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L^q-approach of weak solutions to Stationary Rotating Oseen Equations in exterior domains

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Abstract

We establish the existence and uniqueness of weak solution of the three-dimensional nonhomogeneous stationary Oseen flow around a rotating body in an exterior domain. This article is extension of the previous results on the whole space. For the present extension to the case of exterior domains, we will use the localization procedure (see e.g. KoSo). In this way we combine the previous results (see KNP and FHM) about existence, uniqueness and L^q-boundedness of the solutions for two associated problems, in the whole space and in an appropriate bounded domain.

1 Introduction

The study of Navier - Stokes fluid flows past a rigid body translating with a constant velocity (or past a rotating obstacle with a prescribed constant velocity) is one of the most fundamental questions in theoretical and applied

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Fluid Dynamics. A systematic and rigorous mathematical study was initiated by the fundamental pioneer works of Oseen (1927), Leray (1933, 1934) and then developed by the several others mathematicians with significant contributions.

In the last decade a lot of efforts have been made on the analysis of solutions to different problems, stationary as well nonstationary, linear models as well nonlinear one, in the whole space as well in exterior domains. We refer to [6, 7, 8, 9, 10, 11, 13, 14, 15, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30].

In the present paper we mainly investigate the existence and uniqueness of weak solution to the linear stationary rotating Oseen system in exterior domains in the case of non-integrable right-hand side. Our approach is based on the localization method which decomposed our problem to problems in the whole domain and in the appropriate bounded domain. We consider that our right-hand side is a solution of Bogovskii equation in the corresponding negative homogeneous Sobolev spaces.

Let D be an exterior domain in $\mathbb{R}^3$. We consider the motion of a viscous fluid filling the domain D when the "obstacle" $\Omega = \mathbb{R}^3 \backslash D$, which consists of a finite number of rigid bodies, is rotating about an axis with constant angular velocity ω and moving in the direction of this axis. We assume the fluid with a non-zero velocity $v_\infty = ke_3$ at infinity, and that $\omega = |\omega| e_3 = (0, 0, |\omega|)^T$; we also assume enough regularity for the boundary $\partial\Omega$.

Our aim is to solve the time-periodic Oseen system of equations for the velocity field $v = v(y, t)$ and the associated pressure $q = q(y, t)$ in the time dependent exterior domain

$$D(t) = \{y \in \mathbb{R}^3 : y = O(|\omega|t)x, x \in D\},$$

where

$$O_\omega(t) = \begin{pmatrix} \cos|\omega|\,t & -\sin|\omega|\,t & 0 \\ \sin|\omega|\,t & \cos|\omega|\,t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.1}$$

The complemented boundary conditions on $\partial D(t)$ are taken in the form of a non-slip condition so that the fluid velocity attains $v(y, t) = \omega \wedge y, y \in \partial D(t)$, for all t. The coefficient of viscosity $\nu > 0$ and non necessary integrable external forces $\tilde{f} = \tilde{f}(y, t)$ are given.

Introducing the change of variables

$$x = O_\omega(t)^T y \tag{1.2}$$

and the new functions

$$u(x,t) = O_\omega^T(t)(v(y,t) - v_\infty), \quad p(x,t) = q(y,t), \tag{1.3}$$

as well as the force term $f_1(x,t) = O_\omega(t)^T \tilde{f}(y,t)$, we arrive at the linear system of equations in $D \times (0,\infty)$

$$\begin{aligned} \partial_t u - \nu\Delta u + k\partial_3 u - ((\omega \wedge x)\cdot\nabla)u + \omega \wedge u + \nabla p = f_1 \\ \operatorname{div} u = g_1 \end{aligned} \tag{1.4}$$

but we are interested in the stationary flow in D (so time-periodic solution to (1.4) as well as saying periodic solution to the initial model for $\{v,q\}$).

In the case $k = 0,\ \omega = 0$, this system is a nonhomogeneous Stokes system and if $\omega = 0$ it is a classical nonhomogeneous Oseen system. For simplicity we will consider $\nu = 1$, $k = 1$. We assume that for given f and g our system of equations is the following

$$\left.\begin{aligned} -\Delta u + \partial_3 u - ((\omega \wedge x)\cdot\nabla)u + \omega \wedge u + \nabla p = & \ f \\ \operatorname{div} u = & \ g \end{aligned}\right\} \text{ in D} \tag{1.5}$$

Let's us mention through the relation $\operatorname{div}(((\omega\wedge x)\cdot\nabla)u - \omega\wedge u) = ((\omega\wedge x)\cdot\nabla)\operatorname{div} u = \operatorname{div}((\omega\wedge x)\cdot\operatorname{div} u)$ we define p.

We recall that $D = \mathbb{R}^3 \setminus \Omega$, and that the system of equations (1.5) is now complemented by both homogeneous condition at infinity

$$u \to 0 \text{ as } |x| \to \infty \tag{1.6}$$

and Dirichlet boundary conditions on $\partial\Omega$, either

$$u|_{\partial\Omega} = 0 \tag{1.7}$$

or

$$u|_{\partial\Omega} = \omega \wedge x - e_3,\ \ x \in \partial\Omega. \tag{1.8}$$

If $D = \mathbb{R}^3$, of course $\{u,p\}$ is described by equations (1.5) and condition (1.6) only. The strong solution of the corresponding Cauchy problem (1.5)

