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Newman-Penrose formalism in higher dimensions: vacuum spacetimes with a non-twisting multiple Weyl aligned null direction

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Abstract. Vacuum spacetimes admitting a non-twisting multiple Weyl aligned null direction (WAND) are analyzed in arbitrary dimension using recently developed higher-dimensional Newman-Penrose (NP) formalism. We determine dependence of the metric and of the Weyl tensor on the affine parameter r along null geodesics generated by the WAND for type III and N spacetimes and for a special class of type II and D spacetimes, containing e.g. Schwarzschild-Tangherlini black holes and black strings and branes.

For types III and N, all metric components are at most quadratic polynomials in r while for types II and D the r-dependence of the metric as well as of the Weyl tensor is determined by an integer m corresponding to the rank of the expansion matrix S_{ij}. It is shown that for non-vanishing expansion, all these spacetimes contain a curvature singularity.

As an illustrative example, a shearing expanding type N five-dimensional vacuum solution is also re-derived using higher-dimensional NP formalism. This solution can be however identified with a direct product of a known four-dimensional type N metric with an extra dimension.

1. Introduction

The null frame Newman-Penrose (NP) formalism [1, 2] is a very useful tool for constructing exact solutions of the four-dimensional general relativity. Although the number of equations is considerably larger than in the standard coordinate approach (note, however, that many equations in the NP formalism are redundant, see e.g. [3] and references therein), all differential equations in this formalism are of the first order. Another advantage is that one can also use gauge transformations of the frame in order to simplify the field equations. This is why the formalism is especially powerful when studying algebraically special solutions according to Petrov classification, since in this case some frame components of the Weyl tensor can be set to zero by choosing an appropriate frame.

In recent years solutions to the higher-dimensional Einstein field equations have attracted a lot of interest. Lot of effort went into generalizing basic concepts, properties and results of the four-dimensional general relativity to higher dimensions and there is growing awareness that higher-dimensional gravity contains qualitatively new physics (see e.g. [4] and references therein).

Generalization of the Petrov classification and of the NP formalism to higher dimensions was developed in [5, 6], [7, 8], respectively. Using these methods, it can be shown that in contrast with four dimensions, Goldberg-Sachs theorem is not valid in higher dimensions since multiple Weyl aligned null direction (WAND) in higher-dimensional vacuum algebraically special spacetimes can be shearing [7, 8]. For example, while in four dimensions expanding vacuum type N and III spacetimes are never shearing, in higher dimensions they are always shearing [7]. This presence of shear in higher dimensions can substantially complicate the process of solving the field equations.

In the present paper we apply the higher-dimensional NP formalism to the study of vacuum spacetimes admitting a non-twisting and (possibly) shearing multiple WAND and thus belonging to Weyl types II, D, III or N [5, 6]. After introductory remarks and necessary definitions, in Sec. 3 we study dependence of the metric of the above mentioned classes of spacetimes on the affine parameter r along null geodesics generated by the multiple WAND. It is also pointed out that in fact main results of this section apply also to a special subclass I(a) of the type I. In appropriate coordinates, the r-dependence of all components of the metric except of the component g_{00} turns out to be at most quadratic in r. The component g_{00} is again quadratic in r for types III and N and more complicated for types II and D. These two cases are thus studied separately.

In Sec. 4 the r-dependence of g_{00} and of the Weyl tensor for types III and N is determined. It is also shown that when expansion $\theta \neq 0$ these spacetimes are singular. In type N the second order curvature invariant $I = C^{abcd;rs}C_{amcn;rs}C^{tmun;vw}C_{tbud;vw}$ diverges in arbitrary dimension at a point which can be set to $r = 0$. Similarly, a first order curvature invariant is used for type III expanding spacetimes.

In Sec. 5 we determine the r-dependence of g_{00} and of the Weyl tensor for types II and D. Since the problem of solving corresponding differential equations in arbitrary dimension seems to be too complex, we focus on a special case with all non-vanishing eigenvalues of S_{ij} being equal and 'antisymmetric' part of the Weyl tensor Φ^A_{ij} being zero. These assumptions are satisfied for example for all non-twisting Kerr-Schild spacetimes [9], in particular for Schwarzschild-Tangherlini black holes or corresponding black strings/branes. It also seems to be reasonable to expect that the Weyl tensor in the case with distinct eigenvalues of S_{ij} and $\Phi^A_{ij} = 0$ will have the same behaviour in the leading order asymptotically thanks to (3.1).

It turns out that the r-dependence of g_{00} for Weyl types II and D is determined by an integer m corresponding to the rank of the expansion matrix S_{ij}. In the expanding case, apart from a quadratic polynomial in r, g_{00} also contains a term proportional to r^{1-m} for $m \neq 1$ and $\ln r$ for $m = 1$. ‡ Using similar arguments as in [9] it can be shown that in the expanding case the Kretschmann curvature invariant $R_{abcd}R^{abcd}$ diverges for $r = 0$ and that it is regular there in the non-expanding case. We also briefly discuss the shear-free case which occurs for $m = 0$ (Kundt spacetimes) and for $m = n - 2$ (Robinson-Trautman spacetimes). In contrast with the four-dimensional general relativity, in the $m = n - 2 > 2$ case, boost weight -1 and -2 components of the Weyl tensor necessarily vanish and the spacetime is thus of type D in agreement with [10].

In sec. 6, in order to provide an illustrative example of the use of the higher-

‡ Note that since we do not employ all field equations of the NP formalism, it may in fact turn out that solutions corresponding to the case $m = 1$ do not exist. In four dimensions the case $m = 1$ is forbidden by the Goldberg-Sachs theorem.

dimensional NP formalism, we focus on solving the full set of the field equations for type N. To considerably simplify resulting equations, we make several additional assumptions on the metric and we arrive to an exact vacuum solution. However, after a coordinate transformation it can be found that the resulting solution could be obtained as a direct product of a four-dimensional type N Robinson-Trautman metric with an extra dimension.

The higher-dimensional vacuum Ricci [8] and Bianchi [7] equations, extensively used throughout this paper, are given in a parallelly propagated frame with a multiple WAND in Appendix A and Appendix B, respectively.

2. Preliminaries

2.1. Algebraic classification of the Weyl tensor and Newman-Penrose formalism in higher dimensions

For convenience, let us briefly summarize basic aspects of algebraic classification of the Weyl tensor and the Newman-Penrose formalism in higher dimensions needed in the following sections. More information can be found in original references [5, 6] (classification) and [7, 8] (NP-formalism). Algebraic classification of the Weyl tensor in higher dimensions was also reviewed in [11].

We introduce a null frame with two null vectors $\boldsymbol{m}^{(1)} = \boldsymbol{m}_{(0)} = \boldsymbol{\ell}$, $\boldsymbol{m}^{(0)} = \boldsymbol{m}_{(1)} = \boldsymbol{n}$, and $n-2$ orthonormal spacelike vectors $\boldsymbol{m}^{(i)} = \boldsymbol{m}_{(i)}$ subject to

$$\ell^a \ell_a = n^a n_a = \ell^a m^{(i)}_a = n^a m^{(i)}_a = 0, \qquad \ell^a n_a = 1, \qquad m^{(i)a} m^{(j)}_a = \delta_{ij}. \tag{2.1}$$

The metric reads

$$g_{ab} = 2\ell_{(a} n_{b)} + \delta_{ij} m^{(i)}_a m^{(j)}_b. \tag{2.2}$$

Indices $a, b, \dots$ take values from 0 to $n-1$, while $i, j, \dots$ from 2 to $n-1$. Note also that since indices $i, j, \dots$ are raised/lowered by δ_{ij} there is no need to distinguish between subscripts and superscripts of this type.

