



Higher dimensional Kerr-Schild spacetimes

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Abstract

We investigate general properties of Kerr-Schild (KS) metrics in $n > 4$ spacetime dimensions. First, we show that the Weyl tensor is of type II or more special if the null KS vector \mathbf{k} is geodesic (or, equivalently, if $T_{ab}k^ak^b = 0$). We subsequently specialize to vacuum KS solutions, which naturally split into two families of non-expanding and expanding metrics. After demonstrating that *non-expanding* solutions are equivalent to the known class of vacuum Kundt solutions of Weyl type N, we analyze *expanding* solutions in detail. We show that they can only be of the type II or D, and we characterize optical properties of the multiple Weyl aligned null direction (WAND) \mathbf{k} . In general, \mathbf{k} has caustics corresponding to curvature singularities. In addition, it is generically shearing. Nevertheless, we arrive at a possible ‘weak’ $n > 4$ extension of the Goldberg-Sachs theorem, limited to the KS class, which matches previous conclusions for general type III/N solutions. In passing, properties of Myers-Perry black holes and black rings related to our results are also briefly discussed.

1 Introduction

Kerr-Schild spacetimes [1] have played an important role in four-dimensional general relativity. In particular, all vacuum KS solutions have been known for some time [1–3]. They are a subset of algebraically special spacetimes, and the corresponding KS null congruence is a geodesic, shearfree, repeated principal null direction of the Weyl tensor. Notably, they include the Kerr metric, arguably one of the physically most important known exact solutions of Einstein’s equations in vacuum. Other well-known vacuum solutions which can be put in the KS form are, for example, *pp*-waves and type N Kundt waves. In addition, the KS class can also admit (aligned) electromagnetic or matter fields, so as to include, for instance, the Kerr-Newmann metric, the Vaidya solution and *pp*-waves coupled to

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a null Maxwell field or to pure radiation. (We refer the reader to [4] for a comprehensive review and for a number of original references.)

In recent years, gravity in more than four spacetime dimensions has become an active area of ongoing studies, mainly motivated by the increasing interest in string theory and extra-dimensional scenarios. In this context, the KS ansatz led to the remarkable discovery of rotating vacuum black holes in an arbitrary dimension $n > 4$ [5]. However, several features of higher-dimensional gravity have proven to be substantially different from the four-dimensional case, and more insight would be desirable. In this paper we explore systematically general properties of KS metrics in higher dimensions. The main focus will be on Einstein's theory in vacuum, bearing in mind, however, that some of our results may have applications also in different contexts.

In addition to the intrinsic interest for possible specific KS solutions, an advantage of the KS ansatz is its mathematical tractability, which enables one to study it analytically in full generality. We thus present it also to illustrate a concrete application of the recently developed higher-dimensional classification of the Weyl tensor [6, 7] and Newman-Penrose formalism [8, 9] (cf [10] for another very recent application). This will also enable us to present a partial extension (restricted to KS solutions) of the Goldberg-Sachs theorem [11, 12] to higher dimensions, thus making contact with the previous results of [8]. The plan of the paper is as follows.

In Sec. 2 we consider general KS spacetimes and study constraints on the compatible form of the energy-momentum tensor. In particular, we show that the null KS vector \mathbf{k} is geodesic if and only if $T_{ab}k^ak^b = 0$ (similarly as in four dimensions [4]).

In Sec. 3 we specialize to the case with a geodesic KS vector \mathbf{k} and show that in that case the spacetime is of Weyl type II or more special.

In the rest of the paper we further restrict ourselves to vacuum solutions. These naturally split into a non-expanding and an expanding class. In Sec. 4 we study the non-expanding class and show that it is equivalent to type N Kundt vacuum solutions. As a consequence, we also observe that $n > 4$ pp-waves cannot always be cast in the KS form, as opposed to the four-dimensional case.

The more complex expanding case is analyzed in Sec. 5. From a subset of the vacuum equations we obtain an *optical constraint*, i.e. a purely geometric condition on the KS null congruence \mathbf{k} in the flat background. This is then used to show that expanding vacuum KS spacetimes are necessarily of Weyl type II or D. By integration of the Ricci identities we subsequently determine the dependence of the Ricci rotation coefficients (in particular, of optical quantities) on an affine parameter r along the KS congruence. This enables us also to find the r -dependence of the KS function and thus to study general basic properties of KS geometries. In particular, we discuss a 'weak' extension of the Goldberg-Sachs theorem valid for vacuum KS spacetimes, which are generically shearing. We also demonstrate, in the 'generic' case, the presence of a curvature singularity at a caustic of \mathbf{k} . This is in particular true for all expanding non-twisting spacetimes, on which we comment as a special subset of KS solutions.

We conclude in Sec. 6 with a summary and final remarks. In appendix A frame components of the Riemann tensor are given for KS space-

times with a geodetic \mathbf{k} . Subsequent appendices B–D contain several proofs and technical details related to the results presented in the main text.

1.1 Preliminaries

We use the notation of [8] (see also [9]) and, in particular, we introduce a null frame $\mathbf{m}^{(0)} = \mathbf{n}$, $\mathbf{m}^{(1)} = \boldsymbol{\ell}$, $\mathbf{m}^{(i)}$ ($i = 2, \dots, n-1$). The only non-zero scalar products are $\ell^a n_a = 1$ and $m^{(i)a} m_a^{(j)} = \delta_{ij}$ ($a, b = 0, 1, \dots, n-1$), and the spacetime metric reads $g_{ab} = 2\ell_{(a} n_{b)} + \delta_{ij} m_a^{(i)} m_b^{(j)}$ (sum over i, j). Directional derivatives along frame vectors will be denoted by $D \equiv \ell^a \nabla_a$, $\Delta \equiv n^a \nabla_a$, $\delta_i \equiv m^{(i)a} \nabla_a$. The full set of the corresponding Ricci rotation coefficients is defined in [8,9]. In particular, we will often use the definition $\ell_{a;b} = L_{cd} m_a^{(c)} m_b^{(d)}$ (sum over c, d). In fact, it will be convenient to adapt the frame vector $\boldsymbol{\ell}$ to coincide with the KS null vector \mathbf{k} . When \mathbf{k} is geodetic, we will denote by r the corresponding affine parameter. Quantities that do not depend on r will be denoted by a subscript or superscript index 0 (e.g., \mathcal{H}_0 , $s_{(i)}^0$, etc.).

Let us also anticipate here that we will be using five types of indices with different ranges, namely

$$\begin{aligned} a, b, c, \dots &= 0, 1, \dots, n-1, & i, j, k, \dots &= 2, \dots, n-1, \\ \alpha, \beta, \dots &= 2p+2, \dots, m+1, & \rho, \sigma, \dots &= m+2, \dots, n-1, \\ \mu, \nu &= 1, \dots, p, & & \text{with } 0 \leq 2p \leq m \leq n-2, \end{aligned} \quad (1)$$

where p and m are fixed integers defined later on. Einstein's summation convention is employed except for indices μ, ν (for which summation will be indicated explicitly), or when at least one of the repeated indices is in brackets (e.g., there will be no summation over i in $DA_{ij} = -2s_{(i)} A_{ij}$), unless (only in very few exceptional cases) stated otherwise. Note that for indices i, j, \dots we will not distinguish between subscripts and superscripts since there is no difference between covariant and contravariant components.

When referring to equations presented in previous papers it will be sometimes convenient to denote them by the corresponding equation number followed by the reference number, e.g. eq. (11f, [9]).

2 General case

In this section we define KS spacetimes and we study their general properties, without imposing Einstein's equations and without requiring the KS null congruence to be geodetic.

2.1 Ansatz and geodetic condition

We study an n -dimensional spacetime with a metric in the KS form

$$g_{ab} = \eta_{ab} - 2\mathcal{H}k_a k_b, \quad (2)$$

with η_{ab} being the Minkowski metric $\text{diag}(-1, 1, \dots, 1)$, \mathcal{H} a scalar function and k_a a 1-form that is assumed to be null with respect to η_{ab} , i.e. $\eta^{ab}k_ak_b = 0$ (η^{ab} is defined as the inverse of η_{ab} and $\eta^{ab} \neq g^{ac}g^{bd}\eta_{cd}$). From (2) it follows for the corresponding vector that $k^a \equiv \eta^{ab}k_b = g^{ab}k_b$, so that k^a is null also with respect to g_{ab} , and that the inverse metric has the form

$$g^{ab} = \eta^{ab} + 2\mathcal{H}k^ak^b. \quad (3)$$

One also gets for the metric determinant

$$g = \eta = -1. \quad (4)$$

Straightforwardly from the definition of the Christoffel symbols we obtain

$$\Gamma^a_{bc} = -(\mathcal{H}k^ak_b)_{,c} - (\mathcal{H}k^ak_c)_{,b} + \eta^{ad}(\mathcal{H}k_bk_c)_{,d} + 2\mathcal{H}k^ak^dk(\mathcal{H}k_bk_c)_{,d}, \quad (5)$$

from which

$$\Gamma^a_{bc}k^bk^c = 0, \quad \Gamma^a_{bc}k_ak^b = 0, \quad \Gamma^a_{ab} = 0, \quad (6)$$

and consequently also

$$k_{a;b}k^b = k_{a,b}k^b, \quad k^a_{;b}k^b = k^a_{,b}k^b. \quad (7)$$

In particular, this implies that k^a is geodesic in the flat geometry η_{ab} iff it is geodesic in the full geometry g_{ab} . We also have

$$\Gamma^a_{bc}k_a = (\mathcal{H}k_bk_c)_{;d}k^d, \quad \Gamma^a_{bc}k^b = -(\mathcal{H}k^ak_c)_{;d}k^d, \quad (8)$$

$$k_{a;c}k^a = k_{a,c}k^a = 0, \quad k_{a;bc}k^a + k_{a;b}k^a_{;c} = k_{a,bc}k^a + k_{a,b}k^a_{,c} = 0. \quad (9)$$

Since $\Gamma^a_{ab} = 0$, we can express the Ricci tensor as

$$R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{db}\Gamma^d_{ac}. \quad (10)$$

After employing the above formulas, Einstein's equations for the projected component $R_{ab}k^ak^b$ read

$$R_{ab}k^ak^b = 2\mathcal{H}g^{ab}(k_{a;c}k^c)(k_{b;d}k^d) = \kappa_0 T_{ab}k^ak^b. \quad (11)$$

From now on, when using a null frame ℓ , \mathbf{n} , $\mathbf{m}^{(i)}$, we will make the convenient choice

$$\ell = \mathbf{k}, \quad (12)$$

adapted to the KS ansatz.

Using eq. (11) and (see [8])

$$k_{a;b}k^b = L_{10}k_a + L_{i0}m_a^{(i)} \quad (13)$$

(sum over i), we can formulate the following

Proposition 1 *The null vector \mathbf{k} in the KS metric (2) of an arbitrary dimension is geodesic if and only if the energy-momentum tensor satisfies $T_{ab}k^ak^b = 0$.*

Then, using frame indices, the condition of proposition 1 reads $T_{00} = 0$ (or, equivalently, $R_{00} = 0$), and it holds iff the null frame components of T_{ab} do not include a term proportional to $n_a n_b$. This condition is of course satisfied in the case of *vacuum* spacetimes, also with a possible cosmological constant, or in the presence of matter fields aligned with the KS vector \mathbf{k} , such as an aligned Maxwell field (defined by $F_{ab}k^a \sim k_b$) or aligned pure radiation (i.e., $T_{ab} \sim k_ak_b$).

