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HOLOMORPHIC RETRACTIONS AND BOUNDARY BEREZIN TRANSFORMS

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ABSTRACT. In an earlier paper, the authors have shown that the convolution of a function f continuous on the closure of a Cartan domain and a K-invariant finite measure μ on that domain is again continuous on the closure, and, moreover, its restriction to any boundary face F depends only on the restriction of f to F and is equal to the convolution, in F, of the latter restriction with some measure μ_F on F uniquely determined by μ. In this article, we give an explicit formula for μ_F in terms of F, showing in particular that for measures μ coresponding to the Berezin transforms the measures μ_F again correspond to Berezin transforms, but with a shift in the value of the Wallach parameter. Finally, for the special case of Cartan domains of type I and II we also obtain a nice and simple description of the holomorphic retraction on these domains which arises as the boundary limit of geodesic symmetries.

1. Introduction

Let $\Omega = G/K$ be an irreducible bounded symmetric domain in $\mathbf{C}^d$ in its Harish-Chandra realization (i.e. a Cartan domain), with rank r and characteristic multiplicities a and b. Thus G is the identity connected component of the group of all biholomorphic self-maps of Ω, and $K \subset G$ the subgroup stabilizing the origin $0 \in \Omega$. Under the action of G, the topological boundary $\partial\Omega$ has a decomposition

$$\partial\Omega = \partial_1\Omega \cup \cdots \cup \partial_r\Omega$$

into G-orbits; each $\partial_l\Omega$, $l = 1, \dots, r$, is a disjoint union of *boundary faces,* which are also Cartan domains in their own right, except that they are of lower dimension and have their center not at the origin but at some point $v \in \partial_l\Omega$. The group G acts on $\partial\Omega$ by mapping the face $\Omega(v)$ centered at $v \in \partial_l\Omega$ into $\Omega(\widetilde{v})$ with some $\widetilde{v} \in \partial_l\Omega$. Also, the Cartan domain $\Omega(v)$, $v \in \partial_l\Omega$, has the same multiplicities a, b as Ω, and rank $r - l$; in particular, if $l = r$ then $\Omega(v)$ reduces to a point, and $\partial_r\Omega$ is exactly the Shilov boundary of Ω.

For any K-invariant finite measure μ on Ω which is absolutely continuous with respect to the Lebesgue measure, consider the convolution operator

$$B_\mu : f \mapsto f * \mu \tag{1}$$

acting on functions on Ω. That is,

$$B_\mu f(x) := \int_\Omega f \circ \phi \, d\mu \qquad \text{where } \phi \in G,\ \phi(0) = x. \tag{2}$$

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Owing to the K-invariance of μ, the right-hand side does not depend on the choice of ϕ, so the definition is unambiguous. One can take for ϕ e.g. the geodesic symmetry $\phi_x \in G$ interchanging x and the origin, or the *transvection* g_x defined by $g_x(z) := \phi_x(-z)$. Note that for

$$d\mu = c_\nu h(z,z)^{\nu-p}\, dz, \tag{3}$$

where $h(x,y)$ is the Jordan triple determinant, $p=(r-1)a+b+2$ is the genus of Ω, dz stands for the Lebesgue measure, and c_ν is the normalization constant making $d\mu$ a probability measure, the operator B_ν coincides with the celebrated *Berezin transform* corresponding to the Wallach parameter $\nu > p-1$ [BCZ] [Pe] [UU] [AO] [Zh] [Co] .

It was shown by Kaup and Sauter [KS] that if $x \to a \in \partial\Omega$, then $g_x \to g_a$, locally uniformly on Ω, where the limit g_a is a holomorphic retraction of Ω onto the boundary face containing a. Further, if $a \in \Omega(v)$ and $\alpha = a - v$, then $g_a = g_v g_\alpha = g_\alpha g_v$, where in the last occurrence g_α is understood in the Cartan domain $\Omega(v)$ rather than in Ω.

In [AE], the present authors showed that the existence of the retraction g_a has important consequences for the boundary behaviour of the convolution operators B_μ. Namely, whenever f is a continuous function on Ω which extends continuously to $\Omega\cup\Omega(v)$, then the convolution $B_\mu f = f * \mu$ is also continuous on $\Omega\cup\Omega(v)$; further, the restriction of $B_\mu f$ to $\Omega(v)$ depends only on the restriction of f to $\Omega(v)$, and the operator

$$f\big|_{\Omega(v)} \mapsto (B_\mu f)\big|_{\Omega(v)} \tag{4}$$

is again an operator of the form (1), except that the convolution is taken in $\Omega(v)$ rather than in Ω and in the place of μ there is some K_v-invariant finite measure μ_v on $\Omega(v)$ (K_v being the K-group for $\Omega(v) \cong G_v/K_v$).