Lorentz transformations are generated by null rotations

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \boldsymbol{n} + z_i \boldsymbol{m} i - \tfrac{1}{2} z^2 \boldsymbol{\ell}, \qquad \hat{\boldsymbol{m}}^{(\boldsymbol{i})} = \boldsymbol{m}^{(\boldsymbol{i})} - z_i \boldsymbol{\ell}, \tag{2.3}$$

with $z^2 \equiv z_i z^i$, spins

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(\boldsymbol{i})} = X^i{}_j \boldsymbol{m}^{(\boldsymbol{j})}, \tag{2.4}$$

with $X^i{}_j$ being orthogonal matrices and boosts

$$\hat{\boldsymbol{\ell}} = \lambda \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \lambda^{-1} \boldsymbol{n}, \qquad \hat{\boldsymbol{m}}^{(\boldsymbol{i})} = \boldsymbol{m}^{(\boldsymbol{i})}. \tag{2.5}$$

If a quantity q transforms under a boost (2.5) as $\hat{q} = \lambda^b q$ we say that q has a boost weight b.

The Ricci rotation coefficients L_{ab}, N_{ab} and $\overset{i}{M}_{ab}$ are defined by [7]

$$\ell_{a;b} = L_{cd} m^{(c)}_a m^{(d)}_b \ , \qquad n_{a;b} = N_{cd} m^{(c)}_a m^{(d)}_b \ , \qquad m^{(i)}_{a;b} = \overset{i}{M}_{cd}\, m^{(c)}_a m^{(d)}_b \tag{2.6}$$

and their transformation properties under (2.3)-(2.5) are given in [8]. These quantities satisfy constraints

$$L_{0a} = N_{1a} = 0, \tag{2.7}$$

$$N_{0a} + L_{1a} = 0, \quad \overset{i}{M}_{0a} + L_{ia} = 0, \quad \overset{i}{M}_{1a} + N_{ia} = 0, \quad \overset{i}{M}_{ja} + \overset{j}{M}_{ia} = 0. \tag{2.8}$$

In four dimensions, L_{ab}, N_{ab} and $\overset{i}{M}_{ab}$ are equivalent to standard complex NP spin coefficients κ, σ, ρ, etc. (see [8] for the correspondence).

Covariant derivatives along the frame vectors are defined by

$$D \equiv \ell^a \nabla_a, \qquad \triangle \equiv n^a \nabla_a, \qquad \delta_i \equiv m^{(i)a} \nabla_a. \tag{2.9}$$

By introducing notation

$$T_{\{pqrs\}} = \tfrac{1}{2}(T_{[ab][cd]} + T_{[cd][ab]}), \tag{2.10}$$

we can decompose the Weyl tensor and sort its components by boost weight [6]

$$\begin{aligned}
C_{abcd} = {} & \overbrace{4C_{0i0j}\, n_{\{a} m^{(i)}_{\ b} n_c m^{(j)}_{\ d\,\}}}^{2} \\
& + \overbrace{8C_{010i}\, n_{\{a} \ell_b n_c m^{(i)}_{\ d\,\}} + 4C_{0ijk}\, n_{\{a} m^{(i)}_{\ b} m^{(j)}_{\ c} m^{(k)}_{\ d\,\}}}^{1} \\
& \left.\begin{aligned} & +4C_{0101}\ n_{\{a} \ell_b n_c \ell_{\,d\,\}} + 4C_{01ij}\ n_{\{a} \ell_b m^{(i)}_{\ c} m^{(j)}_{\ d\,\}} \\ & +8C_{0i1j}\ n_{\{a} m^{(i)}_{\ b} \ell_c m^{(j)}_{\ d\,\}} + C_{ijkl}\ m^{(i)}_{\{a} m^{(j)}_{\ b} m^{(k)}_{\ c} m^{(l)}_{\ d\,\}} \end{aligned}\right\}^{0} \\
& + \overbrace{8C_{101i}\, \ell_{\{a} n_b \ell_c m^{(i)}_{\ d\,\}} + 4C_{1ijk}\, \ell_{\{a} m^{(i)}_{\ b} m^{(j)}_{\ c} m^{(k)}_{\ d\,\}}}^{-1} \\
& + \overbrace{4C_{1i1j}\, \ell_{\{a} m^{(i)}_{\ b} \ell_c m^{(j)}_{\ d\,\}}}^{-2},
\end{aligned}$$

where boost weight of various components is indicated by integers (-2,. . . ,2). Note that frame components of the Weyl tensor are subject to constraints [7] following from symmetries of the Weyl tensor

$$\begin{aligned}
& C_{0[i|0|j]} = 0, \\
& C_{0i(jk)} = C_{0ijk} + C_{0kij} + C_{0jki} = 0, \\
& C_{ijkl} = C_{\{ijkl\}}, \quad C_{ijkl} + C_{iljk} + C_{iklj} = 0, \quad C_{01ij} = 2C_{0[i|1|j]}, \\
& C_{1i(jk)} = C_{1ijk} + C_{1kij} + C_{1jki} = 0, \\
& C_{1[i|1|j]} = 0
\end{aligned} \tag{2.11}$$

and from its tracelessness

$$\begin{aligned}
& C_{0i0i} = C_{1i1i} = 0, \\
& C_{010i} = C_{0jij}, \quad C_{101i} = C_{1jij}, \\
& 2C_{0i1j} = C_{01ij} - C_{ikjk}, \quad C_{0101} = -\tfrac{1}{2} C_{ijij}.
\end{aligned} \tag{2.12}$$

We obtain following numbers of independent Weyl tensor frame components of various boost weights [7]

$$\overbrace{2\left(\frac{n(n-3)}{2}\right)}^{2,-2} + \overbrace{2\left(\frac{(n-1)(n-2)(n-3)}{3}\right)}^{1,-1} + \overbrace{\frac{(n-2)^2(n-1)(n-3)}{12} + \frac{(n-2)(n-3)}{2}}^{0},$$

which is in agreement with number of independent components of the Weyl tensor being $(n+2)(n+1)n(n-3)/12$.

We define boost order of a tensor $\boldsymbol{T}$ to be boost weight of its leading term. It turns out that boost order of a tensor depends only on vector $\boldsymbol{\ell}$, being independent on the choice of $\boldsymbol{n}$ and $\boldsymbol{m}^{(i)}$ [6]. Therefore, given a tensor $\boldsymbol{T}$, preferred null directions may exist for which boost order of $\boldsymbol{T}$ is less then for a generic choice of $\boldsymbol{\ell}$. Algebraic classification of tensors in higher dimensions [6] is based on existence (and multiplicity) of these preferred null directions in a given spacetime. In case of the Weyl tensor, we call them Weyl aligned null directions (WANDs) and spacetime is said to be of principal type G (general) if there are no WANDs, and of principal type I, II, III and N if there are WANDS of multiplicity 1, 2, 3, 4, respectively. Therefore in type I, II, III and N spacetimes all Weyl tensor components with boost weight higher or equal to 2, 1, 0, -1, respectively can be transformed away by an appropriate choice of the frame vector $\boldsymbol{\ell}$. In some cases one can also set trailing frame components to zero, and this is the basis of the secondary classification. For instance in type D (principal type II, secondary type *ii*), only boost weight zero components are non-vanishing in an appropriately choosen frame. In four dimensions principal and secondary classification reduce to the well known Petrov classification.