2.2 Optics of the KS vector field

It is interesting to discuss how the optical properties of k_a in the two geometries g_{ab} and η_{ab} are related, i.e. to compare the Ricci rotation coefficients [8,9] L_{ab} and \tilde{L}_{ab} defined with respect to g_{ab} and η_{ab} , respectively. In order to do so one has to set up ‘null’ frames for both metrics. Let \mathbf{k} , \mathbf{n} , $\mathbf{m}^{(i)}$ be the frame for the full spacetime metric g_{ab} as discussed above. Then, from eq. (2) it follows that $\eta_{ab} = 2k_{(a}[n_{b)} + \mathcal{H}k_{b)}] + \delta_{ij}m_a^{(i)}m_b^{(j)}$ (sum over i, j). Thus, \mathbf{k} , $\tilde{\mathbf{n}}$, $\mathbf{m}^{(i)}$ is now a convenient frame for the flat metric η_{ab} , provided one takes

$$\tilde{n}_a = n_a + \mathcal{H}k_a. \quad (14)$$

Now, from eqs. (8) and (13) one gets

$$k_{a;b} = k_{a,b} - (D\mathcal{H} + 2\mathcal{H}L_{10})k_a k_b - 2\mathcal{H}L_{i0}m_a^{(i)}k_b \quad (15)$$

(sum over i). Expanding $k_{a;b}$ on the frame \mathbf{k} , \mathbf{n} , $\mathbf{m}^{(i)}$ and $k_{a,b}$ on the frame \mathbf{k} , $\tilde{\mathbf{n}}$, $\mathbf{m}^{(i)}$ in terms of L_{ab} and \tilde{L}_{ab} , respectively, and using (14) one finds that $L_{ab} = \tilde{L}_{ab}$ except for $L_{11} = \tilde{L}_{11} - D\mathcal{H} - \mathcal{H}\tilde{L}_{10}$ and $L_{1i} = \tilde{L}_{1i} - \mathcal{H}\tilde{L}_{i0}$. In particular, one has $L_{i0} = \tilde{L}_{i0}$ (so that \mathbf{k} is geodetic with respect to g_{ab} iff it is geodetic with respect to η_{ab} , as already observed in section 2), and the matrix

$$L_{ij} \equiv k_{a;b}m^{(i)a}m^{(j)b} = k_{a,b}m^{(i)a}m^{(j)b} \quad (16)$$

is the same with respect to both metrics. The optical scalars shear, twist and expansion [8,9] are thus unchanged¹ (note that when \mathbf{k} is geodetic this statement is independent of the particular choice (14) since L_{ij} is then invariant under null rotations with \mathbf{k} fixed [9]).

3 The case of a geodetic KS vector field

In the rest of the paper we will assume that the null KS vector \mathbf{k} is *geodetic* and *affinely parametrized* (i.e., $L_{i0} = 0 = L_{10}$), which seems to be the simplest and physically most interesting case (recall proposition 1). The following definitions [8,9] will thus be useful:

$$\begin{aligned} S_{ij} &\equiv L_{(ij)} = \sigma_{ij} + \theta\delta_{ij}, & A_{ij} &\equiv L_{[ij]}, \\ \theta &\equiv \frac{1}{n-2}S_{ii}, & \sigma^2 &\equiv \sigma_{ij}\sigma_{ij}, & \omega^2 &\equiv A_{ij}A_{ij}. \end{aligned} \quad (17)$$

We shall refer to S_{ij} , σ_{ij} and A_{ij} as the *expansion*, *shear* and *twist* matrices, respectively, and to θ , σ and ω as the corresponding scalars.

Under the geodetic assumption, eq. (15) reduces to

$$k_{a;b} = k_{a,b} - (D\mathcal{H})k_a k_b, \quad (18)$$

which will be employed in the following. Using $k_a{}^b = \eta^{ac}k_{c,b}$ (and, from now on, denoting $(\)_{,d}\eta^{cd}$ by $(\)^{,c}$, e.g. $k_a{}^b = \eta^{bc}k_{a,c}$ etc.) one easily finds $k_{a,b}k^{a,c} = k_{a;b}k^{a;c}$, $k_{a,b}k^{c,a} = k_{a;b}k^{c;a}$, $k_a{}^b k_{,c}^a = k_a{}^b k^a{}_{,c}$, $k_{a,bc}k^b = -k_{a,b}k^b{}_{,c}$, $k_{a,bc}k^b k^c = 0$ and similar identities, also to be used

¹Cf [13] in the case $n = 4$.

in following calculations. In addition, certain expressions will take a more compact form if we write them in terms of the Ricci rotation coefficients and of directional derivatives as

$$\begin{aligned} k^a{}_{,a} &= L_{ii} = S_{ii}, & k_{a,b}k^{a,b} &= L_{ij}L_{ij}, & k_{a,b}k^{b,a} &= L_{ij}L_{ji}, \\ \mathcal{H}_{,a}k^a &= D\mathcal{H}, & \mathcal{H}_{,ab}k^ak^b &= D^2\mathcal{H}. \end{aligned} \quad (19)$$

From eq. (5) we thus get

$$\begin{aligned} \Gamma^e{}_{ab,e} &= (\mathcal{H}k_ak_b)_{,e}{}^e - (\mathcal{H}k_ak^e)_{,be} - (\mathcal{H}k_bk^e)_{,ae} \\ &\quad + 2[\mathcal{H}D^2\mathcal{H} + (D\mathcal{H} + \mathcal{H}L_{ii})D\mathcal{H}]k_ak_b, \end{aligned} \quad (20)$$

$$\Gamma^e{}_{fb}\Gamma^f{}_{ae} = 2[(D\mathcal{H})^2 - 2\mathcal{H}^2\omega^2]k_ak_b, \quad (21)$$

The Ricci tensor then reads²

$$\begin{aligned} R_{ab} &= (\mathcal{H}k_ak_b)_{,e}{}^e - (\mathcal{H}k_ak^e)_{,be} - (\mathcal{H}k_bk^e)_{,ae} \\ &\quad + 2\mathcal{H}[D^2\mathcal{H} + L_{ii}D\mathcal{H} + 2\mathcal{H}\omega^2]k_ak_b. \end{aligned} \quad (22)$$

Consequently, \mathbf{k} is an eigenvector of the Ricci tensor, i.e.

$$R_{ab}k^b = -[D^2\mathcal{H} + (n-2)\theta D\mathcal{H} + 2\mathcal{H}\omega^2]k_a. \quad (23)$$

For certain applications it may be useful to observe that the mixed Ricci components, i.e. $R^a{}_b = (\mathcal{H}k^ak_b)_{,s}{}^s - (\mathcal{H}k^ak^s)_{,bs} - (\mathcal{H}k_bk^s)_{,as}$, are linear in \mathcal{H} [14, 15]. The Ricci scalar is thus also linear and reads

$$R = -2[D^2\mathcal{H} + 2(n-2)\theta D\mathcal{H} + \mathcal{H}(n-2)(n-3)\theta^2 + \mathcal{H}(\omega^2 - \sigma^2)]. \quad (24)$$

So far we have worked in Minkowski coordinates adapted to the flat background metric η_{ij} . Note, however, that expressions for scalars, such as frame components of tensors, do not depend on that choice.

3.1 Frame components of the Ricci tensor

From eq. (22), one finds the non-vanishing frame components of the Ricci tensor

$$R_{01} = -[D^2\mathcal{H} + (n-2)\theta D\mathcal{H} + 2\mathcal{H}\omega^2], \quad (25)$$

$$R_{ij} = 2\mathcal{H}L_{ik}L_{jk} - 2[D\mathcal{H} + (n-2)\theta\mathcal{H}]S_{ij}, \quad (26)$$

$$\begin{aligned} R_{11} &= \delta_i(\delta_i\mathcal{H}) + (N_{ii} - 2\mathcal{H}L_{ii})D\mathcal{H} + (4L_{1j} - 2L_{j1} - \overset{i}{M}_{ji})\delta_j\mathcal{H} - L_{ii}\Delta\mathcal{H} \\ &\quad + 2\mathcal{H}\left(2\delta_iL_{[1i]} + 4L_{1i}L_{[1i]} + L_{i1}L_{i1} - L_{11}L_{ii}\right. \\ &\quad \left.+ 2L_{[1j]}\overset{j}{M}_{ii} - 2A_{ij}N_{ij} - 2\mathcal{H}\omega^2\right), \end{aligned} \quad (27)$$

$$\begin{aligned} R_{1i} &= -\delta_i(D\mathcal{H}) + 2L_{[i1]}D\mathcal{H} + 2L_{ij}\delta_j\mathcal{H} - L_{jj}\delta_i\mathcal{H} \\ &\quad + 2\mathcal{H}\left(\delta_jA_{ij} + A_{ij}\overset{j}{M}_{kk} - A_{kj}\overset{i}{M}_{kj} - L_{jj}L_{1i} + 3L_{ij}L_{[1j]} + L_{ji}L_{(1j)}\right). \end{aligned} \quad (28)$$

Note that $R_{00} = 0 = R_{0i}$ identically.

²This formula for R_{ab} is equivalent to eq. (32.10) given in [4] for $n = 4$, up to rewriting partial derivatives as covariant ones.

3.2 Algebraic type of the Weyl tensor

The full set of the frame components of the Riemann tensor is given in appendix A for any KS geometry with a geodetic, affinely parametrized KS vector. Thanks to $R_{0i0j} = R_{010i} = R_{0ijk} = 0$ and to $R_{00} = R_{0i} = 0$, for the Weyl tensor we find immediately

$$C_{0i0j} = 0, \quad C_{010i} = 0, \quad C_{0ijk} = 0. \quad (29)$$

The Weyl tensor components with boost order 2 and 1 are thus identically zero, which enables us to conclude

Proposition 2 *Kerr-Schild spacetimes (2) in arbitrary dimension with a geodetic KS vector \mathbf{k} are of Weyl type II (or more special), with \mathbf{k} being a WAND of order of alignment ≥ 1 .*

Note that the KS null vector \mathbf{k} must indeed be geodetic for a wide class of matter fields, in particular in vacuum (cf proposition 1), so that proposition 2 applies in those cases.

Another interesting result follows from the observation [16] that spacetimes (not necessarily of the KS class) which are either static or stationary with ‘expansion’ and ‘reflection symmetry’ can be only of the Weyl types G, I_i , D or O (see [16] for details and precise definitions). Taking the ‘intersection’ of this family with the set of KS spacetimes considered in proposition 2, we can conclude that in arbitrary dimension $n \geq 4$

Proposition 3 *All static spacetimes of the KS class with a geodetic \mathbf{k} are of Weyl type D or O. All stationary spacetimes of the KS class with ‘reflection symmetry’ and with a geodetic, expanding \mathbf{k} are of Weyl type D or O. In both cases, if the Weyl tensor is non-zero, \mathbf{k} is a multiple WAND.*

As an immediate consequence, the higher-dimensional rotating black holes of Myers and Perry (indeed obtained using the KS ansatz [5]), are necessarily of Weyl type D in any dimension, as we anticipated in [16] (this had previously been demonstrated by an explicit calculation of the Weyl tensor in [17]). This also applies to uniform black strings, since adding flat dimensions to a KS metric clearly preserves the KS structure. On the other hand, proposition 2 implies that five-dimensional vacuum black rings [18] do not admit a KS representation, since they are of Weyl type I_i [19]. Let us also emphasize that proposition 3 is not restricted to vacuum KS solutions. For instance, it can also be used to conclude that static black holes with electric charge (and, possibly, a cosmological constant) [20] are also of Weyl type D.

3.3 Vacuum equations

Vacuum solutions must satisfy $R_{ab} = 0$. We note from eqs. (25)–(28) that the Ricci tensor components R_{01} and R_{ij} are simple and do not involve Ricci rotation coefficients other than L_{ij} , which characterize the optical properties of \mathbf{k} . It is thus natural to start from the corresponding vacuum

equations. Imposing $R_{ij} = 0$ gives³

$$(D \ln \mathcal{H})S_{ij} = L_{ik}L_{jk} - (n-2)\theta S_{ij}. \quad (30)$$

Contracting eq. (30) with δ^{ij} we obtain

$$\begin{aligned} (n-2)\theta(D \ln \mathcal{H}) &= L_{ik}L_{ik} - (n-2)^2\theta^2 \\ &= \sigma^2 + \omega^2 - (n-2)(n-3)\theta^2, \end{aligned} \quad (31)$$

while its tracefree part (i.e., $R_{ij} - \frac{R_{kk}}{n-2}\delta_{ij} = 0$) is⁴

$$\begin{aligned} (D \ln \mathcal{H})\sigma_{ij} &= \left(\sigma^2_{ij} - \frac{1}{n-2}\sigma^2\delta_{ij}\right) - \left(A^2_{ij} + \frac{1}{n-2}\omega^2\delta_{ij}\right) \\ &\quad + 2\sigma_{k(i}A_{j)k} - (n-4)\theta\sigma_{ij}. \end{aligned} \quad (32)$$

Next, the equation $R_{01} = 0$ requires

$$D^2\mathcal{H} + (n-2)\theta D\mathcal{H} + 2\mathcal{H}\omega^2 = 0. \quad (33)$$

Using eq. (31), this can also be rewritten as $D^2\mathcal{H} = [-\sigma^2 - 3\omega^2 + (n-2)(n-3)\theta^2]\mathcal{H}$.

Note that eq. (31) involves the function \mathcal{H} in a non-trivial way only when $\theta \neq 0$. It will thus be convenient to study non-expanding and expanding solutions separately in the following sections. The remaining vacuum equations seem to be of little help in a general study, and there is no need to write them down at this stage.