(1.6) has been analyzed in L^q-spaces, $1 < q < \infty$, in [8] proving the *a priori* estimates

$$\|\nabla^2 u\|_q + \|\nabla p\|_q \leq c(\|f\|_q + \|\nabla g + (\omega \wedge x) \cdot g - g e_3\|_q), \tag{1.9}$$

$$\|\partial_3 u\|_q + \| - ((\omega \wedge x) \cdot \nabla) u + \omega \wedge u\|_q \leq c(1 + \frac{1}{|\omega|^2})\|f\|_q \tag{1.10}$$

with the constant $c > 0$ independent of $|\omega|$, the second estimate being written with $g = 0$ just to simplify. Further these results were improved in [6] in weighted spaces, obtaining the following *a priori* estimates (always written with $g = 0$ to simplify)

$$\|\nabla^2 u\|_{q,w} + \|\nabla p\|_{q,w} \leq c\|f\|_{q,w}, \tag{1.11}$$

$$\|\partial_3 u\|_{q,w} + \| - ((\omega \wedge x) \cdot \nabla) u + \omega \wedge u\|_{q,w} \leq c(1 + \frac{1}{|\omega|^{5/2}})\|f\|_{q,w} \tag{1.12}$$

where the weights w belong to the more general Muckenhoupt class $\tilde{A}_q^-$, and with the constant $c > 0$ independent of $|\omega|$. A weak solution to the same Cauchy problem (1.5) (1.6) in L^q setting, $1 < q < \infty$, was investigated in [21] and the following *a priori* estimates was proved (always written with $g = 0$)

$$\|\nabla u\|_q + \|p\|_q + \| - ((\omega \wedge x) \cdot \nabla) u + \omega \wedge u\|_{-1,q} \ \leq \ C\|f\|_{-1,q}, \tag{1.13}$$

where data belong to the dual of nonhomogeneous Sobolev spaces, which will be introduced in this section.

In the work of Galdi [14] the pointwise estimates for Navier-Stokes equations with rotating terms were proved. He obtained that

$$|u_s(x)| \leq \frac{c}{|x|}, \|\nabla u_s(x)\| + \|P_s(x)\| \leq \frac{c}{|x|^2}.$$

Another outlook on the pointwise estimates above in a differential framework by use of functional spaces has been recently proved by Farwig, Hishida [11]. Further Galdi and Silvestre [16] have proved a stability of solution u_s . Generalization in $L_{3,\infty}$ setting was done by Hishida and Shibata [27].

We will study the boundary value problem (1.5) (1.6) (1.7) and applying the so-called localization technique [19], we immediately observe that it combines both systems in the whole space and in a bounded domain : Indeed, choose $\rho > \rho_0 > 0$ so large that $\Omega \subset B_{\rho_0} = \{x \in \mathbb{R}^3 : |x| < \rho_0\}$ and take a cut-off function $\psi \in C_0^\infty(B_\rho; [0,1])$ such that $\psi = 1$ on B_{ρ_0} and $\mathrm{supp}(\nabla\psi) \subset \{x : \rho_0 < |x| < \rho\}$; introducing now $U = (1-\psi)u, V = \psi u, \pi = (1-\psi)p$ and $\tau = \psi p$, we get

$$u = U + V \tag{1.14}$$

$$p = \sigma + \tau \tag{1.15}$$

with in the whole space

$$\left.\begin{aligned} -\Delta U + \partial_3 U - ((\omega \wedge x)\cdot\nabla)U + \omega \wedge U + \nabla\sigma &= F_1(u,p) \\ \operatorname{div} U &= G_1(u) \\ U \to 0 \text{ as } |x| &\to \infty, \end{aligned}\right\} \tag{1.16}$$

where $G_1(u) = -\nabla\psi \cdot u + (1-\psi)g$,

and in the bounded domain $D_\rho = D \cap B_\rho$

$$\left.\begin{aligned} -\Delta V + \partial_3 V + \nabla\tau &= F_2(u,p) \\ \operatorname{div} V = G_2(u) &= \nabla\psi \cdot u + \psi g \\ V|_{\partial D_\rho} &= 0, \end{aligned}\right\} \tag{1.17}$$

where

$$\left.\begin{aligned} F_1(u,p) =& (1-\psi)f + 2(\nabla\psi\cdot\nabla)u + [\Delta\psi + ((\omega\wedge x)\nabla)\psi]u \\ &- (\nabla\psi)p + (\partial_3\psi)u, \\ F_2(u,p) =& \psi f + \psi[((\omega\wedge x)\cdot\nabla)u - \omega\wedge u] - 2(\nabla\psi\cdot\nabla)u \\ &- (\Delta\psi)u + (\nabla\psi)p + (\partial_3\psi)u. \end{aligned}\right\} \tag{1.18}$$

Let us observe that, in the bounded domain D_ρ, we can equivalently write the following nonhomogeneous Stokes problem

$$\left.\begin{aligned} -\Delta V + \nabla\tau &= F_2 \\ \operatorname{div} V = G_2(u) &= \nabla\psi \cdot u + \psi g \\ V|_{\partial D_\rho} &= 0, \end{aligned}\right\} \tag{1.19}$$

modifying $F_2 = \psi f + \psi[((\omega\wedge x)\cdot\nabla)u - \omega\wedge u] - 2(\nabla\psi\cdot\nabla)u - (\Delta\psi)u$
$+ (\nabla\psi)p + \partial_3(\psi u)$.

In section 2, we will give the definition of a weak solution to problem (1.5) (1.6) (1.7) and our main result, existence and uniqueness of its solution. In section 3 we recall intermediate known results for both problems

(1.16) and (1.17). In an appendix, we also recall the general results by Bogovski, Farwig and Sohr [1, 2, 3], Kozono and Sohr [19] and generalization in negative Sobolev spaces by Geissert, Heck, Hieber [18] we used to solve in different domains the equations in the form $\operatorname{div} u = g$ with Dirichlet boundary conditions. Sections 4 and 5 are devoted to the proof of the main result.

Let us fix the notations.
$C_0^\infty(\mathbb{R}^3)$ consists of functions of the class C^∞ with compact supports contained in $\mathbb{R}^3$. By $L^q(\mathbb{R}^3)$ we denote the usual Lebesgue spaces with norm $\|\cdot\|_q$. We define the homogeneous Sobolev spaces

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla\cdot\|_q} = \{v \in L^q_{\text{loc}}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\}/\mathbb{R}, \tag{1.20}$$

$$\widehat{W}^{1,q}(D) = \overline{C_0^\infty(D)}^{\|\nabla\cdot\|_{q,D}}$$

$$= \{v \in L^q_{\text{loc}}(\overline{D});\ \nabla v \in L^q(D)^3, v|_{\partial D} = 0\}\ \text{ for } 3 \le q < \infty, \tag{1.21}$$

$$= \{v \in L^{3q/3-q}(\overline{D});\ \nabla v \in L^q(D)^3, v|_{\partial D} = 0\}\ \text{for } 1 \le q < 3. \tag{1.22}$$

and their dual space

$$\widehat{W}^{-1,q}(\mathbb{R}^3) = (\widehat{W}^{1,q/(q-1)}(\mathbb{R}^3))',$$
$$\widehat{W}^{-1,q}(D) = (\widehat{W}^{1,q/(q-1)}(D))'.$$

$\langle .,.\rangle$ denotes either different duality pairings or the inner product in L^2.

