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Boundedness of the maximal operator and singular integral operator in generalized Morrey spaces

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Abstract. In this paper we provide the conditions on the pair (ω_1, ω_2) which ensures the boundedness of the maximal operator and Calderón-Zygmund singular integral operators from one generalized Morrey space $\mathcal{M}_{p,\omega_1}$ to another $\mathcal{M}_{p,\omega_2}$, $1 < p < \infty$, and from the space $\mathcal{M}_{1,\omega_1}$ to the weak space $W\mathcal{M}_{1,\omega_2}$. As applications, by these results we get some estimates for uniformly elliptic operators on generaized Morrey spaces.

1. Introduction

The theory of boundedness of classical operators of real analysis, such as maximal operator and singular integral operators etc, from one weighted Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with weighted Lebesgue spaces, general Morrey-type spaces also play an important role.

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The maximal operator M is defined by

$$Mf(x) = \sup_{t>0} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)|dy,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

Definition 1.1. *Let $k(x) : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$. We call $k(x)$ a Calderón-Zygmund kernel (C-Z kernel) if*

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(i) $\quad k \in C^{\infty}(\mathbb{R}^n \backslash \{0\})$;

(ii) $\quad k(x)$ *is homogeneous of degree* $-n$;

(iii) $\quad \displaystyle\int_{\Sigma} k(x) d\sigma_{\xi} = 0$, *where* $\Sigma = \{x \in \mathbb{R}^n : |x| = 1\}$ *is the unit sphere in* $\mathbb{R}^n$.

Theorem 1.2 ([9]). *Let k be a real measurable function in $\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ such that*

(i) $\quad k(x,z)$ *is a C-Z kernel for a.a.* $x \in \mathbb{R}^n$;

(ii) $\quad \displaystyle\max_{|j| \le 2n} \|(\partial^j / \partial z^j) k(x,z)\|_{L_{\infty}(\mathbb{R}^n \times \Sigma)} = M < \infty$.

For $\varepsilon > 0$ set

$$T_{\varepsilon} f(x) := \int_{|x-y|>\varepsilon} k(x, x-y) f(y) dy.$$

Then there exists $Tf \in L_p(\mathbb{R}^n)$ such that

$$\lim_{\varepsilon \to 0+} \|T_{\varepsilon} f - Tf\|_{L_p(\mathbb{R}^n)} = 0$$

and, moreover, there exists a positive constant C such that

$$\|Tf\|_{L_p(\mathbb{R}^n)} \le C \|f\|_{L_p(\mathbb{R}^n)}.$$

Morrey spaces $\mathcal{M}_{p,\lambda}$ were introduced by C. Morrey in 1938 [14] and defined as follows. *For $0 \le \lambda \le n$, $1 \le p \le \infty$, $f \in \mathcal{M}_{p,\lambda}$ if $f \in L_p^{\rm loc}(\mathbb{R}^n)$ and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty,$$

where $B(x,r)$ is the open ball centered at x of radius r. Note that $\mathcal{M}_{p,0} = L_p(\mathbb{R}^n)$ and $\mathcal{M}_{p,n} = L_{\infty}(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of solutions to partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

We also denote by $W\mathcal{M}_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space.

F. Chiarenza and M. Frasca [8] studied the boundedness of the maximal operator M in these spaces. Their results can be summarized as follows:

Theorem 1.3. *Let $1 \le p < \infty$ and $0 < \lambda < n$. Then for $1 < p < \infty$ M is bounded from $\mathcal{M}_{p,\lambda}$ to $\mathcal{M}_{p,\lambda}$ and for $p = 1$ M is bounded from $\mathcal{M}_{1,\lambda}$ to $W\mathcal{M}_{1,\lambda}$.*

G.D.Fazio and M.A.Ragusa [9] studied the boundedness of the Calderón-Zygmund singular integral operators in Morrey spaces, and their results imply the following statement for Calderón-Zygmund operators T.

Theorem 1.4. *Let $1 \le p < \infty$, $0 < \lambda < n$. Then for $1 < p < \infty$ Calderón-Zygmund singular integral operator T is bounded from $\mathcal{M}_{p,\lambda}$ to $\mathcal{M}_{p,\lambda}$ and for $p = 1$ T is bounded from $\mathcal{M}_{1,\lambda}$ to $W\mathcal{M}_{1,\lambda}$.*

Note that in the case of the classical Calderón-Zygmund singular integral operators Theorem 1.4 was proved by J. Peetre [18]. If $\lambda = 0$, the statement of Theorem 1.4 reduces to Theorem 1.2 for $L_p(\mathbb{R}^n)$ (see also [6], [21]).