4 Non-expanding vacuum solutions

4.1 Einstein's equations

The case $\theta = 0$ turns out to be somewhat special, since from eq. (31) we immediately get $\sigma = 0 = \omega$, and thus

$$L_{ij} = 0, \quad (34)$$

and from this and eq. (33)

$$D^2\mathcal{H} = 0, \quad (35)$$

i.e., $\mathcal{H} = \mathcal{H}_0 + \mathcal{G}_0 r$ (recall that in our notation r denotes an affine parameter along \mathbf{k} and $D\mathcal{H}_0 = 0 = D\mathcal{G}_0$). The vacuum equations $R_{ij} = 0$ and $R_{01} = 0$ are thus identically satisfied. Note, in particular, that the sole non-expanding condition implies that we are restricted to a subset of the higher-dimensional *Kundt class* of non-expanding, non-shearing and non-twisting vacuum solutions.

Setting $L_{ij} = 0$ in the Ricci components (27) and (28), the remaining vacuum equations read

$$\begin{aligned} \delta_i(\delta_i\mathcal{H}) + N_{ii}D\mathcal{H} + (4L_{1j} - 2L_{j1} - \overset{i}{M}_{ji})\delta_j\mathcal{H} \\ + 2\mathcal{H}\left(2\delta_i L_{[1i]} + 4L_{1i}L_{[1i]} + L_{i1}L_{i1} + 2L_{[1j]}\overset{j}{M}_{ii}\right) = 0, \end{aligned} \quad (36)$$

$$\delta_i(D\mathcal{H}) - 2L_{[i1]}D\mathcal{H} = 0. \quad (37)$$

³Hereafter, for brevity we shall write $\mathcal{H}^{-1}D\mathcal{H} = D \ln \mathcal{H}$, where it is understood that $\ln \mathcal{H}$ should be replaced by $\ln |\mathcal{H}|$ whenever $\mathcal{H} < 0$.

⁴For $n = 4$ this reduces to $(D \ln \mathcal{H})\sigma_{ij} = 2\sigma_{k(i}A_{j)k}$.

4.2 Equivalence with Kundt solutions of type N

4.2.1 Non-expanding KS solutions are of type N

We have seen above that vacuum non-expanding KS solutions are Kundt spacetimes. In general, the higher-dimensional Kundt class is known to be of Weyl type II or more special, provided $R_{00} = 0 = R_{0i}$ (thus in particular in vacuum) [9]. However, here we show that, in arbitrary dimension, vacuum KS spacetimes of the Kundt class are restricted to type N. In fact, it has already been shown in section 3 that the components of the Weyl tensor with boost weight 2 and 1 vanish. Using eqs. (34), (35) and (37) together with eqs. (A2)–(A6) we also find that all Weyl components with boost weight 0 and -1 are zero (since in vacuum $R_{abcd} = C_{abcd}$). The Weyl tensor is thus of type N, q.e.d..

4.2.2 Kundt solutions of type N are KS

We have thus demonstrated that vacuum solutions of the KS class with a non-expanding KS vector are a subset of Kundt spacetimes of Weyl type N. We now show that the converse is also true, namely that vacuum Kundt solutions of Weyl type N are of the KS form, so that the two families of solutions coincide.

Vacuum Kundt solutions of Weyl type N belong to the family of spacetimes with vanishing scalar invariants (VSI) [21, 22]. We can thus begin with the higher-dimensional VSI metric [22, 23]

$$ds^2 = 2du \left[dr + H(r, u, x^k) du + W_i(r, u, x^k) dx^i \right] + \delta_{ij} dx^i dx^j, \quad (38)$$

with $i, j, k = 2, \dots, n-1$, $k_a dx^a = du$ being the multiple WAND.

Similarly as for $n = 4$, higher-dimensional vacuum Kundt spacetimes of Weyl type N consist of two invariantly defined subfamilies [22, 23]: Kundt waves (with $L_{1i} = L_{i1} \neq 0$) and pp -waves (for which $L_{1i} = L_{i1} = 0 = L_{11}$). It is convenient to discuss these subfamilies separately.

The metric functions H and W_i for higher-dimensional type N Kundt waves are [22]

$$\begin{aligned} W_2 &= -2 \frac{r}{x^2}, \\ W_s &= x^q B_{qs}(u) + C_s(u), \\ H &= \frac{r^2}{2(x^2)^2} + H^0(u, x^i), \end{aligned} \quad (39)$$

where $s, q = 3, \dots, n-1$, $B_{qs} = -B_{sq}$ and H^0 must obey a field equation given in [22]. As shown in [22], metric (38), (39) is flat for

$$H^0(u, x^i) = H_{\text{flat}}^0 = \frac{1}{2} \sum_{s=3}^{n-1} W_s^2 + x^2 F_0(u) + x^2 x^i F_i(u), \quad (40)$$

where $F_0(u)$ and $F_i(u)$ are arbitrary functions of u . Therefore metric (38), (39) is in the KS form for an arbitrary choice of $H^0(u, x^i)$, since it can always be rewritten as $ds^2 = ds_{\text{flat}}^2 + (H^0 - H_{\text{flat}}^0) du^2$.

A similar argument can be used to show that also higher-dimensional type N pp -waves belong to the KS class – the corresponding metric functions W_i and H_{flat}^0 entering (38) can be found in [22].⁵

We can thus summarize the results of this section in the following (see theorem 32.6 of [4] for $n = 4$)

Proposition 4 *In arbitrary dimension $n \geq 4$, the Kerr-Schild vacuum spacetimes with a non-expanding KS congruence \mathbf{k} coincide with the class of vacuum Kundt solutions of Weyl type N.*

4.2.3 KS does not include all pp -waves for $n > 4$

A comment on some differences between $n = 4$ and $n > 4$ dimensional pp -waves is now in order. By a natural extension of the $n = 4$ terminology of [4], in any dimension pp -waves are defined as spacetimes (not necessarily of the KS form) admitting a covariantly constant null vector ℓ , i.e. $\ell_{a;b} = 0$. It then follows directly from the definition of the Riemann tensor that

$$R_{abcd}\ell^d = 0. \quad (41)$$

For four-dimensional vacuum spacetimes this is equivalent to the definition of the type N [4]. It is also known that, in addition to being of Weyl type N, all four-dimensional vacuum pp -waves can be cast in the KS form [4].

By contrast, in higher dimensions eq. (41) is only a necessary, but not sufficient condition for type N. It only says that the type is II (or more special) [19], and in fact for $n > 4$ one can easily construct explicit vacuum pp -waves of types III, II and D, as we now briefly discuss.

Vacuum pp -wave metrics of Weyl type III can be obtained directly by specializing results of [22] to the vacuum case. One simple five-dimensional example is metric (38) with

$$W_2 = 0, \quad W_3 = h(u)x^2x^4, \quad W_4 = h(u)x^2x^3, \quad (42)$$

$$H = H_0 = h(u)^2 \left[\frac{1}{24} ((x^3)^4 + (x^4)^4) + h^0(x^2, x^3, x^4) \right], \quad (43)$$

where $h^0(x^2, x^3, x^4)$ is linear in x^2, x^3, x^4 .

In addition, one can also construct $(n_1 + n_2)$ -dimensional vacuum pp -waves of Weyl type II, e.g. by taking a direct product of a n_1 -dimensional vacuum pp -wave of Weyl type N or III with an Euclidean n_2 -dimensional Ricci-flat (but non-flat) metric (with both $n_1, n_2 \geq 4$, and $n_1 \geq 5$ if the n_1 -dimensional pp -wave is of type III).⁶ Similarly, $(n_1 + n_2)$ -dimensional

⁵There is a typo in eq. (96) of [22]: just drop the inequality $m \leq n$ in the second sum. We thank Nicos Pelavas for correspondence on this point.

⁶The proof of these statements is straightforward and we just sketch it. First, all the mentioned products are really pp -waves since a covariantly constant vector field ℓ defined in the n_1 -dimensional geometry can be trivially lifted to the $(n_1 + n_2)$ -dimensional product geometry, in which it will still be covariantly constant. Such products are necessarily of Weyl type II or more special (with the lifted ℓ being a multiple WAND), as follows from the decomposable form of the Weyl tensor of product geometries [6, 16]. They cannot be of Weyl type III or N since they inherit non-zero curvature invariants (e.g., Kretschmann) from the n_2 -dimensional Euclidean space. They cannot be of type D since, starting from a frame adapted

vacuum pp -waves of Weyl type D arise if one takes a direct product of a n_1 -dimensional flat spacetime with an Euclidean n_2 -dimensional curved, Ricci-flat space (with $n_1 \geq 2$, $n_2 \geq 4$).

As a direct consequence of proposition 4, such $n > 4$ pp -waves of Weyl type II, D or III do not admit a KS form. They thus represent a ‘counterexample’ to both the $n = 4$ results, namely they are (higher-dimensional) vacuum pp -waves that are neither of type N nor of the KS form.

Note, in addition, that pp -waves of Weyl type II and D cannot belong to the VSI class of spacetimes (which is compatible only with types III, N, O [21]), and therefore they necessarily possess some non-vanishing curvature invariants. In table 1 we summarize the aforementioned properties of various types of vacuum pp -waves in four (4D) and higher dimensions (HD).⁷

	Weyl type	KS	VSI
4D	N	✓	✓
HD	N	✓	✓
HD	III	X	✓
HD	II (D)	X	X

Table 1: Properties of various types of vacuum pp -waves.

5 Expanding vacuum solutions

Non-expanding vacuum solutions of the KS class have been fully classified according to the discussion of the previous section. Let us now consider solutions with an expanding k , i.e. $\theta \neq 0$.

5.1 Consequences of Einstein’s equations

For $\theta \neq 0$, from (31) one can write $D \ln \mathcal{H}$ as

$$D \ln \mathcal{H} = \frac{L_{ik}L_{ik}}{(n-2)\theta} - (n-2)\theta. \quad (44)$$

Substituting into (30) one gets

$$L_{ik}L_{jk} = \frac{L_{lk}L_{lk}}{(n-2)\theta} S_{ij}. \quad (45)$$

to the canonical form of the (type N or III) Weyl tensor of the n_1 -dimensional spacetime, the components of negative boost weight turn out to be unchanged under null rotations, and thus cannot be set to zero. The only possible Weyl type is thus indeed II.

⁷In fact, in view of our previous comments a similar table applies to all Kundt solutions, in which case type III appears also in 4D [4], again VSI but not KS.

Interestingly, this equation is independent of the KS function \mathcal{H} and it is thus a purely geometric condition on the KS null congruence \mathbf{k} in the Minkowskian ‘background’ η_{ab} (recall the discussion of subsection 2.2). We will thus refer to it as the *optical constraint*.⁸ Its consequences on the form of L_{ij} will be studied in detail in appendix C and D, and they will be employed in the following sections.

Using (17), eq. (45) can also be rewritten in terms of the shear and twist matrices as

$$(n-2)\theta \left[\left(\sigma^2_{ij} - \frac{1}{n-2} \sigma^2 \delta_{ij} \right) - \left(A^2_{ij} + \frac{1}{n-2} \omega^2 \delta_{ij} \right) + 2\sigma_{k(i} A_{j)k} \right] = [\sigma^2 + \omega^2 - (n-2)\theta^2] \sigma_{ij}. \quad (46)$$

This traceless equation is in fact equivalent to (32), after using (44) (since the trace of eq. (45) is obviously an identity).⁹ Contracting (46) with σ_{ij} one finds the scalar constraint

$$\sigma^2 [\sigma^2 + \omega^2 - (n-2)\theta^2] - (n-2)\theta [\sigma^2_{ij} \sigma_{ij} - \sigma_{ij} A^2_{ji}] = 0. \quad (47)$$

So far we have discussed consequences of the vacuum equation $R_{ij} = 0$. Next, one has to make sure that eq. (33) (i.e., $R_{01} = 0$) is now compatible with (44). In fact, taking the D -derivative of (44) and using the scalar Sachs equations of [9] and eq. (47), one exactly recovers eq. (33). This is thus automatically satisfied, provided the preceding equations hold. Of course, when looking for an explicit solution one should also solve the remaining Einstein equations, namely $R_{11} = 0$ and $R_{1i} = 0$. These are, however, too involved unless one makes some further assumptions (such as the presence of symmetries, etc.). In any case, they are not needed in the following discussion.

5.2 Algebraic type

From the general result of proposition 2, we already know that the Weyl tensor of any KS metric with a geodesic \mathbf{k} is of type II or more special. Here we show that, in fact, the types III and N are not possible for vacuum expanding solutions.

By *reductio ad absurdum*, let us thus assume that all components of the Riemann (Weyl) tensor with boost weigh zero, i.e. (A2)–(A4), vanish.