Lemma 1.1.

- *For $1 < r < n$ we have $\widehat{W}^{1,r}(R^3) = \{u \in L^s(R^3) : \nabla u \in L^r(R^3)\}$, where $s = \frac{3r}{3-r}$.*
- *Let $r \ge n$. Suppose $u_k \in C_0^\infty(R^3), k = 1, 2, ..$ is a Cauchy sequence in $\widehat{W}^{1,r}(R^3)$. Then there is a Cauchy sequence $w_k \in C_0^\infty$ with $\nabla u \in L^r(R^3)$ satisfying*

$$\|\nabla u_k - \nabla w_k\|_{L^r(R^3)} \to 0,$$

$$w_k \to u \text{ in } L^r_{loc}(R^3), \tag{1.23}$$

$$\nabla w_k \to \nabla u \text{ in } L^r(R^3) \text{ as } k \to \infty.$$

Such u is unique up to additive constants. In this case, we have the inclusion $\widehat{W}_0^{1,r}(R^3) \subset \{[u] \in L^r_{loc}(R^3)/R^1 : \nabla u \in L^r(R^3)\}$ *where* $[u] = \{w \in L^r_{loc}(R^3) : w - u \in R^1\}$.

Remark 1.1 Another possibility of the definition of the homogeneous Sobolev spaces we can found in the work of Galdi [12]. He defines the homogeneous Sobolev spaces by the following way

$$\widehat{W}^{1,q}(\mathbb{R}^3) = \overline{C_0^\infty(\mathbb{R}^3)}^{\|\nabla \cdot\|_q}$$

and from Theorem II.6.3, and Remark II.6.2 [12] he gives the following characterization of the spaces

$$\begin{aligned} \widehat{W}^{1,q}(\mathbb{R}^3) &= \{v \in L^1_{\rm loc}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3\},\ q \geq 3, \\ &= \{v \in L^1_{\rm loc}(\mathbb{R}^3);\ \nabla v \in L^q(\mathbb{R}^3)^3), v \in L^{\frac{3q}{3-q}}(\mathbb{R}^3)\},\ q < 3. \end{aligned} \tag{1.24}$$

Remark 1.2 We would like to mention that definition (1.20) and (1.24) are equivalent in the following sense. In definition (1.20) the elements of space are classes of functions since we factorized the homogeneous spaces $\widehat{W}^{1,r}$ by constant. In definition (1.24) we divide to two cases : first one - the case $1 < r < n$ where Sobolev imbedding is valid and the case of $r \geq n$ where we have that limit of Cauchy sequences are unique up to constant see previous Lemma 1.1.

2 The main result

We consider the problem (1.5) (1.6) (1.7). Let $1 < q < \infty$, $f \in (\widehat{W}^{-1,q}(D))^3$, $g \in L^q(D)$ such that ∇g and $((\omega \wedge x) \cdot \nabla) g$ belong to $(\widehat{W}^{-1,q}(D))^3$.

Definition 2.1 *We call* $\{u, p\}$ *a weak solution to* (1.5) (1.6) (1.7) *if*

$$\begin{aligned}
&(1) \qquad \{u,p\} \in (\widehat{W}_0^{1,q}(D))^3 \times L^q(D),\\
&(2) \qquad \operatorname{div} u = g \quad in \quad L^q(D),\\
&(3) \qquad ((\omega \wedge x)\cdot\nabla)u - \omega \wedge u \quad in \quad (\widehat{W}^{-1,q}(D))^3,\\
&(4) \qquad \langle \nabla u, \nabla\varphi\rangle + \langle \partial_3 u, \varphi\rangle - \langle((\omega \wedge x)\cdot\nabla)u - \omega\wedge u, \varphi\rangle = \langle p, \operatorname{div}\varphi\rangle + \langle f, \varphi\rangle\\
&\qquad \text{for all } \varphi \quad in \ (\widehat{W}_0^{1,q/(q-1)}(D))^3,\\
&\qquad \text{where } p \text{ is obtained from } \Delta p = \operatorname{div} f + \Delta g - \partial_3 g + ((\omega \wedge x)\cdot\nabla)g.
\end{aligned} \tag{2.1}$$

Remark 2.1 Since we make use of operator $\mathcal{B}$ solving the equation (2) with $u \in (\widehat{W}_0^{1,q}(D))^3$, see Appendix, we can ask for u in the form $\mathcal{B}g + u_0$ with the homogeneous divergence condition $\operatorname{div} u_0 = 0$, solution to

$$-\Delta u_0 + \partial_3 u_0 - ((\omega \wedge x)\cdot\nabla)u_0 + \omega \wedge u_0 + \nabla p_0 = f_0$$

where

$$f_0 = f + (\Delta - \partial_3 + ((\omega \wedge x)\cdot\nabla) - \omega \wedge \cdot)\mathcal{B}g,$$

with

$$u_0|_{\partial\Omega} = 0 \text{ and } u_0 \to 0 \text{ as } |x| \to \infty.$$

Therefore the study of the previous solenoidal problem is sufficient.

Our main result is

Theorem 2.1. *Let $3/2 < q < 3$, and suppose f and g given as previously. Then there exists an unique weak solution $\{u, p\}$ to* (1.5) (1.6) (1.7) *(uniqueness up to a constant multiple of ω for u), which satisfies the estimate*

$$\left.\begin{array}{l} \|\nabla u\|_{q,D} + \|p\|_{q,D} + \| - ((\omega \wedge x)\cdot\nabla)u + \omega \wedge u\|_{-1,q,D} \\ \le c_q(\|f\|_{-1,q,D} + \|g\|_{q,D} + \|(\omega \wedge x)\cdot g\|_{-1,q,D}), \end{array}\right\} \tag{2.2}$$

with some constant $c_q > 0$ independent of $|\omega|$.