The actual determination of the measures μ_v from μ and v remained an open problem in [AE]; in particular, it was conjectured there that for the case (3) of the Berezin transforms, the operators (1) are also of that type, though possibly with different ν.

The aim of this note is to prove the last conjecture in full: we exhibit an explicit formula relating μ and μ_v, which implies in particular that if μ is of the form (3) and $v \in \partial_l\Omega$, then μ_v is also of the form (3) — taken in the Cartan domain $\Omega(v)$ instead of Ω — except that ν gets replaced by $\nu - \frac{la}{2}$. The proof goes by transferring everything, via the Cayley transform, from the bounded domain Ω into its unbounded realization as Siegel domain of type II, where additional computational machinery is available. This is done in Section 2.

Our second result is that for the special case of matrix balls and symmetric matrix balls — i.e. of Cartan domains of types I and II in the notation of Hua's book [Hu] — the unbounded realization also yields a very simple formula for the holomorphic retraction g_v: namely, upon conjugation with the Cayley transform, g_v becomes simply the orthogonal projection onto (the image of) the corresponding boundary face $\Omega(v)$. This is proved in Section 3. We conjecture that the last result remains in force also for Cartan domains of the other types (III–VI).

2. The measures μ_v

As in the Introduction let $\Omega = G/K$ be a Cartan domain in $\mathbf{C}^d$ of type (r,a,b), given in its Harish-Chandra realization; and let ϕ_x and g_x, $x \in \Omega$, be the geodesic symmetries interchanging x and the origin and the transvections $g_x(z) = \phi_x(-z)$, respectively.

We will use the language of Jordan theory, see e.g. [Lo], [FK] or [Ar] for the details and notation. In particular, we let Z ($\cong \mathbf{C}^d$) stand for the JB^*-triple whose unit ball is Ω, $\{xyz\}$ for the triple product of Z, $D(x,y)$ for the multiplication operators $D(x,y)z = \{xyz\}$, and $Q(x)$ for the (antilinear) quadratic operator $Q(x)z = \{xzx\}$. An element $v \in Z$ is a *tripotent* if $\{vvv\} = v$. Two tripotents u, v are said to be *orthogonal* if $D(u,v) = 0$ (this is equivalent to $D(v,u) = 0$). Associated to a tripotent v is the *Peirce decomposition*

$$Z = Z_1(v) \oplus Z_{1/2}(v) \oplus Z_0(v), \tag{5}$$

with $Z_\nu(v) = \operatorname{Ker}(D(v,v) - \nu)$, $\nu = 0, \frac{1}{2}, 1$. The subspace $Z_1(v)$ is a JB^*-algebra under the product $x \circ y = \{xvy\}$, with unit v and involution $z^* = \{vzv\}$. A tripotent is called *minimal* if $\dim Z_1(v) = 1$.

To a system $e_1, \dots, e_m$ of pairwise orthogonal tripotents, there is similarly associated the *joint Peirce decomposition*

$$Z = \bigoplus_{0 \le i \le j \le m} Z_{ij} \tag{6}$$

of Z into orthogonal subspaces

$$Z_{ij} = \{z \in \mathbf{C}^d : \ D(e_k, e_k)z = \frac{\delta_{ik} + \delta_{jk}}{2} z_0 \ \forall k = 1, \dots, m\}, \tag{7}$$

of which (5) is a special case (for $m = 1$).

Any maximal system of pairwise orthogonal minimal tripotents $e_1, \dots, e_r$ is called a *Jordan frame*; its cardinality r is the same for all Jordan frames and equal to the rank of Ω. Any $z \in Z$ can be written in the form

$$z = k(t_1 e_1 + \dots + t_r e_r),$$

where $k \in K$ and $t_1 \ge t_2 \ge \dots \ge 0$; the numbers $t_1, \dots, t_r$ are determined by z uniquely (but k need not be). Further, z belongs to Ω, $\partial\Omega$ or $\partial_l\Omega$ ($l = 1, \dots, r$), respectively, if and only if $t_1 < 1$, $t_1 = 1$, or $t_1 = t_2 = \dots = t_l = 1 > t_{l+1}$; and z is a tripotent in $\partial_l\Omega$ (or, a tripotent of *rank* l) if and only if $t_1 = \dots = t_l = 1$, $t_{l+1} = \dots = t_r = 0$. For any such tripotent, the intersection

$$\Omega_0(v) := \Omega \cap Z_0(v)$$

is a Cartan domain of type $(r - l, a, b)$, and its translate

$$\Omega(v) := v + \Omega_0(v)$$

is precisely the boundary face centered at v. It is a face both in the sense of convex geometry (i.e. intersection of $\overline{\Omega}$ with a supporting hyperplane in $\mathbf{C}^d \cong \mathbf{R}^{2d}$) and in the sense of being a complex variety wholly contained in $\partial\Omega$. All boundary faces arise in this way.

