} \sum_{i\in I'(s)} R(i,s)[d_i^{(s)}]^{n-2} = \lim_{s\to\infty} \sum_{i\in I'(s)} [d_i^{(s)}]^{n-3} = 0, \tag{27}$$

$$\lim_{s\to\infty} \sum_{i\in I''(s)} \lambda_s^{-2} P(i,s)[\rho_i^{(s)}]^{n-2} = 0. \tag{28}$$

In order to investigate of behavior of $\mathbf{r}^{(s)}(x)$ and $\mathbf{r}_j^{(s)}(x)$ as $s \to \infty$ from asymptotic expansion (25) we use the pointwise and integral estimates from Theorem 1, and the construction and properties of cut - off functions. In such way we can prove the following results:

Theorem 3 *Suppose that conditions* $(\mathbf{i})-(\mathbf{v})$ *are satisfied. Then the sequence of functions* $\{\mathbf{r}_j^{(s)}(x)\}_{s=1}^{\infty}$, $j = 1,2,3,4$, *converge to zero strongly in* $H(\Omega)$ *as* $s \to \infty$.

Theorem 4 *Suppose that conditions* $(\mathbf{i}) - (\mathbf{v})$ *are satisfied. Then the sequence of solutions* $\{\mathbf{r}^{(s)}(x)\}_{s=1}^{\infty}$ *converges to zero strongly in* $W^{1,\vartheta}(\Omega)^n$ *for any* $0 < \vartheta < 2$ *and weakly in* $H(\Omega)$ *as* $s \to \infty$.

The proofs of these theorems are similar to proofs of analogous results in [18] and they are omitted here.

Using Theorems 3,4 and integral identity (2), we can study the behavior of the reminder term $\boldsymbol{\omega}_s(x)$ of asymptotic expansion (25). As a result, we obtain:

Theorem 5 *Suppose that conditions* $(\mathbf{i})-(\mathbf{v})$ *are satisfied. Then the sequence of functions* $\{\boldsymbol{\omega}_s(x)\}_{s=1}^{\infty}$ *converges to zero strongly in* $H(\Omega)$ *as* $s \to \infty$.

Proof It follows from construction of function (25) that $\mathbf{w}_s(x)$ converges to zero weakly as $s \to \infty$. We will prove that $\mathbf{w}_s(x)$ converges to zero strongly in $H(\Omega)$. Testing the integral identity (2) by $\boldsymbol{\varphi}_s(x) = \mathbf{w}_s(x)$, we get

$$\int_{\Omega_s} \nu \nabla \mathbf{u}^{(s)} \nabla \mathbf{w}_s \, dx - \int_{\Omega_s} (\mathbf{u}^{(s)} \cdot \nabla)\mathbf{w}_s \cdot \mathbf{u}^{(s)} dx = -\int_{\Omega_s} \mathbf{f}\mathbf{w}_s \, dx \tag{29}$$

From the weak convergence $\mathbf{w}_s(x)$ to zero in $H(\Omega)$ we have

$$\lim_{s\to\infty}\int\limits_{\Omega_s}\mathbf{f}\mathbf{w}_s dx = 0.$$

Let us consider the second integral in the left-hand side of (29)

$$\int\limits_{\Omega_s}(\mathbf{u}^{(s)}\cdot\nabla)\mathbf{w}_s\cdot\mathbf{u}^{(s)}dx$$

$$=\int\limits_{\Omega_s}((\mathbf{u}^{(0)}+\mathbf{r}^{(s)}+\sum_{j=1}^{4}\mathbf{r}_j^{(s)}+\boldsymbol{\omega}_s)\cdot\nabla)\mathbf{w}_s\cdot(\mathbf{u}^{(0)}+\mathbf{r}^{(s)}+\sum_{j=1}^{4}\mathbf{r}_j^{(s)}+\boldsymbol{\omega}_s)dx.$$

Then we obtain

$$\lim_{s\to\infty}\int\limits_{\Omega_s}(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{w}_s\cdot\mathbf{u}^{(0)}dx = 0$$

since $\mathbf{w}_s(x)$ converges to zero weakly in $H(\Omega)$ as $s\to\infty$.