In the present work, we study the boundedness of maximal operator M and Calderón-Zygmund singular integral operators T from one generalized Morrey space $\mathcal{M}_{p,\omega_1}$ to another $\mathcal{M}_{p,\omega_2}$, $1 < p < \infty$, and from the space $\mathcal{M}_{1,\omega_1}$ to the weak space $W\mathcal{M}_{1,\omega_2}$. As applications, by these results we get some estimates for uniformly elliptic operators on generaized Morrey spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Generalized Morrey spaces

For the sake of completeness we recall the definition of the spaces and some properties of the spaces we are going to use.

If in place of the power function r^λ in the definition of $\mathcal{M}_{p,\lambda}$ we consider any positive measurable weight function $\omega(x,r)$, then it becomes generalized Morrey space $\mathcal{M}_{p,\omega}$.

Definition 2.1. *Let $\omega(x,r)$ be a positive measurable weight function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $\mathcal{M}_{p,\omega}$ the generalized Morrey space, the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \omega(x,r)^{-\frac{1}{p}} \|f\|_{L_p(B(x,t))}.$$

Definition 2.2. *We say that (ω_1,ω_2) belong to the class $\mathcal{Z}_{p,n}$, $p \in [0,\infty)$, if there is a constant C such that, for any $x \in \mathbb{R}^n$ and for any $t > 0$*

$$\left(\int_t^\infty \left(\frac{\operatorname{ess\,inf}_{r<s<\infty} \omega_1(x,s)}{r^n}\right)^{\frac{1}{p}} \frac{dr}{r}\right)^p \le C\,\frac{\omega_2(x,t)}{t^n}, \ \textit{if } p \in (0,\infty) \tag{2.1}$$

and

$$\operatorname*{ess\,sup}_{t<r<\infty} \frac{\operatorname{ess\,inf}_{r<s<\infty} \omega_1(x,s)}{r^n} \le C\frac{\omega_2(x,t)}{t^n}, \ \textit{if } p = 0. \tag{2.2}$$

Definition 2.3. *We say that* (ω_1,ω_2) *belong to the class* $\widetilde{\mathcal{Z}}_{p,n}$, $p\in[0,\infty)$, *if there is a constant* C *such that, for any* $x\in\mathbb{R}^n$ *and for any* $t>0$,

$$\left(\int_t^\infty \left(\frac{\omega_1(x,r)}{r^n}\right)^{\frac{1}{p}}\frac{dr}{r}\right)^p \le C\frac{\omega_2(x,t)}{t^n},\ \ \text{if } p\in(0,\infty) \tag{2.3}$$

and

$$\operatorname*{ess\,sup}_{t<r<\infty}\frac{\omega_1(x,r)}{r^n}\le C\frac{\omega_2(x,t)}{t^n},\ \ \text{if } p=0. \tag{2.4}$$

Note that $\widetilde{\mathcal{Z}}_{p,n}\subset\mathcal{Z}_{p,n}$ for $p\in[0,\infty)$.

The following property for the class $\mathcal{Z}_{p,n}$, $p\in[0,\infty)$ is valid.

Lemma 2.4. *Let* $0<p<\infty$. *Then*

$$\mathcal{Z}_{p,n}\subset\mathcal{Z}_{0,n}.$$

Proof. Let $p\in(0,\infty)$. Assume that $(\omega_1,\omega_2)\in\mathcal{Z}_{p,n}$. Then for any $s\in(t,\infty)$

$$\begin{aligned}\frac{\omega_2(x,t)}{t^n} &\gtrsim \left(\int_t^\infty\left(\frac{\operatorname{ess\,inf}_{r<\tau<\infty}\omega_1(x,\tau)}{r^n}\right)^{\frac{1}{p}}\frac{dr}{r}\right)^p\\ &\gtrsim \left(\int_s^\infty\left(\frac{\operatorname{ess\,inf}_{r<\tau<\infty}\omega_1(x,\tau)}{r^n}\right)^{\frac{1}{p}}\frac{dr}{r}\right)^p\\ &\gtrsim \operatorname*{ess\,inf}_{s<\tau<\infty}\omega_1(x,\tau)\left(\int_s^\infty\frac{dr}{r^{\frac{n}{p}+1}}\right)^p\\ &\approx \frac{\operatorname{ess\,inf}_{s<\tau<\infty}\omega_1(x,\tau)}{s^n}.\end{aligned}$$

Thus

$$\frac{\omega_2(x,t)}{t^n}\gtrsim\operatorname*{ess\,sup}_{t<s<\infty}\frac{\operatorname{ess\,inf}_{s<\tau<\infty}\omega_1(x,\tau)}{s^n}.$$

It proves that

$$\mathcal{Z}_{p,n}\subset\mathcal{Z}_{0,n}.$$

□

Remark 2.5. *Let* $\omega(t)=t^n$. *Then* $(\omega,\omega)\in\mathcal{Z}_{0,n}$, *but* $(\omega,\omega)\notin\mathcal{Z}_{p,n}$ *for any* $p\in(0,\infty)$.

