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Regular Variation on Time Scales and Dynamic Equations

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Abstract

The purpose of this paper is twofold. First, we want to initiate a study of regular variation on time scales by introducing this concept in such a way that it unifies and extends well studied continuous and discrete cases. Some basic properties of regularly varying functions on time scales will be established as well. Second, we give conditions under which certain solutions of linear second order dynamic equations are regularly varying. Open problems and possible directions for a future research are discussed, too.

Keywords: Time scale; regularly varying function; Karamata function; regularly varying sequence; differential equation; difference equation; dynamic equation.

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1 Introduction

The concept of regular variation has been shown to be extremely useful in many fields of mathematics, both, in the continuous and the discrete setting,

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see e.g. [2, 5, 6, 8, 9, 12, 13, 14, 15, 16]. In this paper we introduce the concept of regular variation for real functions defined on an arbitrary time scale $\mathbb{T}$. It will be shown that our definition is a generalization and unification of the continuous and the discrete case, in a certain sense. From this point of view, our paper can be understood as the one which wants to initiate study of this important concept in a general time scale setting. Recall that in addition to the classical differential and difference calculi, the calculus on time scales includes as a special case also the so-called quantum calculus, see e.g. [11]. In the second part of this paper, we provide information about asymptotic behavior of positive decreasing solutions of linear second order dynamic equations (which include an one-dimensional Schrödinger differential equation). We give sufficient and necessary conditions under which the solutions are regularly (or slowly) varying. For related results concerning linear second order differential and difference equations see [14] and [15], respectively.

The paper is organized as follows. First we recall basic facts about time scales. Then we define regularly varying functions on time scales and prove some of its important properties. In particular, we establish a representation theorem for such functions using the fact that they are related to solutions of certain linear first order dynamic equations. Connections of regularly varying functions with positive solutions of linear second order dynamic equations will be shown in Section 4. Open problems and possible directions for a future research are discussed in the last section.

2 Preliminaries

In 1988, Stefan Hilger [10] introduced the calculus on time scales which unifies continuous and discrete analysis. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We define the *forward jump operator* σ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the *graininess* μ of the time scale $\mathbb{T}$ by $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is said to be *right-dense*, *right-scattered*, if $\sigma(t) = t$, $\sigma(t) > t$, respectively. We denote $f^\sigma := f \circ \sigma$. Throughout this paper we assume that $\mathbb{T}$ is a time scale which is unbounded above. For a function $f : \mathbb{T} \to \mathbb{R}$ the *delta derivative* is defined by

$$f^\Delta(t) := \lim_{s \to t, \sigma(s) \neq t} \frac{f^\sigma(s) - f(t)}{\sigma(s) - t}.$$

Here are some useful formulas involving delta derivative: $f^\sigma = f + \mu f^\Delta$, $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = f^\Delta g^\sigma + f g^\Delta$, $(f/g)^\Delta = (f^\Delta g - f g^\Delta)/g g^\sigma$, where f, g are delta differentiable and $g g^\sigma \neq 0$ in the last formula. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$ (left-dense and left-scattered points are defined similarly as their "right counterparts"). The classes of real rd-continuous functions and real rd-continuously delta differentiable functions on a time scale interval I will be denoted by $\mathcal{C}_{\mathrm{rd}}(I)$ and by $\mathcal{C}^1_{\mathrm{rd}}(I)$, respectively. For $a, b \in \mathbb{T}$ and a delta differentiable function f, the *Newton integral* is defined by $\int_a^b f^\Delta(t)\,\Delta t = f(b) - f(a)$. Note that every rd-continuous function has an antiderivative. For the concept of the *Riemann delta integral* and the *Lebesgue delta integral* see [4, Chapter 5]. Note that we have

$$\sigma(t) = t,\ \mu(t) \equiv 0,\ f^\Delta = f',\ \int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt,\ \text{when } \mathbb{T} = \mathbb{R},$$