From strong convergence of $(\mathbf{r}^{(s)}+\sum_{j=1}^{4}\mathbf{r}_j^{(s)})$ and $\mathbf{w}_s(x)$ in $L_2(\Omega)^n$ to zero and boundedness of $|\mathbf{u}^{(0)}(x)|$ and $\|\nabla\mathbf{w}_s\|_{L_2(\Omega)^n}$ by constant independent of s we deduce:

$$\lim_{s\to\infty}\int\limits_{\Omega_s}(\mathbf{u}^{(0)}\cdot\nabla)\mathbf{w}_s\cdot(\mathbf{r}^{(s)}+\sum_{j=1}^{4}\mathbf{r}_j^{(s)})dx = 0,$$

$$\lim_{s\to\infty}\int\limits_{\Omega_s}((\mathbf{r}^{(s)}+\sum_{j=1}^{4}\mathbf{r}_j^{(s)})\cdot\nabla)\mathbf{w}_s\cdot\mathbf{u}^{(0)}dx = 0,$$

$$\lim_{s\to\infty}\int\limits_{\Omega_s}(\mathbf{w}_s(x)\cdot\nabla)\mathbf{w}_s\cdot\mathbf{u}^{(0)}dx\le C\lim_{s\to\infty}\Big(\int\limits_{\Omega_s}|\mathbf{w}_s(x)|^2\,dx\Big)^{1/2}\Big(\int\limits_{\Omega_s}|\nabla\mathbf{w}_s(x)|^2\,dx\Big)^{1/2}=0$$

$$\lim_{s\to\infty}\int\limits_{\Omega_s}((\mathbf{r}^{(s)}+\sum_{j=1}^{4}\mathbf{r}_j^{(s)})\cdot\nabla)\mathbf{w}_s\cdot(\mathbf{r}^{(s)}+\sum_{j=1}^{4}\mathbf{r}_j^{(s)})dx$$

$$\le C\lim_{s\to\infty}\Big(\int\limits_{\Omega_s}|\mathbf{r}^{(s)}+\sum_{j=1}^{4}\mathbf{r}_j^{(s)}|^4\,dx\Big)^{1/2}\Big(\int\limits_{\Omega_s}|\nabla\mathbf{w}_s(x)|^2\,dx\Big)^{1/2}=0,$$

$$\lim_{s\to\infty}\int\limits_{\Omega_s}(\mathbf{u}^{(s)}\cdot\nabla)\mathbf{w}_s\cdot\mathbf{u}^{(s)}dx\le\lim_{s\to\infty}\Big(\int\limits_{\Omega_s}|\mathbf{w}_s(x)|^4\,dx\Big)^{1/2}\Big(\int\limits_{\Omega_s}|\nabla\mathbf{w}_s(x)|^2\,dx\Big)^{1/2}=0.$$

Now we consider the first integral in the left-hand side of (29)

$$\int\limits_{\Omega_s}\nabla\mathbf{u}^{(s)}\nabla\mathbf{w}_s\,dx\le C\int\limits_{\Omega_s}(\nabla\mathbf{u}^{(0)}+\nabla\mathbf{r}^{(s)}+\nabla\left(\sum_{j=1}^{4}\mathbf{r}_j^{(s)}\right)+\nabla\mathbf{w}_s)\nabla\mathbf{w}_s\,dx.$$

Since $|\nabla \mathbf{u}^{(0)}(x)|$ is bounded in Ω, and $\nabla(\sum_{j=1}^{4} \mathbf{r}_j^{(s)})$ converges strongly to zero, $\nabla \mathbf{w}_s(x)$ converges weakly to zero in $L_2(\Omega)^n$ as $s \to \infty$, we obtain

$$\lim_{s\to\infty} \int\limits_{\Omega_s} \nabla \mathbf{u}^{(0)} \nabla \mathbf{w}_s \, dx = 0,$$

$$\lim_{s\to\infty} \int\limits_{\Omega_s} \nabla \left(\sum_{j=1}^{4} \mathbf{r}_j^{(s)} \right) \nabla \mathbf{w}_s \, dx = 0.$$