The element

$$e := e_1 + e_2 + \dots + e_r$$

is a *maximal tripotent*: it satisfies $Z_0(e) = \{0\}$.

Recall that in any Jordan algebra, an element z is called *invertible* if $Q(z)$ is an invertible operator, and the *inverse* z^{-1} is then defined by $z^{-1} := Q(z)^{-1} z^*$. It is known [Up, Chapter 4] that the inverse map is a rational map written in exact (i.e. reduced) form as $z^{-1} = p(z)/N(z)$, where $p(z)$ is a polynomial (taking values in the Jordan algebra) which plays the role of the matrix adjoint of z, and $N(z)$ is a polynomial (taking nonzero complex values) called the *determinant polynomial*, or *Koecher norm*, of the Jordan algebra. In particular, fixing a Jordan frame $e_1, \dots, e_r$ of Z the above applies to the Jordan algebras $Z_1(e_1 + \dots + e_j)$, $1 \le j \le r$; we denote the corresponding determinant polynomials by N_j and extend them to all of Z by defining $N_j(z) := N_j(P_{1j}(z))$, where P_{1j} is the projection of Z onto $Z_1(e_1 + \dots + e_j)$ given by the Peirce decomposition (5). For an r-tuple $\mathbf{m} = (m_1, \dots, m_r)$ of integers satisfying $m_1 \ge m_2 \ge \dots \ge m_r \ge 0$, the *conical polynomial* $N_{\mathbf{m}}$ associated with $\mathbf{m}$ is

$$N^{\mathbf{m}} := N_1^{m_1 - m_2} N_2^{m_2 - m_3} \dots N_r^{m_r}.$$

If $z \in Z$ is such that $N_j(z) > 0$ for all j, then we can even define *conical functions*

$$N^{\boldsymbol{\lambda}} := N_1^{\lambda_1 - \lambda_2} N_2^{\lambda_2 - \lambda_3} \dots N_r^{\lambda_r}$$

for any $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in \mathbf{C}^r$. This applies, in particular, to all z in the convex cone

$$\Lambda := \{x \circ x : \ x = x^* \in Z_1(e), \ N_r(x) \neq 0\}$$

of *positive elements* in $Z_1(e)$. Note that if $\lambda_{l+1} = \lambda_{l+2} = \dots = \lambda_r = 0$ for some l, then

$$N^{\boldsymbol{\lambda}} = N_1^{\lambda_1 - \lambda_2} \dots N_l^{\lambda_l}$$

makes sense for all $(\lambda_1, \dots, \lambda_l) \in \mathbf{C}^l$ even for every $z \in Z$ satisfying

$$N_j(z) > 0 \qquad \forall j = 1, \dots, l. \tag{8}$$

The *Cayley transform* $\gamma = \gamma^{[\Omega, e]}$ associated with the domain Ω and its maximal tripotent e is defined by

$$\begin{aligned} \gamma(z) &= (e - z_1) \circ (e + z_1)^{-1} - 2\{(e + z_1)^{-1}, e, z_2\} \\ &= \{e - z_1 - 2z_2, e, (e + z_1)^{-1}\}, \end{aligned} \tag{9}$$

for $z = z_1 + z_2 \in Z_1(e) \oplus Z_{1/2}(e) = Z$. Let $F : Z_{1/2}(e) \times Z_{1/2}(e) \to Z_1(e)$ be the $Z_1(e)$-valued Hermitian form

$$F(x, y) := \{x, y, e\},$$

and define

$$\tau(z) := \frac{z_1 + z_1^*}{2} - F(z_2, z_2).$$

Then the Cayley transform (9) maps Ω biholomorphically onto the *Siegel domain*

$$\mathcal{S} := \{z \in Z : \ \tau(z) \in \Lambda\}. \tag{10}$$

Its inverse is given by

$$\begin{aligned} \gamma^{-1}(w) &= (e - w_1) \circ (e + w_1)^{-1} - 4\{(e + w_1)^{-1}, e, w_2\} \\ &= \{e - w_1 - 4w_2, e, (e + w_1)^{-1}\}. \end{aligned}$$

The following lemma is immediate from the definition (9) of γ. Note that γ maps e into 0 and 0 into e.