T. Mizuhara [13], E. Nakai [16] and V. S. Guliyev [10] (see also [11]) generalised Theorem 1.4 and obtained sufficient conditions on functions ω_1 and ω_2 ensuring the boundedness of M and T from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$. In [16] the following statement was proved, containing the result in [13].

Theorem 2.6. *Let $1 \le p < \infty$. Moreover, let w be a positive measurable function satisfying the following conditions: there exists $c > 0$ such that*

$$0 < r \le t \le 2r \Rightarrow c^{-1}w(r) \le w(t) \le cw(r) \tag{2.5}$$

and $(\omega, \omega) \in \widetilde{\mathcal{Z}}_{1,n}$.

Then for $1 < p < \infty$ the operators M and T are bounded from $\mathcal{M}_{p,w}$ to $\mathcal{M}_{p,w}$ and for $p = 1$ M and T are bounded from $\mathcal{M}_{1,w}$ to $W\mathcal{M}_{1,w}$.

The following statement, containing the results in [13], [16] was proved in [10] (see also [11]).

Theorem 2.7. *Let $1 \le p < \infty$ and $(\omega_1, \omega_2) \in \widetilde{\mathcal{Z}}_{p,n}(\mathbb{R}^n)$. Then for $1 < p < \infty$ the operator T is bounded from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$ and for $p = 1$ the operator T is bounded from $\mathcal{M}_{1,w}$ to $W\mathcal{M}_{1,w}$.*

In [1]-[5], [10] and [11] the boundedness of the maximal operator and the singular integral operators in local and global Morrey-type spaces has been investigated. Note that the global Morrey-type space is a more general space than generalized Morrey space.

3. Boundedness of the maximal operator in generalized Morrey spaces

Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_\infty(t,\infty)},\ t \in (0,\infty).$$

The following Theorem was proved in [4].

Theorem 3.1. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\theta(t,\infty)} < \infty$ for any $t > 0$ and u be a continuous non-negative function on $(0,\infty)$*

Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{3.1}$$

Sufficient conditions on ω for the boundedness of M in generalized Morrey spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ have been obtained in [1], [2], [4], [5], [13], [16].

The following lemma is true.

Lemma 3.2. *Let $1 < p < \infty$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$ the inequality*

$$\|Mf\|_{L_p(B(x,r))} \lesssim \|f\|_{L_p(B(x,2r))} + r^{\frac{n}{p}} \sup_{t>2r} t^{-n} \|f\|_{L_1(B(x,t))} \tag{3.2}$$

holds for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, the inequality

$$\|Mf\|_{WL_1(B(x,r))} \lesssim \|f\|_{L_1(B(x,2r))} + r^{n} \sup_{t>2r} t^{-n} \|f\|_{L_1(B(x,t))} \tag{3.3}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$. It is obvious that for any ball $B = B(x,r)$

$$\|Mf\|_{L_p(B)} \le \|M(f\chi_{(2B)})\|_{L_p(B)} + \|M(f\chi_{\mathbb{R}^n\setminus(2B)})\|_{L_p(B)}.$$

By the continuity of the operator $M : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$, $1 < p < \infty$ we have

$$\|M(f\chi_{(2B)})\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)}.$$

Let y be an arbitrary point from B. If $B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\} \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\}$, then $t > |y-z| \ge |x-z| - |x-y| > 2r - r = r$.

On the other hand $B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\} \subset B(x,2t)$. Indeed, $z \in B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\}$, then we get $|x-z| \le |y-z| + |x-y| < t + r < 2t$.

Hence

$$\begin{aligned} M(f\chi_{\mathbb{R}^n\setminus(2B)})(y) &= \sup_{t>0} \frac{1}{|B(y,t)|} \int_{B(y,t)\cap\{\mathbb{R}^n\setminus(2B)\}} |f(y)|dy \\ &\le 2^n \sup_{t>r} \frac{1}{|B(x,2t)|} \int_{B(x,2t)} |f(y)|dy \\ &= 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)|dy. \end{aligned}$$

Therefore, for all $y \in B$ we have

$$M(f\chi_{\mathbb{R}^n\setminus(2B)})(y) \le 2^n \sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)|dy. \tag{3.4}$$

Thus

$$\|Mf\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + |B|^{\frac{1}{p}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)dy \right)$$

Let $p = 1$. It is obvious that for any ball $B = B(x,r)$

$$\|Mf\|_{WL_1(B)} \le \|M(f\chi_{(2B)})\|_{WL_1(B)} + \|M(f\chi_{\mathbb{R}^n\setminus(2B)})\|_{WL_1(B)}.$$

By the continuity of the operator $M : L_1(\mathbb{R}^n) \to WL_1(\mathbb{R}^n)$ we have

$$\|M(f\chi_{(2B)})\|_{WL_1(B)} \lesssim \|f\|_{L_1(2B)}.$$

Then by (3.4), we get the inequality (3.3).