In agreement with [7] we introduce notation appropriate for type III and N spacetimes

$$\Psi_i \equiv C_{101i}, \quad \Psi_{ijk} \equiv \tfrac{1}{2} C_{1kij}, \quad \Psi_{ij} \equiv \tfrac{1}{2} C_{1i1j}, \tag{2.13}$$

where from (2.11), (2.12) Ψ_i, Ψ_{ijk} and Ψ_{ij} satisfy

$$\Psi_i = 2\Psi_{ijj}, \quad \Psi_{ijk} = -\Psi_{jik}, \quad \Psi_{ijk} + \Psi_{kij} + \Psi_{jki} = 0, \quad \Psi_{ij} = \Psi_{ji}, \quad \Psi_{ii} = 0. \tag{2.14}$$

Thus e.g. in type N spacetimes, the Weyl tensor is given by

$$C_{abcd} = 8\Psi_{ij}\, \ell_{\{a} m^{(i)}_{\ b} \ell_c m^{(j)}_{\ d\}} \tag{2.15}$$

and is determined by $\frac{n(n-3)}{2}$ components of the symmetric traceless $(n-2)\times(n-2)$ matrix Ψ_{ij}.

For describing boost weight zero components of the Weyl tensor we will introduce real matrix Φ_{ij} as in [12]

$$\Phi_{ij} \equiv C_{0i1j}. \tag{2.16}$$

Then from (2.11), (2.12)

$$C_{01ij} = 2C_{0[i|1|j]} = 2\Phi^A_{ij}, \quad C_{0(i|1|j)} = \Phi^S_{ij} = -\tfrac{1}{2} C_{ikjk}, \quad C_{0101} = -\tfrac{1}{2} C_{ijij} = \Phi, \tag{2.17}$$

with Φ^S_{ij}, Φ^A_{ij}, and $\Phi \equiv \Phi_{ii}$ being the symmetric and antisymmetric parts of Φ_{ij} and its trace, respectively. Boost weight zero components of the Weyl tensor are thus determined by Φ_{ij} and C_{ijkl}.

2.2. Spacetimes admitting non-twisting WANDs

We consider an n-dimensional vacuum spacetime admitting a non-twisting null congruence generated by a multiple WAND $\boldsymbol{\ell}$. Thus $\boldsymbol{\ell}$ is normal and tangent to null hypersurfaces $u =$const ($g^{ab}u_{,a}\, u_{,b} = 0$, $a, b = 0 \dots\ n-1$) and the WAND $\ell^a = g^{ab}u_{,b}$ is necessarily geodetic and affinely parameterized, $\ell^a{}_{;b}\, \ell^b = 0$.

Similarly as in [1, 10], we choose a coordinate $x^0 \equiv u$, a coordinate $x^1 \equiv r$, where r is an affine parameter along null geodesics generated by $\boldsymbol{\ell}$, and 'transverse' coordinates x^α ($\alpha = 2 \dots n-1$) labeling the null geodesics on hypersurfaces $u =$const and being constant along each geodesic. For the contravariant components of the metric tensor it follows that $g^{01} = 1$, $g^{00} = 0 = g^{0\alpha}$. Then the frame $\boldsymbol{\ell}$, $\boldsymbol{n}$, and $\boldsymbol{m}^{(i)} = \boldsymbol{m}_{(i)}$ satisfying (2.1) can be given as

$$\ell^a = [0, 1, 0, \dots, 0], \qquad \ell_a = [1, 0, \dots, 0], \tag{2.18}$$

$$n^a = [1, U, X^\alpha], \qquad n_a = [V, 1, Y_\alpha], \tag{2.19}$$

$$m^a_{(i)} = [0, \omega_i, \xi^\alpha_i], \qquad m^{(i)}_a = [\Omega^i, 0, \eta^i_\alpha]. \tag{2.20}$$

Eqs. (2.1) implies

$$0 = U + V + X^\alpha Y_\alpha, \tag{2.21}$$

$$0 = \omega_i + \xi^\alpha_i Y_\alpha, \tag{2.22}$$

$$0 = \Omega^i + \eta^i_\alpha X^\alpha, \tag{2.23}$$

$$\delta^j_i = \xi^\alpha_i \eta^j_\alpha. \tag{2.24}$$

By multiplying (2.24) by η^i_β we get $\delta^j_i \eta^i_\beta = \eta^j_\beta = (\eta^i_\beta \xi^\alpha_i)\eta^j_\alpha$ which gives

$$\delta^\alpha_\beta = \xi^\alpha_i \eta^i_\beta. \tag{2.25}$$

Since $\boldsymbol{\ell}$ is geodetic and affinely parameterized, $L_{i0} = 0 = L_{10}$. Let us choose a frame that is parallelly propagated, i.e. $N_{i0} = 0 = \overset{i}{M}_{j0}$. For geodetic $\boldsymbol{\ell}$, L_{ij} can be decomposed [7] (cf also [8]) into shear σ_{ij} (trace-free symmetric part), expansion θ (trace) and twist A_{ij} (antisymmetric part) as

$$L_{ij} = \sigma_{ij} + \theta\delta_{ij} + A_{ij}. \tag{2.26}$$

We will also often denote symmetric part of L_{ij} as expansion matrix S_{ij}. Obviously $S_{ij} = \sigma_{ij} + \theta\delta_{ij}$.

When acting on a function f, the operators (2.9) and their commutators [13] can be expressed as

$$D = \partial_r, \quad \triangle = \partial_u + U\partial_r + X^\alpha \partial_\alpha, \quad \delta_i = \omega_i \partial_r + \xi^\alpha_i \partial_\alpha \tag{2.27}$$

and

$$(\triangle D - D\triangle)f = L_{11} Df + L_{i1}\delta_i f, \tag{2.28}$$

$$(\delta_i D - D\delta_i)f = L_{1i} Df + L_{ji}\delta_j f, \tag{2.29}$$

$$(\delta_i \triangle - \triangle \delta_i)f = N_{i1} Df + (L_{i1} - L_{1i}) \triangle f + (N_{ji} - \overset{i}{M}_{j1})\delta_j f, \tag{2.30}$$

$$(\delta_i\delta_j - \delta_j\delta_i)f = (N_{ij} - N_{ji})Df + (L_{ij} - L_{ji}) \triangle f + (\overset{j}{M}_{ki} - \overset{i}{M}_{kj})\delta_k f. \tag{2.31}$$

Apart from Bianchi equations [7] and Ricci equations [8] we need relations between metric components and the Ricci rotation coefficients. Such relations may be obtained by applying the commutators (2.28)–(2.31) on coordinates u, r, x^α. For $f = u$, (2.30) and (2.31) imply

$$0 = L_{i1} - L_{1i}, \tag{2.32}$$

$$0 = L_{ij} - L_{ji}. \tag{2.33}$$

For $f = r$, (2.28)–(2.31) lead to

$$-DU = L_{11} + L_{i1}\omega_i, \tag{2.34}$$

$$-D\omega_i = L_{1i} + L_{ji}\omega_j, \tag{2.35}$$

$$\delta_i U - \triangle\omega_i = N_{i1} + (N_{ji} - \overset{i}{M}_{j1})\omega_j, \tag{2.36}$$

$$\delta_i\omega_j - \delta_j\omega_i = N_{ij} - N_{ji} + (\overset{j}{M}_{ki} - \overset{i}{M}_{kj})\omega_k, \tag{2.37}$$

and for $f = x^\alpha$, (2.28)–(2.31) give

$$-DX^\alpha = L_{j1}\xi^\alpha_j, \tag{2.38}$$

$$-D\xi^\alpha_i = L_{ji}\xi^\alpha_j, \tag{2.39}$$

$$\delta_i X^\alpha - \triangle\xi^\alpha_i = (N_{ji} - \overset{i}{M}_{j1})\xi^\alpha_j, \tag{2.40}$$

$$\delta_i\xi^\alpha_j - \delta_j\xi^\alpha_i = (\overset{j}{M}_{ki} - \overset{i}{M}_{kj})\xi^\alpha_k. \tag{2.41}$$

2.3. Indices

For convenience let us summarize types of the indices used throughout this paper. Apart from indices $a, b, \ldots = 0, \ldots, n-1$, and $i, j, \ldots = 2, \ldots, n-1$ introduced in Sec. 2.1, we also introduce indices $\alpha, \beta = 2, \ldots, n-1$ numbering spacelike coordinates and corresponding components in Sec. 2.2.