Lemma 1. *Let $e = e_1 + \dots + e_r$ be a maximal tripotent and $v = e_1 + \dots + e_l$ a tripotent of rank l. Then for any $x \in Z_0(v)$,*

$$\begin{aligned} \gamma(v + x) &= 0 + \gamma^{[\Omega_0(v), e - v]}(x), \\ \gamma(0 + x) &= v + \gamma^{[\Omega_0(v), e - v]}(x), \\ \gamma^{-1}(v + x) &= 0 + (\gamma^{[\Omega_0(v), e - v]})^{-1}(x), \\ \gamma^{-1}(0 + x) &= v + (\gamma^{[\Omega_0(v), e - v]})^{-1}(x), \end{aligned}$$

where $\gamma^{[\Omega_0(v),e-v]}$ *stands for the Cayley transform associated with the Cartan domain* $\Omega_0(v)$ *and its maximal tripotent* $e-v$.

In particular, γ *maps the boundary face* $\Omega(v)$ *of* Ω *biholomorphically onto the boundary face* $\mathcal{S}_0(v)$ *of* $\mathcal{S}$*, where*

$$\mathcal{S}_0(v) = \textit{the interior of } \overline{\mathcal{S}} \cap Z_0(v)$$

(*where the interior is understood in* $Z_0(v)$*, and the bar over* $\mathcal{S}$ *denotes the closure.*)

It has been proved in [UU] and [AZ] that for "any" linear operator T on $C^\infty(\Omega)$ which commutes with G, i.e.

$$T(f\circ\phi) = (Tf)\circ\phi \qquad \forall\phi\in G,$$

the functions

$$e_{\boldsymbol\lambda}(z) := N^{\boldsymbol\lambda}(\tau(\gamma(z))), \qquad z\in\Omega,\ \boldsymbol\lambda\in\mathbf{C}^r,$$

are eigenfunctions of T:

$$Te_{\boldsymbol\lambda} = \widetilde{T}(\boldsymbol\lambda)e_{\boldsymbol\lambda}, \tag{11}$$

for any $\boldsymbol\lambda$ for which $e_{\boldsymbol\lambda}$ belongs to the domain of T. This applies, in particular, to all G-invariant differential operators T on Ω, as well as to all convolution operators $T=B_\mu$ of the form (1) with K-invariant finite measures μ. For the former, $Te_{\boldsymbol\lambda}$ is defined for any $\boldsymbol\lambda\in\mathbf{C}^r$ and the map $T\mapsto\widetilde{T}(\boldsymbol\lambda+\boldsymbol\rho)$, where

$$\rho_j = \frac{j-1}{2}a + \frac{b+1}{2}, \qquad j=1,\dots,r,$$

is known as the *Harish-Chandra isomorphism*; its image consists precisely all of polynomials on $\mathbf{C}^r$ invariant under the group $\mathfrak{W}$ generated by all permutations of the coordinates $\lambda_1,\dots,\lambda_r$ and all sign changes $\lambda_j\mapsto\pm\lambda_j$. For $T=B_\mu$, we will write just $\widetilde{\mu}(\boldsymbol\lambda)$ instead of $\widetilde{T}(\boldsymbol\lambda)$; note that in view of the K-invariance of μ, we then have $Te_{\boldsymbol\lambda}=T\phi_{\boldsymbol\lambda}$ where $\phi_{\boldsymbol\lambda}$ are the *spherical functions*

$$\phi_{\boldsymbol\lambda}(z) := \int_K e_{\boldsymbol\lambda}(kz)\,dk,$$

dk being the normalized Haar measure on K. It is known that, unlike $e_{\boldsymbol\lambda}$ which is always unbounded, $\phi_{\boldsymbol\lambda}$ is a bounded function on Ω whenever $\boldsymbol\lambda$ belongs to the set

$$W := \text{the closed convex hull of } \{\boldsymbol\rho+\pi\boldsymbol\rho:\ \pi\in\mathfrak{W}\}.$$

Since μ is assumed to be finite, $\widetilde{\mu}$ is thus defined at least for $\boldsymbol\lambda\in W$ (in particular — on some open set containing $\boldsymbol\rho$). Finally, for μ of the form (3) (so that B_μ is a Berezin transform), the eigenvalues were computed explicitly by Unterberger and Upmeier [UU]: the results is

$$\widetilde{\mu}(\boldsymbol\lambda) = \prod_{j=1}^r \frac{\Gamma(\nu+\lambda_j-\frac{d}{r}-\frac{j-1}{2}a)\Gamma(\nu+2\rho_j-\lambda_j-\frac{d}{r}-\frac{j-1}{2}a)}{\Gamma(\nu-\frac{d}{r}-\frac{j-1}{2}a)\Gamma(\nu+2\rho_j-\frac{d}{r}-\frac{j-1}{2}a)}. \tag{12}$$

Theorem 2. *Let* μ *be a* K*-invariant measure on* Ω*, absolutely continuous with respect to the Lebesgue measure,* v *a tripotent of rank* l*,* $\Omega(v)$ *the boundary face with center* v *and* μ_v *the associated measure on the Cartan domain* $\Omega_0(v)$*. Then*

$$\widetilde{\mu}_v(\lambda_1,\dots,\lambda_{r-l}) = \widetilde{\mu}(\lambda_1,\dots,\lambda_{r-l},0,\dots,0)$$

for all $(\lambda_1,\dots,\lambda_{r-l},0,\dots,0)\in W$.

