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Convergence of solutions of a non-local phase-field system

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1 Introduction

This paper is devoted to the study of asymptotic properties and convergence to equilibria of a two-phase model involving non-local terms. Considering a binary alloy with components A and B occupying a spatial domain Ω, and denoting by u and $1-u$ the local concentrations of A and B respectively, Gajewski and Zacharias [5] studied a model describing also long range interaction of particles. This phenomenon is represented by spatial convolution with a suitable kernel, cf. Chen and Fife [2]. The system in question reads:

$$u_t - \nabla \cdot (\mu \nabla v) = 0 \text{ in } (0,T)\times\Omega, \tag{1.1}$$

$$v = f'(u) + \int_\Omega K(|x-y|)(1-2u(t,y))dy, \quad (t,x)\in(0,T)\times\Omega. \tag{1.2}$$

$$\mu\nu\cdot\nabla v = 0 \text{ in } (0,T)\times\partial\Omega, \tag{1.3}$$

$$u(0,x) = u_0,\ u_0\in L^\infty(\Omega), \quad 0\le u_0(x)\le 1, \quad 0 < \int_\Omega u_0 \,\mathrm{d}x = u_\alpha < 1. \tag{1.4}$$

Gajewski and Zacharias [5] proved global existence, uniqueness of solutions and compactness of trajectories in the space $L^2(\Omega)$ under assumptions stated below. However, convergence of trajectories of this system to equilibria was proved only in the case when the norm of the convolution operator is smaller than 2, which means that the global interactions must be small compared with the convexity of f. This condition ensures that the equilibrium state is uniquely defined, which need not be the case in general.

The convergence of solutions of various phase-field systems to equilibria have been proved by many authors with help of the Łojasiewicz inequality. In our case, we have compactness of trajectories in $L^2(\Omega)$ space only, where the energy functional is not twice continuously differentiable, so we have to use the non-smooth version of the Simon-Łojasiewicz theorem which was proved in [6] and generalized in [4]. This version is formulated in Section 4.

Also, boundedness od solutions was proved in [5] on compact time intervals only. The aim of the present paper is to show that any solution with initial datum bounded

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away from "pure states" stabilizes to a single stationary state, and any solution starting from u_0 satisfying (1.4) separates from 0 and 1 in the sense that

$$\max\left\{\|\ln u(t)\|_{L^r(\Omega)},\ \|\ln(1-u(t))\|_{L^r(\Omega)}\right\} \le Cr^2 \text{ for all } t\ge 1,\ r\ge 1, \tag{1.5}$$

and there is a sequence of times $\{t_r\},\ t_r\to\infty$, such that

$$\max\left\{\|\ln u(t)\|_{L^r(\Omega)},\ \|\ln(1-u(t))\|_{L^r(\Omega)}\right\} \le C \text{ for all } t\ge t_r. \tag{1.6}$$

We will proceed as follows. First, we start with the initial value such that

$$c\le u(0,x)\le 1-c \text{ for a.a. } x\in\Omega,\ \text{ and some }\ 0<c<1, \tag{1.7}$$

and prove that u remains bounded away from 0 and 1 for all $t\ge 0$. To this end, we apply the method of Alikakos [1] in a bit different way than in [5]. Then we prove (1.5), (1.6) (Lemma 3.3, Lemma 3.5). Finally, we apply a generalized version of the Łojasiewicz-Simon theorem to show that the time derivative of u belongs to $L^1(T,+\infty;H^1(\Omega)^*)$ which in turn allows us to show convergence of u in $L^2(\Omega$.

2 Assumptions and Preliminaries

We assume that $\Omega\subset\mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$. The existence of global weak solutions of the problem (1.1)-(1.4) in the class

$$u\in C(0,T;L^\infty(\Omega))\cap L^2(0,T;H^1(\Omega)),\ u_t\in L^2(0,T);H^1(\Omega)^*), \tag{2.1}$$

$$w=\int_\Omega K(|x-y|)(1-2u(t,y))dy\in C(0,T;H^{1,\infty}(\Omega)), \tag{2.2}$$

$$v=f'(u)+w, \tag{2.3}$$

was proved in [5] under the following assumptions:

$$f(u)=u\log u+(1-u)\log(1-u), \tag{2.4}$$

$$\mu=\frac{a(x,|\nabla v|)}{f''(u)},\ a \text{ satisfies some monotonocity conditions}, \tag{2.5}$$

$$\int_\Omega\int_\Omega |K(|x-y|)|\ \mathrm{d}x\ \mathrm{d}y=k_0<\infty,\ \sup_{x\in\Omega}\int_\Omega|K(|x-y|)|\mathrm{d}y=k_1<\infty, \tag{2.6}$$

and the operator $\mathcal{J}$ defined by $\mathcal{J}z=\int_\Omega K(|y-x|)z(x)\ \mathrm{d}x$ satisfies

$$\|\mathcal{J}z\|_{H^{1,p}}\le r_p\|z\|_{L^p(\Omega)},\quad 1\le p\le\infty. \tag{2.7}$$

In addition, the existence of a triple (u^*,v^*,w^*) and a sequence of times $t_n\to\infty$ such that

$$u(t_n)\to u^*\ \text{ strongly in } L^2(\Omega) \tag{2.8}$$

$$w(t_n)\to w^*\ \text{ strongly in } H^1 \tag{2.9}$$

$$\arctan(e^{-v(t_n)/2}) \to \arctan(e^{-v^*/2}) \ \text{ strongly in } H^1, \ v^* = const. \tag{2.10}$$

with

$$u^* = \frac{1}{1+\exp(w^*-v^*)}, \quad v^* = const, \quad w^* = \int_\Omega K(|x-y|)(1-2u^*(t,y))\mathrm{d}y \tag{2.11}$$

was proved.