In four dimensions, expansion matrix S_{ij} is of rank 2 in the expanding case due to Goldberg-Sachs theorem. However, in higher dimensions $m \leq n-2$, where m is rank of S_{ij}. In next sections we will often need to distinguish between indices corresponding to non-vanishing ($o, p, q, s = 2, \ldots, m+1$) and vanishing ($v, w, y, z = m+2, \ldots, n-1$) eigenvalues of S_{ij}.

In following calculations it also turns out to be practical to modify Einstein's summation convention for indices o, p, q, s: in an expression there is summation over repeated indices if there are two indices without brackets among them (thus e.g. in $\eta^{p0}_\alpha\eta^{p0}_\beta X^{\beta 0}(r + a_{(p)})^2$ there is summation over p while in $\Phi_{pq}s_{(p)}$ we do not sum over p).

3. Radial integration for non-twisting vacuum Weyl type II, D, III, N spacetimes

In the present paper we study r-dependence of the metric functions, the Ricci rotation coefficients and the Weyl tensor, which, however, is in general different for various algebraic types. In order to avoid repetition, in this section we focus on those metric functions and Ricci rotation coefficients that have the same r-dependence for all algebraic types studied. Note that in contrast with sec. 5, here we do not assume that all non-vanishing eigenvalues of the expansion matrix S_{ij} are equal.

Without loss of generality we choose the frame (2.18)–(2.20) in such a way that S_{ij} is diagonal, $S_{ij} = \text{diag}\{s_{(2)}, \ldots, s_{(m+1)}, 0, \ldots, 0\}$, where m denotes number of non-zero eigenvalues of S_{ij}. As is shown in [14], this assumption is compatible with the frame being parallelly transported. As mentioned in sec. 2.3, indices o, p, q, s corresponding to non-vanishing eigenvalues of S_{ij} run from 2 to $m+1$ and indices v, w, y, z corresponding to vanishing eigenvalues of S_{ij} run from $m+2 \ldots n-1$.

In our case, from Ricci eqs. (A.7) for non-vanishing eigenvalues of S_{ij}, $s_{(p)} \neq 0$, it follows

$$s_{(p)} = \frac{1}{r + a^0_{(p)}}, \tag{3.1}$$

where $a^0_{(p)}$ is an arbitrary function of u and x^α, independent on r. Similarly, throughout the paper, the superscript ^{0} will suggest that the function under consideration does not depend on r.

Ricci eqs. (A.2)=(A.5), $DL_{1i} = -L_{1i}s_{(i)}$, lead to

$$L_{1p} = \frac{l^0_{1p}}{r + a^0_{(p)}}, \quad L_{1w} = l^0_{1w}. \tag{3.2}$$

There is still freedom to perform a null rotation with fixed $\boldsymbol{\ell}$ (2.3). To preserve parallel propagation of the frame, z_i is subject to

$$Dz_i = 0. \tag{3.3}$$

Choosing $z_p = -l^0_{1p}$, we can set L_{1p} to zero by (see [8])

$$\hat{L}_{1p} = L_{1p} + z_j L_{ji} = 0. \tag{3.4}$$

In what follows we omit the hat symbol. Note that parameters z_w can be used to further simplify the metric, e.g. one can set ω^0_w to zero as in sec. 5.2.1 and sec. 6.

From Ricci eqs. (A.14), reduced to $D \overset{j}{M}_{ki} = - \overset{j}{M}_{ki} s_{(i)}$, (2.39) and (2.35), we obtain

$$\overset{j}{M}_{kp} = \frac{\overset{j}{m}_{kp}{}^0}{r + a^0_{(p)}}, \quad \overset{j}{M}_{kw} = \overset{j}{m}_{kw}{}^0, \tag{3.5}$$

$$\xi^\alpha_p = \frac{\xi^{\alpha 0}_p}{r + a^0_{(p)}}, \quad \xi^\alpha_w = \xi^{\alpha 0}_w, \tag{3.6}$$

$$\omega_p = \frac{\omega^0_p}{r + a^0_{(p)}}, \quad \omega_w = -l^0_{1w} r + \omega^0_w, \tag{3.7}$$

respectively and from (2.38)

$$X^\alpha = -l^0_{1w} \xi^{\alpha 0}_w r + X^{\alpha 0}. \tag{3.8}$$

To compute the covariant components of the metric one has to solve (2.21)–(2.24) for η^i_α, Y_α, Ω^p, V. From (2.21)–(2.24) using also (2.25) and (3.6)–(3.8), it follows

$$\eta^p_\alpha = \eta^{p0}_\alpha (r + a_{(p)}), \quad \eta^w_\alpha = \eta^{w0}_\alpha, \tag{3.9}$$

$$Y_\alpha = -\eta^i_\alpha \omega_i = l^0_{1w} \eta^{w0}_\alpha r - (\eta^{p0}_\alpha \omega^0_p + \eta^{w0}_\alpha \omega^0_w), \tag{3.10}$$

$$\Omega^p = -\eta^p_\alpha X^\alpha = -\eta^{p0}_\alpha X^{\alpha 0}(r + a_{(p)}), \tag{3.11}$$

$$\Omega^w = -\eta^w_\alpha X^\alpha = l^0_{1w} r - \eta^{w0}_\alpha X^{\alpha 0}, \tag{3.12}$$

$$V = -U + l^0_{1w} l^0_{1w} r^2 - (\omega^0_w + \eta^{w0}_\alpha X^{\alpha 0}) l^0_{1w} r + X^{\alpha 0}(\eta^{p0}_\alpha \omega^0_p + \eta^{w0}_\alpha \omega^0_w). \tag{3.13}$$

As will be discussed below, the r-dependence of the function U has to be studied separately for types II, D and III, N.