Proof. Choosing a suitable Jordan frame, we may assume that $v = e_{r-l+1} + \cdots + e_r$. Let γ and $\mathcal{S}$ be the Cayley transform and the Siegel domain, respectively, associated to the maximal tripotent $e = e_1 + \cdots + e_r$. We will use the subscript $_{[v]}$ or the superscript $^{[v]}$ to denote objects corresponding to the Cartan domain $\Omega_0(v)$ and the boundary face $\Omega(v) = v + \Omega_0(v)$ instead of Ω; in particular, the ambient complex space is $Z_{[v]} = Z_0(v)$, the element

$$e_{[v]} := e - v = e_1 + \cdots + e_{r-l}$$

is a maximal tripotent of $\Omega_0(v)$, and the Cayley transform $\gamma_{[v]}$ associated to $\Omega_0(v)$ and $e_{[v]}$ coincides with the $\gamma^{[\Omega_0(v), e-v]}$ from Lemma 1, which also relates it to the Cayley transform (9) associated with Ω and e. The same lemma also shows that the corresponding Siegel domain $\mathcal{S}_{[v]} = \gamma_{[v]}(\Omega_0(v))$ coincides with the boundary face $\gamma(\Omega(v)) = \mathcal{S}_0(v)$ of $\mathcal{S}$. Further, $Z^{[v]}_{1/2}(e_{[v]}) = Z_0(v) \cap Z_{1/2}(e)$ and $F(x,y) = \{x, y, e\} = \{x, y, e_{[v]}\} = F_{[v]}(x,y)$ for $x, y \in Z_0(v) \cap Z_{1/2}(e)$; it follows that $\tau_{[v]}$ is just the restriction of τ to $Z_0(v)$, and its image is contained in $Z_0(v) \cap Z_1(e)$. Finally, the Jordan algebras $Z_1(e_1 + \cdots + e_j)$, $1 \le j \le r-l$, are the same for $\Omega_0(v)$ as for Ω, since they are contained in $Z_0(v) \cap Z_1(e)$. It follows that

$$N_j^{[v]}(\tau_{[v]}(w)) = N_j(\tau(w)) > 0 \qquad \forall j = 1, \dots, r-l \ \forall w \in \mathcal{S}_0(v), \tag{13}$$

and

$$N_{[v]}^{(\lambda_1, \dots, \lambda_{r-l})}(\tau_{[v]}(w)) = N^{(\lambda_1, \dots, \lambda_{r-l}, 0, \dots, 0)}(\tau(w)) \qquad \forall w \in \mathcal{S}_0(v)$$

where the right-hand side makes sense in view of (8) and (13).

Combining this with the facts about the Cayley transforms mentioned a few lines above and with Lemma 1, we get

$$\begin{aligned} e^{[v]}_{\boldsymbol{\lambda}}(x) &= N^{\boldsymbol{\lambda}}_{[v]}(\tau_{[v]}(\gamma_{[v]}(x))) \\ &= N^{(\boldsymbol{\lambda}, 0)}(\tau(\gamma(v + x))) = e_{(\boldsymbol{\lambda}, 0)}(v + x) \qquad \forall x \in \Omega_0(v), \end{aligned} \tag{14}$$

where, for brevity, we write $(\lambda_1, \dots, \lambda_{r-l}) = \boldsymbol{\lambda}$ and $(\lambda_1, \dots, \lambda_{r-l}, 0, \dots, 0) = (\boldsymbol{\lambda}, 0)$, and $e_{(\boldsymbol{\lambda}, 0)}$ is extended continuously to $\Omega(v)$ via (13). Consequently, for any $x \in \Omega_0(v)$,

$$\begin{aligned} \widetilde{\mu}_v(\boldsymbol{\lambda}) e^{[v]}_{\boldsymbol{\lambda}}(x) &= (B^{[v]}_{\mu_v} e^{[v]}_{\boldsymbol{\lambda}})(x) \qquad \text{by (11)} \\ &= (B_\mu e_{(\boldsymbol{\lambda}, 0)})(v + x) \qquad \text{by the definition of } \mu_v \\ &= \widetilde{\mu}(\boldsymbol{\lambda}, 0) e_{(\boldsymbol{\lambda}, 0)}(v + x) \qquad \text{by (11) again} \\ &= \widetilde{\mu}(\boldsymbol{\lambda}, 0) e^{[v]}_{\boldsymbol{\lambda}}(x). \end{aligned}$$