$\square$

Lemma 3.3. *Let $1 < p < \infty$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$, the inequality*

$$\|Mf\|_{L_p(B(x,r))} \lesssim r^{\frac{n}{p}} \sup_{t>2r} t^{-\frac{n}{p}} \|f\|_{L_p(B(x,t))}, \tag{3.5}$$

holds for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Moreover, the inequality

$$\|Mf\|_{WL_1(B(x,r))} \lesssim r^n \sup_{t>2r} t^{-n} \|f\|_{L_1(B(x,t))} \tag{3.6}$$

holds for all $f \in L_1^{\rm loc}(\mathbb{R}^n)$.

Proof. Let $1 < p < \infty$. Denote by

$$\mathcal{M}_1 := |B|^{\frac{1}{p}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|} \int_{B(x,t)} |f(y)| dy \right),$$
$$\mathcal{M}_2 := \|f\|_{L_p(2B)}.$$

Applying Hölder's inequality, we get

$$\mathcal{M}_1 \lesssim |B|^{\frac{1}{p}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{\frac{1}{p}}} \left(\int_{B(x,t)} |f(y)|^p dy \right)^{\frac{1}{p}} \right).$$

On the other hand,

$$\begin{aligned} &|B|^{\frac{1}{p}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{\frac{1}{p}}} \left(\int_{B(x,t)} |f(y)|^p dy \right)^{\frac{1}{p}} \right) \\ &\qquad \gtrsim |B|^{\frac{1}{p}} \left(\sup_{t>2r} \frac{1}{|B(x,t)|^{\frac{1}{p}}} \right) \|f\|_{L_p(2B)} \approx \mathcal{M}_2. \end{aligned}$$

Since by Lemma 3.2

$$\|Mf\|_{L_p(B)} \leq \mathcal{M}_1 + \mathcal{M}_2,$$

we arrive at (3.5).

Let $p = 1$. The inequality (3.6) directly follows from (3.3).

$\square$

Theorem 3.4. *Let $p \in [1,\infty)$ and $(\omega_1,\omega_2) \in \mathcal{Z}_{0,n}(\mathbb{R}^n)$. Then for $p > 1$ M is bounded from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$ and for $p = 1$ M is bounded from $\mathcal{M}_{1,\omega_1}$ to $W\mathcal{M}_{1,\omega_2}$.*

Proof. By Lemma 3.3 and Theorem 3.1 we get

$$\begin{aligned} \|Mf\|_{\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_2(x,r)^{-\frac{1}{p}} r^{\frac{n}{p}} \left(\sup_{t>r} t^{-\frac{n}{p}} \|f\|_{L_p(B(x,t))} \right) \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_1(x,r)^{-\frac{1}{p}} \|f\|_{L_p(B(x,t))} = \|f\|_{\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)}, \end{aligned}$$

if $p \in (1,\infty)$ and

$$\begin{aligned}\|Mf\|_{W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_2(x,r)^{-1} r^n \left(\sup_{t>r} t^{-n}\|f\|_{L_1(B(x,t))}\right) \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_1(x,r)^{-1}\|f\|_{L_1(B(x,t))} = \|f\|_{\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)},\end{aligned}$$

if $p=1$. □

Corollary 3.5. *Let $p \in [1,\infty]$ and $\omega : (0,\infty) \to (0,\infty)$ is an increasing function. Assume that the mapping $t \mapsto \frac{\omega(t)}{t^n}$ is almost decreasing (there exists a constant c such that for $s<t$, we have $\frac{\omega(s)}{s^n} \geq c\frac{\omega(t)}{t^n}$). Then there exists a constant $C>0$ such that*

$$\|Mf\|_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}_{p,\omega}(\mathbb{R}^n)}, \ \textit{if } 1<p\leq\infty,$$

and

$$\|Mf\|_{W\mathcal{M}_{1,\omega}(\mathbb{R}^n)} \leq C\|f\|_{\mathcal{M}_{1,\omega}(\mathbb{R}^n)}.$$

4. Singular integrals and Hardy operator

In this section we are going to use the following statement on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)dr, \ 0<t<\infty.$$

Theorem 4.1. ([7]) *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \leq c \operatorname*{ess\,sup}_{t>0} v(t)g(t) \tag{4.1}$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A := \sup_{t>0}\frac{w(t)}{t}\int_0^t \frac{ds}{\operatorname{ess\,sup}_{0<y<s} v(y)} < \infty, \tag{4.2}$$

and $c \approx A$.