while

$$\sigma(t) = t + 1,\ \mu(t) \equiv 1,\ f^\Delta = \Delta f,\ \int_a^b f(t)\,\Delta t = \sum_{t=a}^{b-1} f(t),\ \text{when } \mathbb{T} = \mathbb{Z},$$

and

$$\sigma(t) = qt,\ \mu(t) = (q-1)t,\ f^\Delta = D_q y := (y(qt) - y(t))/(qt - t),$$

$$\int_1^t f(s)\,\Delta s = \sum_{j=0}^{n-1} f(q^j)\mu(q^j),\ t = q^n,\ \text{when } \mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$$

with $q > 1$ We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided $1 + \mu(t)p(t) \neq 0$ for $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T})$. Define the set of *positively regressive* functions $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T})$ as the set consisting of those $p \in \mathcal{R}$ satisfying $1 + \mu(t)p(t) > 0$ for $t \in \mathbb{T}$. Define the *generalized exponential function* $e_p(t, s)$ as the unique solution $e_p(\cdot, t_0)$ of the initial value problem $y^\Delta = p(t)y$, $y(t_0) = 1$, where $p \in \mathcal{R}$. In fact, $e_p(t, s)$ is defined by means of a cylinder transformation, e.g. in [3], but here we prefer a simpler equivalent definition. By an interval $[a, b]$, where $a, b \in \mathbb{T}$, we mean the set $\{t \in \mathbb{T} : a \leq t \leq b\}$, if it is not said otherwise; similarly we define other types of time scale-intervals. The monographs [3, 4] are very good sources for searching many other information concerning time scales and dynamic equations on time scales.

3 Regular variation on time scales

We start with the definition of the central concept.

Definition 1. A positive function $f \in \mathcal{C}_{\mathrm{rd}}([a,\infty))$ is said to be *regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if there exists a positive function $\alpha \in \mathcal{C}^1_{\mathrm{rd}}([a,\infty))$ satisfying

$$f(t) \sim C\alpha(t) \quad \text{and} \quad \lim_{t\to\infty} \frac{t\alpha^\Delta(t)}{\alpha(t)} = \vartheta, \tag{1}$$

C being a positive constant. If $\vartheta = 0$, then f is said to be *slowly varying*.

The totality of regularly varying functions of index ϑ is denoted by $\mathcal{RV}(\vartheta)$. The totality of slowly varying functions is denoted by $\mathcal{SV}$.

The next statement is a representation theorem. Clearly, it suffices if the conditions in the theorem hold eventually (for large t). Without loss of generality we can assume that they are satisfied on the interval $[a,\infty)$.

Theorem 1. *A positive function $f \in \mathcal{C}_{rd}([a,\infty))$ belongs to $\mathcal{RV}(\vartheta)$ if and only if it has the representation*

$$f(t) = \varphi(t)e_\delta(t,a), \tag{2}$$

where $\varphi \in \mathcal{C}_{rd}([a,\infty))$ is a positive function tending to a positive constant and $\delta \in \mathcal{C}_{rd}([a,\infty))$ satisfies $\delta \in \mathcal{R}^+ = \mathcal{R}^+([a,\infty))$ and $\lim_{t\to\infty} t\delta(t) = \vartheta$.

Proof. "Only if": Let $f \in \mathcal{RV}(\vartheta)$. Then there is $\delta \in \mathcal{C}_{\mathrm{rd}}([a,\infty))$ such that $\delta = \alpha^\Delta/\alpha$ and $\lim_{t\to\infty} t\delta(t) = \vartheta$. Moreover, α satisfies the first order linear dynamic equation $\alpha^\Delta = \delta(t)\alpha$, and so it has the form $\alpha(t) = \alpha_0 e_\delta(t,a)$ with $\alpha_0 > 0$. Since α is positive, $e_\delta(t,a)$ is positive as well, and hence $\delta \in \mathcal{R}^+$. From the first condition in (1) we now have that there is a positive function φ tending to a positive constant such that (2) holds.