In what follows, for the sake of simplicity, and without loss of generality, we will assume that

$$a = const, \quad |\Omega| = 1. \tag{2.12}$$

Then

$$\mu = \frac{a}{f''(u)} = a\ u(1-u), \quad v = \ln\frac{u}{1-u} + w. \tag{2.13}$$

3 Global boundedness

Assume that

$$0 < c \le u(0,x) \le 1-c, \quad \text{for a.a. } x \in \Omega. \tag{3.1}$$

Then there is $t_0 > 0$ such that $\frac{1}{u} \in L^2(0,t_0;H^1(\Omega))$. It follows that time derivative of $\int_\Omega \ln u \ \mathrm{d}x$ is L^1-function and we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\ln u(t)|\ \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \ln u(t)\ \mathrm{d}x = \int_\Omega \frac{1}{u^2}\nabla u\ a\nabla u(t) - \frac{1}{u^2}\nabla u(t)\ au(1-u)\nabla w(t)\ \mathrm{d}x \\ = -\int_\Omega a|\nabla \ln u(t)|^2\ \mathrm{d}x - \int_\Omega a(1-u)\nabla \ln u \nabla w(t)\ \mathrm{d}x \\ \le -\frac{1}{2}\int_\Omega a|\nabla \ln u|^2\ \mathrm{d}x + \frac{1}{2}\int_\Omega a|\nabla w|^2\ \mathrm{d}x.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega -\ln(1-u)\ \mathrm{d}x = -\int_\Omega \frac{1}{(1-u)^2}\nabla u\ a\nabla u - \frac{1}{(1-u)^2}\nabla u\ au(1-u)\nabla w\ \mathrm{d}x \\ = -\int_\Omega a|\nabla \ln(1-u)|^2\ \mathrm{d}x + \int_\Omega au\nabla\ln(1-u)\nabla w\ \mathrm{d}x \\ \le -\frac{1}{2}\int_\Omega a|\nabla \ln(1-u)|^2\ \mathrm{d}x + \frac{1}{2}\int_\Omega a|\nabla w|^2\ \mathrm{d}x.\end{aligned}$$

Denote

$$C_1 = \frac{a}{2}\operatorname*{ess\,sup}_{t\ge 0} \||\nabla w(t)|\|_\infty^2, \tag{3.2}$$

$$\Omega_1^t = \{x \in \Omega; u(t,x) \ge \frac{1}{2}u_\alpha\}. \tag{3.3}$$

Then, necessarily,

$$|\Omega_1^t| \ge \frac{1}{2}u_\alpha \ \text{ for all } t \ge 0. \tag{3.4}$$

Indeed, if it is not the case, then we have

$$u_\alpha = \int_\Omega u(t,x)\ \mathrm{d}x = \int_{\Omega_1} + \int_{\Omega\setminus\Omega_1} < \frac{u_\alpha}{2}\cdot 1 + \frac{u_\alpha}{2}|\Omega\setminus\Omega_1| < u_\alpha,$$

a contradiction.

To estimate $\int_\Omega a|\nabla \ln u|^2\ \mathrm{d}x$, we use the following lemma, which is a particular case of Theorem 4.2.1 in [7].

Lemma 3.1 *Let Ω be a connected, Lipschitz domain and suppose $u \in H^1(\Omega)$. If $L \in [H^1(\Omega)]^*$ and $L(\chi_\Omega) = 1$, then*

$$\|u - L(u)\|_{L^2(\Omega)} \leq C_2\|L\|\|\nabla u\|_{L^2(\Omega)}, \tag{3.5}$$

where $C_2 = C_2(\Omega)$.

(Here we denoted by $L(u)$ both the value of the functional and the corresponding constant function). We apply Lemma 3.1 with the functional L given by

$$Lz = \frac{1}{|\Omega_1|}\int_{\Omega_1} z(x)\ \mathrm{d}x, \quad \Omega_1 \subset \Omega.$$