Remark 2.2

- Similar results were obtained by Hishida for the stationary Stokes problem [24].
- It is possible to avoid some of the restrictions $3/2 < q < 3$ see [30]. In this way, we can consider the null space of the problem which is given by

$$K = \Big\{u \in \widehat{W}^{1,q}(D) | \operatorname{div}\, u = 0, u|_{\partial\Omega} = 0,$$
$$(u, p) \text{ is a solution of (1.5) for some } p \in L^q(D)\Big\}.$$

  Then the solution u will be unique in $W^{1,q}(D)/K$ for $3/2 < q < \infty$.

3 Intermediate known results

In accordance with Remark 2.1 and with the localization procedure, we can start with the analysis of the homogeneous problem in the whole space, then precisely the problem (1.5) in the divergence free case. So we use the notations $\{U_0, \pi_0\}$, and preliminary results from [21, 23] are :

Definition 3.1 *Let $1 < q < \infty$. Given $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, we call $\{U_0, \pi_0\} \in \widehat{W}^{1,q}_{0,\sigma}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ a weak solution to (1.16) with $G_1 = 0$ if*

$$
\begin{array}{ll}
(1) & \operatorname{div} U_0 = 0 \quad in \quad L^q(\mathbb{R}^3), \\
(2) & ((\omega \wedge x)\cdot\nabla)U_0 - \omega\wedge U_0 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3, \\
(3) & \langle \nabla U_0, \nabla\varphi\rangle - \langle((\omega\wedge x)\cdot\nabla)U_0 - \omega\wedge U_0, \varphi\rangle \\
 & \quad +\langle \partial_3 U_0, \varphi\rangle - \langle \pi_0, \operatorname{div}\varphi\rangle = \langle F_1, \varphi\rangle \quad for\ all \quad \varphi \in C_0^\infty(\mathbb{R}^3)^3.
\end{array}
$$

In fact, as usual, the integral equation (3) in Definition 3.1 holds by continuity for all $\varphi \in \widehat{W}^{1,q/(q-1)}_0(\mathbb{R}^3)^3$.

Theorem 3.1. *([21]) Let $1 < q < \infty$ and let $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ be given, then the problem (1.16) with $G_1 = 0$ possesses a weak solution $\{U_0, \pi_0\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ which satisfies*

$$
\|\nabla U_0\|_q + \|\pi_0\|_q + \| - ((\omega\wedge x)\cdot\nabla)U_0 + \omega\wedge U_0\|_{-1,q} \;\leq\; c\|F_1\|_{-1,q}, \quad (3.1)
$$

with some $c > 0$ depending on q.

Theorem 3.2. *([21]) The solution $\{U_0, \pi_0\}$ given by Theorem 3.1 is unique.*

Corollary 3.3. *([21]) Let $1 < q < 4$, $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ and let $U_0 \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be the unique weak solution to problem (1.16) with $G_1 = 0$. Then there exists $\alpha \in \mathbb{R}$ such that*

$$
U_0 - \alpha e_3 \in L^s(R^3)^3 \textit{ for all } s > 1, \frac{1}{s} \in \frac{1}{q} - \left[\frac{1}{4}, \frac{1}{3}\right].
$$

Moreover

$$
\|U_0 - \alpha e_3\|_s \leq c\|F_1\|_{-1,q}
$$

with a constant $c = c(s, |\omega|) > 0$.

Remark 3.4([21]) Let $1 < q < 3$, $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$, and let $U_0 \in \widehat{W}^{1,q}(\mathbb{R}^3)^3$ be the unique weak solution to problem (1.16) with $G_1 = 0$. Then we also have the weighted estimate

$$\|\frac{1}{|x|}U_0\|_q \leq c\|F_1\|_{-1,q}$$

with $c = c(q, \omega) > 0$. R

Concerning now the nonhomogeneous problem in the whole space, we recall the following result

Theorem 3.4. *([23]) Let $1 < q < \infty$ and let $F_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)^3$ be given. Suppose $G_1 \in L^q(\mathbb{R}^3)$ such that $(\omega \wedge x) \cdot G_1 \in \widehat{W}^{-1,q}(\mathbb{R}^3)$. Then the problem* (1.16) *possesses a weak solution $\{U, \pi\} \in \widehat{W}^{1,q}(\mathbb{R}^3)^3 \times L^q(\mathbb{R}^3)$ which satisfies*

$$\begin{aligned} &\|\nabla U\|_q + \|\pi\|_q + \| - ((\omega \wedge x) \cdot \nabla)U + \omega \wedge U\|_{-1,q} \\ &\qquad\leq\ c(\|F_1\|_{-1,q} + \|G_1\|_q + \|(\omega \wedge x) \cdot G_1\|_{-1,q}), \end{aligned} \tag{3.2}$$

with some $c > 0$ depending on q.

We now recall the well known result, e.g. from [19], about the nonhomogeneous Stokes problem in bounded domains : Indeed in the proof of our main theorem we will be interested by $\{V, \tau\}$ which solves the problem (1.19) in the domain D_ρ. So, with the notations $\{V, \tau\}$ the theorem reads

Theorem 3.5. *Let $1 < q < \infty$. Suppose that*

$$F_2 \in W^{-1,q}(\ D_\rho)^3,\ G_2 \in L^q(\ D_\rho),\ \int_{D_\rho} G_2(x)dx = 0.$$

Then the problem (1.19) *possesses an unique (up to an additive constant for τ) weak solution $\{V, \tau\} \in W_0^{1,q}(\ D_\rho)^3 \times L^q(\ D_\rho)$, which satisfies the estimate*

$$\|\nabla V\|_{q,\ D_\rho} + \|\tau - \bar{\tau}\|_{q,\ D_\rho} \leq C(\|F_2\|_{-1,q,\ D_\rho} + \|G_2\|_{q,\ D_\rho}), \tag{3.3}$$

where $\bar{\tau} = \frac{1}{|\ D_\rho|}\int_{D_\rho} \tau(x)dx$.