"If": Let (2) hold with $\delta \in \mathcal{R}^+$ and $\lim_{t\to\infty} t\delta(t) = \vartheta$. Put $\alpha(t) = e_\delta(t,a)$. Then α is a positive function such that $\lim_{t\to\infty} t\alpha^\Delta(t)/\alpha(t) = \lim_{t\to\infty} t\delta(t) = \vartheta$. Since $f(t) \sim C\alpha(t)$, where $C = \lim_{t\to\infty}\varphi(t)$, $f \in \mathcal{RV}(\vartheta)$. □

Remark 1. If $\mathbb{T} = \mathbb{R}$, then (2) reduces to

$$f(t) = \varphi(t)\exp\left\{\int_a^t \frac{\psi(s)}{s}\,ds\right\}, \tag{3}$$

where $\lim_{t\to\infty} \varphi(t) = C > 0$ and $\lim_{t\to\infty} \psi(t) = \vartheta$. If $\mathbb{T} = \mathbb{Z}$, then (2) reduces to

$$f_t = \varphi_t \prod_{j=a}^{t-1} \left(1 + \frac{\psi_j}{j}\right), \tag{4}$$

where $\lim_{t\to\infty} \varphi_t = C > 0$ and $\lim_{t\to\infty} \psi_t = \vartheta$. In both these cases the obtained formulas coincide with the known representation formula in the continuous case (see [16]), resp. in the discrete case (see [15]). Hence our definition can be understood as a generalization and unification of that in the continuous and the discrete case, in a certain sense.

The next defined normalized regular variation will be of particular interest in our subsequent theory.

Definition 2. A positive function $f \in \mathcal{C}^1_{\mathrm{rd}}([a, \infty))$ is said to be *normalized regularly varying of index* ϑ, $\vartheta \in \mathbb{R}$, if it satisfies

$$\lim_{t\to\infty} \frac{t f^\Delta(t)}{f(t)} = \vartheta.$$

If $\vartheta = 0$, then f is said to be *normalized slowly varying*.

The totality of normalized regularly varying functions of index ϑ is denoted by $\mathcal{NRV}(\vartheta)$. The totality of normalized slowly varying functions is denoted by $\mathcal{NSV}$.

It is easy to see that $f \in \mathcal{NRV}(\vartheta)$ if and only if it has the representation (2), where $\varphi(t)$ is replaced by a positive constant C, i.e.,

$$f(t) = C e_\delta(t, a). \tag{5}$$

Let $f \in \mathcal{NRV}(\vartheta)$. Then $f^\Delta(t) \gtreqless 0$ if and only if $\delta(t) \gtreqless 0$ for $t \in [a, \infty)$ in the representation (5). This follows from the inequality $f^\Delta(t)/f(t) = \delta(t)$. Further note that if $\delta(t) \geq 0$, then clearly $\delta \in \mathcal{R}^+$. If $f \in \mathcal{NSV}$ is decreasing, then for δ from (5) being in $\mathcal{R}^+$ it is sufficient to assume $\mu(t) = O(t)$. Indeed, if $\mu(t)/t$ is bounded and $\lim_{t\to\infty} t\delta(t) = 0$, then $\lim_{t\to\infty} \mu(t)\delta(t) = 0$. Similarly, if $f \in \mathcal{NRV}(\vartheta)$ (with $\vartheta < 0$) is decreasing, then for $\delta \in \mathcal{R}^+$ it is sufficient to assume $\mu(t) = o(t)$.

4 Regularly varying decreasing solutions of second order linear dynamic equations

Consider the linear dynamic equation

$$y^{\Delta\Delta} - p(t)y^{\sigma} = 0, \tag{6}$$

where $p \in \mathcal{C}_{\mathrm{rd}}([a,\infty))$ is positive. Basic properties of (6) can be found e.g. in [3] or [7]. By a solution we will mean a nontrivial solution. Recall that a *solution* y of (6) is called *nonoscillatory* if $yy^{\sigma} > 0$ eventually. Otherwise, it is called *oscillatory*. In view of the Sturm type separation theorem, one solution of (6) is oscillatory if and only if every solution is so. Hence we may speak about *(non)oscillation of equation* (6). Since $y^{\Delta\Delta} = 0$ is nonoscillatory (such an equation is readily explicitly solvable), then (6) is nonoscillatory as well by the Sturm type comparison theorem. Moreover, if $\mathbb{M}$ denotes the set of all (nontrivial) solutions of (6), then any $y \in \mathbb{M}$ is eventually monotone and belongs to one of the two classes