The covariant components of the metric tensor (cf (2.2)) thus read

$$g_{11} = 0, \quad g_{01} = 1, \quad g_{1\alpha} = 0, \tag{3.14}$$

$$\begin{aligned} g_{00} &= 2V + \Omega^i \Omega^i = 2V + \eta^{p0}_\alpha X^{\alpha 0} \eta^{p0}_\beta X^{\beta 0} (r + a_{(p)})^2 \\ &\quad + (l^0_{1w} r - \eta^{w0}_\alpha X^{\alpha 0})(l^0_{1w} r - \eta^{w0}_\beta X^{\beta 0}), \end{aligned} \tag{3.15}$$

$$\begin{aligned} g_{0\alpha} &= Y_\alpha + \Omega^j \eta^j_\alpha \\ &= -\eta^{p0}_\alpha \eta^{p0}_\beta X^{\beta 0}(r + a_{(p)})^2 + 2 l^0_{1w} \eta^{w0}_\alpha r - (\eta^{p0}_\alpha \omega^0_p + \eta^{w0}_\alpha \omega^0_w + \eta^{w0}_\alpha \eta^{w0}_\beta X^{\beta 0}) \\ &= \gamma^2_\alpha r^2 + \gamma^1_\alpha r + \gamma^0_\alpha, \end{aligned} \tag{3.16}$$

$$g_{\alpha\beta} = \eta^k_\alpha \eta^k_\beta = \eta^{p0}_\alpha \eta^{p0}_\beta (r + a_{(p)})^2 + \eta^{w0}_\alpha \eta^{w0}_\beta = \gamma^2_{\alpha\beta} r^2 + \gamma^1_{\alpha\beta} r + \gamma^0_{\alpha\beta}, \tag{3.17}$$

therefore *the vacuum metric with a non-twisting multiple WAND has the form*

$$\mathrm{d}s^2 = g_{00}\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + 2\big(\gamma^2_\alpha r^2 + \gamma^1_\alpha r + \gamma^0_\alpha\big)\mathrm{d}u\mathrm{d}x^\alpha + \big(\gamma^2_{\alpha\beta} r^2 + \gamma^1_{\alpha\beta} r + \gamma^0_{\alpha\beta}\big)\mathrm{d}x^\alpha \mathrm{d}x^\beta, \quad (3.18)$$

where functions $\gamma^N_{\alpha\beta}$ *and* γ^N_α, $N = 0, 1, 2$, *introduced in (3.16), (3.17) do not depend on* r.

Differentiating eq. (2.34) with respect to r and using (2.35), (3.13) and the Ricci equation (A.1) for L_{11}, we arrive to

$$C_{0101} = -V_{,rr}. \quad (3.19)$$

Consequently, for type III and N spacetimes (where C_{0101} has to vanish) V is linear in r, while for type II and D spacetimes the r-dependence of V (and hence of U) can be more complicated. Types II, D and III, N will be thus discussed separately in the following sections. Note that for deriving the metric (3.18) only assumptions $C_{0i0j} = C_{010i} = 0$ on the Weyl tensor are necessary and it was not necessary to assume $C_{0kij} = 0$. Therefore the metric (3.18) applies also to the special class of type I spacetimes with $C_{010i} = 0$ denoted by I(a) in [5]. As for the Ricci tensor, in fact up to now we have assumed only $R_{00} = R_{0i} = 0$.

Note that it was shown that for type III and N expanding vacuum spacetimes $m = 2$ in arbitrary dimension and that $s_{(2)} = s_{(3)}$ [7]. If all non-vanishing eigenvalues of S_{ij} are equal, i.e. from (3.1) $s_{(p)} = 1/(r + a^0(u, x^\alpha))$ for all p, one can perform a coordinate transformation [10] that leaves unchanged null hypersurfaces $u =$const and preserves the affine character of the parameter r

$$\tilde{r} = r + a^0(u, x^\alpha). \quad (3.20)$$

Then from Ricci eqs. (A.11) (for $i = k = q$, $j = p$)

$$\omega^0_p = 0. \quad (3.21)$$

In the following, for simplicity we omit the tilde symbol over r and over absolute terms, such as ω^0_w, $X^{\alpha 0}$, l^0_{11}, U^0, $\overset{i}{m}_{j1}{}^0$, n^0_{i1}.

4. Type III, N

In this section, vacuum type III and N spacetimes are considered and r-dependence of the remaining metric component g_{00}, the Ricci rotation coefficients L_{11}, N_{ij}, and $\overset{i}{M}_{j1}$ and the Weyl tensor is determined. These spacetimes are either non-expanding (Kundt class) with $m = 0$ or expanding with $m = 2$ [7], where, in appropriate coordinates $s_{(2)} = s_{(3)} = 1/r$, as mentioned above.

From Ricci eqs. (A.1) and (2.34) it follows

$$L_{11} = -\, l^0_{1w} l^0_{1w} r + l^0_{11}, \quad (4.1)$$

$$U \;= l^0_{1w} l^0_{1w} r^2 - (l^0_{11} + l^0_{1w}\omega^0_w) r + U^0. \quad (4.2)$$

For future reference let us note that one can still perform a null rotation with fixed $\boldsymbol{\ell}$ (2.3) with $z_p = 0$ for $p = 2, 3$, z_w arbitrary and subject to (3.3)

$$\hat{L}_{1p} = 0,\ \hat{L}_{1w} = L_{1w}, \quad (4.3)$$

$$\hat{\omega}_p = 0,\ \hat{\omega}_w = -l^0_{1w} r + \omega^0_w - z_w = -l^0_{1w} r + \hat{\omega}^0_w, \quad (4.4)$$

$$\hat{\overset{j}{M}}_{ki} = \overset{j}{M}_{ki} + 2 z_{[k} L_{j]i}, \quad (4.5)$$

$$\hat{\xi}^\alpha_i = \xi^\alpha_i, \quad (4.6)$$

$$\hat{L}_{11} = L_{11} + z_i(L_{1i} + L_{i1}) + z_i z_j L_{ij} = -l^0_{1w} l^0_{1w} r + l^0_{11} + 2 z_w l^0_{1w} = -l^0_{1w} l^0_{1w} r + \hat{l}^0_{11} \quad (4.7)$$

$$\hat{U} = l^0_{1w} l^0_{1w} r^2 - \left[l^0_{11} + l^0_{1w}(\omega^0_w + z_w)\right] r + U^0 + z_w \omega^0_w - \tfrac{1}{2} z_w z_w$$
$$= l^0_{1w} l^0_{1w} r^2 - (\hat{l}^0_{11} + l^0_{1w} \hat{\omega}^0_w) r + \hat{U}^0. \quad (4.8)$$

By choosing appropriate z_w, $w = 4, \cdots n-1$, one can simplify ω_w, U or l_{11} (see sec. 5.2.1 and 6).

From Ricci eqs. (A.10), (A.13)

$$N_{ip} = \frac{n^0_{ip}}{r}, \quad N_{iw} = n^0_{iw}, \quad (4.9)$$

$$\overset{i}{M}_{j1} = -\overset{i}{m}_{jw}{}^0 l^0_{1w} r + \overset{i}{m}_{j1}{}^0. \quad (4.10)$$

Let us conclude this section by writing down the metric for the Weyl types III, N. From (3.13), using (4.2), we arrive at

$$V = (l^0_{11} - l^0_{1v} \eta^{v0}_\alpha X^{\alpha 0}) r - U^0 + X^{\alpha 0} \eta^{w0}_\alpha \omega^0_w. \quad (4.11)$$

Substituting the metric component

$$g_{00} = 2V + \Omega^i \Omega^i = (\eta^{p0}_\alpha \eta^{p0}_\beta X^{\alpha 0} X^{\beta 0} + l^0_{1w} l^0_{1w}) r^2 + 2r[l^0_{11} - 2 l^0_{1v} \eta^{v0}_\alpha X^{\alpha 0}] - 2U^0$$
$$+ 2 X^{\alpha 0} \eta^{w0}_\alpha \omega^0_w + \eta^{w0}_\alpha \eta^{w0}_\beta X^{\alpha 0} X^{\beta 0} = \gamma^2 r^2 + \gamma^1 r + \gamma^0, \quad (4.12)$$

into (3.15), from (3.18) we find that *vacuum type III or N metric with non-twisting multiple WAND has the form*

$$\mathrm{d}s^2 = (\gamma^2 r^2 + \gamma^1 r + \gamma^0)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + 2(\gamma^2_\alpha r^2 + \gamma^1_\alpha r + \gamma^0_\alpha)\mathrm{d}u\mathrm{d}x^\alpha + (\gamma^2_{\alpha\beta} r^2 + \gamma^0_{\alpha\beta})\mathrm{d}x^\alpha \mathrm{d}x^\beta, \quad (4.13)$$

where the functions γ^N, γ^N_α and $\gamma^N_{\alpha\beta}$, $N = 0, 1, 2$, are introduced in (4.12), (3.16) and (3.17), respectively.