4 Uniqueness in the main theorem : Proof

We shall state the conditional uniqueness result precisely : We assume the existence of $\{u_g, p_g\}$ a weak solution to problem (1.5) (1.6) (1.7) in the sense of definition 2.1. For any $f \in (\widehat{W}^{-1,q}(D))^3$ and for an appropriate choice of $g \in L^q(D)$, we have

$$\left.\begin{array}{r} -\Delta u_g + \partial_3 u_g - ((\omega \wedge x)\cdot\nabla)u_g + \omega \wedge u_g = f - \nabla p_g \\ \operatorname{div} u_g = g \end{array}\right\} \tag{4.1}$$

with $u_g \in (\widehat{W}_0^{1,q}(D))^3$, and $p_g \in L^q(D)$.

In particular we know $\{u_0, p_0\}$ for $g = 0$ (anyway, in the spirit of Remark 2.1, it will be the natural first part of the existence proof, see Section 5). Therefore the adjoint model also will admit a weak solution, say $\{u_0^*, p_0^*\}$, so given any $h \in (\widehat{W}^{-1,q}(D))^3$ we have

$$\left.\begin{array}{r} -\Delta u_0^* - \partial_3 u_0^* + ((\omega \wedge x)\cdot\nabla)u_0^* - \omega \wedge u_0^* = h - \nabla p_0^* \\ \operatorname{div} u_0^* = 0 \end{array}\right\} \tag{4.2}$$

with $u_0^* \in (\widehat{W}_{0,\sigma}^{1,2}(D))^3 \bigcap (\widehat{W}^{1,r}(D))^3$, $\frac{3}{2} < r < 6$, and $p_0^* \in L^q(D)$.

Let $\{u_g^1, p_g^1\}$, $\{u_g^2, p_g^2\}$ be two weak solutions to problem (1.5) (1.6) (1.7), computing $u = u_g^1 - u_g^2$ and $p = p_g^1 - p_g^2$, we get

$$\left.\begin{array}{r} -\Delta u + \partial_3 u - ((\omega \wedge x)\cdot\nabla)u + \omega \wedge u = -\nabla p, \\ \operatorname{div} u = 0. \end{array}\right\} \tag{4.3}$$

To prove the uniqueness, i.e. $u = 0$ and $p = 0$, we use the duality method. That's why we need to assume the knowledge of $\{u_0^*, p_0^*\}$. Taking u_0^* as a test function in $(\widehat{W}_{0,\sigma}^{1,2}(D))^3 \bigcap (\widehat{W}^{1,r}(D))^3$, $\frac{3}{2} < r < 3$, we get

$$\langle \nabla u, \nabla u_0^* \rangle + \langle \partial_3 u, u_0^* \rangle - \langle ((\omega \wedge x)\cdot\nabla)u - \omega \wedge u, u_0^* \rangle = 0.$$

Taking now u as a test function in $(\widehat{W}_0^{1,q}(D))^3$ for the problem (4.2) we obtain

$$\langle \nabla u_0^*, \nabla u \rangle - \langle \partial_3 u_0^*, u \rangle + \langle ((\omega \wedge x)\cdot\nabla)u_0^* - \omega \wedge u_0^*, u \rangle = \langle h, u \rangle$$

$$= \langle \nabla u_0^*, \nabla u \rangle + \langle u_0^*, \partial_3 u \rangle - \langle u_0^*, ((\omega \wedge x)\cdot\nabla)u - \omega \wedge u \rangle.$$

Then $\langle h, u\rangle = 0$ for any h, so $u = 0$ in $(\widehat{W}_0^{1,q}(D))^3$, it means u is constant, necessary a multiple of ω. As a consequence, in (4.3), $\nabla p = 0$, and because of $p \in L^q(D)$, we get $p = 0$.

Remark 4.1 L^2-uniqueness can be established directly, as we shall see in the first step of the next section. We here have easily extend to our problem the duality method used by Hishida [26] for the stationary Stokes problem, the solenoidality being essential.

5 Existence in the main theorem : Proof

Step 1: Existence (homogeneous divergence case)
Let $f = \operatorname{div} F$ with $F \in C_0^\infty(\ D)^9$. In the domain $\ D_R$ according to the support of F, we apply the classical approach to solve

$$-\Delta u + \partial_3 u - ((\omega \wedge x) \cdot \nabla)u + \omega \wedge u + \nabla p = f = \operatorname{div} F$$

with $\operatorname{div} u = 0$ and with homogeneous Dirichlet boundary conditions on $\partial\ D_R$.

The bilinear form $b(u, \varphi) = \langle \nabla u, \nabla \varphi\rangle + \langle \partial_3 u, \varphi\rangle - \langle((\omega \wedge x) \cdot \nabla)u - \omega \wedge u, \varphi\rangle$ is coercive on $(\widehat{W}_{0,\sigma}^{1,2}(D_R))^3 \times (\widehat{W}_{0,\sigma}^{1,2}(D_R))^3$: $\langle ., .\rangle$ here stands for the L^2-inner product. One can easily verify that $b(u_R, u_R) = \|\nabla u_R\|^2_{2,\ D_R} = \langle \operatorname{div} F, u_R\rangle$.

Using the Lax-Milgram theorem we justify the existence of an unique solution $u_R \in (\widehat{W}_{0,\sigma}^{1,2}(D_R))^3$, which satisfies the estimate

$$\|\nabla u_R\|_{2,\ D_R} \le \|F\|_{2,\ D_R} = \|F\|_{2,D}.$$

We can extend u_R by zero in $\ D \setminus\ D_R$. Then we obtain $\widetilde{u_R} \in (\widehat{W}_{0,\sigma}^{1,2}(\ D))^3$ satisfying the same estimate, uniform as $R \to +\infty$.

We now choose a sequence of numbers $\{R_n\}$, tending to infinity, so that $\widetilde{u_{R_n}}$ converge weakly in $(\widehat{W}_{0,\sigma}^{1,2}(\ D))^3$. The limit u is unique (always up to an additive multiple of ω) such that

$$\langle \nabla u, \nabla \varphi\rangle + \langle \partial_3 u, \varphi\rangle - \langle((\omega \wedge x) \cdot \nabla)u - \omega \wedge u, \varphi\rangle - \langle \operatorname{div} F, \varphi\rangle = 0$$

for all $\varphi \in C_0^\infty(D)^3$, then for all $\varphi \in (\widehat{W}_0^{1,2}(\ D))^3$.