Since $e^{[v]}_{\boldsymbol{\lambda}}$ does not vanish identically on $\Omega_0(v)$ (for instance, $e^{[v]}_{\boldsymbol{\lambda}}(0) = 1$), we must have $\widetilde{\mu}_v(\boldsymbol{\lambda}) = \widetilde{\mu}(\boldsymbol{\lambda}, 0)$, which proves the theorem. □

Corollary 3. *If μ is of the form (3) for some $\nu > p-1$, then μ_v is of the same form (with respect to $\Omega_0(v)$) only with ν replaced by $\nu - \frac{la}{2}$, where $l = \operatorname{rank} v$.*

Proof. For μ as in (3) we have by (12)

$$\begin{aligned} \widetilde{\mu}(\lambda_1, \dots, \lambda_{r-l}, 0, \dots, 0) &= \prod_{j=1}^{r} \frac{\Gamma(\nu + \lambda_j - \frac{d}{r} - \frac{j-1}{2}a)\Gamma(\nu + 2\rho_j - \lambda_j - \frac{d}{r} - \frac{j-1}{2}a)}{\Gamma(\nu - \frac{d}{r} - \frac{j-1}{2}a)\Gamma(\nu + 2\rho_j - \frac{d}{r} - \frac{j-1}{2}a)} \\ &= \prod_{j=1}^{r-l} \frac{\Gamma(\nu + \lambda_j - \frac{d}{r} - \frac{j-1}{2}a)\Gamma(\nu + 2\rho_j - \lambda_j - \frac{d}{r} - \frac{j-1}{2}a)}{\Gamma(\nu - \frac{d}{r} - \frac{j-1}{2}a)\Gamma(\nu + 2\rho_j - \frac{d}{r} - \frac{j-1}{2}a)}. \end{aligned} \tag{15}$$

On the other hand, for $\Omega_0(v) =: \Omega^{[v]}$ in the place of Ω we have $r^{[v]} = r - l$, $a^{[v]} = a$, $b^{[v]} = b$, so $\rho_j^{[v]} = \rho_j$ while

$$\frac{d^{[v]}}{r^{[v]}} = \frac{r^{[v]}-1}{2}a^{[v]} + b^{[v]} + 1 = \frac{r-l-1}{2}a + b + 1 = \frac{d}{r} - \frac{la}{2}.$$

Thus for a measure η of the form (3) but on $\Omega_0(v)$ and with σ in the place of ν we have

$$\begin{aligned}\widetilde{\eta}(\lambda_1,\dots,\lambda_{r-l}) &= \prod_{j=1}^{r-l} \frac{\Gamma(\sigma+\lambda_j-\frac{d^{[v]}}{r^{[v]}}-\frac{j-1}{2}a^{[v]})\Gamma(\sigma+2\rho_j^{[v]}-\lambda_j-\frac{d^{[v]}}{r^{[v]}}-\frac{j-1}{2}a^{[v]})}{\Gamma(\sigma-\frac{d^{[v]}}{r^{[v]}}-\frac{j-1}{2}a^{[v]})\Gamma(\sigma+2\rho_j^{[v]}-\frac{d^{[v]}}{r^{[v]}}-\frac{j-1}{2}a^{[v]})} \\ &= \prod_{j=1}^{r-l} \frac{\Gamma(\sigma+\lambda_j-\frac{d}{r}+\frac{la}{2}-\frac{j-1}{2}a)\Gamma(\sigma+2\rho_j-\lambda_j-\frac{d}{r}+\frac{la}{2}-\frac{j-1}{2}a)}{\Gamma(\sigma-\frac{d}{r}+\frac{la}{2}-\frac{j-1}{2}a)\Gamma(\sigma+2\rho_j-\frac{d}{r}+\frac{la}{2}-\frac{j-1}{2}a)}. \qquad (16)\end{aligned}$$

Comparing (15) and (16), we see that $\widetilde{\mu} = \widetilde{\eta}$ if $\nu = \sigma + \frac{la}{2}$. Since the function $\widetilde{\eta}$ determines η uniquely, this completes the proof. $\square$

Note that since $\boldsymbol{\rho}^{[v]} = \boldsymbol{\rho}$, we have $W_{[v]} = W \cap (\mathbf{C}^{r-l} \times \{0\})$, so that $(\boldsymbol{\lambda}, 0) \in W$ whenever $\boldsymbol{\lambda} \in W_{[v]}$.