$$\mathbb{M}^+ = \{y \in \mathbb{M} : \exists T \in [a,\infty) \text{ such that } y(t)y^{\Delta}(t) > 0 \text{ for } t \in [T,\infty)\},$$
$$\mathbb{M}^- = \{y \in \mathbb{M} : y(t)y^{\Delta}(t) < 0 \text{ for } t \in [a,\infty)\}.$$

These classes are nonempty. Basic asymptotic properties of solutions of (6) in the class $\mathbb{M}^-$ was studied in [1] (in fact, there was studied a more general equation than (6), namely a quasilinear dynamic equation). In our paper we study asymptotic properties from a different (and somehow deeper) point of view — we establish necessary and sufficient conditions under which positive decreasing solutions of (6) are normalized slowly/regularly varying. Note that considering just positive elements of $\mathbb{M}^-$ is without loss of generality, in view of the homogeneity of the solution space.

One of the main tools used in the subsequent proofs is based on the Riccati like transformation. Note that the below described technique works no matter what the sign of p is. If y is a solution of (6) with $y(t)y^{\sigma}(t) > 0$ for large t, say $t \in [a,\infty)$, (in particular, (6) is nonoscillatory), then w defined by $w = y^{\Delta}/y$ satisfies the Riccati dynamic equation

$$w^{\Delta}(t) - p(t) + \frac{w^2(t)}{1 + \mu(t)w(t)} = 0 \tag{7}$$

with $w \in \mathcal{R}^+$ for $t \in [a,\infty)$. The opposite implication holds as well, and this technique is usually referred to as the *Riccati technique*.

In the next two theorems we give conditions guaranteeing the existence of regularly varying solutions of (6). We will see that the index of regular variation depends on the value of the limit of certain expression involving the coefficient p.

Theorem 2. *Let y be any positive decreasing solution of* (6) *and* $\mu(t) = O(t)$. *Then $y \in \mathcal{NSV}$ if and only if*

$$\lim_{t\to\infty} t \int_t^\infty p(s)\,\Delta s = 0. \tag{8}$$

Proof. "Only if": Let $y \in \mathcal{NSV}$ be a positive decreasing solution of (6) on $[a, \infty)$. Set $w = y^\Delta/y$. Then $w(t) < 0$ and satisfies (7) with $w \in \mathcal{R}^+$ for $t \in [a, \infty)$. Since $y \in \mathcal{NSV}$, we have $\lim_{t\to\infty} tw(t) = 0$ and $\lim_{t\to\infty} w(t) = 0$. First we show that

$$\int_a^\infty \frac{w^2(t)}{1+\mu(t)w(t)}\,\Delta t < \infty.$$

In view of $\mu(t) = O(t)$, there exists $N > 0$ such that $\mu(t)/t \le N$ for $t \in [a, \infty)$. Since $\lim_{t\to\infty} t|w(t)| = 0$, there exists $M \le 1/(2N)$ such that $|w(t)| \le M/t$ for large t, say again $t \in [a, \infty)$, without loss of generality. Then

$$\mu(t)|w(t)| \le \frac{\mu(t)M}{t} \le \frac{\mu(t)}{2Nt} \le \frac{1}{2}$$

for $t \in [a, \infty)$. Hence,

$$\begin{aligned}
\int_a^\infty \frac{w^2(t)}{1+\mu(t)w(t)}\,\Delta t &\le \int_a^\infty \frac{M^2/t^2}{1-\mu(t)|w(t)|}\,\Delta t \\
&= M^2 \int_a^\infty \frac{1}{t\sigma(t)} \cdot \frac{1+\mu(t)/t}{t-\mu(t)|w(t)|}\,\Delta t \\
&\le M^2 \int_a^\infty \frac{1}{t\sigma(t)} \cdot \frac{1+N}{1-1/2}\,\Delta t \\
&= 2M^2(1+N)a.
\end{aligned}$$