Sufficient conditions on ω for the boundedness of T in generalized Morrey spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$ have been obtained in [3], [10], [11], [13], [16].

The following Lemma has been proved in [10]. For the sake of completeness we give the proof.

Lemma 4.2. *Let $p \in [1,\infty)$, $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and for any $x_0 \in \mathbb{R}^n$*

$$\int_1^\infty t^{-\frac{n}{p}+1}\|f\|_{L_p(B(x_0,t))}dt < \infty.$$

Then Calderón-Zygmund singular integral $Tf(x)$ exists for a.a. $x \in \mathbb{R}^n$ and for any $x_0 \in \mathbb{R}^n$, $r > 0$ and $p \in (1, \infty)$

$$\|Tf\|_{L_p(B(x_0,r))} \le C\, r^{\frac{n}{p}} \int_{2r}^{\infty} t^{-\frac{n}{p}+1} \|f\|_{L_p(B(x_0,t))} dt, \tag{4.3}$$

where constant $C > 0$ does not depend on x_0, r and f.

Moreover, for any $x_0 \in \mathbb{R}^n$ and $r > 0$

$$\|Tf\|_{WL_1(B(x_0,r))} \le C\, r^{n} \int_{2r}^{\infty} t^{-n+1} \|f\|_{L_1(B(x_0,t))} dt, \tag{4.4}$$

where constant $C > 0$ does not depend on x_0, r and f.

Proof. Let $p \in (1, \infty)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus (2B)}$. Since $f_1 \in L_p(\mathbb{R}^n)$, $Tf_1(x)$ exists for a.a. $x \in \mathbb{R}^n$ and from the boundedness of T in $L_p(\mathbb{R}^n)$ ([9]) it follows that:

$$\|Tf_1\|_{L_p(B)} \le \|Tf_1\|_{L_p(\mathbb{R}^n)} \le C\|f_1\|_{L_p(\mathbb{R}^n)} = C\|f\|_{L_p(2B)},$$

where constant $C > 0$ is independent of f.

Let us prove that the non-singular integral $Tf_2(x)$ exists for all $x \in B$.

It's clear that $x \in B$, $y \in \mathbb{R}^n \setminus (2B)$ implies $\frac{1}{2}|x_0 - y| \le |x - y| \le \frac{3}{2}|x_0 - y|$. We get

$$|Tf_2(x)| \le 2^n \int_{\mathbb{R}^n \setminus (2B)} \frac{|f(y)|}{|x_0 - y|^n} dy.$$

By Fubini's theorem we have

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus (2B)} \frac{|f(y)|}{|x_0 - y|^n} dy &\approx \int_{\mathbb{R}^n \setminus (2B)} |f(y)| \int_{|x_0-y|}^{\infty} \frac{dt}{t^{n+1}} dy \\
&\approx \int_{2r}^{\infty} \int_{2r \le |x_0-y| < t} |f(y)| dy \frac{dt}{t^{n+1}} \\
&\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{n+1}}
\end{aligned}$$

Applying Hölder's inequality, we get

$$\int_{\mathbb{R}^n \setminus (2B)} \frac{|f(y)|}{|x_0 - y|^n} dy \lesssim \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Therefore $Tf_2(x)$ exists for all $x \in B$. Since $\mathbb{R}^n = \bigcup_{r>0} B(x_0, r)$, we get existence of $Tf(x)$ for a.a. $x_0 \in \mathbb{R}^n$.

Moreover, for all $p \in [1, \infty)$ the inequality

$$\|Tf_2\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}. \tag{4.5}$$

is valid. Thus

$$\|Tf\|_{L_p(B)} \lesssim \|f\|_{L_p(2B)} + r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

On the other hand,

$$\begin{aligned}\|f\|_{L_p(2B)} &\approx r^{\frac{n}{p}} \|f\|_{L_p(2B)} \int_{2r}^{\infty} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.\end{aligned}$$

Thus

$$\|Tf\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \|f\|_{L_p(B(x_0,t))} \frac{dt}{t^{\frac{n}{p}+1}}.$$

Let $p=1$. From the weak (1,1) boundedness of T ([9]) it follows that:

$$\|Tf_1\|_{WL_1(B)} \le \|Tf_1\|_{WL_1(\mathbb{R}^n)} \le C\|f_1\|_{L_1(\mathbb{R}^n)} = C\|f\|_{L_1(2B)},$$

where the constant $C>0$ is independent of f.