In fact to derive the metric (4.13) only the following assumptions on the Ricci tensor have been made: $R_{00} = R_{0i} = 2R_{01} - R/(n-1) = 0$.

Note that in the non-expanding case, i.e. for $m = 0$, γ^2_α and $\gamma^2_{\alpha\beta}$ vanish (see (3.16), (3.17)) and the metric (4.13) is compatible with higher-dimensional Kundt metrics given in [15, 16]. In the expanding case, i.e. $m = 2$, the metric (4.13) is compatible with four-dimensional vacuum type III and N Robinson-Trautman solutions (see e.g. [2]) and with direct products of these metrics with a flat space.

In the following sections we study r-dependence of the Weyl tensor separately for types N and III.

4.1. The Weyl tensor for type N

In this section r-dependence of the remaining quantities entering the Ricci and Bianchi equations is derived for vacuum type N spacetimes. In an appropriately chosen frame there are only Weyl components of boost weight -2, $\Psi_{ij} \equiv \frac{1}{2} C_{1i1j}$. As was shown in [7], Ψ_{ij} can be diagonalized together with S_{ij} and admits a form $\Psi_{ij} = \mathrm{diag}.\{p,\ -p,\ 0,\ \cdots 0\}$. Similarly as in [14], it can be shown that the condition of both Ψ_{ij} and S_{ij} being diagonal is compatible with the frame being parallelly propagated.

Eqs. (A.6) and (B.4) lead to

$$N_{i1} = -(n^0_{iw} l^0_{1w}) r + n^0_{i1}, \quad (4.14)$$

$$p = \frac{p^0}{r}. \quad (4.15)$$

As was shown in [13] the curvature invariant

$$I_N \equiv C^{a_1 b_1 a_2 b_2;c_1 c_2} C_{a_1 d_1 a_2 d_2;c_1 c_2} C^{e_1 d_1 e_2 d_2;f_1 f_2} C_{e_1 b_1 e_2 b_2;f_1 f_2} \tag{4.16}$$

reduces for non-twisting type N vacuum spacetimes to

$$I_N = 36(n-2)^8 \theta^8 (\Psi_{ij}\Psi_{ij})^2. \tag{4.17}$$

I_N clearly diverges at $r=0$ in the expanding case and therefore a curvature singularity is located there. The non-expanding (Kundt) case belongs to VSI spacetimes [13], i.e. spacetimes with vanishing curvature invariants of all orders, and therefore curvature invariants cannot be used for locating possible singularities.

4.2. The Weyl tensor for type III

Now let us examine r-dependence of the Weyl tensor for type III vacuum spacetimes. In an appropriately chosen frame, there are only Weyl tensor components of boost weight -1 and -2, i.e. Ψ_i, Ψ_{ijk} and Ψ_{ij}, respectively (see (2.13), (2.14)).

Bianchi eqs. (B.1), (B.9), and (B.4) read (note that in our case (B.6) is equivalent to (B.9))

$$D\Psi_i = -2\Psi_e L_{ei} = -2\Psi_i s_{(i)}, \tag{4.18}$$

$$D\Psi_{jki} = \Psi_{kei}L_{ej} - \Psi_{jei}L_{ek} = -\Psi_{jki}(s_{(j)} + s_{(k)}), \tag{4.19}$$

$$2D\Psi_{ij} - \delta_j \Psi_i = 2\Psi_{jei}L_{e1} - 2\Psi_{ie}L_{ej} + \Psi_e \overset{e}{M}_{ij} \;. \tag{4.20}$$

Equations (4.18), (4.19) imply

$$\Psi_p = \frac{\Psi_p^0}{r^2}, \qquad \Psi_w = \Psi_w^0, \tag{4.21}$$

$$\Psi_{wvi} = \Psi_{wvi}^0, \quad \Psi_{pwi} = \frac{\Psi_{pwi}^0}{r}, \quad \Psi_{pri} = \frac{\Psi_{pri}^0}{r^2}. \tag{4.22}$$

From (4.21), (4.22) and (2.14) it follows

$$\Psi_{prw} = \Psi_{wvp} = 0, \quad \Psi_{wrp} = \Psi_{wpr}, \quad \Psi_{pvw} = \Psi_{pwv}, \quad \Psi_{pww} = 0 = \Psi_{wpp}. \tag{4.23}$$

Note that some of the Bianchi identities reduce to algebraical equations, studied in detail in [7]. Here we use results of [7] to simplify the Weyl tensor (4.21), (4.22). Namely, eqs. (54) in [7] for $(i=w,\, j=v,\, k=p)$ lead to

$$\Psi_{pwv} = 0 \tag{4.24}$$

and for $i,j,k = v,w,z$ in the expanding case $\theta \neq 0$ eqs. (58) in [7] give

$$\Psi_{vwz} = 0 \quad \Rightarrow \quad \Psi_w = 0. \tag{4.25}$$

To summarize: non-vanishing boost weight -1 Weyl tensor components for $\theta \neq 0$ are (cf (C.20) in [7])

$$\Psi_2 = 2\Psi_{233} = \tfrac{\Psi_2^0}{r^2}, \qquad \Psi_3 = 2\Psi_{322} = \tfrac{\Psi_3^0}{r^2},$$

$$\Psi_{w22} = -\Psi_{w33} = \tfrac{\Psi_{w22}^0}{r}, \quad \Psi_{w23} = \Psi_{w32} = \tfrac{\Psi_{w23}^0}{r},$$

while for the non-expanding case $\Psi_w = \Psi_w^0$ and $\Psi_{wvz} = \Psi_{wvz}^0$.

From eqs. (4.20) in the non-expanding case $\theta = 0$ the boost weight -2 components of the Weyl tensor are

$$\Psi_{wv} = \tfrac{r}{2}\left(\xi_v^{\alpha 0}\Psi_{w,\alpha}^0 + 2\Psi_{vzw}^0 l_{1z}^0 + \Psi_z^0 \overset{z}{m}_{wv}{}^0\right) + \Psi_{wv}^0, \tag{4.26}$$

while in the expanding case with (4.25)

$$\Psi_{wv} = \Psi^0_{wv} - \tfrac{1}{2r}\Psi^0_p \overset{p}{m}_{wv}{}^0, \tag{4.27}$$

$$\begin{aligned}\Psi_{wp} &= \tfrac{1}{r}\Psi^0_{wp} - \tfrac{1}{2r^2}\Psi^0_q \overset{q}{m}_{wp}{}^0 \\ &= \Psi_{pw} = \Psi^0_{pw} - \tfrac{1}{2r}(2l^0_{1w}\Psi^0_p + \xi^{\alpha 0}_w \Psi^0_{p,\alpha} + \Psi^0_q \overset{q}{m}_{pw}{}^0) + \tfrac{1}{2r^2}\omega^0_w \Psi^0_p,\end{aligned} \tag{4.28}$$

$$\Psi_{pq} = -\Psi^0_{wqp} l^0_{1w} + \tfrac{\Psi^0_{pq}}{r} - \tfrac{1}{2r^2}(\Psi^0_o \overset{o}{m}_{pq}{}^0 + \xi^{\alpha 0}_q \Psi^0_{p,\alpha}). \tag{4.29}$$