Now, integrating (7) from t to ∞ and multiplying by t we get

$$-tw(t) + t\int_t^\infty \frac{w^2(s)}{1+\mu(s)w(s)}\,\Delta s = t\int_t^\infty p(s)\,\Delta s. \tag{9}$$

Next we show that

$$\lim_{t\to\infty} t \int_t^\infty \frac{w^2(s)}{1+\mu(s)w(s)}\,\Delta s = 0.$$

Using the time scale L'Hospital rule and the above derived estimates, we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{\int_t^\infty [w^2(s)/(1+\mu(s)w(s))]\,\Delta s}{1/t} &= \lim_{t\to\infty} \frac{t\sigma(t)w^2(t)}{1+\mu(t)w(t)} \\
&= \lim_{t\to\infty} (tw(t))^2 \frac{1+\mu(t)/t}{1-\mu(t)|w(t)|} \\
&\le \lim_{t\to\infty} (tw(t))^2 \frac{1+N}{1-\mu(t)M/t} \\
&\le \lim_{t\to\infty} (tw(t))^2 \frac{1+N}{1-MN} \\
&\le 2(1+N) \lim_{t\to\infty} (tw(t))^2 \\
&= 0
\end{aligned}$$

From (9) we now get (8).

"If": Let y be a positive decreasing solution of (6) for $t \in [a,\infty)$. We claim that $\lim_{t\to\infty} y^\Delta(t) = 0$. If not, then there is $M > 0$ such that $y^\Delta(t) \le -M$ for $t \in [a,\infty)$, and so $y(t) \le y(a) - (t-a)M$. Letting $t \to \infty$ we have $\lim_{t\to\infty} y(t) = -\infty$, a contradiction. Hence integration of (6) from t to ∞ yields $y^\Delta(t) = -\int_t^\infty p(s)y^\sigma(s)\,\Delta s$. Multiplying this equality by $t/y(t)$ and using a monotone nature of y we obtain

$$-\frac{ty^\Delta(t)}{y(t)} = \frac{t}{y(t)} \int_t^\infty p(s)y^\sigma(s)\,\Delta s \le \frac{ty(t)}{y(t)} \int_t^\infty p(s)\,\Delta s = t\int_t^\infty p(s)\,\Delta s.$$

Hence, $\lim_{t\to\infty} ty^\Delta(t)/y(t) = 0$ by (8), and so $y \in \mathcal{NSV}$. □

Remark 2. A closer examination of the proof shows that the condition $\mu(t) = O(t)$ is not needed to prove the "if" part.

Theorem 3. *Let y be any positive decreasing solution of* (6) *and $\mu(t) = o(t)$. Then $y \in \mathcal{NRV}(\vartheta)$ if and only if*

$$\lim_{t\to\infty} t\int_t^\infty p(s)\,\Delta s = A > 0, \tag{10}$$

where ϑ is the negative root of the equation $\lambda^2 - \lambda - A = 0$, i.e., $\vartheta = \left(1-\sqrt{1+4A}\right)/2$.

Proof. "Only if": Let $y \in \mathcal{NRV}(\vartheta)$ be a positive solution of (6) for $t \in [a,\infty)$. Set $w = y^\Delta/y$. Then $w(t) < 0$ and satisfies (7) with $w \in \mathcal{R}^+$ for $t \in [a,\infty)$. Since $y \in \mathcal{NRV}(\vartheta)$, we have $\lim_{t\to\infty} tw(t) = \vartheta$ and $\lim_{t\to\infty} w(t) = 0$. We show that