⁸This constraint is of course identically satisfied in the trivial case $\sigma = 0 = \omega$ (i.e., $L_{ij} = S_{ij} = \theta \delta_{ij}$) which includes, e.g., the Schwarzschild-Tangherlini solution [20]. Less trivial examples are provided by the KS congruence of static black strings, for which $\sigma \neq 0$ and $\omega = 0$, or of rotating black strings and Myers-Perry black holes (which were indeed constructed in [5] using the KS ansatz), both having $\sigma \neq 0 \neq \omega$. (To explicitly verify these statements one may use the corresponding optical quantities calculated in any dimension in [8, 16].)

On the other hand, a simple example of a geodesic null congruence *violating* the optical constraint is provided by the vector field $\mathbf{k} = \sqrt{1 + f^2(\phi)} \partial_t + f(\phi) \partial_\rho + \partial_z$ in the flat geometry $ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2 + \delta_{AB} dy^A dy^B$ (with $A, B = 1, \dots, n-4$). This geodesic null congruence is expanding and shearing for $f \neq 0$, and violates the optical constraint if $f, \phi \neq 0$, in which case it is also twisting.

⁹In odd dimensions, it implies that one cannot have $\sigma_{ij} = 0$ if $A_{ij} \neq 0$ (this was already known in a more general context [9]).

In particular, we can multiply R_{0i1j} by L_{lj} . Using the optical constraint (45), the condition $R_{0i1j}L_{lj} = 0$ gives

$$S_{il}D \ln \mathcal{H} = -2A_{ki}S_{kl}, \quad (48)$$

where we have dropped an overall factor $L_{jk}L_{jk} = \sigma^2 + \omega^2 + (n-2)\theta^2 > 0$. Using as a ‘spatial’ basis an eigenframe of S_{ij} , we have $S_{ij} = \text{diag}(s_{(2)}, s_{(3)}, s_{(4)}, \dots)$. At least one eigenvalue must be non-zero, say $s_{(2)} \neq 0$. Then from (48) with $i = l = 2$ we get

$$s_{(2)}D \ln \mathcal{H} = -2s_{(2)}A_{22}, \quad (49)$$

which obviously can never be satisfied since $A_{22} = 0$ (and since the case $D\mathcal{H} = 0$ is ‘forbidden’, cf appendix B). This contradiction completes the proof of

Proposition 5 *In arbitrary dimension $n \geq 4$, Kerr-Schild vacuum spacetimes with an expanding KS congruence \mathbf{k} are of algebraic type II or D.*

For a similar result for $n = 4$ cf [4, 24].

5.3 Fixing the r -dependence

As detailed in appendices C and D, the optical constraint (45) and the Sachs equations [9] imply that there exists an appropriate frame, satisfying $M_{j0}^i = 0$, such that the matrix L_{ij} takes the block diagonal form

$$L_{ij} = \begin{pmatrix} \boxed{\mathcal{L}_{(1)}} & & & \\ & \ddots & & \\ & & \boxed{\mathcal{L}_{(p)}} & \\ & & & \boxed{\tilde{\mathcal{L}}} \end{pmatrix}. \quad (50)$$

The first p blocks are 2×2 and the last block $\tilde{\mathcal{L}}$ is a $(n-2-2p) \times (n-2-2p)$ -dimensional diagonal matrix. They are given by

$$\mathcal{L}_{(\mu)} = \begin{pmatrix} s_{(2\mu)} & A_{2\mu, 2\mu+1} \\ -A_{2\mu, 2\mu+1} & s_{(2\mu)} \end{pmatrix} \quad (\mu = 1, \dots, p),$$

$$s_{(2\mu)} = \frac{r}{r^2 + (a_{(2\mu)}^0)^2}, \quad A_{2\mu, 2\mu+1} = \frac{a_{(2\mu)}^0}{r^2 + (a_{(2\mu)}^0)^2}, \quad (51)$$

$$\tilde{\mathcal{L}} = \frac{1}{r} \text{diag}(\underbrace{1, \dots, 1}_{(m-2p)}, \underbrace{0, \dots, 0}_{(n-2-m)}), \quad (52)$$

with $0 \leq 2p \leq m \leq n-2$. The integer m denotes the rank of L_{ij} , so that L_{ij} is non-degenerate when $m = n-2$.

Accordingly, the optical scalars are given by

$$(n-2)\theta = 2 \sum_{\mu=1}^p \frac{r}{r^2 + (a_{(2\mu)}^0)^2} + \frac{m-2p}{r}, \quad (53)$$

$$\omega^2 = 2 \sum_{\mu=1}^p \left(\frac{a_{(2\mu)}^0}{r^2 + (a_{(2\mu)}^0)^2} \right)^2, \quad (54)$$

$$\begin{aligned} \sigma^2 = 2 \sum_{\mu=1}^p \left(\frac{r}{r^2 + (a_{(2\mu)}^0)^2} - \theta \right)^2 + (m-2p) \left(\frac{1}{r} - \theta \right)^2 \\ + (n-2-m)\theta^2. \end{aligned} \quad (55)$$

Note, in particular, that expansion θ indicates the presence of a caustic at $r = 0$, except when $2p = m$ (with m even). Note also that shear is generically non-zero (the special case $\sigma = 0$ will be discussed below). The twist is zero if and only if $p = 0$ (or, equivalently, all $a_{(2\mu)}^0$ vanish).

One can similarly fix the r -dependence of all Ricci rotation coefficients (at least with the additional ‘gauge’ condition $N_{i0} = 0$). Since this is not needed in the following discussion, corresponding results are relegated to appendix D.

Knowing the form of L_{ij} enables us to solve the vacuum equation (44), where now $L_{ik}L_{ik} = (n-2)\theta r^{-1}$. The r -dependence of \mathcal{H} is thus given by

$$\mathcal{H} = \frac{\mathcal{H}_0}{r^{m-2p-1}} \prod_{\mu=1}^p \frac{1}{r^2 + (a_{(2\mu)}^0)^2}. \quad (56)$$

This includes in particular the solution of (44) in the case when its rhs vanishes, i.e. $D\mathcal{H} = 0$, which happens for $m = 1$ (implying $p = 0$). This is, however, incompatible with the Bianchi identities [8] as explained in appendix B. Hence, in the following we shall restrict to

$$2 \leq m \leq n-2. \quad (57)$$

For example, the r -dependence in the special case of Myers-Perry solutions is obtained by setting $m = n-2$ in (56) (note, however, that function (56) is more readily compared with that of [5] using the parametrization (6) of [25] for the direction cosines). Black holes with only one non-zero spin [5], in particular, correspond to the further choice $p = 1$, and static black holes to $p = 0$ (see also subsection 5.7).

In general, the asymptotic behavior of \mathcal{H} for $r \rightarrow \infty$ is given by

$$\mathcal{H} = \frac{\mathcal{H}_0}{r^{m-1}} + \mathcal{O}(r^{-m-1}), \quad (58)$$

i.e., the function \mathcal{H} behaves as a Newtonian potential in $(m+1)$ space dimensions with r as a radial coordinate.

5.4 On the Goldberg-Sachs theorem

Let us now use the above results to comment on a partial extension to higher dimensions (but restricted to KS solution) of the Goldberg-Sachs

theorem. In four dimensions, this is a well-known theorem stating that in a vacuum (non-flat) spacetime, a null congruence is geodesic and shearfree if and only if it is a multiple principal null direction of the Weyl tensor [4, 11, 12].

For $n = 4$, the matrix L_{ij} associated with a generic null vector ℓ [8] is 2×2 and can be written in terms of the standard Newman-Penrose notation [4] as [9]¹⁰

$$L_{ij} = -\frac{1}{2} \begin{pmatrix} (\rho + \bar{\rho}) + (\sigma + \bar{\sigma}) & -i(\rho - \bar{\rho}) + i(\sigma - \bar{\sigma}) \\ i(\rho - \bar{\rho}) + i(\sigma - \bar{\sigma}) & (\rho + \bar{\rho}) - (\sigma + \bar{\sigma}) \end{pmatrix}. \quad (59)$$

In vacuum, when ℓ is a multiple principal null direction of the Weyl tensor the Goldberg-Sachs theorem implies $\sigma = 0$, so that L_{ij} reduces to

$$L_{ij} = -\frac{1}{2} \begin{pmatrix} \rho + \bar{\rho} & -i(\rho - \bar{\rho}) \\ i(\rho - \bar{\rho}) & \rho + \bar{\rho} \end{pmatrix}. \quad (60)$$

We have observed in previous sections that for vacuum KS spacetimes the KS vector $\ell = \mathbf{k}$ is indeed a (geodesic) multiple WAND for any $n \geq 4$. Accordingly, in the special case $n = 4$, eqs. (50)–(52) must be of the shearfree form (60). It is easy to verify that this is indeed the case, since for $n = 4$ we necessarily have $m = 2$ (cf (57)) and therefore L_{ij} either coincides with one of the blocks $\mathcal{L}_{(\mu)}$ (for $p = 1$, i.e. $\rho - \bar{\rho} \neq 0$) or is proportional to the two-dimensional identity matrix (for $p = 0$, i.e. $\rho - \bar{\rho} = 0$).

However, it has been pointed out [5, 8, 9, 16, 31] that the Goldberg-Sachs theorem cannot be extended to higher dimensions in the most direct formulation ‘multiple WANDs are geodesic and shearfree in vacuum’. Indeed, we have already noticed that \mathbf{k} is generically shearing. Nevertheless, for any $n > 4$ our result (50)–(52) can be viewed as a ‘weaker’ higher-dimensional Goldberg-Sachs theorem, albeit *restricted to KS spacetimes*: in a suitable basis, *the matrix L_{ij} (associated with the geodesic, multiple WAND \mathbf{k}) consists of 2×2 blocks that are ‘shearfree’,* i.e. reflecting the four-dimensional shearfree condition (60) in various orthogonal 2-planes.¹¹ The only ‘exceptions’ to this of course arise in odd spacetime dimensions (there will be an isolated one-dimensional block) and in the degenerate case $\det L = 0$, in which one may add an arbitrary number of zeros along the diagonal (most simply, by just taking a direct product with flat dimensions). This ‘generalized Goldberg-Sachs condition’ can also be expressed in a basis-independent form simply as (dropping the matrix indices)

$$[S, A] = 0, \quad A^2 = S^2 - \mathcal{F}S, \quad (61)$$

where $S = (L + L^T)/2$ and $A = (L - L^T)/2$ are the symmetric and antisymmetric parts of L , and \mathcal{F} can be fixed as in (C1) by taking the trace. The first equation in (61) is clearly equivalent to the condition

¹⁰Note that, exceptionally in this subsection only, the complex shear σ does not coincide with the real shear scalar defined in section 3 and used throughout the paper (although they are simply related), but it is the usual Newman-Penrose spin coefficient.

¹¹Let us recall that we have been focusing on the expanding case $\theta \neq 0$, since $\theta = 0 \Rightarrow L_{ij} = 0$ for vacuum KS spacetimes (see section 4).

$[L, L^T] = 0$, i.e. L is a (real) *normal* matrix. The second equation in (61) is just a consequence of the first one when $n = 4$.

Let us also emphasize that the above ‘weak Goldberg-Sachs theorem’ has, in fact, been proven in more generality (i.e., without the KS assumption) for higher-dimensional vacuum spacetimes of Weyl type III and N: the WANDs of such spacetimes must be geodetic, and the associated matrix L_{ij} is indeed of the form (50), with only one non-zero 2×2 block and zeros elsewhere [8].¹²

For a possible complete extension of the Goldberg-Sachs theorem, one should study the remaining possibilities, i.e. $n > 4$ vacuum spacetimes of Weyl type II and D that do not belong to the KS family. In addition, one should also analyze the inverse implication, that is, whether in vacuum the existence of a geodetic null congruence with an associated matrix L_{ij} of the form (50) implies that the algebraic type is II or more special (this is already known in the simple Kundt case $L_{ij} = 0$ [9]). Let us recall that the ‘geodetic part’ of the $n > 4$ Goldberg-Sachs theorem has been discussed in [16].

It is worth mentioning that a $n > 4$ extension of the Robinson theorem has been proven in even dimensions [26] assuming a generalization of the shearfree condition (see also [27–30]) different from ours. The relation to our work will be discussed elsewhere.