Then

$$\|L\| = \frac{1}{|\Omega_1|},$$

and we have for a.a. $t \geq 0$:

$$\int_\Omega |\nabla \ln u(t)|^2\ \mathrm{d}x \geq \Big(\frac{|\Omega_1^t|}{C_2}\Big(\|\ln u(t) - L(\ln u(t))\|_{L^2(\Omega)}\Big)\Big)^2$$

$$\geq \frac{|\Omega_1^t|^2}{2C_2^2}\Big(\int_\Omega |\ln u(t)|\ \mathrm{d}x\Big)^2 - \frac{|\Omega_1^t|}{C_2^2}\Big|\ln\frac{u_\alpha}{2}\Big|^2. \tag{3.6}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\ln u(t)|\ \mathrm{d}x + \beta^2\Big(\int_\Omega |\ln u(t)|\ \mathrm{d}x\Big)^2 \leq N^2$$

where

$$\beta^2 = \frac{a}{2C_2^2}\Big(\frac{u_\alpha}{2}\Big)^2, \quad N^2 = \frac{a}{2C_2^2}\Big|\ln\frac{u_\alpha}{2}\Big|^2 + C_1.$$

Then $\int_\Omega |\ln u(t)|\ \mathrm{d}x$ is dominated by a solution b of the equation

$$\dot{b}(t) + \beta^2 b^2(t) = N^2, \quad b(0) = \int_\Omega |\ln u(0)|\ \mathrm{d}x. \tag{3.7}$$

The solution of this equation is bounded by $\frac{N}{\beta}$ if the initial value $b(0) \leq \frac{N}{\beta}$, and it is given by

$$b(t) = \frac{N}{\beta}\frac{\exp(2N\beta(t+k)) + 1}{\exp(2N\beta(t+k)) - 1} \tag{3.8}$$

for $b(0) > \frac{N}{\beta}$, where k is chosen such that the initial condition is satisfied. We see that for $t \geq 1$ and any $k \geq 0$, the estimate

$$\|\ln u(t)\|_1 \leq m_1 = \frac{N}{\beta}\frac{\exp(2N\beta)+1}{\exp(2N\beta)-1} \tag{3.9}$$

holds true, where m_1 depends only on u_α, the integral mean of u_0.

If $u(0)$ satisfies (1.4) but not (3.1), we find a sequence of functions $u^n(0)$ satisfying (3.1) such that

$$u^n(0) \to u(0) \text{ in } L^\infty(\Omega),$$

and use the following lemma on continuous dependence of solutions on the initial data:

Lemma 3.2 *Let u_1, u_2 be two solutions of (1.1), (1.2). Then*

$$\|(u_1-u_2)(t)\|^2_{L^2(\Omega)} \leq C(t)\|(u_1-u_2)(0)\|^2_{L^2(\Omega)}. \tag{3.10}$$

Proof: . We subtract the corresponding equations (1.1) and multiply by $u_1 - u_2$. We get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u_1-u_2\|^2_{L^2(\Omega)} = -\int_\Omega a|\nabla u_1 - \nabla u_2|^2 - (\mu_1\nabla w_1 - \mu_2\nabla w_2(\nabla u_1 - \nabla u_2)\ \mathrm{d}x$$

$$\leq -\int_\Omega \frac{a}{2}|\nabla u_1-\nabla u_2|^2+\frac{a}{2}\Big[u_1(1-u_1)(\nabla w_1-\nabla w_2)+(u_1(1-u_1)-u_2(1-u_2))\nabla w_2(t)\Big]^2\ \mathrm{d}x$$

$$\leq \frac{a}{16}\|\nabla w_1-\nabla w_2\|^2_{L^2(\Omega)} + a\|\nabla w_2\|^2_{L^\infty(\Omega)}\|u_1-u_2\|^2 \leq C\|u_1-u_2\|^2_{L^2(\Omega)}.$$

Hence (3.10) follows.

q.e.d.

Consequently, $u^n(t) \to u(t)$ in $L^2(\Omega)$, for any $t > 0$, and also in $L^r(\Omega)$ for any $r > 0$ because $\|u(t)\|_{L^\infty(\Omega)} \leq 1$. Moreover, $\int_\Omega |\ln u^n(t)|\ \mathrm{d}x \leq m_1$ for any n and any $t > 1$, which allows us to deduce

$$\int_\Omega |\ln u(t)|\ \mathrm{d}x \leq m_1,\ t > 1. \tag{3.11}$$

The same procedure applies to $\int_\Omega |\ln(1-u)|\ \mathrm{d}x$, which, together with (2.7) yields:

Lemma 3.3 *Let u_0 satisfy (1.4), (u, v, w) be a solution of (1.1)-(1.4). Then*

$$\|v(t)\|_{L^1(\Omega)} \leq m_1 + r_\infty \quad \textit{for all}\ \ t \geq 1, \tag{3.12}$$

$$\|w(t)\|_{H^{1,\infty}} \leq r_\infty \quad \textit{for}\ \ t \geq 0, \tag{3.13}$$

where $m_1,\ r_\infty$ are given by (3.9), (2.7) respectively.

Next, we derive estimates of the norm of $\ln u(t)$ in the space $L^r(\Omega),\ r \geq 2$.