$$\int_a^\infty \frac{w^2(t)}{1+\mu(t)w(t)}\,\Delta t < \infty.$$

There exists $M > 0$ such that $w(t) \le M/t$. Further, in view of the condition $\lim_{t\to\infty} \mu(t)/t = 0$, we have $\mu(t)|w(t)| \le M\mu(t)/t \le 1/2$ for large t, say $t \in [a,\infty)$. Hence,

$$\begin{aligned}
\int_a^\infty \frac{w^2(t)}{1+\mu(t)w(t)}\,\Delta t &\le \int_a^\infty \frac{M^2/t^2}{1-\mu(t)|w(t)|}\,\Delta t \\
&\le 2M^2 \int_a^\infty \frac{1}{t^2}\,\Delta t \\
&= 2M^2 \int_a^\infty \frac{1}{t\sigma(t)}\left(1+\frac{\mu(t)}{t}\right)\Delta t \\
&\le 2M^2\left(1+\frac{1}{2M}\right)\int_a^\infty \frac{1}{t\sigma(t)}\,\Delta s \\
&= (2M^2+M)a.
\end{aligned}$$

Thus as in the previous proof we get (9). Further we show that

$$\lim_{t\to\infty} t\int_t^\infty \frac{w^2(s)}{1+\mu(s)w(s)}\,\Delta s = \vartheta^2.$$

Using the time scale L'Hospital rule and the above derived estimates, since

$$\lim_{t\to\infty} \mu(t)|w(t)| \le \lim_{t\to\infty} \frac{M\mu(t)}{t} = 0,$$

we have

$$\begin{aligned}
\lim_{t\to\infty} \frac{\int_t^\infty [w^2(s)/(1+\mu(s)w(s))]\,\Delta s}{1/t} &= \lim_{t\to\infty} \frac{t\sigma(t)w^2(t)}{1+\mu(t)w(t)} \\
&= \lim_{t\to\infty} (tw(t))^2 \frac{1+\mu(t)/t}{1-\mu(t)|w(t)|} \\
&= \vartheta^2.
\end{aligned}$$

From (9), $\lim_{t\to\infty} t\int_t^\infty p(s)\,\Delta s = -\vartheta + \vartheta^2 = A$.

“If”: Let y be a positive decreasing solution of (6) for $t \in [a, \infty)$. Similarly as in the previous proof, we have $\lim_{t\to\infty} y^\Delta(t) = 0$. Set $\eta(t) = ty^\Delta(t)/y(t)$. Then

$$0 < -\eta(t) = \frac{t}{y(t)} \int_t^\infty p(s) y^\sigma(s)\, \Delta s \le t \int_t^\infty p(s)\, \Delta s,$$

and so η is bounded. Further, η satisfies the modified Riccati dynamic equation

$$\left(\frac{\eta(t)}{t}\right)^\Delta - p(t) + \frac{\eta^2(t)/t^2}{1 + \mu(t)\eta(t)/t} = 0 \tag{11}$$

with $\eta(t)/t \in \mathcal{R}^+$ for $t \in [a, \infty)$. Since η is bounded, we have $\lim_{t\to\infty} \eta(t)/t = 0$, and so integration of (11) from t to ∞ yields

$$-\frac{\eta(t)}{t} = \int_t^\infty p(s)\, \Delta s - \int_t^\infty \frac{\eta^2(s)/s^2}{1 + \mu(s)\eta(s)/s}\, \Delta s. \tag{12}$$

Let us write condition (10) as

$$t \int_t^\infty p(s)\, \Delta s = A + \varepsilon_1(t) = \vartheta^2 - \vartheta + \varepsilon_1(t), \tag{13}$$

where $\lim_{t\to\infty} \varepsilon_1(t) = 0$. Multiplying (12) by t with the use of (13) we get

$$-\eta(t) = \vartheta^2 - \vartheta + \varepsilon_1(t) - t \int_t^\infty \frac{\eta^2(s)/s^2}{1 + \mu(s)\eta(s)/s}\, \Delta s. \tag{14}$$

Denote

$$\varepsilon_2(t) = t \int_t^\infty \eta^2(s) \left(\frac{1}{s^2 + \mu(s)\eta(s)s} - \frac{1}{s\sigma(s)} \right).$$