5.5 Weyl tensor components of boost weight zero

The r -dependence of the matrix L_{ij} determines explicitly also the r -dependence of the Weyl components with boost weight 0 (cf appendix A). In order to express these components compactly, it is convenient to introduce a $(n - 2) \times (n - 2)$ matrix $\Phi_{ij} \equiv C_{0i1j}$ along with its symmetric and antisymmetric parts $\Phi_{ij}^S \equiv \Phi_{(ij)}$ and $\Phi_{ij}^A \equiv \Phi_{[ij]}$. All boost order zero components of the Weyl tensor are then determined in terms of the $(n - 2)(n - 3)/2$ independent components of Φ_{ij}^A and the $(n - 1)(n - 2)^2(n - 3)/12$ independent components of C_{ijkl} , since $\Phi_{ij}^S = -\frac{1}{2}C_{ikjk}$, $C_{0i1j} = 2\Phi_{ij}^A$, $C_{0101} = \Phi \equiv \Phi_{ii}$ [8, 16].

It follows from appendix A that, using the adapted frame of appendix C, the matrix Φ_{ij} inherits the block structure of L_{ij} , with the only non-zero components and trace given by

$$\Phi_{2\mu, 2\mu} = \Phi_{2\mu+1, 2\mu+1} = -2\mathcal{H}A_{2\mu, 2\mu+1}^2 - s_{(2\mu)}D\mathcal{H}, \quad (62)$$

$$\Phi_{2\mu, 2\mu+1} = \Phi_{2\mu+1, 2\mu}^A = -D(\mathcal{H}A_{2\mu, 2\mu+1}), \quad (63)$$

$$\Phi_{\alpha\beta} = -\frac{1}{r}\delta_{\alpha\beta}D\mathcal{H}, \quad \Phi = D^2\mathcal{H} = -(n - 2)\theta D\mathcal{H} - 2\mathcal{H}\omega^2. \quad (64)$$

Up to index permutations, the non-zero C_{ijkl} components are

$$C_{2\mu, 2\mu+1, 2\mu, 2\mu+1} = 2\mathcal{H}(3A_{2\mu, 2\mu+1}^2 - s_{(2\mu)}^2), \quad (65)$$

$$\begin{aligned} C_{2\mu, 2\mu+1, 2\nu, 2\nu+1} &= 2C_{2\mu, 2\nu, 2\mu+1, 2\nu+1} = -2C_{2\mu, 2\nu+1, 2\mu+1, 2\nu} \\ &= 4\mathcal{H}A_{2\mu, 2\mu+1}A_{2\nu, 2\nu+1}, \end{aligned} \quad (66)$$

¹²More precisely, this has been proven for all $n > 4$ solutions of Weyl type N, for $n > 4$ non-twisting solutions of type III, and for all $n = 5$ solutions of type III. For twisting type III solutions with $n > 5$ an extra assumption on the Weyl tensor was necessary [8].

$$C_{2\mu, 2\nu, 2\mu, 2\nu} = C_{2\mu, 2\nu+1, 2\mu, 2\nu+1} = -2\mathcal{H}s_{(2\mu)}s_{(2\nu)}, \quad (67)$$

$$C_{(\alpha)(i)(\alpha)(i)} = -2\mathcal{H}s_{(i)}r^{-1}, \quad (68)$$

where $\nu \neq \mu$.

Let us observe that all the above components fall off at $r \rightarrow \infty$ as $1/r^{m+1}$ or faster (in the non-twisting case, i.e. when all $A_{2\mu, 2\mu+1}$ vanish, only $1/r^{m+1}$ terms are present, cf also [10]).

Since (as shown above) expanding KS spacetime can be only of Weyl type II or D in vacuum, Weyl components of boost weight 0 fully determine the Kretschmann scalar, i.e.

$$R_{abcd}R^{abcd} = 4(R_{0101})^2 + R_{ijkl}R_{ijkl} + 8R_{0j1i}R_{0i1j} - 4R_{01ij}R_{01ij}. \quad (69)$$

In vacuum $R_{abcd} = C_{abcd}$, and we can use the above compact notation to reexpress

$$R_{abcd}R^{abcd} = 4\Phi^2 + C_{ijkl}C_{ijkl} + 8\Phi_{ij}^S\Phi_{ij}^S - 24\Phi_{ij}^A\Phi_{ij}^A. \quad (70)$$

This will be useful soon in the discussion of singularities.

To conclude, let us observe that Weyl components of boost weight -1 and -2 contain also derivatives different from D , and we cannot study their r -dependence at this general level.

5.5.1 Type D KS spacetimes: also the second multiple WAND is geodetic

Type D KS spacetimes are a special subclass of general KS metrics. By definition of type D, only boost weight 0 components of the Weyl tensor (as given above) are non-zero in an adapted frame, i.e. they admit a second multiple WAND not parallel to the KS (geodetic) vector \mathbf{k} . Here we show that this second WAND is also geodetic (as already known in the special case of Myers-Perry black holes [17], see also [16]).

First, note that, from (63), one has $\Phi_{ij}^A = 0 \Leftrightarrow \omega^2 = 0$ (cf eqs. (51) and (56)). In addition, since $\Phi_{ij}^S \neq 0$ (cf (64) and appendix B), for $\omega = 0$ (i.e., $p = 0$) it is easy to see from (64) that $\Phi_{(i)(i)} \neq -\Phi$ for all values of i . Thus, for type D KS spacetimes proposition 6 of [16] implies that the second multiple WAND is indeed geodetic (since one can always align the frame vector \mathbf{n} to it, and this is compatible with the assumed condition $\overset{i}{M}_{j0} = 0$). This is a general consequence of the Goldberg-Sachs theorem in four dimensions, but it is a non-trivial result for $n > 4$ (cf [16]). Therefore, it can be viewed as another partial extension of the Goldberg-Sachs theorem, again limited to KS spacetimes.

5.6 Singularities

The form (56) of the KS function suggests there may be singularities at $r = 0$, except in the cases $2p = m$ (m even) and $2p = m - 1$ (m odd). In order to invariantly identify possible curvature singularities, let us analyze the behavior of the Kretschmann scalar (70).

Eq. (70) clearly consists of a sum of squares, except for the last term, which is negative. For our purposes, it suffices to focus on the first and the last terms determined by (63) and (64) with (51) and (56).

5.6.1 ‘Generic’ case ($2p \neq m$, $2p \neq m - 1$)

Excluding for now the special cases $2p = m$ and $2p = m - 1$, from (56) one easily finds that for $r \sim 0$

$$\mathcal{H} \sim r^{-(m-2p-1)}, \quad D\mathcal{H} \sim r^{-(m-2p)}, \quad D^2\mathcal{H} \sim r^{-(m-2p+1)}. \quad (71)$$

Inserting this into (63) and the second of eqs. (64), it is clear that the first term in (70) will dominate over the last term near $r \sim 0$, and therefore the Kretschmann scalar will diverge at $r = 0$, thus confirming the presence of a curvature singularity. Note that this ‘generic’ case includes, in particular, all non-twisting solutions, for which $p = 0$ (and $\Phi_{ij}^A = 0$).

As an explicit example, $n > 5$ Myers-Perry black holes with only one non-zero spin ($m = n - 2$, $p = 1$) fall in this subclass, and in appropriate coordinates [5] one has $2\mathcal{H} = -\mu r^{5-n}/(r^2 + a^2 \cos^2 \theta)$ (a is the angular momentum parameter). See [5] for a detailed discussion, including other possible examples with more than one rotation parameter.

5.6.2 Special cases $2p = m$ and $2p = m - 1$

Let us now comment on the cases $2p = m$ (m even) and $2p = m - 1$ (m odd), for which $\mathcal{H} = r\mathcal{H}_0 \prod_{\mu=1}^{m/2} [r^2 + (a_{(2\mu)}^0)^2]^{-1}$ and $\mathcal{H} = \mathcal{H}_0 \prod_{\mu=1}^{(m-1)/2} [r^2 + (a_{(2\mu)}^0)^2]^{-1}$, respectively. With these assumptions \mathcal{H} and its D -derivatives are clearly non-singular at $r = 0$ (since p denotes the number of non-vanishing terms $a_{(2\mu)}^0$ in (56)). However, in general, $a_{(2\mu)}^0$ may be functions of spacetime coordinates different from r . Given a specific KS solution, there may thus exist ‘special points’ where some of the $a_{(2\mu)}^0$ vanish. Let us say, e.g., that q (with $q \geq 1$) of such terms $a_{(2\mu)}^0$ vanish simultaneously there. Then, proceeding as above one finds that at those special points, when $r \rightarrow 0$

$$\begin{aligned} \mathcal{H} &\sim r^{-2q+1}, & D\mathcal{H} &\sim r^{-2q}, & D^2\mathcal{H} &\sim r^{-2q-1} & (2p = m), & (72) \\ \mathcal{H} &\sim r^{-2q}, & D\mathcal{H} &\sim r^{-2q-1}, & D^2\mathcal{H} &\sim r^{-2q-2} & (2p = m - 1). & (73) \end{aligned}$$

Again, the first term in (70) will dominate over the last one and the Kretschmann scalar will diverge at the ‘special points’ when $r \rightarrow 0$.

For example, for both the $n = 4$ Kerr solution ($m = 2$, $p = 1$) and the $n = 5$ Myers-Perry metric with only one spin ($m = 3$, $p = 1$) the only non-zero $a_{(2\mu)}^0$ function is determined by $(a_{(2)}^0)^2 = a^2 \cos^2 \theta$. Hence, there is a curvature singularity at $r = 0$ and $\theta = \pi/2$ (this is the well-known ring-shaped singularity of the Kerr spacetime). By contrast, the $n = 5$ Myers-Perry solution with two non-zero spins (again, $m = 3$, $p = 1$) contains $(a_{(2)}^0)^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$ (in the notation of, e.g., [31]), which never vanishes. Possible singularities of rotating black hole spacetimes are studied in more generality in [5].

5.7 Special subfamilies

We now briefly comment on special subfamilies of expanding KS solutions, characterized by a non-twisting or a non-shearing KS vector \mathbf{k} . In relation to the first possibility, let us note that general properties of

higher-dimensional non-twisting vacuum spacetimes (not necessarily KS) have been recently studied in [10], and that all such solutions are explicitly known with the additional assumptions of non-zero expansion and vanishing shear (‘Robinson-Trautman spacetimes’) [32, 33].

5.7.1 Non-twisting solutions

The KS vector \mathbf{k} is non-twisting if and only if L_{ij} is a symmetric matrix, i.e. for $p = 0$. This clearly gives

$$L_{ij} = S_{ij} = \frac{1}{r} \text{diag}(\underbrace{1, \dots, 1}_m, \underbrace{0, \dots, 0}_{(n-2-m)}), \quad (74)$$

so that (cf (53)–(55))

$$\theta = \frac{1}{r} \frac{m}{n-2}, \quad \omega = 0, \quad \sigma^2 = \frac{1}{r^2} \frac{m(n-2-m)}{n-2}. \quad (75)$$

From eq. (56), the r -dependence of \mathcal{H} becomes simply

$$\mathcal{H} = \frac{\mathcal{H}_0}{r^{m-1}}. \quad (76)$$

For Weyl components with boost weight 0 we now have the non-zero components

$$\Phi_{\alpha\beta} = \Phi_{\alpha\beta}^S = \frac{\mathcal{H}_0}{r^{m+1}} (m-1) \delta_{\alpha\beta}, \quad (77)$$

$$C_{(\alpha)(\beta)(\alpha)(\beta)} = -\frac{2\mathcal{H}_0}{r^{m+1}}. \quad (78)$$

As mentioned above, all non-twisting KS solutions contain a curvature singularity at $r = 0$.

Explicit n -dimensional examples of non-twisting KS spacetimes with $m \geq 2$ are static (uniform) black strings and black branes constructed as a direct product of a $(m+2)$ -dimensional Schwarzschild black hole with $(n-m-2)$ -dimensional flat space (which are of Weyl type D [16]). In suitable coordinates, these can be written as

$$ds^2 = \left[-(2dr + Kdu)du + r^2 d\Omega_m^2 + \delta_{AB} dy^A dy^B \right] + \frac{\mu}{r^{m-1}} du^2, \quad (79)$$

where the metric in square brackets represents a flat n -dimensional spacetime (with $A, B = 1, \dots, n-m-2$, and $d\Omega_m^2$ being the line element of a m -dimensional space of constant curvature K), $k_a dx^a = du$ and $2\mathcal{H} = -\mu r^{-m+1}$.

The above non-twisting KS solutions are, in addition, non-shearing iff $m = n - 2$ (the Kundt case $m = 0$ is ruled out here by the assumption $\theta \neq 0$). In this case, they must belong to the family of higher-dimensional Robinson-Trautman spacetimes [32]. In fact, one can see from the results of [32] that the only Robinson-Trautman solutions that are also KS are given by static Schwarzschild-Tangherlini black holes, i.e. by metric (79) with $m = n - 2$ (in which case a cosmological constant can also be added [32] while still preserving the KS form).¹³

¹³The $n > 4$ Robinson-Trautman family contains, in addition, generalized black holes with

5.7.2 Non-shearing solutions (n even)

Non-shearing, non-twisting solutions belong to the already discussed KS-Robinson-Trautman class, so that we can now focus on non-shearing but twisting solutions. Recall [9] that $\sigma = 0$ and $\omega \neq 0$ can occur only for even n (and for $\theta \neq 0$).