Then by (4.5) we get the inequality (4.4). $\square$

Theorem 4.3. *Let $p \in [1,\infty)$ and $(\omega_1,\omega_2) \in \mathcal{Z}_{p,n}$. Then Calderón-Zygmund singular integral $Tf(x)$ exists for a.a. $x \in \mathbb{R}^n$ and for $p>1$ the operator T is bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$ and for $p=1$ the operator T is bounded from $\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)$ to $W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)$. Moreover, for $p>1$*

$$\|Tf\|_{\mathcal{M}_{p,\omega_2}} \lesssim \|f\|_{\mathcal{M}_{p,\omega_1}},$$

and for $p=1$

$$\|Tf\|_{W\mathcal{M}_{1,\omega_2}} \lesssim \|f\|_{\mathcal{M}_{1,\omega_1}}.$$

Proof. By Lemma 4.2 and Theorem 4.1 we have for $p>1$

$$\begin{aligned}\|Tf\|_{\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-\frac{1}{p}} r^{\frac{n}{p}} \int_r^{\infty} \|f\|_{L_p(B(x,t))} \frac{dt}{t^{\frac{n}{p}+1}} \\ &\approx \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-\frac{1}{p}} r^{\frac{n}{p}} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L_p(B(x,t^{-\frac{p}{n}}))} dt \\ &= \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r^{-\frac{p}{n}})^{-\frac{1}{p}} \frac{1}{r} \int_0^{r} \|f\|_{L_p(B(x,t^{-\frac{p}{n}}))} dt \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_1(x,r^{-\frac{p}{n}})^{-\frac{1}{p}} \|f\|_{L_p(B(x,r^{-\frac{p}{n}}))} = \|f\|_{\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)}\end{aligned}$$

and for $p=1$

$$\begin{aligned}\|Tf\|_{W\mathcal{M}_{1,\omega_2}(\mathbb{R}^n)} &\lesssim \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-1} r^n \int_r^\infty \|f\|_{L_1(B(x,t))} \frac{dt}{t^{n+1}} \\ &\approx \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r)^{-1} r^n \int_0^{r^{-n}} \|f\|_{L_1(B(x,t^{-n}))} dt \\ &= \sup_{x\in\mathbb{R}^n,\, r>0} \omega_2(x,r^{-\frac{1}{n}})^{-1} \frac{1}{r} \int_0^r \|f\|_{L_1(B(x,t^{-\frac{1}{n}}))} dt \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \omega_1(x,r^{-\frac{1}{n}})^{-1} \|f\|_{L_1(B(x,r^{-\frac{1}{n}}))} = \|f\|_{\mathcal{M}_{1,\omega_1}(\mathbb{R}^n)}.\end{aligned}$$

□

Corollary 4.4. *Let $p\in[1,\infty)$ and $(\omega_1,\omega_2)\in\widetilde{\mathcal{Z}}_{p,n}(\mathbb{R}^n)$. Then for $p>1$ T is bounded from $\mathcal{M}_{p,\omega_1}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\omega_2}(\mathbb{R}^n)$ and for $p=1$ T is bounded from $\mathcal{M}_{1,\omega_1}$ to $W\mathcal{M}_{1,\omega_2}$.*

Note that Theorem 2.7 and Corollary 4.4 coincide.

5. Estimates for uniformly elliptic operators on generaized Morrey spaces

In this section we consider uniformly elliptic operators

$$L = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x)\partial_j\right) + V(x)$$

with non-negative potentials V on $\mathbb{R}^n$ ($n\geq 3$) which belong to certain reverse Hölder class. We show several estimates for VL^{-1}, $V^{\frac{1}{2}}\nabla L^{-1}$ and $\nabla^2 L^{-1}$ on generalized Morrey spaces under certain assumptions on $a_{ij}(x)$, V and p. Our results generalize some results of K. Kurata and S. Sugano [12].

For the Schrödinger operators $-\Delta+V(x)$ with nonnegative polynomials V, several authors ([20], [23], [24]) studied L_p boundedness for $1<p<\infty$ of $\nabla(-\Delta+V)^{-\frac{1}{2}}$, $(-\Delta+V)^{-\frac{1}{2}}\nabla$, and $\nabla(-\Delta+V)^{-1}\nabla$, $V^{\frac{1}{2}}\nabla(-\Delta+V)^{-1}$, and $\nabla^2(-\Delta+V)^{-1}$. In particular, J. Zhong [24] proved that if V is a non-negative polynomial, $\nabla^2(-\Delta+V)^{-1}$, $\nabla(-\Delta+V)^{-\frac{1}{2}}$, and $\nabla(-\Delta+V)^{-1}\nabla$ are Calderón-Zygmund operators. Recently, Z. Shen [19] generalized these results. He proved that $\nabla(-\Delta+V)^{-\frac{1}{2}}$, $(-\Delta+V)^{-\frac{1}{2}}\nabla$, and $\nabla(-\Delta+V)^{-1}\nabla$ are Calderón-Zygmund operators, if V belongs to the reverse Hölder class B_n (see Definition 6.1), which includes non-negative polynomials and allows some non-smooth potentials. Moreover, Z. Shen also showed L_p boundedness for $V(-\Delta+V)^{-1}$, and $\nabla^2(-\Delta+V)^{-1}$ when $V\in B_{n/2}$ and $V^{\frac{1}{2}}\nabla(-\Delta+V)^{-1}$ when $V\in B_n$.