Lemma 3.4 *Let u be a solution of (1.1)-(1.4). Then there exist constants $B_1,\ B_2,\ B_3$, depending only on u_α, and a sequence of times $\{t_r\}$ such that the following estimates hold for $r \geq 2$:*

(i) $\|\ln u(t)\|_{L^r(\Omega)} \le B_1 \|\ln u(0)\|_{L^r(\Omega)}$ for all $t \ge 0$,
(ii) $\|\ln u(t)\|_{L^r(\Omega)} \le B_2 r^2$ for all $t \ge 1$,
(iii) $\|\ln u(t)\|_{L^r(\Omega)} \le B_3$ for all $t \ge t_r$.

Proof. For $r \ge 2$ we denote

$$\mathcal{M}_r(t) = \int_\Omega (-\ln u(t))^r \mathrm{d}x, \tag{3.14}$$

and estimate its time derivative:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_r(t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (-\ln u(t))^r \ \mathrm{d}x = -r\int_\Omega \frac{(-\ln u)^{r-1}}{u} u_t(t) \ \mathrm{d}x = r\int_\Omega \nabla\Big(\frac{(-\ln u)^{r-1}}{u}\Big)\mu\nabla v(t) \ \mathrm{d}x \\
&= -r\int_\Omega \frac{(r-1)(-\ln u)^{r-2}\nabla u + (-\ln u)^{r-1}\nabla u}{u^2} a(\nabla u + u(1-u)\nabla w) \ \mathrm{d}x \\
&= -r\int_\Omega a\Big[(r-1)(-\ln u)^{r-2} + (-\ln u)^{r-1}\Big]\Big[|\nabla \ln u|^2 + \nabla(\ln u)(1-u)\nabla w\Big] \ \mathrm{d}x \\
&\le -r\int_\Omega a\Big[(r-1)(-\ln u)^{r-2} + (-\ln u)^{r-1}\Big]\Big[\frac{1}{2}|\nabla \ln u|^2 - \frac{1}{2}(1-u)^2|\nabla w|^2\Big] \ \mathrm{d}x \\
&\le -r\int_\Omega a(r-1)(-\ln u)^{r-2}\frac{1}{2}|\nabla \ln u|^2 \ \mathrm{d}x + \int_\Omega \Big[r(r-1)(-\ln u)^{r-2} + r(-\ln u)^{r-1}\Big]C_1 \ \mathrm{d}x \\
&= -\frac{2a(r-1)}{r}\int_\Omega \Big|\nabla(-\ln u)^{\frac{r}{2}}\Big|^2 \ \mathrm{d}x + C_1\int_\Omega r(r-1)(-\ln u)^{r-2} + r(-\ln u)^{r-1} \ \mathrm{d}x \\
&\le -\frac{2a(r-1)}{r}\Big[\varepsilon^{-1}\int_\Omega (-\ln u(t))^r \ \mathrm{d}x - C\varepsilon^{\frac{-n-2}{2}}\Big(\int_\Omega (-\ln u(t))^{\frac{r}{2}} \ \mathrm{d}x\Big)^2\Big] \\
&\quad + C_1\int_\Omega r(r-1)(-\ln u(t))^{r-2} + r(-\ln u(t))^{r-1} \ \mathrm{d}x,
\end{aligned}$$

where we used the inequality

$$\|\xi\|_{L^2}^2 \le \varepsilon\|\nabla\xi\|_{L^2}^2 + C\varepsilon^{-n/2}\|\xi\|_{L^1}^2.$$

With the notation (3.14) we have $\mathcal{M}_s(t) \le \mathcal{M}_r(t)$ whenever $s \le r$ and $\mathcal{M}_r(t) \ge 1$. Then, taking $\varepsilon = \frac{a}{C_1 r^2}$, we arrive at

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_r(t) \le &\ -C_1 r(r-2)\mathcal{M}_r(t) + 2C_1 r C a^{-\frac{n}{2}} C_1^{\frac{n}{2}}(r-1)r^n\Big(\mathcal{M}_{\frac{r}{2}}(t)\Big)^2 \\
\le &\ -2C_1 r\mathcal{M}_r(t) + 2C_1 r A r^{n+1}\Big(\mathcal{M}_{\frac{r}{2}}(t)\Big)^2,
\end{aligned} \tag{3.15}$$

provided that $r \ge 4$ and $A = Ca^{-n/2}C_1^{n/2}$. This yields

$$\mathcal{M}_r(t) \le 2\max\{1, \ \operatorname*{ess\,sup}_{t\in(0,t_0)} A r^{n+1}\Big(\mathcal{M}_{\frac{r}{2}}(t)\Big)^2, \ \mathcal{M}_r(0)\}. \tag{3.16}$$

Consequently, choosing $r = 2^k$, we get

$$\mathcal{M}_{2^k}(t) \le A2^{k(n+2)} \cdot \Big(A2^{(k-1)(n+2)}\Big)^2 \cdots \Big(A2^{(k-(k-1))(n+2)}\Big)^{2^{k-1}} \cdot \Big(\mathcal{M}_{1,2^k}\Big)^{2^k}, \tag{3.17}$$

where

$$\mathcal{M}_{1,r} = \max\{1,\ \operatorname*{ess\,sup}_{t>0} \mathcal{M}_1(t),\ M_r(0)\}.$$

The right hand side of (3.17) becomes

$$A^{2^k-1}\Big(\mathcal{M}_{1,2^k}\Big)^{2^k} \cdot 2^{[n+2][k+2(k-1)+2^2(k-2)+\ldots+2^{k-1}(k-(k-1))]}$$

$$= A^{2^k-1}\Big(\mathcal{M}_{1,2^k}\Big)^{2^k} \cdot 2^{(n+2)(-k+2^{k+1}-2)}.$$

Taking the $1/2^k$ power of both sides of (3.17) we obtain

$$\|\ln u(t)\|_{L^r(\Omega)} \leq A\mathcal{M}_{1,r} \cdot 2^{2(n+2)}, \quad r = 2^k, \tag{3.18}$$

which implies (i).