We see from (55) that $\sigma = 0$ requires $m = n - 2$ (i.e., L_{ij} must be non-degenerate), so that m is also necessarily even. Then, we have to consider two possible situations.

1. If $2p = n - 2$, \mathbf{k} is shearfree iff $s_{(2\mu)} = \theta$ for any $\mu = 1, 2, \dots, (n-2)/2$ (cf (55)), so that $a_{(2)}^0 = a_{(4)}^0 = \dots a_{(n-2)}^0 \equiv a_0$, and

$$\theta = \frac{r}{r^2 + a_0^2}, \quad \omega = \sqrt{n-2} \frac{a_0}{r^2 + a_0^2}. \quad (80)$$

This agrees with the general behavior found in [9] (note also that $A_{ik}A_{jk} = \frac{\omega^2}{n-2}\delta_{ij}$, as expected from the optical constraint (45) with $\sigma = 0$).

The function \mathcal{H} reduces to

$$\mathcal{H} = \frac{r\mathcal{H}_0}{(r^2 + a_0^2)^{\frac{n-2}{2}}}. \quad (81)$$

This includes, for instance, the $n = 4$ Kerr solution. Explicit solutions for $n > 4$ seem to be presently unknown.

2. If $2p < n - 2$, we have the additional condition $\theta = 1/r$. This implies $\omega = a_0 = 0$, which is the already-discussed non-twisting case.

6 Conclusions

We have presented a systematic study of geometric properties of KS metrics in higher dimensions. We have preliminary discussed general results that do not require any further assumptions. Namely, the KS vector \mathbf{k} is geodesic if and only if the energy-momentum tensor satisfies $T_{ab}k^a k^b = 0$ (proposition 1). When this happens (e.g., in vacuum) the Weyl tensor is of type II (or more special) and \mathbf{k} is a multiple WAND (proposition 2). Furthermore, we have shown that optical properties of \mathbf{k} are the same with respect to both the flat background and the full metric. Subsequently, our analysis has focused on vacuum solutions.

For non-expanding metrics the most general KS solution is now known, since they are equivalent to the type N Kundt class in vacuum (proposition 4). Expanding solutions required a more detailed analysis. Again, they turned out to be algebraically special, but the only possible types are II and D (proposition 5). We have also shown that the choice of a possible KS congruence is restricted in vacuum by an ‘‘optical constraint’’. In $n = 4$ dimensions this requires \mathbf{k} to be a shearfree congruence, in agreement with the standard Goldberg-Sachs theorem. For $n > 4$ we have

non-constant curvature horizons [32], and ‘exceptional’ solutions with ‘zero mass’ $\mu = 0$ [32,33] (see also [16]). These cannot be KS spacetimes because their Weyl components C_{ijkl} contain an r^{-2} term which is absent from (78).

proven a partial and apparently weaker extension of this theorem, which however naturally reduces to the familiar result when $n = 4$. This extension has been derived here only for KS solutions but, interestingly, it agrees with previous results for general vacuum type III/N spacetimes [8]. Moreover, by integration of the Ricci identities we have fixed the dependence of the optical matrix L_{ij} and of the KS function \mathcal{H} on the affine parameter r along \mathbf{k} . This enabled us to prove the presence of a curvature singularity in “generic” expanding KS spacetimes. As a side remark, let us note that we were interested in the $n > 4$ case, but our analysis also applies to the $n = 4$ case, for which we have rederived various known results previously scattered in several publications (many reviewed, however, in [4]).

In future work, it will be interesting to employ our results to possibly find new expanding KS solutions, and to understand to what extent the Myers-Perry metrics exhaust the expanding vacuum KS class. In addition, further investigation will admit non-zero matter fields and *generalized* KS solutions, in which the background is not necessarily flat. These include other important spacetimes such as rotating black holes in (A)dS [34, 35] and their NUT extensions [25].

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A Frame components of the Riemann tensor (\mathbf{k} geodetic)

When \mathbf{k} is geodesic and affinely parametrized we find the following frame components of the Riemann tensor corresponding to the line element (2)

$$R_{0i0j} = 0, \quad R_{010i} = 0, \quad R_{0ijk} = 0, \quad (\text{A1})$$

$$R_{0101} = D^2\mathcal{H}, \quad R_{01ij} = 2A_{ji}D\mathcal{H} + 4\mathcal{H}S_{k[j}A_{i]k}, \quad (\text{A2})$$

$$R_{0i1j} = -L_{ij}D\mathcal{H} - 2\mathcal{H}A_{ki}L_{kj}, \quad (\text{A3})$$

$$R_{ijkl} = 4\mathcal{H}(A_{ij}A_{kl} + A_{k[j}A_{i]l} + S_{l[i}S_{j]k}), \quad (\text{A4})$$

$$R_{011i} = -\delta_i(D\mathcal{H}) + 2L_{[i1]}D\mathcal{H} + L_{ji}\delta_j\mathcal{H} + 2\mathcal{H}(2L_{ji}L_{[1j]} + L_{j1}A_{ji}), \quad (\text{A5})$$

$$R_{1ijk} = 2L_{[j|i}\delta_{|k]}\mathcal{H} + 2A_{jk}\delta_i\mathcal{H} - 2\mathcal{H}\left(\delta_i A_{kj} + L_{1j}L_{ki} - L_{1k}L_{ji} - L_{j1}A_{ki} + L_{k1}A_{ji} + 2L_{[1i]}A_{kj} + A_{lj}M_{ki}^l - A_{lk}M_{ji}^l\right), \quad (\text{A6})$$

$$R_{1i1j} = \delta_{(i}(\delta_{j)}\mathcal{H}) + M_{(ij)}^k\delta_k\mathcal{H} + (2L_{1j} - L_{j1})\delta_i\mathcal{H} + (2L_{1i} - L_{i1})\delta_j\mathcal{H} + N_{(ij)}D\mathcal{H} - S_{ij}\Delta\mathcal{H} + 2\mathcal{H}\left(\delta_{(i|}L_{1|j)} - \Delta S_{ij} - 2L_{1(i}L_{j)1} + 2L_{1i}L_{1j} - L_{k(i}N_{k|j)} - 2\mathcal{H}L_{k(i}A_{j)k}\right)$$

$$+ L_{1k} \overset{k}{M}_{(ij)} - 2\mathcal{H}A_{ik}A_{jk} - L_{k(i} \overset{k}{M}_{j)1} - L_{(i|k} \overset{k}{M}_{|j)1}). \quad (\text{A7})$$

For certain calculations, it may be useful to note that, using the Ricci identities (11k, [9]), the above component R_{1ijk} can also be transformed to the somewhat different form

$$\begin{aligned} R_{1ijk} = & 2A_{jk}\delta_i\mathcal{H} - L_{ki}\delta_j\mathcal{H} + L_{ji}\delta_k\mathcal{H} + 2\mathcal{H}\left(2\delta_{[k}S_{j]i} + 2\overset{l}{M}_{[jk]}S_{il}\right. \\ & \left. - 2\overset{l}{M}_{i[j}S_{k]l} + 2L_{1i}A_{jk} - 2L_{1[j}A_{k]i}\right). \end{aligned} \quad (\text{A8})$$

Let us finally emphasise that throughout sections 4 and 5 and in appendix D we restrict to vacuum spacetimes, so that $C_{abcd} = R_{abcd}$ and the Weyl tensor is there given simply by the above expressions.

B Expanding solutions with $D\mathcal{H} = 0$ do not exist in vacuum

In the special case when $D\mathcal{H} = 0$, from the vacuum equation (33) we get $\omega = 0$, i.e. $L_{ij} = S_{ij}$. Then eq. (30) reads

$$S_{ik}S_{jk} = (n-2)\theta S_{ij}. \quad (\text{B1})$$

Using an eigenframe of S_{ij} , it is easy to see that the only possible solution has the form

$$S_{ij} = s_{(2)} \text{diag}(1, 0, 0, \dots), \quad (\text{B2})$$

with $s_{(2)} = (n-2)\theta$. Now, putting $D\mathcal{H} = 0$, $A_{ij} = 0$, $L_{ij} = S_{ij}$ and eq. (B2) into eqs. (A2)–(A4), we find that all components of the Riemann (Weyl) tensor with non-negative boost weight vanish. The algebraic type must thus be III or N. However, by analyzing the Bianchi identities it has been proven in [8] that the canonical form of the expansion matrix S_{ij} for type N and (non-twisting) type III spacetimes in vacuum is $S_{ij} = s_{(2)} \text{diag}(1, 1, 0, 0, \dots)$ (cf eqs. (50) and (C.20) of [8]). This is clearly incompatible with (B2). Therefore, there do not exist vacuum solutions of the KS class with $\theta \neq 0$ and $D\mathcal{H} = 0$.

C Solving the optical constraint

In this appendix we provide a solution of the optical constraint (45) for the matrix L_{ij} , in the case of an expanding KS vector, i.e. $\theta \neq 0$. We introduce the compact notation

$$\mathcal{F} = \frac{L_{ik}L_{lk}}{(n-2)\theta} = \frac{\sigma^2 + \omega^2 + (n-2)\theta^2}{(n-2)\theta}, \quad (\text{C1})$$

so that (45) simply reads

$$L_{il}L_{jl} = \mathcal{F}S_{ij}. \quad (\text{C2})$$

In the following, it will be convenient to analyze separately the two possible cases $\det L \neq 0$ and $\det L = 0$.

C.1 Non-degenerate case

Let us start with the non-degenerate case $\det L \neq 0$, so that there exists a matrix inverse of L_{ij} . Let us denote by L_{ij}^{-1} such inverse matrix (or, sometimes, its (ij) -element – there will be no ambiguity according to the context). It is also convenient to define symbols for its antisymmetric and symmetric parts, i.e.

$$B_{ij} = L_{[ij]}^{-1}, \quad C_{ij} = L_{(ij)}^{-1}. \quad (\text{C3})$$

Now, multiplying (C2) by L_{kj}^{-1} one gets

$$L_{ik} = \mathcal{F} S_{ij} L_{kj}^{-1}, \quad (\text{C4})$$

and by further multiplication by L_{ii}^{-1} we find $\delta_{lk} = \mathcal{F} C_{lk}$, so that

$$L_{ij}^{-1} = \mathcal{F}^{-1} \delta_{ij} + B_{ij}. \quad (\text{C5})$$

Eq. (C4) thus becomes

$$L_{ik} = S_{ik} + \mathcal{F} S_{ij} B_{kj}. \quad (\text{C6})$$

The symmetric and anti-symmetric parts of this equation give rise, respectively, to

$$B_{kj} S_{ji} + B_{ij} S_{jk} = 0, \quad (\text{C7})$$

$$2A_{ik} = \mathcal{F}(B_{kj} S_{ji} - B_{ij} S_{jk}). \quad (\text{C8})$$

The symmetric matrix S_{ij} defines a natural frame of orthonormal eigenvectors. It is thus convenient to identify our basis vectors $\mathbf{m}^{(i)}$ with such eigenvectors,¹⁴ so that $S_{ij} = \text{diag}(s_{(2)}, s_{(3)}, s_{(4)}, \dots)$. Eqs. (C7) and (C8) then read

$$B_{ki}(s_{(i)} - s_{(k)}) = 0, \quad (\text{C9})$$

$$2A_{ik} = \mathcal{F} B_{ki}(s_{(i)} + s_{(k)}). \quad (\text{C10})$$

If $B_{ki} = 0$ for all values of the indices, then eqs. (C9) and (C10) are identically satisfied and, by (C5), $L_{ij} = \mathcal{F} \delta_{ij}$, and all eigenvalues of S_{ij} take the same value $s_{(i)} = \mathcal{F}$.