In this section we consider uniformly elliptic operators

$$L = L_0 + V(x) = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}(x)\partial_j\right) + V(x)$$

with certain non-negative potentials V on $\mathbb{R}^n$ $(n \geq 3)$, where $a_{ij}(x)$ is a measurable function satisfying the conditions:

($\mathbf{A_1}$) There exists a constant $\lambda \in (0,1]$ such that

$$a_{ij}(x) = a_{ji}(x), \ \lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \lambda^{-1}|\xi|^2, \ x, \xi \in \mathbb{R}^n;$$

($\mathbf{A_2}$) There exist constants $\alpha \in (0,1]$ and $K > 0$ such that

$$\|a_{ij}\|_{C^\alpha(\mathbb{R}^n)} \leq K.$$

Throughout this section we use the following notation:

$$\partial_j = \nabla_j = \nabla_{x_j} = \frac{\partial}{\partial x_j}, \ |\nabla u(x)|^2 = \sum_{j=1}^{n} |\nabla_j u(x)|^2$$

The purpose of this section is to show boundedness of the operators $T_1 = VL^{-1}$, $T_2 = V^{\frac{1}{2}}\nabla L^{-1}$ and $T_3 = \nabla^2 L^{-1}$ from one generalized Morrey space $\mathcal{M}_{p,\omega_1}$ to another $\mathcal{M}_{p,\omega_2}$. Although it is known T_1 and T_3 are Calderón-Zygmund operators for the case $L = -\Delta + V$ with non-negative polynomials V, it is not known that whether T_j $(j = 1,2,3)$ are Calderón-Zygmund operators or not under the general condition $V \in B_\infty$. We show, under the same conditions as in [19] for V , boundedness of $T_1 = VL^{-1}$ and $T_2 = V^{\frac{1}{2}}\nabla L^{-1}$ on generalized Morrey spaces $\mathcal{M}_{p,\omega}(\mathbb{R}^n)$. Actually, we used pointwise estimates of $T_k f(x)$, $k = 1,2$, by the Hardy-Littlewood maximal function (see [12], Theorem 1.3). We also show boundedness of $T_3 = \nabla^2 L^{-1}$ on generalized Morrey spaces under the additional assumption

($\mathbf{A_3}$) There exist a constant $\alpha \in (0,1]$ such that

$$a_{ij} \in C^{1+\alpha}(\mathbb{R}^n), \ a_{ij}(x+z) = a_{ij}(x), \ \text{for all } x \in \mathbb{R}^n, \ \text{for all } z \in \mathbb{Z}^n,$$

and

$$\sum_{i}^{n} \partial_i \left(a_{ij}(x)\right) = 0, \ j = 1, \ldots, n.$$

Here L^{-1} is the integral operator with the fundamental solution (or the minimal Green function (see e. g. [15])) of L as its integral kernel. We can also define $L^{-1}f$ for $f \in C_0^\infty(\mathbb{R}^n)$ as the unique solution of $Lu = f$ on certain generalized Morrey space $\mathcal{M}_{2,\omega}(\mathbb{R}^n)$, and can see it is a bounded operator on certain generalized Morrey spaces $\mathcal{M}_{2,\omega}(\mathbb{R}^n)$ (see e. g. [20]).

Definition 5.1. *Let $V(x) \geq 0$.*

(1) *A nonnegative locally L_q integrable function V on $\mathbb{R}^n$ is said to belong to the reverse Hölder class B_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality*

$$\left(\frac{1}{|B|}\int_B V(x)^q dx\right)^{\frac{1}{q}} \leq \frac{C}{|B|}\int_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

(2) *We say $V \in B_\infty$, if there exists a constant $C > 0$ such that*

$$\|V\|_{L_\infty(B)} \leq \frac{C}{|B|}\int_B V(x)dx$$

holds for every ball B in $\mathbb{R}^n$.

Clearly, $B_\infty \subset B_q$ for $1 < q < \infty$. But it is important that the B_q class has a property of "self-improvement"; that is, if $V \in B_q$, then $V \in B_{q+\varepsilon}$ for some $\varepsilon > 0$ (see [17]).