Considering $\Psi_{ij} = \Psi_{ji}$, we get

$$\Psi^0_{wv} = \Psi^0_{vw}, \quad \Psi^0_{pw} = 0, \quad \Psi^0_{pq} = \Psi^0_{qp}, \tag{4.30}$$

$$\Psi^0_{wp} = -\tfrac{1}{2}(2l^0_{1w}\Psi^0_p + \xi^{\alpha 0}_w \Psi^0_{p,\alpha} + \Psi^0_q \overset{q}{m}_{pw}{}^0), \tag{4.31}$$

$$\Psi^0_p \overset{p}{m}_{wv}{}^0 = \Psi^0_p \overset{p}{m}_{vw}{}^0, \tag{4.32}$$

$$\Psi^0_q \overset{q}{m}_{wp}{}^0 = -\omega^0_w \Psi^0_p, \tag{4.33}$$

$$\Psi^0_o \overset{o}{m}_{pq}{}^0 + \xi^{\alpha 0}_q \Psi^0_{p,\alpha} = \Psi^0_o \overset{o}{m}_{qp}{}^0 + \xi^{\alpha 0}_p \Psi^0_{q,\alpha}. \tag{4.34}$$

From (A.6) one can also determine the remaining Ricci rotation coefficients

$$N_{w1} = (-n^0_{wv} l^0_{1v} + \Psi^0_w \delta_{m0})r + n^0_{w1}, \quad N_{p1} = -n^0_{pv} l^0_{1v} r + n^0_{p1} - \tfrac{\Psi^0_p}{r}. \tag{4.35}$$

As was shown in [13], the curvature invariant

$$I_{III} = C^{a_1 b_1 a_2 b_2 ; e_1} C_{a_1 c_1 a_2 c_2 ; e_1} C^{d_1 c_1 d_2 c_2 ; e_2} C_{d_1 b_1 d_2 b_2 ; e_2} \tag{4.36}$$

can be expressed as (74,[13])S

$$I_{III} = 64 S^4 \left[9\psi^4 + 27\psi^2(\mathcal{O}_{PP} + \mathcal{O}_{PF}) + 28(\mathcal{O}_{PP} + \mathcal{O}_{PF})^2 \right] \tag{4.37}$$

$$= 4(n-2)^4 \theta^4 \left[9\psi^4 + 27\psi^2({\Psi_{w22}}^2 + {\Psi_{w23}}^2) + 28({\Psi_{w22}}^2 + {\Psi_{w23}}^2)^2 \right], \tag{4.38}$$

where $\psi^2 = \Psi_i \Psi_i$. Note that all terms entering (4.38) are non-negative and thus singularity in one of these terms implies that the curvature invariant I_{III} is singular. For non-vanishing expansion this is always the case for $r = 0$ and thus a curvature singularity is located there. For type III Kundt spacetimes, the invariant I_{III} (and in fact all curvature invariants of all orders) identically vanishes [13].

5. Type D and II

5.1. Type D

In an adapted frame, type D Weyl tensor has only boost weight zero components determined by Φ_{ij} and C_{ijkl}, see (2.16), (2.17).

For vacuum type D spacetimes with a parallelly propagated frame and with the matrix S_{ij} set to a diagonal form, Bianchi eqs. (B.3), (B.5) and (B.12) can be rewritten using (2.16), (2.17), cf also eqs. (24), (25) in [12]

$$2D\Phi^A_{ij} = -3\Phi^A_{ij}(s_{(i)} + s_{(j)}) - \Phi^S_{ij}(s_{(j)} - s_{(i)}), \tag{5.1}$$

$$2D\Phi^S_{ij} = 3\Phi^A_{ij}(s_{(i)} - s_{(j)}) - \Phi^S_{ij}(s_{(j)} + s_{(i)}) - 2\Phi s_{(i)}\delta_{ij}, \tag{5.2}$$

$$\begin{aligned} DC_{ijkm} = &-\Phi_{kj} s_{(i)} \delta_{im} - \Phi_{mi} s_{(j)} \delta_{jk} + \Phi_{ki} s_{(j)} \delta_{jm} \\ &+ \Phi_{mj} s_{(i)} \delta_{ik} - C_{ijkm}(s_{(m)} + s_{(k)}). \end{aligned} \tag{5.3}$$

S Eq. (4.37) is expressed using notation of [13], while in (4.38) it is rewritten in terms of the quantities introduced in the present paper. Note also there is a misprint in eq. (74) in [13]. It was obtained in Maple using definition $\psi = \Psi_i \Psi_i$, while standard definition, used also in [13] and in the present paper, is $\psi^2 = \Psi_i \Psi_i$. Therefore ψ in eq. (74) from [13] has to be replaced by ψ^2.

Eqs. (5.1) imply $\Phi^A_{wv} = \Phi^{A0}_{wv}$. For simplicity let us assume $\Phi^A_{ij} = 0$ and in what follows we thus identify Φ_{ij} with Φ^S_{ij}. Note that for Kerr-Schild spacetimes $A_{ij} = 0 \Rightarrow \Phi^A_{ij} = 0$ [9], however, this implication need not hold for general spacetimes. Then eqs. (5.1) yield

$$\Phi_{pw} = 0, \tag{5.4}$$

$$\Phi_{pq}(s_{(q)} - s_{(p)}) = 0, \tag{5.5}$$

thus $\Phi_{pq} = 0$ for $s_{(q)} \neq s_{(p)}$.

From eqs. (5.2), (5.5), for $p \neq q$ and $s_{(q)} = s_{(p)}$

$$2D\Phi_{wv} = 0 \quad \Rightarrow \quad \Phi_{wv} = \Phi^0_{wv}, \tag{5.6}$$

$$\begin{aligned} 2D\Phi_{pq} &= -\Phi_{pq}(s_{(p)} + s_{(q)}) = -2\Phi_{pq}s_{(p)} \\ &\Rightarrow \quad \Phi_{pq} = \frac{\Phi^0_{pq}}{r + a^0_{(p)}} \quad \text{for} \quad p \neq q, \quad s_{(q)} = s_{(p)}. \end{aligned} \tag{5.7}$$

Trace of eqs. (5.2) together with (5.6) leads to

$$D\Phi = D\Phi_{pp} = -\Phi S_{ii} - \Phi_{ii}s_{(i)} = -(\Phi_{pp} + \Phi^0_{ww})S_{ii} - \Phi_{pp}s_{(p)}, \tag{5.8}$$

while diagonal terms of (5.2) read

$$D\Phi_{(p)(p)} = -(\Phi_{pp} + \Phi^0_{ww} + \Phi_{(p)(p)})s_{(p)}. \tag{5.9}$$

From now on we assume that $s_{(p)} = 1/r$ for all p ‖. Then eq. (5.8) reduces to

$$D\Phi = D\Phi_{pp} = -\left(\Phi_{pp} + \Phi^0_{ww}\right)\frac{m}{r} - \Phi_{pp}\frac{1}{r} \quad \Rightarrow \quad \Phi_{pp} = \frac{\Phi^0}{r^{m+1}} - \frac{m\Phi^0_{ww}}{m+1} \tag{5.10}$$

and thus

$$\Phi = \frac{\Phi^0}{r^{m+1}} + \frac{\Phi^0_{ww}}{m+1}. \tag{5.11}$$

Then eqs. (5.9) imply

$$\begin{aligned} D\Phi_{(p)(p)} &= -\left(\frac{\Phi^0}{r^{m+1}} + \frac{\Phi^0_{ww}}{m+1} + \Phi_{(p)(p)}\right)s_{(p)} \\ \Rightarrow \ \Phi_{(p)(p)} &= \frac{\Phi^0}{mr^{m+1}} + \frac{\Phi^0_{(p)(p)}}{r} - \frac{\Phi^0_{ww}}{m+1}. \end{aligned} \tag{5.12}$$