To get estimates independent of the size of the initial value $\|\ln u(0)\|_{L^r(\Omega)}$, we proceed in a similar way as in the proof of Lemma 3.3. Dominating the equation for $\mathcal{M}_r^{\frac{1}{r}}$ by a quadratic differential equation, we get an estimate which does not depend on the size of the initial datum, but it grows as r^2. It is sufficient to show (ii) for some $t_0 \in (0,1]$, and then proceed as in the proof of (i) starting at t_0. We denote

$$M_r(t) = \mathcal{M}_r^{\frac{1}{r}}(t) = \|\ln u(t)\|_{L^r(\Omega)},$$

and estimate its time derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} M_r = \frac{1}{r}\mathcal{M}_r^{\frac{1}{r}-1} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_r.$$

We proceed in the same way as above but this time we do not neglect the term

$$-ar(-\ln u)^{r-1}\frac{1}{2}|\nabla \ln u|^2.$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t} M_r =$$

$$-\frac{2a(r-1)}{r^2}\mathcal{M}_r^{\frac{1}{r}-1} \cdot \int_\Omega \Big|\nabla(-\ln u)|^{\frac{r}{2}}\Big|^2 \,\mathrm{d}x - \frac{2a}{(r+1)^2}\mathcal{M}_r^{\frac{1}{r}-1} \cdot \int_\Omega \Big|\nabla(-\ln u)^{\frac{r+1}{2}}\Big|^2 \,\mathrm{d}x$$

$$+\mathcal{M}_r^{\frac{1}{r}-1} \cdot C_1\Big[(r-1)M_{r-2} + M_{r-1}\Big]$$

Now, we apply Lemma 3.1 with

$$z = |\ln u|^{\frac{r}{2}}, \quad z = |\ln u|^{\frac{r+1}{2}},$$

respectively. Taking (3.4) and (3.6) into account, we get

$$\int_\Omega \Big|\nabla(-\ln u)|^{\frac{r}{2}}\Big|^2 \,\mathrm{d}x \geq \frac{u_\alpha^2}{8C_2^2}\mathcal{M}_r - \frac{u_\alpha}{C^2}|\ln\frac{u_\alpha}{2}|^r,$$

$$\int_\Omega \left|\nabla(-\ln u)|^{\frac{r+1}{2}}\right|^2\,\mathrm{d}x \geq \frac{u_\alpha^2}{8C_2^2}\mathcal{M}_{r+1} - \frac{u_\alpha}{C^2}|\ln\frac{u_\alpha}{2}|^{r+1}.$$

If

$$\frac{1}{2}\frac{u_\alpha^2}{8C_2^2}\mathcal{M}_r \leq \frac{u_\alpha}{C^2}|\ln\frac{u_\alpha}{2}|^r,\quad \frac{1}{2}\frac{u_\alpha^2}{8C_2^2}\mathcal{M}_{r+1} \leq \frac{u_\alpha}{C^2}|\ln\frac{u_\alpha}{2}|^{r+1},$$

at some point $t_0 \in (0,1)$ then we can start at this point and proceed as in the proof of (i) to show that $M_r(t)$, $M_{r+1}(t)$ are bounded for all $t \geq t_0$. If it is not the case, we arrive at the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}M_r \leq -\frac{au_\alpha^2}{4C_2^2}\frac{r-1}{r^2}M_r - \frac{au_\alpha^2}{16C_2^2}\frac{1}{(r+1)^2}M_r^2 + C_1((r-1)M_r^{-1}+1). \tag{3.19}$$

Again, we are done if we can find a constant $C_3 > 0$ such that $M_r(t_1) \leq C_3 r$ for some $t_1 \in (0,1)$. Otherwise we have

$$\frac{au_\alpha^2}{4C_2^2}\frac{r-1}{r^2}M_r \geq C_1((r-1)M_r^{-1}+1)$$

for $t \in (0,1)$, which implies that M_r satisfies a quadratic differential inequality, and we deduce that

$$M_r(1) \leq C_4 r^2,\quad C_4 = \frac{32C_2^2}{au_\alpha^2}. \tag{3.20}$$

Hence (ii) follows.