If, instead, for some components $B_{\bar{k}\bar{i}} \neq 0$, eq. (C9) implies that the corresponding eigenvalues of S_{ij} coincide (i.e., $s_{(\bar{i})} = s_{(\bar{k})}$). After ordering the basis vectors so as to have multiple eigenvalues next to each other on the diagonal of S_{ij} , the matrix B_{ki} must thus be composed of (anti-symmetric) blocks with their diagonal on the main diagonal, each block corresponding to a repeated $s_{(\bar{i})}$ (e.g., $B_{23} \neq 0$ implies $s_{(3)} = s_{(2)}$ and then, if $s_{(4)} \neq s_{(2)}$, we have $B_{24} = 0$, etc.). Moreover, there can be at most one eigenvalue of S_{ij} with multiplicity 1, and this must equal \mathcal{F} (supposing, e.g., that $s_{(2)} \neq s_{(\hat{k})}$ for $\hat{k} = 3, \dots, n-1$, then one gets $B_{2\hat{k}} = 0$,

¹⁴A *priori*, there is no reason to expect that this frame is parallelly propagated along \mathbf{k} , a condition that we do not need here. However, we shall see in the following appendix that such an eigenframe is, in fact, compatible with parallel transport.

so that, by (C5), $L_{2\bar{k}}^{-1} = 0 = L_{\bar{k}2}^{-1}$, $L_{22}^{-1} = \mathcal{F}^{-1}$, and thus $s_{(2)} = \mathcal{F}$; and of course there cannot be distinct eigenvalues both equal to \mathcal{F} .

Indices \bar{i} , \bar{j} , \bar{k} , ... take values within a given block (and thus $s_{(\bar{i})} = s_{(\bar{k})}$), so that eq. (C10) now splits into separate equations for each block of the matrix B_{ki} , i.e.

$$A_{\bar{i}\bar{k}} = s_{(\bar{i})} \mathcal{F} B_{\bar{k}\bar{i}}, \quad (\text{C11})$$

and components of A_{ij} with indices referring to different blocks necessarily vanish (since thus does B_{ij}). The matrix A_{ij} has thus the same block-structure of B_{ij} (or simpler, if some of the $s_{(i)}$ vanish). Eq. (C11) also implies that all eigenvalues $s_{(i)}$ (for all blocks) must be non-zero, since we are now considering the case $\det L \neq 0$.

Using (C5), (C6), and (C11), within each block we can thus write

$$L_{\bar{i}\bar{j}}^{-1} = \mathcal{F}^{-1} \delta_{\bar{i}\bar{j}} + B_{\bar{i}\bar{j}}, \quad L_{\bar{i}\bar{j}} = s_{(\bar{i})} (\delta_{\bar{i}\bar{j}} + \mathcal{F} B_{\bar{j}\bar{i}}). \quad (\text{C12})$$

But since these two blocks must be inverse to each other (and since $s_{(\bar{i})} \neq 0$ takes the same value for any \bar{i} within a block), we necessarily have

$$B_{\bar{i}\bar{k}} B_{\bar{j}\bar{k}} = \frac{\mathcal{F} - s_{(\bar{i})}}{\mathcal{F}^2 s_{(\bar{i})}} \delta_{\bar{i}\bar{j}}. \quad (\text{C13})$$

This implies that any block-matrix $B_{\bar{i}\bar{j}}$ must be even-dimensional (and that $(\mathcal{F} - s_{(\bar{i})})s_{(\bar{i})} > 0$), unless it vanishes. When this happens, one has simply $L_{\bar{i}\bar{j}}^{-1} = \mathcal{F}^{-1} \delta_{\bar{i}\bar{j}}$ and $L_{\bar{i}\bar{j}} = s_{(\bar{i})} \delta_{\bar{i}\bar{j}}$, so that

$$s_{(\bar{i})} = \mathcal{F} \quad (\Leftrightarrow B_{\bar{i}\bar{j}} = 0). \quad (\text{C14})$$

Hence, if there exist more than one block satisfying $B_{\bar{i}\bar{j}} = 0$, to such values of indices there corresponds only one possible eigenvalue $s_{(\bar{i})} = \mathcal{F}$ of the matrix S_{ij} , i.e. S_{ij} contains a diagonal block given by $\mathcal{F} \text{diag}(1, 1, \dots, 1)$.

Within each even-dimensional block with $B_{\bar{i}\bar{j}} \neq 0$, $S_{\bar{i}\bar{j}} = s_{(\bar{i})} \delta_{\bar{i}\bar{j}}$ and we can thus finally perform a rotation so as to put this anti-symmetric matrix in a canonical form with two-dimensional anti-symmetric blocks (constrained by (C13)) along the diagonal and zeros elsewhere.

To summarize, we have shown that the matrix L_{ij}^{-1} can always be written in a block-diagonal form. If its antisymmetric part B_{ij} does not vanish identically, there is a number $p \leq (n-2)/2$ of two-dimensional blocks of the form

$$\mathcal{L}_{(\mu)}^{-1} = \begin{pmatrix} \mathcal{F}^{-1} & -\mathcal{F}^{-1} \left(\frac{\mathcal{F} - s_{(2\mu)}}{s_{(2\mu)}} \right)^{1/2} \\ \mathcal{F}^{-1} \left(\frac{\mathcal{F} - s_{(2\mu)}}{s_{(2\mu)}} \right)^{1/2} & \mathcal{F}^{-1} \end{pmatrix}, \quad (\text{C15})$$

$$\mu = 1, \dots, p, \quad 0 \leq p \leq \frac{n-2}{2}.$$

$s_{(2\mu)}$ need not be all distinct and thus some two-blocks may be identical.

In addition, there is a $(n-2-2p) \times (n-2-2p)$ -dimensional diagonal block

$$\tilde{\mathcal{L}}^{-1} = \mathcal{F}^{-1} \text{diag}(1, 1, \dots, 1). \quad (\text{C16})$$

Since each block can be inverted separately, finding the explicit form of the matrix L_{ij} is now straightforward. This consists of p blocks of the form

$$\mathcal{L}_{(\mu)} = \begin{pmatrix} s_{(2\mu)} & [s_{(2\mu)}(\mathcal{F} - s_{(2\mu)})]^{1/2} \\ -[s_{(2\mu)}(\mathcal{F} - s_{(2\mu)})]^{1/2} & s_{(2\mu)} \end{pmatrix}, \quad (\text{C17})$$

and of one $(n - 2 - 2p) \times (n - 2 - 2p)$ -dimensional diagonal block

$$\tilde{\mathcal{L}} = \mathcal{F} \underbrace{\text{diag}(1, 1, \dots, 1)}_{(n-2-2p)}. \quad (\text{C18})$$

More explicitly, the matrix L_{ij} thus takes the form

$$L_{ij} = \begin{pmatrix} \boxed{\mathcal{L}_{(1)}} & & & & & \\ & \boxed{\mathcal{L}_{(2)}} & & & & \\ & & \ddots & & & \\ & & & \boxed{\mathcal{L}_{(p)}} & & \\ & & & & \boxed{\tilde{\mathcal{L}}} & \end{pmatrix}. \quad (\text{C19})$$

C.2 Degenerate case

Let us now study the degenerate case $\det L = 0 = \det L^T$. This implies that there exists a non-zero vector v (living in the ‘transverse’ space spanned by vectors $\mathbf{m}^{(i)}$) such that $L^T v = 0$. By (C2) this gives also $Lv = 0$. If we now choose an orthonormal basis such that, say, $\mathbf{m}^{(n-1)}$ corresponds to v , we can rewrite $L^T v = 0 = Lv$ as

$$L_{(n-1),i} = 0 = L_{i,(n-1)}. \quad (\text{C20})$$

In addition, the optical constraint (C2) now reduces to $L_{i\bar{k}} L_{\bar{j}k} = \mathcal{F} S_{i\bar{j}}$, with the barred indices ranging from 2 to $(n - 2)$ only. In other words, we have the same equation in a reduced space. We can then distinguish the two cases $\det L_{i\bar{j}} \neq 0$ and $\det L_{i\bar{j}} = 0$, and similarly proceed until we find a subspace with a non-degenerate reduced matrix L . Finally, we can proceed exactly as in the non-degenerate case. It is thus obvious that the general form of L_{ij} will have the same block diagonal form (C19), except that now the diagonal block contains also $(n - 2 - m)$ zeros,

$$\tilde{\mathcal{L}} = \mathcal{F} \text{diag}(\underbrace{1, \dots, 1}_{(m-2p)}, \underbrace{0, \dots, 0}_{(n-2-m)}), \quad (\text{C21})$$

where $m \equiv \text{rank}(L)$, and $0 \leq 2p \leq m$. The non-degenerate case clearly corresponds to $m = n - 2$.

D Integration of the Ricci identities

The general form of the Ricci identities in higher dimensions has been given in [9]. Here we consider them only in the case $\theta \neq 0$, and in a form simplified by the fact that $\ell = \mathbf{k}$ is geodetic and affinely parametrized. Further simplification could be achieved by using a null frame that is parallelly transported along \mathbf{k} (which is always a possible choice [9]), i.e., $M_{j0}^i = 0 = N_{i0}$. For certain purposes, however, it is desirable to retain the full freedom of null rotations of \mathbf{n} , and for now we thus only impose

$$M_{j0}^i = 0. \quad (\text{D1})$$

We also employ the special form that the Riemann tensor takes in case of KS spacetimes, as given in appendix A, and use the compact notation (62)–(64) (strictly speaking, this is defined for the Weyl tensor, but in vacuum we can equivalently apply it to the Riemann tensor). Moreover, we keep into account consequences of the vacuum equations discussed in appendix C.

D.1 Sachs equations: form of L_{ij}

Under the above assumptions, eq. (11g, [9]) becomes

$$DL_{ij} = -L_{ik}L_{kj}. \quad (\text{D2})$$

D.1.1 Non-degenerate case

By methods similar to those of [12], the solution to (D2) can be written in terms of its inverse as [36]

$$L_{ij}^{-1} = r\delta_{ij} + (L^0)_{ij}^{-1}, \quad (\text{D3})$$

where $(L^0)_{ij}^{-1}$ represents L_{ij}^{-1} at $r = 0$. For compatibility with the optical constraint, the ‘initial value’ $(L^0)_{ij}^{-1}$ must be of the canonical form obtained in appendix C. The solution for any r can then be obtained immediately by adding a term $r\delta_{ij}$ to eqs. (C15) and (C16) evaluated at $r = 0$.

Hence, the inverse matrix L_{ij}^{-1} consists of p two-blocks and of one $(n - 2 - 2p) \times (n - 2 - 2p)$ -dimensional block of the form, respectively,

$$\mathcal{L}_{(\mu)}^{-1} = \begin{pmatrix} \mathcal{F}_0^{-1} + r & -\mathcal{F}_0^{-1} \left(\frac{\mathcal{F}_0 - s_{(2\mu)}^0}{s_{(2\mu)}^0} \right)^{1/2} \\ \mathcal{F}_0^{-1} \left(\frac{\mathcal{F}_0 - s_{(2\mu)}^0}{s_{(2\mu)}^0} \right)^{1/2} & \mathcal{F}_0^{-1} + r \end{pmatrix}, \quad (\text{D4})$$

$$\tilde{\mathcal{L}}^{-1} = (\mathcal{F}_0^{-1} + r)\text{diag}(1, 1, \dots, 1). \quad (\text{D5})$$

Correspondingly, the matrix L_{ij} (C19) consists of blocks

$$\mathcal{L}_{(\mu)} = \frac{1}{1 + 2rs_{(2\mu)}^0 + r^2s_{(2\mu)}^0\mathcal{F}_0}$$

$$\times \begin{pmatrix} s_{(2\mu)}^0(1+r\mathcal{F}_0) & [s_{(2\mu)}^0(\mathcal{F}_0-s_{(2\mu)}^0)]^{1/2} \\ -[s_{(2\mu)}^0(\mathcal{F}_0-s_{(2\mu)}^0)]^{1/2} & s_{(2\mu)}^0(1+r\mathcal{F}_0) \end{pmatrix}, \quad (\text{D6})$$

$$\tilde{\mathcal{L}} = \frac{\mathcal{F}_0}{1+r\mathcal{F}_0} \text{diag}(1, 1, \dots, 1). \quad (\text{D7})$$

By construction, this matrix L_{ij} satisfies the optical constraint at $r=0$. However, note that the last two equations are of the same form of (C17) and (C18), respectively, provided one rewrites them using

$$s_{(2\mu)} = \frac{s_{(2\mu)}^0(1+r\mathcal{F}_0)}{1+2rs_{(2\mu)}^0+r^2s_{(2\mu)}^0\mathcal{F}_0}, \quad \mathcal{F} = \frac{\mathcal{F}_0}{1+r\mathcal{F}_0}. \quad (\text{D8})$$

This implies that L_{ij} automatically satisfies the optical constraint for any r . In other words, the canonical frame determined by the optical constraint is compatible with the condition $\overset{i}{M}_{j0} = 0$ (and, in particular, with parallel transport along \mathbf{k} if $N_{i0} = 0$ is also required).