Since η is bounded and $\mu(t) = o(t)$, using the time scale L’Hospital rule we have

$$\begin{aligned}
\lim_{t\to\infty} \varepsilon_2(t) &= \lim_{t\to\infty} \eta^2(t) \left(\frac{t\sigma(t)}{t^2 + \mu(t)\eta(t)t} - \frac{t\sigma(t)}{t\sigma(t)} \right) \\
&= \lim_{t\to\infty} \eta^2(t) \left(\frac{1 + \mu(t)/t}{1 + \eta(t)\mu(t)/t} - 1 \right) \\
&= 0.
\end{aligned}$$

Hence,

$$t \int_t^\infty \frac{\eta^2(s)/s^2}{1 + \mu(s)\eta(s)/s}\, \Delta s = t \int_t^\infty \frac{\eta^2(s)}{s\sigma(s)}\, \Delta s + \varepsilon_2(t),$$

where $\lim_{t\to\infty} \varepsilon_2(t) = 0$. Thus from (14) we obtain

$$-\eta(t) = \vartheta^2 - \vartheta - t \int_t^\infty \frac{\eta^2(s)}{s\sigma(s)} \Delta s + \varepsilon(t),$$

where $\varepsilon(t) = \varepsilon_1(t) - \varepsilon_2(t)$. Consequently,

$$-\eta(t) = \vartheta^2 - \vartheta - tG(t) \int_t^\infty \frac{1}{s\sigma(s)} \Delta s + \varepsilon(t),$$

where $m(t) \le G(t) \le M(t)$ with $m(t) = \inf_{s\ge t} \eta^2(s)$, $M(t) = \sup_{s\ge t} \eta^2(s)$, or

$$G(t) - \eta(t) = \vartheta^2 - \vartheta + \varepsilon(t). \tag{15}$$

We claim that $\lim_{t\to\infty} \eta(t) = \vartheta$. Recall that $-\eta$ is a bounded positive function. First we assume that there exists $\lim_{t\to\infty}(-\eta(t)) = L \ge 0$. Then from (15) we get $L^2 + L = \vartheta^2 - \vartheta$. If $L > -\vartheta$, then $\vartheta^2 = L^2 + L + \vartheta > L^2$, contradiction. Similarly we get contradiction if $L < -\vartheta$. Next we assume that $\liminf_{t\to\infty}(-\eta(t)) = L_* < L^* = \limsup_{t\to\infty}(-\eta(t))$. Introduce L_1 by

$$L_1 = \sqrt{\liminf_{t\to\infty} G(t)}.$$

Clearly, $0 \le L_* \le L_1$. We distinguish the following three cases that give an exhaustive description of the whole situation.

(a) $L_* \le L_1 < -\vartheta$: Then from (15), $L_1^2 + L_* = \vartheta^2 - \vartheta$. But we have $L_* + \vartheta < 0$, hence $\vartheta^2 = L_1^2 + L_* + \vartheta < L_1^2$, contradiction with $L_1 < -\vartheta$.

(b) $L_* < L_1 \le -\vartheta$: Then from (15), $\vartheta^2 < L_1^2$, contradiction with $L_1 \le -\vartheta$.

(c) $L_1 > -\vartheta$ (or $L_* > -\vartheta$ which implies $L_1 > -\vartheta$): Introduce L_2 by

$$L_2 = \sqrt{\limsup_{t\to\infty} G(t)}.$$

Then clearly $-\vartheta < L_1 \le L_2 \le L^*$ and (15) yields $L_2^2 + L^* = \vartheta^2 - \vartheta$. But we have $L^* + \vartheta > 0$, hence $\vartheta^2 = L_2^2 + L^* + \vartheta > L_2^2$, contradiction with $L_2 > -\vartheta$.