Using definition of the function $\mathbf{v}_{k,i,m}^{(s)}(x)$ and asymptotic properties (26), (28), it can be shown that

$$\lim_{s\to\infty} \int\limits_{\Omega_s} \nabla \mathbf{r}^{(s)} \nabla \mathbf{w}_s \, dx = 0.$$

This means that

$$\lim_{s\to\infty} \int\limits_{\Omega_s} (\nabla \mathbf{w}_s, \nabla \mathbf{w}_s) \, dx = 0,$$

which prove the theorem.

Now we shall present the method of construction of the problem for limit function $\mathbf{u}^{(0)}(x)$. Let $\mathbf{h}(x,t)$ be an arbitrary function of class $H(\Omega)$. Let us introduce a sequence

$$\mathbf{h}_s(x) = \mathbf{h}^{(s)}(x) - \sum_{j=1}^{4} \mathbf{h}_j^{(s)}(x), \tag{30}$$

where

$$\mathbf{h}^{(s)}(x) = \sum_{i\in I_s''} \sum_{m=1}^{M(i,s)} \operatorname{curl} \Big\{ \sum_{k=1}^{n} \mathbf{v}_{k,i,m}^{(s)}(x) \mathbf{h}_{i,m,k}^{(s)} \varphi_{i,m}^{(s)}(x) \Big\},$$

$$\mathbf{h}_1^{(s)}(x) = \sum_{i\in I_s'} \sum_{r=1}^{R(i,s)} \operatorname{curl} \Big\{ \big([\bar{\mathbf{h}}_{i,r}^{(s)} - \mathbf{h}(x)] \big) \omega_{i,r}^{(s)}(x) \Big\}$$

$$+ \sum_{i\in I_s''} \sum_{m=1}^{M(i,s)} \left(\operatorname{curl} \Big\{ \big([\mathbf{h}_{i,m}^{(s)} - \mathbf{h}(x)] \big) \varphi_{i,m}^{(s)}(x) \Big\} + \sum_{t=1}^{2} \operatorname{curl} \Big\{ \big([\mathbf{h}_{i,m}^{(s,t)} - \mathbf{h}(x)] \big) \psi_{i,m}^{(s,t)}(x) \Big\} \right)$$

$$+ \sum_{i\in I_s''} \sum_{p=1}^{P(i,s)} \operatorname{curl} \Big\{ \big([\hat{\mathbf{h}}_{i,p}^{(s)} - \mathbf{h}(x)] \big) \chi_{i,p}^{(s)}(x) \Big\},$$

$$\mathbf{h}_2^{(s)}(x) = - \sum_{i\in I_s'} \sum_{r=1}^{R(i,s)} \operatorname{curl} \Big\{ \sum_{k=1}^{n} \bar{\mathbf{v}}_{k,i,r}^{(s)}(x) \bar{\mathbf{h}}_{i,r,k}^{(s)} \omega_{i,r}^{(s)}(x) \Big\},$$

$$\mathbf{h}_3^{(s)}(x) = - \sum_{i\in I_s''} \sum_{m=1}^{M(i,s)} \sum_{t=1}^{2} \operatorname{curl} \Big\{ \sum_{k=1}^{n} \mathbf{v}_{k,i,m}^{(s,t)}(x) \mathbf{h}_{i,m,k}^{(s,t)} \psi_{i,m}^{(s,t)}(x) \Big\},$$

$$\mathbf{h}_4^{(s)}(x) = -\sum_{i\in I''(s)} \sum_{p=1}^{P(i,s)} \operatorname{curl}\Big\{\sum_{k=1}^{n} \hat{\mathbf{v}}_{k,i,p}^{(s)}(x)\tilde{\mathbf{h}}_{i,p,k}^{(s)}\chi_{i,p}^{(s)}(x)\Big\},$$