By density of $\{\operatorname{div} F\}_{F\in(C_0^\infty(D))^9}$ in $(\widehat{W}^{-1,q}(\ D))^3$, the previous integral equation holds with $f \in (\widehat{W}^{-1,q}(D))^3$: Thus we have

$$\langle \nabla u, \nabla \varphi\rangle + \langle \partial_3 u, \varphi\rangle - \langle ((\omega \wedge x)\cdot \nabla)u - \omega \wedge u, \varphi\rangle - \langle f, \varphi\rangle = 0. \tag{5.1}$$

for all $\varphi \in (\widehat{W}^{1,2}_{0,\sigma}(\ D))^3$.

Therefore there exists $p \in L^2_{loc}(\ D)$ (unique up to an additive constant) such that

$$-\Delta u + \partial_3 u - ((\omega \wedge x)\cdot \nabla)u + \omega \wedge u - f = -\nabla p.$$

Step 2 : $\{u, p\}$ in $W^{1,q} \times L^q$ and localization
Having $\{u, p\}$ from Step 1, and applying the localization technique we will get the problem (1.16) in the whole space and, either the problem (1.17), or its variant (1.19) in a certain bounded domain D_ρ.

Let $\phi \in (C_0^\infty(\mathbb{R}^3))^3$, and let ψ be always the same cut-off function. We can successively choose the following test functions φ in (5.1) :

- $\varphi = (1-\psi)\phi$, so we can read all integrals over $\mathbb{R}^3$ and interpret the solved problem in the whole space by $\{U, \pi\} = \{(1-\psi)u, (1-\psi)p\}$ as in (1.16); the formulas describing $F_1 = F_1(u, p)$ and $G_1 = G_1(u)$ are given in Section 1.

- $\varphi = \psi\phi$, so we can interpret in D_ρ the solved problem by $\{V, \tau\} = \{\psi u, \psi p\}$ as in (1.17) or (1.19); see in Section 1 the detailed formulas for $F_2 = F_2(u, p)$ and $G_2 = G_2(u)$.

Theorem 3.5 and Theorem 3.6 solve respectively these problems under the following hypothesis

$$F_1 \in (\widehat{W}^{-1,q}(\mathbb{R}^3))^3, G_1 \in L^q(\mathbb{R}^3), (\omega \wedge x)\cdot G_1 \in W^{-1,q}(\mathbb{R}^3),$$

$$F_2 = F_2(u, p) \in (\widehat{W}^{-1,q}(\ D_\rho))^3, G_2 = G_2(u) \in L^q_0(\ D_\rho),$$

with estimates (3.2) resp. (3.3).

To exploit the previous estimates, it remains essentially to control all terms we have from formulas (1.18) in $\|F_j(u, p)\|_{-1,q,\mathbb{R}^3 or D_\rho}, j = 1, 2$ for appropriate q. In this way, we recall that $\nabla\psi$ and $\Delta\psi$ have a compact support in D_ρ at the most, (precisely in the annulus $\{x : \rho_0 < |x| < \rho\}$ closed to the "obstacle" Ω), so we have

- either $\|\phi\|_{q/(q-1),\ D_\rho} \leq c(|\ D_\rho|)\|\nabla\phi\|_{q/(q-1),\ D_\rho}$

  by Friedrichs-Poincaré inequality,

  or $\|\phi\|_{q/(q-1),\ D_\rho} \leq |\ D_\rho|^{1/3}\|\phi\|_{r,\ D_\rho}$

  by Holder inequality, with $\frac{1}{r}+\frac{1}{3}=\frac{q-1}{q}$, so necessary $\frac{q-1}{q}>\frac{1}{3}$ and $q>\frac{3}{2}$.

  Thus $\|\phi\|_{q/(q-1),\ D_\rho} \leq |\ D_\rho|^{1/3}\|\phi\|_{r,\mathbb{R}^3} \leq c(|\ D_\rho|)\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$
- $|\langle(1-\psi)f,\phi\rangle| \leq c\|f\|_{-1,q,\ D}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$
- $|\langle 2(\nabla\psi\cdot\nabla)u+[\Delta\psi+((\omega\wedge x)\cdot\nabla)\psi]u,\phi\rangle|$

  $\leq |\langle(\Delta\psi)\phi,u\rangle|+|\langle 2(\nabla\psi\cdot\nabla)\phi,u\rangle|+|\langle((\omega\wedge x)\cdot\nabla)\psi]\phi,u\rangle|$

  $\leq c\|u|_{D_\rho}\|_{q,\ D_\rho}\|\nabla\phi\|_{q/(q-1),\mathbb{R}^3}$
- $|\langle(\nabla\psi)p,\phi\rangle| \leq c\|p|_{D_\rho}\|_{-1,q,\ D_\rho}\|\nabla\phi\|_{q/q-1,\mathbb{R}^3}$
- $|\langle(\partial_3\psi)u,\phi\rangle| \leq c\|u|_{D_\rho}\|_{q,\ D_\rho}\|\nabla\phi\|_{q/q-1,\mathbb{R}^3}$
- $\|(\omega\wedge x)(u\cdot\nabla\psi)\|_{-1,q,R^3} \leq C\|u\|_{q,D_q}\|\nabla\phi\|_{q/(q-1),R^3}$

Then, by means of the embedding $W_0^{1,2}(D_\rho)\subset L^q(D_\rho)$ where $\frac{1}{6}\leq\frac{1}{q}\leq\frac{2}{3}$, we know that $\|u|_{D_\rho}\|_{q,\ D_\rho}\leq c$.

Since $G_1=\nabla\psi u+(1-\psi)g$ and $G_2=\nabla\psi u+\psi g$ it implies

- $|<\nabla\psi u,\phi>| \leq C\|u\|_{q,D_\rho}\|\nabla\phi\|_{q/(q-1),R^3}$
- $|<(1-\psi)g,\phi>| \leq \|g\|_{q,R^3}\|\nabla\phi\|_{q/(q-1),R^3}$
- $|<\psi g,\phi>| \leq \|g\|_{q,R^3}\|\nabla\phi\|_{q/(q-1),R^3}$
- $|<(\omega\wedge x)(1-\psi)g,\phi>| \leq c\|(\omega\wedge x)g\|_{-1,q,R^3}\|\nabla\phi\|_{q/(q-1),R^3}$.