To prove (iii), we use (ii), (2.8), and the interpolation inequality. There is a sequence of times $\{t_n)\} \to \infty$ such that

$$u(t_n) \to u^* \text{ strongly in } L^2(\Omega),$$

and $\|\ln u(t_n)\|_{L^2(\Omega)} \leq 4B_1$. Hence, we get

$$\ln(u_{t_n}) \to \ln(u^*) \text{ strongly in } L^2(\Omega),$$

where, due to (2.11),

$$\max\{\|u^*\|_{L^\infty(\Omega)}, \|1-u^*\|_{L^\infty(\Omega)}\} = m < 1$$

and, subsequently,

$$\max\{\|\ln u^*\|_{L^\infty(\Omega)}, \|\ln(1-u^*)\|_{L^\infty(\Omega)} \leq C_5 = -\ln m.$$

Now, we find a sequence $\{\varepsilon_r\}$ such that

$$\varepsilon_r \leq \Big(\frac{1}{4B_2 r^2 + C_5}\Big)^{r-1},$$

and a corresponding sequence of times $\{t_r\}$ such that

$$\|\ln u(t_r) - \ln u^*\|_{L^2(\Omega)} \leq \varepsilon_r.$$

It follows that

$$\|M_r(t_r)\|_{L^r(\Omega)} \le \| \ln u(t_r) - \ln u^* \|_{L^2(\Omega)}^{\frac{1}{r-1}} \cdot \| \ln u(t_r) - \ln u^* \|_{L^{2r}(\Omega)}^{\frac{r-1}{r-2}} + C_5$$

$$\le \varepsilon_r^{\frac{1}{r-1}} (4B_2 r^2 + C_5) + C_5.$$

Again, starting at t_r, we repeat the proof of (i) to get (iii).

Remark 1. This procedure applied to $\| \ln(1-u)\|_r^r$ yields the same estimates also in this case. With Lemma 3.3 at hand, we can also deduce the convergence of a sequence $v(t_n)$ to v^* in $L^2(\Omega)$, in addition to (2.10).

Remark 2. Assuming that

$$f'(u_0) \in L^\infty(\Omega), \tag{3.21}$$

we can take the limit as $k \to \infty$ of both sides of (3.18) to infer that there is a constant B (which does not depend on time) such that

$$\|v(t)\|_{L^\infty(\Omega)} \le B \quad \text{for all} \quad t \ge 0, \tag{3.22}$$

which extends the assertion of Theorem 3.5 in [5]. We also have the L^∞-estimate for u, namely, there exists a constant $0 < k < 1$ depending only on u_α such that

$$k \le u(t,x) \le 1 - k \quad \text{for a.a. } x \in \Omega,\ t \ge 1. \tag{3.23}$$

4 Łojasiewicz-Simon Theorem

In this section, we state the generalized version of the Łojasiewicz-Simon Theorem proved in [4].

Let V and W be Banach spaces densely and continuously embedded into the Hilbert space H and its dual H^*, respectively. Assume that the restriction of the duality map $J \in L(H, H^*)$ to V is an isomorphism from V onto $W = J(V)$. Moreover, let $H = H_0 + H_1$ where $H_1 \subset V$ is a finite-dimensional subspace and H_0 is its orthogonal complement in H. Denote by H_0^0 the anihilator of H_0:

$$H_0^0 = \{g \in H^*; \langle g, z\rangle = 0 \text{ for all } z \in H_0\}.$$

Let

$$F = \Phi + \Psi, \tag{4.1}$$

with $\Phi,\ \Psi$ satisfying the following conditions:

Φ is a Fréchet differentiable functional from an open set $U \subset V \to R$. Moreover, assume that the Fréchet derivative $D\Phi : U \to W$ is a real analytic operator which satisfies

$$\langle D\Phi(u) - D\Phi(v), u - v\rangle \ge \alpha \|u - v\|_H^2, \quad \|D\Phi(u) - D\Phi(v)\|_{H^*} \le \gamma \|u - v\|_H,$$

for all $u, v \in U$ and some constants $\alpha, \gamma > 0$. In addition, the second Fréchet derivative $D^2\Phi(u) \in L(V, W)$ is assumed to be an isomorphism for all $u \in U$.

$$\Psi(u) = \frac{1}{2}\langle Tu, u\rangle + \langle l, u\rangle + d, u \in H,$$

where $T \in L(H, H^*)$ be a self-adjoint and completely continuous operator such that its restriction to V is a completely continuous operator in $L(V, W)$. $l \in W$ and $d \in \mathbb{R}$ are fixed.

Theorem 4.1 *Let F be given by (4.1) and the above assumptions be satisfied. Let $(u^*, v^*) \in U \times H_0^0$ satisfy $DF(u^*) = v^*$. Then we can find constants δ, $\lambda > 0$, and $\theta \in (0, \frac{1}{2}]$ such that for all $u \in U$ which satisfy $u - u^* \in H_0$ and $\|u - u^*\|_H \leq \delta$ we have the following inequality:*

$$|F(u) - F(u^*)|^{1-\theta} \leq \lambda \inf\{\|DF(u) - f\|_{H^*};\ f \in H_0^0\}. \tag{4.2}$$