This proves that $\lim_{t\to\infty} \eta(t) = \vartheta$, and the proof of the theorem is complete. $\square$

Remark 3. Theorem 2 and Theorem 3 can be unified into one statement, where $A \ge 0$ and ϑ is assumed to be the nonpositive root of $\lambda^2 - \lambda - A = 0$. The condition on the graininess can be expressed e.g. as that there is $B > 0$ such that $\limsup_{t\to\infty} \mu(t)/t \le AB$.

5 Concluding remarks

In this last section we indicate some directions for a future research related to the topic of this paper.

Usually, in the continuous case, a regularly varying function f of index ϑ, $\vartheta \in \mathbb{R}$, is defined as one which is positive and measurable on the real interval $[a, \infty)$, and for all $\lambda > 0$ it satisfies

$$\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta}. \tag{16}$$

This definition is due to Karamata [13]. Such functions have the representation (3) or

$$f(t) = \varphi(t) t^{\vartheta} \exp\left\{ \int_a^t \frac{\tilde{\psi}(s)}{s} \Delta s \right\},$$

where $\lim_{t \to \infty} \tilde{\psi}(t) = 0$. In the basic theory of regularly varying sequences two main approaches are known. First, the approach by Karamata [12] based on a counterpart of the continuous definition: A positive sequence $\{f_t\}$, $t \in \{a, a+1, \dots\} \subset \mathbb{Z}$ is said to be regularly varying of index ϑ, $\vartheta \in \mathbb{R}$, if

$$\lim_{t \to \infty} \frac{f_{[\lambda t]}}{f_t} = \lambda^{\vartheta} \tag{17}$$

for all $\lambda > 0$, where $[u]$ denotes the integer part of u. Second, the approach by Galambos and Seneta [8] based on a purely sequential definition: A positive sequence $\{f_t\}$, $t \in \{a, a+1, \dots\} \subset \mathbb{Z}$ is said to be regularly varying of index ϑ, $\vartheta \in \mathbb{R}$, if there exists a positive sequence $\{\alpha_t\}$ satisfying

$$f_t \sim C\alpha_t \quad \text{and} \quad \lim_{t \to \infty} t\left(1 - \frac{\alpha_{t-1}}{\alpha_t}\right) = \vartheta, \tag{18}$$

C being a positive constant. In [5] it was shown that these two definitions are equivalent. In [15] is was shown that the second condition in (18) can be replaced by $\lim_{t \to \infty} t\Delta\alpha_t/\alpha_t = \vartheta$, cf. (1). Moreover (see [15]), $\{f_t\}$ has the representation (4) or

$$f_t = \varphi_t t^{\vartheta} \prod_{j=1}^{t-1} \left(1 + \frac{\tilde{\psi}_j}{j}\right),$$

where $\lim_{t \to \infty} \tilde{\psi}_t = 0$.

Taking into account the above facts, it is natural to look for a general definition for $f : \mathbb{T} \to \mathbb{R}$ in the sense of Karamata, i.e., unifying (16) and (17). One possible candidate for such a definition could be the condition

$$\lim_{t\to\infty} \frac{f(\tau(\lambda t))}{f(t)} = \lambda^{\vartheta},$$

where $\tau : \mathbb{R} \to \mathbb{T}$ is defined as $\tau(t) = \max\{s \in \mathbb{T} : \ s \leq t\}$. Is this definition equivalent to Definition 1? For this, are some additional conditions needed? Is there a significant role played by the graininess of $\mathbb{T}$, which is not known from the "classical" cases? Some of the first computations in this direction show that there could be. Note that a Karamata type definition could relax the assumption on the smoothness of f.

In the theory of regular variation in the continuous case, many interesting properties of regularly varying functions have been established, see [2, 9, 14, 16]. In addition, there is also the theory of rapidly varying functions and of other similar objects. Although not so deep, an analogous discrete theory has been developed. As noticed in [5], this development is not generally close and sometimes far from a simple imitation of arguments of the continuous considerations. Both these theories have been shown to be extremely useful in many applications concerning various fields of mathematics. In view of all these facts, one can claim that the study of regular variation on time scales promises interesting and nontrivial adventures with receiving useful results at their ends. One of the purposes of our paper was to initiate such a study.

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