Then applying the Theorem 3.4 together with previous estimates we get

$$\begin{aligned}\|\nabla U\|_{q,\mathbb{R}^3}+\|\pi\|_{q,\mathbb{R}^3} \leq\ & c(\|f\|_{-1,q}+\|u|_{D_\rho}\|_{q,D_\rho}+\|p|_{D_\rho}\|_{-1,q,D_\rho}+\\ &+\|(\omega\wedge x)g\|_{-1,q,R^3}+\|g\|_{q,D_\rho})\end{aligned} \tag{5.2}$$

$$
\begin{aligned}
\|\nabla V\|_{q,D_\rho}+\|\tau\|_{q,D_\rho} \quad \leq \quad & c\big(\|f\|_{-1,q}+\|u|_{D_\rho}\|_{q,D_\rho}+\|p|_{D_\rho}\|_{-1,q,D_\rho}+ \\
& +\Big|\int_{D_\rho}\psi(x)p(x)dx\Big|+\|g\|_{q,D_\rho}\big).
\end{aligned} \tag{5.3}
$$

Finally $(u=U+V, p=\pi+\tau)$,

$$
\begin{aligned}
&\|\nabla u\|_{q,D}+\|p\|_{q,D}\leq c(\|f\|_{-1,q}+\|u|_{D_\rho}\|_{q,D_\rho}+\|p|_{D_\rho}\|_{-1,q,D_\rho}+\Big|\int_{D_\rho}p(x)dx\Big| \\
&\|-((\omega\wedge x)\cdot\nabla)u+\omega\wedge u\|_{-1,q}\leq\|f\|_{-1,q}+\|\nabla u|_{D_\rho}\|_q+\|p|_{D_\rho}\|_q.
\end{aligned} \tag{5.4}
$$

Therefore our weak solution $\{u,p\}$ from Step 1 verifies $u\in\widehat{W}_0^{1,q}(D)^3$ and $p\in L^q(D)$, $3/2<q<6$.

Step 3: Proof of the estimate (2.2) **for** $\{u,p\}$
Estimate (2.2) reads

$$
\begin{aligned}
&\|\nabla u\|_{q,D}+\|p\|_{q,D}+\|-((\omega\wedge x)\cdot\nabla)u+\omega\wedge u\|_{-1,q,D} \\
&\leq c_q(\|f\|_{-1,q,D}+\|g\|_{q,D}+\|(\omega\wedge x)\cdot g\|_{-1,q,D}).
\end{aligned}
$$

Let $3/2<q<3$. Suppose on the contrary the existence of two sequences $\{f_k\}$ in $\widehat{W}^{-1,q}(D)^3$ and $\{g_k\}$ in $L^q(D)$ tending to $f_\infty=0$ and $g_\infty=0$ as k tends to infinity, such that for the corresponding sequence of solutions $\{u_k,p_k\}$ in $\widehat{W}_0^{1,q}(D)^3\times L^q(D)$

$$
\|\nabla u_k\|_{q,D}+\|p_k\|_{q,D}+\|-((\omega\wedge x)\cdot\nabla)u_k+\omega\wedge u_k\|_{-1,q,D}=1. \tag{5.5}
$$

We know from Step 2 that

$$
\|\nabla u_k\|_{q,D}+\|p_k\|_{q,D}\leq c(\|f_k\|_{-1,q}+\|u_k|_{D_\rho}\|_{q,D_\rho}+\|p_k|_{D_\rho}\|_{-1,q,D_\rho}+\Big|\int_{D_\rho}p_k(x)dx\Big|
$$

and

$$
\|-((\omega\wedge x)\cdot\nabla)u_k+\omega\wedge u_k\|_{-1,q,D}\leq\|f_k\|_{-1,q}+\|\nabla u_k|_{D_\rho}\|_{q,D_\rho}+\|p_k|_{D_\rho}\|_{q,D_\rho}.
$$

On the other hand we have (with $q<3$)

$$
\begin{aligned}
&\|u_k|_{D_\rho}\|_{1,q,D_\rho}\leq\|\nabla u_k|_{D_\rho}\|_{q,D_\rho}+c\|u_k|_{D_\rho}\|_{3q/(3-q),D_\rho}\leq c\|\nabla u_k\|_{q,D}\leq c \\
&\|p_k|_{D_\rho}\|_{q,D_\rho}\leq 1,
\end{aligned}
$$

thus we can extract subsequences $\{u'_k\}$, $\{p'_k\}$ weakly convergent in $W^{1,q}(D_\rho)\times L^q(D_\rho)$, strongly convergent in $L^q(D_\rho)\times W^{-1,q}(D_\rho)$ (by Rellich's theorem), say $\{u_\infty, p_\infty\}$ the limit.

Then $\{u_\infty, p_\infty\} \in \widehat{W}_0^{1,q}(D)^3 \times L^q(D)$ is the unique weak solution to problem (1.5) (1.6) (1.7) with $f = f_\infty = 0$ and $g = g_\infty = 0$. From Section 2, $u_\infty = \alpha|\omega|e_3$ and $p_\infty = 0$, leading in a contradiction with (5.5).

We have completed the proof of theorem 2.1.

6 Nonhomogeneous boundary conditions

If we replace the homogeneous Dirichlet boundary conditions by nonhomogeneous ones in the form of (1.8), we also have the following theorem

Theorem 6.1. *Let $3/2 < q < 3$, and suppose f and g given as previously. Then there exists a unique weak solution $\{u,p\}$ to* (1.5) (1.6) (1.8) *(uniqueness up to a constant multiple of ω for u), which satisfies the estimate*

$$\left.\begin{array}{l} \|\nabla u\|_{q,D} + \|p\|_{q,D} + \|(\omega\wedge x)\cdot\nabla u - \omega\wedge u\|_{-1,q,D} \\ \le c_q(\|f\|_{-1,q,D} + \|g\|_{q,D} + \|(\omega\wedge x)\cdot g\|_{-1,q,D} + |\omega| + |\omega|^2 + 1), \end{array}\right\} \tag{6.1}$$

with some constant $c_q > 0$ independent of $|\omega|$.