5 Convergence

In this section, we prove that there is $T > 0$ such that $u_t \in L^1(T, \infty; (H^1)^*)$, which enables us to show convergence of the whole trajectory of u to u^*, a stationary solution given by (2.11). We will apply Theorem 4.1 to the energy functional associated with our system, i.e.,

$$F(u) = \int_\Omega f(u) + u\mathcal{J}(u) + u \cdot K * 1 \ \mathrm{d}x, \tag{5.1}$$

the corresponding spaces beeing

$$H = H^* = L^2(\Omega),\ H^0 = \{u \in H, \int_\Omega u \ \mathrm{d}x = 0\},\ H_0^0 = \{v = const\},\ V = L^\infty(\Omega),$$

$$\Phi(u) = \int_\Omega f(u) \ \mathrm{d}x,\ T(u) = -2\mathcal{J}(u),\ l = K * 1,\ d = 0.$$

Multiplying (1.1) by v and (1.2) by u_t, integrating over Ω and subtracting, we obtain the energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}F(u(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f(u(t)) - u(t)J(u(t)) + u(t)l\mathrm{d}x = -\int_\Omega \mu|\nabla v|^2\mathrm{d}x \tag{5.2}$$

As $u(t)$ stays bounded away from zero and one, the functional F is bounded from below and the hypotheses in Theorem 4.1 are fulfilled.

The limit energy

$$F_\infty = \lim_{t\to\infty} F(u(t)) = F(u^*)$$

is the same for any u^* in the $\omega-$limit set of u.

The Fréchet derivative of $F(u(t))$ is represented by

$$F'(u(t)) = f'(u(t)) - 2J(u(t)) + l = v(t).$$

Now, let (u^*, v^*, w^*) belong to the ω-limit set and satisfy (2.11). (Existence of such solutions was proved in [5]). Then

$$F'(u^*) = v^*,$$

and integrating (5.2) from t to ∞, we get

$$\int_t^\infty \int_\Omega \mu |\nabla v|^2 \, \mathrm{d}x\mathrm{d}t = F(u(t)) - F_\infty = F(u(t)) - F(u^*). \tag{5.3}$$

By virtue of Theorem 4.1, we have

$$|F(u(t)) - F(u^*)|^{1-\theta} \leq \lambda \inf\{\|v(t) - z\|_{L^2(\Omega)};\ z = const\} = \lambda \|v(t) - \overline{v(t)}\|_{L^2(\Omega)}$$

provided that

$$\|u(t) - u^*\|_{L^2(\Omega)} \leq \delta. \tag{5.4}$$

This, combined with (5.2) and taking into account (2.12), (3.21), yields

$$\frac{4}{a}\int_t^\infty \int_\Omega (\mu|\nabla v|)^2 \mathrm{d}x\mathrm{d}s \leq \int_t^\infty \int_\Omega \mu|\nabla v|^2 \, \mathrm{d}x\mathrm{d}s \leq \lambda \|v(t) - \overline{v}(t)\|_{L^2(\Omega)}^{\frac{1}{1-\theta}} \tag{5.5}$$

$$\leq \lambda \Big(ak^2\Big)^{\frac{1}{\theta - 1}} \|\mu|\nabla v|(t)\|_{L^2(\Omega)}^{\frac{1}{1-\theta}},$$

where k is the bound from (3.23).

At this point, we employ the following lemma, the proof of which can be found in [3].

Lemma 5.1 *Let $Z \geq 0$ be a measurable function on $(0, \infty)$ such that*

$$Z \in L^2(0, \infty),\ \|Z\|_{L^2(0,\infty)} \leq Y$$

and there exist $\alpha \in (1, 2)$, $\xi > 0$ and an open set $\mathcal{M} \subset (0, \infty)$ such that

$$(\int_t^\infty Z^2(s)\ ds)^\alpha \leq \xi\ Z^2(t) \text{ for a.a. } t \in \mathcal{M}.$$

Then $Z \in L^1(\mathcal{M})$ and there exists a constant $c = c(\xi, \alpha, Y)$ independent of $\mathcal{M}$ such that

$$\int_\mathcal{M} Z(s)\ ds \leq c.$$

Setting $Z(t) = \|\mu|\nabla v|(t)\|_{L^2(\Omega)}$ in Lemma 5.1, we get

$$\int_\mathcal{M} \|\mu \nabla v(s)\|_{L^2(\Omega)} \mathrm{d}s < \infty, \tag{5.6}$$

where

$$\mathcal{M} = \cup_J \{J \mid J \text{ is an open interval where (5.4) holds}\}.$$

Since $u^* \in \omega[u]$, $\mathcal{M}$ is non-empty, and we get

$$\int_{\mathcal{M}} \|\partial_t u(t)\|_{(H^1)^*(\Omega)}\, \mathrm{d}t < \infty. \tag{5.7}$$

Our next goal is to show that there exists τ such that $(\tau, +\infty) \subset \mathcal{M}$. To begin with, realize that from the energy inequality (5.2) we deuce that

$$u_t \in L^2(0, +\infty; H^1(\Omega)^*)$$

$$|\nabla v| \in L^2(0, +\infty; L^2(\Omega)).$$

Denote

$$N = \|u\|_{L^\infty(0,+\infty;L^2(\Omega))} + \|\nabla w\|_{L^\infty(0,+\infty;L^2(\Omega))}. \tag{5.8}$$