For practical purposes it will be convenient to simplify the above equations by shifting the affine parameter as $r = \tilde{r} - 1/\mathcal{F}_0$, and simultaneously defining the new quantities $a_{(2\mu)}^0 = [s_{(2\mu)}^0(\mathcal{F}_0 - s_{(2\mu)}^0)]^{1/2}/(s_{(2\mu)}^0\mathcal{F}_0)$. Then one has (dropping tildes over r)

$$\mathcal{L}_{(\mu)} = \frac{1}{r^2 + (a_{(2\mu)}^0)^2} \begin{pmatrix} r & a_{(2\mu)}^0 \\ -a_{(2\mu)}^0 & r \end{pmatrix}, \quad \tilde{\mathcal{L}} = \frac{1}{r} \text{diag}(1, 1, \dots, 1). \quad (\text{D9})$$

This general structure of L_{ij} agrees with the results of [16] (up to reordering the basis vectors) for the specific case of Myers-Perry black holes.

D.1.2 Degenerate case

When $\det L = 0$, it is easy to see that the vector v (that satisfies $Lv = 0$) can be parallelly transported along \mathbf{k} [36]. We can therefore proceed as in the non-degenerate case to solve the Sachs equations in the ‘non-zero block’ of L_{ij} , the remaining part of the matrix keeping its zero form. We thus do not repeat the results here, just replace the last $(n-2-m)$ entries of $\tilde{\mathcal{L}}$ by zeros in (D9).

D.1.3 Summary

In the following, we will often need to refer to the elements of the matrices S_{ij} and A_{ij} , and we will need a notation that enables us to handle both the non-degenerate and the degenerate case simultaneously. It will also be convenient to distinguish between indices α, β, \dots referring to the non-zero eigenvalues of $\tilde{\mathcal{L}}$ and ρ, σ, \dots to its zero eigenvalues. For further compactness, we also use a complex notation (with i being the imaginary unit) and we summarize the non-zero elements of L_{ij} as

$$s_{(2\mu)} + iA_{2\mu, 2\mu+1} = \frac{r + ia_{(2\mu)}^0}{r^2 + (a_{(2\mu)}^0)^2}, \quad (\mu = 1, \dots, p) \quad (\text{D10})$$

$$s_{(\alpha)} = \frac{1}{r} \quad (\alpha = 2p + 2, \dots, m + 1), \quad (\text{D11})$$

$$s_{(\rho)} = 0 \quad (\rho = m + 2, \dots, n - 1). \quad (\text{D12})$$

D.2 Remaining Ricci rotation coefficients

The above results for L_{ij} enable us to find the r dependence of all Ricci rotation coefficients by integrating the corresponding Ricci identities [9] that contain a derivative along \mathbf{k} . From now on, however, in addition to (D1) we also use the null rotation freedom on \mathbf{n} to set

$$N_{i0} = 0, \quad (\text{D13})$$

i.e. we use a null frame that is *parallelly transported* along \mathbf{k} .

D.2.1 Form of L_{1i} and L_{i1}

Let us start from eq. (11b, [9]), which now takes the form

$$DL_{1i} = -L_{1j}L_{ji}. \quad (\text{D14})$$

These equations decouple according to the block structure of L_{ij} , and the resulting coefficients can be compactly written in complex form as

$$L_{1,2\mu} + iL_{1,2\mu+1} = (l_{1,2\mu}^0 + il_{1,2\mu+1}^0)(s_{(2\mu)} + iA_{2\mu,2\mu+1}), \quad (\text{D15})$$

where $l_{1,2\mu}^0$ and $l_{1,2\mu+1}^0$ are real integration ‘constants’, and $s_{(2\mu)}$ and $A_{2\mu,2\mu+1}$ are given in (D10).

The solution for coefficients L_{1i} with i corresponding to the diagonal block of L_{ij} is simply

$$L_{1\alpha} = \frac{l_{1\alpha}^0}{r}, \quad L_{1\rho} = l_{1\rho}^0. \quad (\text{D16})$$

Ricci equations (11e, [9]), i.e.

$$DL_{i1} = -L_{ij}L_{j1}, \quad (\text{D17})$$

lead to similar expressions for the coefficients L_{i1} , namely

$$L_{2\mu,1} + iL_{2\mu+1,1} = (l_{2\mu,1}^0 + il_{2\mu+1,1}^0)(s_{(2\mu)} - iA_{2\mu,2\mu+1}), \quad (\text{D18})$$

$$L_{\alpha 1} = \frac{l_{\alpha 1}^0}{r}, \quad L_{\rho 1} = l_{\rho 1}^0. \quad (\text{D19})$$

D.2.2 Form of $\overset{i}{M}_{jk}$

Next, the coefficients $\overset{i}{M}_{jk}$ are analogously determined by eq. (11n, [9]), which here takes the form

$$D\overset{i}{M}_{jk} = -\overset{i}{M}_{jl}L_{lk}. \quad (\text{D20})$$

Again, the solutions can be given in natural pairs, each for each block of L_{ij} , as distinguished by the value of the last index k , i.e.

$$\overset{i}{M}_{j,2\mu} + i\overset{i}{M}_{j,2\mu+1} = (\overset{i}{m}_{j,2\mu}^0 + i\overset{i}{m}_{j,2\mu+1}^0)(s_{(2\mu)} + iA_{2\mu,2\mu+1}), \quad (\text{D21})$$

with real integration ‘constants’. When k corresponds to the diagonal part of L_{ij} we have simply

$$M_{j\alpha}^i = \frac{\dot{m}_{j\alpha}^0}{r}, \quad M_{j\rho}^i = \dot{m}_{j\rho}^0. \quad (\text{D22})$$

Because of the index symmetries of M_{jk}^i [8], we require $\dot{m}_{jk}^0 + \dot{m}_{ik}^0 = 0$ for any $i, j, k = 2, \dots, n-1$.

D.2.3 Form of M_{j1}^i

Eq. (11m, [9]) here becomes

$$DM_{j1}^i = -M_{jk}^i L_{k1} - 2\Phi_{ij}^A. \quad (\text{D23})$$

From (D10), one has $DA_{ij} = -2s_{(i)}A_{ij}$ and thus, by (A2), $\Phi_{ij}^A = -D(\mathcal{H}A_{ij})$.

Using also previous results for M_{jk}^i we find

$$\begin{aligned} M_{j1}^i = & \sum_{\mu=1}^p \left[\frac{r}{r^2 + (a_{(2\mu)}^0)^2} (\dot{m}_{j,2\mu}^0 l_{2\mu,1}^0 + \dot{m}_{j,2\mu+1}^0 l_{2\mu+1,1}^0) \right. \\ & \left. + \frac{a_{(2\mu)}^0}{r^2 + (a_{(2\mu)}^0)^2} (\dot{m}_{j,2\mu}^0 l_{2\mu+1,1}^0 - \dot{m}_{j,2\mu+1}^0 l_{2\mu,1}^0) \right] \\ & + \frac{\dot{m}_{j\alpha}^0 l_{\alpha 1}^0}{r} - \dot{m}_{j\rho}^0 l_{\rho 1}^0 r + 2A_{ij}\mathcal{H} + \dot{m}_{j1}^0, \end{aligned} \quad (\text{D24})$$

where the real ‘constants’ satisfy $\dot{m}_{j1}^0 + \dot{m}_{i1}^0 = 0$.

D.2.4 Form of L_{11}

One can also easily integrate (11a, [9]), here reduced to

$$DL_{11} = -L_{1i}L_{i1} - R_{0101}, \quad (\text{D25})$$

to fix the r -dependence of L_{11} , i.e.

$$\begin{aligned} L_{11} = & -D\mathcal{H} + l_{11}^0 + \sum_{\mu=1}^p \left[(l_{1,2\mu}^0 l_{2\mu,1}^0 + l_{1,2\mu+1}^0 l_{2\mu+1,1}^0) s_{(2\mu)} \right. \\ & \left. + (l_{1,2\mu}^0 l_{2\mu+1,1}^0 - l_{1,2\mu+1}^0 l_{2\mu,1}^0) A_{2\mu,2\mu+1} \right] \\ & + \frac{l_{1\alpha}^0 l_{\alpha 1}^0}{r} - l_{1\rho}^0 l_{\rho 1}^0 r. \end{aligned} \quad (\text{D26})$$

D.2.5 Form of N_{ij}

Eq. (11j, [9]) here reads

$$DN_{jk} = -N_{jl}L_{lk} - \Phi_{kj}. \quad (\text{D27})$$

Note that, for $k = 2\mu, 2\mu + 1$, it is more natural to deal with a complex unknown, namely $N_{j,2\mu} + iN_{j,2\mu+1}$. Then, using (62), (63) and (D10)–(D12), eq. (D27) can be solved obtaining

$$N_{j,2\mu} + iN_{j,2\mu+1} = (n_{j,2\mu}^0 + in_{j,2\mu+1}^0)(s_{(2\mu)} + iA_{2\mu,2\mu+1}) + \mathcal{H}(s_{(2\mu)} - iA_{2\mu,2\mu+1})(\delta_{j,2\mu} + i\delta_{j,2\mu+1}), \quad (\text{D28})$$

$$N_{j\alpha} = \frac{1}{r}(n_{j\alpha}^0 + \mathcal{H}\delta_{j\alpha}), \quad N_{j\rho} = n_{j\rho}^0, \quad (\text{D29})$$

where n_{ij}^0 are real ‘constants’.

D.2.6 Form of N_{i1} in an adapted parallelly transported frame

So far, the frame vector \mathbf{n} has not been specified, except for the requirement that it be parallelly transported along \mathbf{k} . While retaining the latter condition, we can still perform a null rotation

$$\hat{\mathbf{k}} = \mathbf{k}, \quad \hat{\mathbf{n}} = \mathbf{n} + z_i \mathbf{m}^{(i)} - \frac{1}{2} z_k z_k \boldsymbol{\ell}, \quad \hat{\mathbf{m}}^{(i)} = \mathbf{m}^{(i)} - z_i \boldsymbol{\ell}, \quad (\text{D30})$$

provided $Dz_i = 0$, so as to simplify some of the Ricci rotation coefficients. In particular, if we choose

$$z_i = -l_{i1}^0 \quad (\text{for } i = 2, \dots, m+1), \quad z_\rho = 0, \quad (\text{D31})$$

we obtain (see the transformation properties given in [9] – we drop hats from the transformed coefficients)

$$L_{i1} = 0 \quad (i = 2, \dots, m+1), \quad (\text{D32})$$

which is equivalent to $l_{2\mu,1}^0 = l_{2\mu+1,1}^0 = l_{\alpha 1}^0 = 0$, while the $L_{\rho 1} = l_{\rho 1}^0$ are unchanged. (Alternatively, one can also set $L_{1i} = 0$, but this appears to be less convenient in what follows.)

With this choice, eq. (11f, [9]) becomes

$$DN_{i1} = -n_{i\rho}^0 l_{\rho 1}^0 + R_{101i}. \quad (\text{D33})$$

In addition, using eq. (D14) and the commutator (22, [21]), eq. (A5) reduces to

$$R_{011i} = -D(\delta_i \mathcal{H}) - 2D(L_{1i} \mathcal{H}) \quad (i = 2, \dots, m+1), \quad (\text{D34})$$

$$R_{011\rho} = -D(\delta_\rho \mathcal{H}) - 2D(L_{1\rho} \mathcal{H}) + l_{\rho 1}^0 D\mathcal{H}. \quad (\text{D35})$$

We can thus straightforwardly integrate eq. (D33) and find

$$N_{i1} = -n_{i\sigma}^0 l_{\sigma 1}^0 r + \delta_i \mathcal{H} + 2L_{1i} \mathcal{H} + n_{i1}^0, \quad (i = 2, \dots, m+1) \quad (\text{D36})$$

$$N_{\rho 1} = -n_{\rho\sigma}^0 l_{\sigma 1}^0 r + \delta_\rho \mathcal{H} + (2l_{1\rho}^0 - l_{\rho 1}^0) \mathcal{H} + n_{\rho 1}^0, \quad (\text{D37})$$

where \mathcal{H} and L_{1i} are specified by (56) and (D15), (D16). For certain applications it may be useful to recall that the coefficients N_{i1} vanish if and only if the frame vector \mathbf{n} is geodetic (an affine parameter corresponding to $L_{11} = 0$).

Note finally that, when $l_{2\mu,1}^0 = l_{2\mu+1,1}^0 = l_{\alpha 1}^0 = 0$, eq. (D24) simplifies to

$$M_{j1}^i = -m_{j\rho}^i l_{\rho 1}^0 r + 2A_{ij}\mathcal{H} + m_{j1}^i, \quad (\text{D38})$$

and eq. (D26) becomes

$$L_{11} = -D\mathcal{H} + l_{11}^0 - l_{1\rho}^0 l_{\rho 1}^0 r. \quad (\text{D39})$$

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