here we keep for the function $\mathbf{h}(x)$ all notations from (25) and

$$\mathbf{h}_{i,m}^{(s)} - \mathbf{h}(x) = \operatorname{curl}\tilde{\mathbf{h}}_{i,m}^{(s)}(x),\ \ \mathbf{h}_{i,m}^{(s,t)} - \mathbf{h}(x) = \operatorname{curl}\tilde{\mathbf{h}}_{i,m}^{(s,t)}(x),$$

$$\bar{\mathbf{h}}_{i,r}^{(s)} - \mathbf{h}(x) = \operatorname{curl}\tilde{\bar{\mathbf{h}}}_{i,r}^{(s)}(x),\ \ \hat{\mathbf{h}}_{i,p}^{(s)} - \mathbf{h}(x) = \operatorname{curl}\tilde{\hat{\mathbf{h}}}_{i,p}^{(s)}(x).$$

Substituting the function $\boldsymbol{\varphi}^{(s)}(x) = \mathbf{h}_s(x)$ into integral identity (2), we get:

$$\int_{\Omega} (\nu\nabla\mathbf{u}^{(0)}\nabla\mathbf{h} - (\mathbf{u}^{(0)}\cdot\nabla)\mathbf{h}\cdot\mathbf{u}^{(0)})\,dx + \int_{\Omega}(\mathbf{f},\mathbf{h})\,dx$$

$$+\nu\sum_{i\in I''_s}\sum_{m=1}^{M(i,s)}\sum_{k,\ell=1}^{n}\int_{\Omega}(\nabla\mathbf{v}_{k,i,m}^{(s)},\nabla\mathbf{v}_{\ell,i,m}^{(s)})\mathbf{h}_{i,m,k}^{(s)}\mathbf{u}_{i,m,\ell}^{(s)}dx = \Upsilon(m,s), \tag{31}$$

where

$$|\Upsilon(m,s)| \le \gamma_1^{(s)} + \gamma_2^{(m)},$$

and sequences $\gamma_1^{(s)}$, $\gamma_2^{(m)}$ are such that $\lim_{s\to\infty}\gamma_1^{(s)} = 0$, $\lim_{m\to\infty}\gamma_2^{(m)} = 0$. Finally, using the condition (**vi**) in the left-hand side of (31) and passing to the limit as $s\to\infty$, $m\to\infty$, we obtain that the limit function $\mathbf{u}^{(0)}(x)$ is a weak solution of the following averaged problem:

$$\begin{gathered}\nu\Delta\mathbf{u}(x) - (\mathbf{u}\cdot\nabla)\,\mathbf{u} - \nu c(x)\mathbf{u}(x) = \nabla p(x) + \mathbf{f}(x),\quad x \text{ in } \Omega,\\ \operatorname{div}\mathbf{u}(x) = 0,\quad x \text{ in } \Omega,\\ \mathbf{u}(x)|_{\partial\Omega} = 0.\end{gathered} \tag{32}$$

The weak solution we understand in the sense of the following definition:

Definition 2 *We say that* $\mathbf{u}(x) \in H(\Omega)$ *is a weak solution of problem (32), if the integral identity*

$$\int_{\Omega}(\nu\nabla\mathbf{u}\nabla\boldsymbol{\varphi} - (\mathbf{u}\cdot\nabla)\boldsymbol{\varphi}\cdot\mathbf{u})\,dx + \int_{\Omega}\nu c(x)\mathbf{u}\boldsymbol{\varphi}\,dx = -\int_{\Omega}(\mathbf{f},\boldsymbol{\varphi})\,dx$$

is satisfied for every $\boldsymbol{\varphi} \in H(\Omega)$.

The main result of this paper is the following:

Theorem 6 *Suppose that conditions* (**i**)$-$(**vi**) *are satisfied. Then the sequence of solutions* $\{\mathbf{u}^{(s)}(x)\}_{s=1}^{\infty}$ *of problems (1) converges to function* $\mathbf{u}^{(0)}(x)$ *strongly in* $W^{1,\vartheta}(\Omega)^n$ *for any* $0 < \vartheta < 2$ *as* $s\to\infty$ *and the function* $\mathbf{u}^{(0)}(x)$ *is a weak solution of problem (32).*

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