To any $\delta > 0$ we find $T(\delta) > 0$ such that

$$\|u_t\|_{L^1(\mathcal{M}\cap(T(\delta),+\infty;H^1(\Omega)^*)} < \delta \tag{5.9}$$

$$\|u_t\|_{L^2((T(\delta),+\infty;H^1(\Omega)^*)} < \delta \tag{5.10}$$

$$\|\nabla v\|_{L^2((T(\delta),+\infty;L^2(\Omega))} < \delta \tag{5.11}$$

Next, let $(t_1, t_2) \subset \mathcal{M}$, $t_i \geq T(\delta)$ for some $\delta < 1$. In view of (5.11), (5.8) we find $t_3 \in [t_1, t_1 + 1]$ such that $\|u(t_3)\|_{H^1(\Omega)} \leq N + 1$. Then

$$\|u(t_1) - u(t_2)\|^2_{L^2(\Omega)} \leq 2\Big[\|u(t_1) - u(t_3)\|^2_{L^2(\Omega)} + \|u(t_3) - u(t_2)\|^2_{L^2(\Omega)}\Big]$$

and we have

$$\frac{1}{2}\|u(t_1) - u(t_3)\|^2_{L^2(\Omega)} = \int_{t_1}^{t_3} \langle u_t(s), u(t_3) - u(s)\rangle$$

$$\leq \int_{t_1}^{t_3} \|u_t(s)\|_{H^1(\Omega)^*}\Big[\|u(t_3)\|_{H^1(\Omega)} + \|u(s)\|_{L^2(\Omega)} + \|\nabla w(s)\|_{L^2(\Omega)} + \|\nabla v(s)\|_{L^2(\Omega)}\Big]$$

$$\leq \|u_t\|_{L^1((t_1,t_1+1);H^1(\Omega)^*)}\Big[N + 1 + \|u\|_{L^\infty(0,+\infty;L^2(\Omega))} + \|\nabla w\|_{L^\infty(0,+\infty;L^2(\Omega))}\Big]$$

$$+\|u_t\|_{L^2(T(\delta),+\infty;H^1(\Omega)^*)}\|\nabla v\|_{L^2(T(\delta),+\infty;L^2(\Omega))} \leq \delta(2N + 1 + \delta).$$

The same estimate holds for $\|u(t_3) - u(t_2)\|_{L^2(\Omega)}$ provided that $t_3 \geq t_2$, and also for $t_3 < t_2$, where we use (5.9). Summing up, we have

$$\|u(t_1) - u(t_2)\|^2_{L^2(\Omega)} \leq 8\delta(2N + 1 + \delta) \tag{5.12}$$

and we can find δ and the corresponding $T(\delta) = \tau$ such that

$$\left.\begin{array}{c} \|u(t_1) - u(t_2)\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \\ \text{whenever} \\ \|u(t) - u^*\|_{L^2(\Omega)} < \varepsilon \text{ for all } t \in (t_1, t_2) \text{ where } \tau \leq t_1 < t_2. \end{array}\right\} \tag{5.13}$$

Since $u^* \in \omega[u]$, a large τ can be chosen so that

$$\|u(\tau) - u^*\|_{L^2(\Omega)} < \frac{\varepsilon}{3}, \tag{5.14}$$

and then (5.13) yields $[\tau, \infty) \subset M$. Indeed taking

$$\bar{t} = \inf\{t > \tau \mid \|u(t) - u^*\|_{L^2(\Omega)} \geq \varepsilon\},$$

we have $\bar{t} > \tau$ and

$$\|u(\bar{t}) - u^*\|_{L^2(\Omega)} \geq \varepsilon \text{ if } \bar{t} \text{ is finite.} \tag{5.15}$$

On the other hand, by virtue of (5.13), (5.14),

$$\|u(t) - u^*\|_{L^2(\Omega)} \leq \|u(t) - u(\tau)\|_{L^2(\Omega)} + \|u(\tau) - u^*\|_{L^2(\Omega)} < \frac{2}{3}\varepsilon \text{ for all } \tau \leq t < \bar{t}$$

which, together with (5.15), yields $\bar{t} = \infty$.

We have proved the following result.

Theorem 5.1 *Let (u, v, w) be a solution of the problem (1.1)-(1.4) with the data given by (2.4),(2.6),(2.7),(2.12), and let (3.21 hold. Then there is (u^*, v^*, w^*) satisfying (2.11) such that,*

$$u(t) \to u^* \ \ \textit{strongly in} \ \ L^2(\Omega),$$
$$v(t) \to v^* \ \ \textit{strongly in} \ \ L^2(\Omega),$$
$$w(t) \to w^* \ \ \textit{strongly in} \ \ H^1(\Omega),$$

as time goes to infinity.

Remark 3. It is still an open question whether any solution with the initial datum u_0 satisfying (1.4) stabilizes to a single stationary state as time tends to infinity even in the case that there is a continuum of equilibria.

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