We also remark that the relation $\sigma = \nu - \frac{la}{2}$ can be rewritten as

$$\nu - \frac{p-1}{2} = \sigma - \frac{p^{[v]}-1}{2}.$$

In particular, $\nu > p-1$ implies that $\sigma > \frac{p^{[v]}+p}{2} - 1 = p^{[v]} - 1 + la > p - 1$ as well.

3. The retraction ρ_v on domains of type I and II

In this section, we only consider Cartan domains of type I and II in Hua's notation [Hu]. Domains of type I are the matrix unit balls

$$I_{rR} := \{z \in \mathbf{C}^{r\times R} : \|z\|_{\mathbf{C}^R\to\mathbf{C}^r} < 1\}, \qquad R \geq r.$$

Domains of type II are

$$II_r := \{z \in I_{rr} : z^t = z\}, \qquad r \geq 2,$$

where z^t is the transpose of z. The restrictions $R \geq r$ and $r \geq 2$ stem from the isomorphisms $I_{rR} \cong I_{Rr}$ and $II_1 = I_{1,1}$, respectively. The rank of I_{rR} and II_r is equal to r, the characteristic multiplicity a to 2, and the characteristic multiplicity b to $R-r$ and 0, respectively.

In both cases, the Jordan triple product is given by

$$\{x,y,z\} = \frac{xy^*z + zy^*x}{2}.$$

The matrices e_j, $1 \leq j \leq r$, whose (j,j)-entries equal one and all other entries are equal to zero, form a Jordan frame. The corresponding maximal tripotent e is given by

$$\{e\}_{jk} = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k, \end{cases}$$

where $1 \leq j \leq r$, $1 \leq k \leq R$ for type I_{rR} and $1 \leq j,k \leq r$ for type II_r.

Writing

$$z = [X|Y] \qquad (17)$$

where X is a square matrix of size r and Y is an $r \times (R-r)$ matrix for type I_{rR} and absent for type II_r, the Peirce decomposition $Z = Z_1(e) \oplus Z_{1/2}(e)$ is given by

$$z = [X|0] \oplus [0|Y].$$

The Jordan algebra $Z_1(e)$ is thus isomorphic to I_{rr} with the usual matrix multiplication as the product, and the usual operation of taking the adjoint (i.e. complex-conjugate transpose) as the involution $z \mapsto z^*$. It follows that the Cayley transform associated with e is given by

$$\gamma(z) = (I+X)^{-1} \cdot [I - X| - Y]. \tag{18}$$

Here and below, I will always stand for the identity matrix of the appropriate size.

Let now v be the tripotent $v = e_1 + \dots + e_l$, $1 \le l \le r$. Splitting in (17) the rows and the first r columns into two blocks of sizes l and $r-l$, respectively, we may write v and e as

$$v = \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \qquad e = \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & I & 0 \end{array}\right];$$

similarly, for the general matrix z we refine (17) to

$$z = \left[\begin{array}{cc|c} A & B & E \\ C & D & F \end{array}\right]. \tag{19}$$

The joint Peirce decomposition (6) corresponding to the tripotents v (as e_1) and $e-v$ (as e_2) is then given by

$$\begin{gathered} z_{11} = \left[\begin{array}{cc|c} A & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \quad z_{12} = \left[\begin{array}{cc|c} 0 & B & 0 \\ C & 0 & 0 \end{array}\right], \quad z_{22} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & D & 0 \end{array}\right], \\ z_{01} = \left[\begin{array}{cc|c} 0 & 0 & E \\ 0 & 0 & 0 \end{array}\right], \qquad z_{02} = \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & F \end{array}\right]. \end{gathered} \tag{20}$$

Let P_{0v} stand for the projection onto $Z_0(v) = Z_{02} \oplus Z_{22}$:

$$P_{0v} \left[\begin{array}{cc|c} A & B & E \\ C & D & F \end{array}\right] := \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & D & F \end{array}\right].$$

Theorem 4. *The holomorphic retraction $g_v : \Omega \to \Omega(v)$ corresponding to v (i.e. the limit*

$$g_v := \lim_{\Omega \ni x \to v} g_x$$

of the transvections g_x on Ω) satisfies

$$\gamma(g_v(z)) = P_{0v}(\gamma(z)). \tag{21}$$

Proof. Recall that g_v is in general given by (see [KS] or [AE])

$$g_v(z) = v + (z_{22} + z_{02}) - \{z_{12} + z_{01}, (v + z_{11}^*)^{-1}, z_{12} + z_{01}\}.$$