K. Kurata and S. Sugano [12] proved the following pointwise estimate for T_1 and T_2 which generalize the results in [24], Lemma 3.2 to uniformly elliptic operators with general potentials $V \in B_\infty$.

Theorem A. *Suppose that $A(x)$ satisfies $(\mathbf{A_1})$ for T_1, $(\mathbf{A_1})-(\mathbf{A_2})$ for T_2, and $V \in B_\infty$. Then there exist positive constants C_k, $k = 1, 2$ such that*

$$|T_k f(x)| \leq Mf(x), \;\; f \in C_0^\infty(\mathbb{R}^n), \;\; k = 1, 2.$$

Hence Theorem A and Theorem 3.4 in Section 2 imply

Corollary 5.2. *Let $A(x)$ and $V(x)$ satisfy the same assumptions as in Theorem A.*

(1) *Suppose $1 < p < \infty$, and $(\omega_1, \omega_2) \in \mathcal{Z}_{0,n}$. Then VL^{-1} and $V^{\frac{1}{2}}\nabla L^{-1}$ are bounded from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$.*

(2) *Suppose $1 < p < \infty$, $(\omega_1, \omega_2) \in \mathcal{Z}_{0,n}$ and $(\mathbf{A_3})$ for $A(x)$. Then $\nabla^2 L^{-1}$ is bounded from $\mathcal{M}_{p,\omega_1}$ to $\mathcal{M}_{p,\omega_2}$.*

Theorem B. (1) *Suppose $A(x)$ satisfies $(\mathbf{A_1})$ and $V \in B_q$, $q > n/2$. Then there exist a positive constant C such that*

$$|T_1^* f(x)| \leq CM\big(|f|^{q'}\big)^{1/q'}(x), \;\; f \in C_0^\infty(\mathbb{R}^n),$$

where $1/q + 1/q' = 1$.

(2) *Suppose $A(x)$ satisfies $(\mathbf{A_1}) - (\mathbf{A_2})$. When $V \in B_q$ with $n > q > n/2$ we have*

$$|T_2^* f(x)| \leq CM\big(|f|^{p_1}\big)^{1/p_1}(x), \;\; f \in C_0^\infty(\mathbb{R}^n),$$

where $1/p_1 = 1 + (1/n) - (3/2q)$.

When $V \in B_q$ with $q > n$ we have

$$|T_2^* f(x)| \leq CM\big(|f|^{p_1}\big)^{1/p_1}(x), \;\; f \in C_0^\infty(\mathbb{R}^n),$$

where $1/p_1 = 1 - (1/2q)$.

Hence Theorem B and Theorems 3.4 and 4.3 imply

Corollary 5.3. (1) *Suppose* $A(x)$ *satisfies* $(\mathbf{A_1})$. *Suppose* $V \in B_q$ *with* $q > n/2$, *and* $(\omega_1, \omega_2) \in \mathcal{Z}_{0,n}$ *and* $q' < p < \infty$. *Then* T_1 *is bounded from* $\mathcal{M}_{p,\omega_1}$ *to* $\mathcal{M}_{p,\omega_2}$.

(2) *Suppose* $A(x)$ *satisfies* $(\mathbf{A_1}) - (\mathbf{A_2})$. *Suppose* $V \in B_q$ *with* $n/2 < q < n$, $p_1 < p < \infty$, $1/p_1 = 1 + (1/n) - (3/2q)$ *and* $(\omega_1, \omega_2) \in \mathcal{Z}_{0,n}$. *Then* T_2^* *is bounded from* $\mathcal{M}_{p,\omega_1}$ *to* $\mathcal{M}_{p,\omega_2}$.

(3) *Suppose* $A(x)$ *satisfies* $(\mathbf{A_1}) - (\mathbf{A_2})$. *Suppose* $V \in B_q$ *with* $q > n$, $p_1 < p < \infty$, $1/p_1 = 1 - (1/2q)$ *and* $(\omega_1, \omega_2) \in \mathcal{Z}_{0,n}$. *Then* T_2^* *is bounded from* $\mathcal{M}_{p,\omega_1}$ *to* $\mathcal{M}_{p,\omega_2}$.

(4) *Suppose* $A(x)$ *satisfies* $(\mathbf{A_1}) - (\mathbf{A_3})$, $1 < p < \infty$ *and* $(\omega_1, \omega_2) \in \mathcal{Z}_{p,n}$. *Then* $\nabla^2 L^{-1}$ *is bounded from* $\mathcal{M}_{p,\omega_1}$ *to* $\mathcal{M}_{p,\omega_2}$.

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