Proof: The result is a corollary of Theorem 2.1. Choose a cut-off function $\xi \in C_0^\infty(R^3;[0,1])$ satisfying $\xi = 1$ near the boundary $\partial\Omega$ and set

$$b(x) = \tfrac{1}{2}\text{curl }(\xi(x)|x|^2\omega - \tfrac{1}{2}e_3\wedge\nabla|x|^2)$$

$$b|_{\partial\Omega}(x) = \omega\wedge x - e_3.$$

Let $v = u - b$, $\operatorname{div} v = 0$ since $\operatorname{div} u = 0$ and $\operatorname{div} b = 0$. So we obtain

$$\left.\begin{array}{rcll} -\Delta v + \partial_3 v - ((\omega\wedge x)\cdot\nabla)v + \omega\wedge v + \nabla p & = & f + f_b & \text{in } D \\ \operatorname{div} v & = & 0 & \text{in } D \\ v & = & 0 & \text{on } \partial\Omega \\ v & \to & 0 & \text{at } \infty, \end{array}\right. \tag{6.2}$$

where

$$f_b = -\Delta b + \partial_3 b - ((\omega\wedge x)\nabla)b + (\omega\wedge b).$$

Applying Theorem 2.1 we get the existence of the unique weak solution (v, p) satisfying $\Delta p = \operatorname{div} f$ and the following estimate

$$
\begin{aligned}
\|\nabla v\|_{q,D} + \|p\|_{q,D} \quad &+ \quad \| - ((\omega \wedge x)\nabla)v + \omega \wedge v\|_{-1,q,D} \\
&\leq \quad c(\|f\|_{-1,q,D} + \|f_b\|_{-1,q,D}) \qquad (6.3) \\
&\leq \quad c(\|f\|_{-1,q,D} + |\omega| + |\omega|^2 + 1).
\end{aligned}
$$

Appendix - Bogovski operator

Let us formulate the geometrical assumptions and the properties we will used to take into account a non zero divergence vectorial field. We refer e.g. to [1, 2, 12, 19] for the details.

Geometrical assumptions:

Let $1 < q < +\infty$. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a domain with boundary $\partial\Omega \in C^{1,1}$ and suppose one of the following cases
(i) Ω is bounded
(ii) Ω is an exterior domain, i.e., a domain having a compact nonempty complement.

In the bounded situation, Bogovski [1, 2] has constructed a bounded linear operator $\mathcal{B} : L^q_0(\Omega) \to W^{1,q}_0(\Omega)^N$ such that $u = \mathcal{B}g$ is a solution to

$$
\begin{aligned}
\operatorname{div} u &= g \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial\Omega.
\end{aligned}
\qquad (6.4)
$$

satisfying $\|\mathcal{B}g\|_{W^{1,q}(\Omega)} \leq c\|g\|_q$. The problem (6.4) is not uniquely solved, given $g \in L^q(\Omega)$, $\int_\Omega g(x)dx = 0$ is always assumed.

If we consider less smooth boundary the problem of solvability of Bogovskii operator was solved for star-shaped domain see Galdi [12].
Additionally $\mathcal{B}$ maps $W^{1,q}_0(\Omega) \cap L^q_0(\Omega)$ into $W^{2,q}_0(\Omega)$, see [1].

There are many applications in Fluid Dynamics with the use of Bogovski's operator also in Sobolev spaces of negative order, so $\mathcal{B}$ is a bounded linear operator from $W^{r,q}_0(\Omega)$ in $W^{(r+1),q}_0(\Omega)^N$, $r + 2 > \frac{1}{q}$. They define

$$
W^{s,p}_0 = \widehat{C_c(\Omega)}^{\|\cdot\|_{W^{s,p}(\Omega)}}.
$$

For $s < 0$ they denote

$$W^{s,p}(\Omega) := (W_0^{-s,p'}(\Omega))', \; W_0^{s,p}(\Omega) := (W^{-s,p'}(\Omega))',$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.
For more details see [18]. Also we would like to mention comments concerning Sobolev spaces of negative order in work of Galdi [12] and Farwig, Sohr [4].

Farwig and Sohr [3] have observed $\mathcal{B}$ as a bounded linear operator in domain satisfying one of the assumptions (i)-(ii) that

- from $\widehat{W}^{1,q}(\Omega) \cap L_0^q(\Omega)$ in $W_0^{1,q}(\Omega)^N \cap W^{2,q}(\Omega)^N$, if Ω is bounded
- from $\widehat{W}^{-1,q}(\Omega)$ in $L^q(\Omega)^N$, also if Ω is bounded
- from $W^{1,q}(\Omega) \cap \widehat{W}^{-1,q}(\Omega)$ in $W_0^{1,q}(\Omega)^N \cap W^{2,q}(\Omega)^N$, if Ω is unbounded

and $u = \mathcal{B}g$ solves always $\operatorname{div} \mathcal{B}g = g$ with $\mathcal{B}g|_{\partial\Omega} = 0$ under the condition $\int_\Omega g(x)dx = 0$, and satisfies the estimates

$$\begin{aligned} \|u\|_q &\leq c\|g\|_{-1,q}, \\ \|u\|_{2,q} &\leq c(\|\nabla g\|_q + \|g\|_{-1,q}), \end{aligned}$$

where $c = c(\Omega, q) > 0$ is a constant.

In the work of Kozono and Sohr [19] we find the following lemma

Lemma 6.2. *(Kozono- Sohr) [[19], Lemma 2.2, Corollary 2.3]*
Let $\Omega \subset R^n$ $n \geq 2$ be any domain and let $1 < q < \infty$. For all $f \in \widehat{W}^{-1,q}(\Omega)$, there is $F \in L^(\Omega)^n$ such that

$$\nabla \cdot F = f, \quad \|F\|_{q,\Omega} \leq C\|f\|_{-1,q,\Omega}$$

with some $C > 0$. As a result, the space $\{\nabla \cdot F; F \in C_0^\infty(\Omega)^n\}$ is dense in $\widehat{W}^{-1,q}(\Omega)$.

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