Using (20) therefore gives, for z as in (19),

$$\begin{aligned} g_v(z) &= \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & D & F \end{array}\right] - \left[\begin{array}{cc|c} 0 & B & E \\ C & 0 & 0 \end{array}\right] \left[\begin{array}{cc|c} (I+A)^{-1} & 0 & 0 \\ 0 & I & 0 \end{array}\right]^t \left[\begin{array}{cc|c} 0 & B & E \\ C & 0 & 0 \end{array}\right] \\ &= \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & D-d & F-f \end{array}\right], \end{aligned}$$

where

$$d = C(I+A)^{-1}B, \qquad f = C(I+A)^{-1}E. \tag{22}$$

Thus by (18)

$$
\begin{aligned}
\gamma(g_v(z)) &= \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & (I+D-d)^{-1} \end{bmatrix} \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & I-D+d & -F+f \end{array}\right] \\
&= \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & (I+D-d)^{-1}(I-D+d) & -(I+D-d)^{-1}(F-f) \end{array}\right]. \qquad (23)
\end{aligned}
$$

On the other hand, by (18) again,

$$
\gamma(z) = \begin{bmatrix} I+A & B \\ C & I+D \end{bmatrix}^{-1} \left[\begin{array}{cc|c} I-A & -B & -E \\ -C & I-D & -F \end{array}\right].
$$

Setting

$$
\begin{bmatrix} I+A & B \\ C & I+D \end{bmatrix}^{-1} =: \begin{bmatrix} * & * \\ \alpha & \beta \end{bmatrix}
$$

(where the stars denote entries which will not be needed), we thus have

$$
\begin{aligned}
P_{0v}(\gamma(z)) &= \begin{bmatrix} 0 & 0 \\ \alpha & \beta \end{bmatrix} \left[\begin{array}{cc|c} 0 & -B & -E \\ 0 & I-D & -F \end{array}\right] \\
&= \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & -\alpha B + \beta(I-D) & -\alpha E - \beta F \end{array}\right].
\end{aligned}
$$

Comparing with (23), we thus see that we need to prove that

$$
\begin{aligned}
-\alpha B + \beta(I-D) &= (I+D-d)^{-1}(I-D+d), \\
\alpha E + \beta F &= (I+D-d)^{-1}(F-f).
\end{aligned} \qquad (24)
$$

For brevity, let us write temporarily $U := I+A$ and $V := I+D$. Then

$$
\begin{bmatrix} U & B \\ C & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V - CU^{-1}B \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}
$$

where

$$
X = CU^{-1} \qquad \text{and} \qquad Y = U^{-1}B,
$$

provided U^{-1} exists. Consequently,

$$
\begin{aligned}
\begin{bmatrix} U & B \\ C & V \end{bmatrix}^{-1} &= \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & (V-CU^{-1}B)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \\
&= \begin{bmatrix} U^{-1} + Y(V-CU^{-1}B)^{-1}X & -Y(V-CU^{-1}B)^{-1} \\ -(V-CU^{-1}B)^{-1}X & (V-CU^{-1}B)^{-1} \end{bmatrix}.
\end{aligned}
$$

Hence

$$
\beta = [I+D-C(I+A)^{-1}B]^{-1}, \qquad \alpha = -\beta C(I+A)^{-1}.
$$

By (22), $\beta = (I+D-d)^{-1}$. Consequently,

$$
\begin{aligned}
-\alpha B + \beta(I-D) &= \beta C(I+A)^{-1}B + \beta(I-D) \\
&= \beta(I-D+C(I+A)^{-1}B) \\
&= \beta(I-D+d) \\
&= (I+D-d)^{-1}(I-D+d),
\end{aligned}
$$

while

$$
\begin{aligned}
\alpha E + \beta F &= -\beta C(I+A)^{-1}E + \beta F \\
&= -\beta f + \beta F \\
&= (I+D-d)^{-1}(F-f).
\end{aligned}
$$

Thus (24) holds, which finishes the proof. □

Denote

$$\rho_v(z) := g_v(z) - v.$$

This is a holomorphic retraction of Ω onto $\Omega_0(v)$; see [AE].

Corollary 5. *Using again the notation $\gamma_{[v]}$ for the Cayley transform in $Z_0(v)$ associated with the Cartan domain $\Omega_0(v)$ and its maximal tripotent $e-v$, we have*

$$\rho_v(z) = \gamma_{[v]}^{-1}(P_{0v}(\gamma(z))).$$

That is,

$$\rho_v = \gamma_{[v]}^{-1} \circ P_{0v} \circ \gamma.$$

Proof. Immediate from (21) and Lemma 1. □

We expect that Theorem 4 and Corollary 5 remain in force on any Cartan domain.

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