Comparing (5.12) with (5.10) yields

$$\Phi^0_{pp} = 0. \tag{5.13}$$

Now we can combine (5.12) with (5.7) in

$$\Phi_{pq} = \frac{\Phi^0_{pq}}{r} + \delta_{pq}\left(\frac{\Phi^0}{mr^{m+1}} - \frac{\Phi^0_{ww}}{m+1}\right). \tag{5.14}$$

‖ In fact under this assumption from eqs. (5.1) $\Phi^A_{pq} = \Phi^{A0}_{pq}/r^3$, however, in what follows we still assume $\Phi^A_{ij} = 0$.

From eqs. (5.3) for various combinations of indices we get

$$C_{ijwv} = C^0_{ijwv}, \tag{5.15}$$

$$C_{ijwq} = \frac{C^0_{ijwq}}{r}, \quad i,j \neq q, \tag{5.16}$$

$$C_{p(q)w(q)} = \frac{C^0_{p(q)w(q)}}{r}, \tag{5.17}$$

$$C_{v(q)w(q)} = \frac{C^0_{v(q)w(q)}}{r} + \Phi^0_{wv}, \tag{5.18}$$

$$C_{wvpq} = \frac{C^0_{wvpq}}{r^2}, \tag{5.19}$$

$$C_{wopq} = \frac{C^0_{wopq}}{r^2}, \tag{5.20}$$

$$\begin{aligned} C_{sopq} = & -2(\delta_{sp}\delta_{oq} - \delta_{op}\delta_{sq})\left(\Phi^0 \frac{F_m(r)}{r^2} + \frac{\Phi^0_{ww}}{2(m+1)}\right) \\ & + \frac{C^0_{sopq}}{r^2} + \frac{1}{r}(\Phi^0_{ps}\delta_{oq} + \Phi^0_{qo}\delta_{sp} - \Phi^0_{po}\delta_{sq} - \Phi^0_{qs}\delta_{op}), \end{aligned} \tag{5.21}$$

where

$$F_m(r) = -\ln r \ \text{ for } \ m=1, \qquad F_m(r) = \frac{1}{m(m-1)r^{m-1}} \ \text{ for } \ m \neq 1. \tag{5.22}$$

Note that some of the equations (5.15)–(5.21) are not compatible with symmetries of the Weyl tensor unless corresponding components vanish, thus

$$C_{wpvz} = C_{vzwp} = 0, \tag{5.23}$$

$$C_{wvpq} = C_{pqwv} = 0, \tag{5.24}$$

$$C_{wopq} = C_{pqwo} = 0 \tag{5.25}$$

and from eqs. (2.11) and (5.24)

$$C^0_{vpwq} = C^0_{vqwp}. \tag{5.26}$$

Let us point out that for expanding type D (and in general not for type II) spacetimes, Bianchi eqs. (B.6), with $\Phi^0 \neq 0$, lead to

$$l^0_{1w} = 0. \tag{5.27}$$

However, we will not use this relation further in this section in order to obtain expressions valid also for type II.

Using the identity $\Phi_{ij} = -\frac{1}{2}C_{ikjk}$ (2.12) for the Weyl tensor we arrive to

$$C^0_{vpwp} = 0, \tag{5.28}$$

$$C^0_{wzvz} = -(m+2)\Phi^0_{wv}, \tag{5.29}$$

$$C^0_{poqo} = 0 \quad \text{for} \ \ m \neq 1, \tag{5.30}$$

$$\Phi^0 = 0 \quad \text{for} \ \ m = 1, \tag{5.31}$$

$$C^0_{wpwq} = -m\Phi^0_{pq}. \tag{5.32}$$

Note that when $m = n-2$ (i.e. there are no 'w-type' indices), then $C^0_{pwqw} = 0$ and thus from (5.32) $\Phi^0_{pq} = 0$.

To summarize: non-vanishing boost weight zero components of the Weyl tensor for type D (and II, see sec. 5.2) vacuum spacetimes with non-twisting multiple WAND under assumption $\Phi^A_{ij}=0$ are Φ_{wv} and Φ_{pq} given in (5.6), (5.14), respectively,

$$C_{vwyz} = C^0_{vwyz}, \tag{5.33}$$

$$C_{vpwq} = \frac{C^0_{vpwq}}{r} + \Phi^0_{wv}\delta_{pq}, \tag{5.34}$$

and C_{sopq} given in (5.21) with (5.22), subject to (5.13), (5.26), (5.28)–(5.32).

From Ricci eqs. (A.1) and (2.34) with (5.11)

$$L_{11} = -\left(l^0_{1w}l^0_{1w} + \tfrac{1}{m+1}\Phi^0_{ww}\right) r + \frac{\Phi^0}{mr^m} + l^0_{11}, \tag{5.35}$$

$$U = \left(l^0_{1w}l^0_{1w} + \tfrac{1}{2(m+1)}\Phi^0_{ww}\right) r^2 - (l^0_{11} + l^0_{1w}\omega^0_w)r + \Phi^0 F_m(r) + U^0 \tag{5.36}$$

and from (3.13)

$$V = -\tfrac{1}{2(m+1)}\Phi^0_{ww}r^2 + r(l^0_{11} - l^0_{1w}\eta^{w0}_{\alpha}X^{\alpha 0}) - \Phi^0 F_m(r) - U^0 + X^{\alpha 0}\eta^{w0}_{\alpha}\omega^0_w. \tag{5.37}$$

Then the metric component g_{00} (3.15) read

$$\begin{aligned} g_{00} = {} & (-\tfrac{1}{(m+1)}\Phi^0_{ww} + \eta^{p0}_{\alpha}X^{\alpha 0}\eta^{p0}_{\beta}X^{\beta 0} + l^0_{1w}l^0_{1w})r^2 + 2r(l^0_{11} - 2l^0_{1w}\eta^{w0}_{\alpha}X^{\alpha 0}) \\ & - 2\Phi^0 F_m(r) - 2U^0 + 2X^{\alpha 0}\eta^{w0}_{\alpha}\omega^0_w + \eta^{w0}_{\alpha}X^{\alpha 0}\eta^{w0}_{\beta}X^{\beta 0} \\ = {} & \left(\gamma^2 - \tfrac{1}{(m+1)}\Phi^0_{ww}\right) r^2 + \gamma^1 r + \gamma^0 - 2\Phi^0 F_m(r), \end{aligned} \tag{5.38}$$

where γ^N, $N=0,1,2$ are defined in (4.12). *The metric for type D vacuum spacetimes with a non-twisting multiple WAND then has the form (3.18) with (5.38), (3.16), (3.17) with $a^0_{(p)}=0$ and $l^0_{1w}=0$.* Note that (5.38) is valid for type II spacetimes as well (see sec. 5.2).

Let us now examine the Kretschmann scalar in vacuum

$$\begin{aligned} R_{abcd}R^{abcd} &= 4R^2_{0101} + R_{ijkl}R_{ijkl} + 8R_{0j1i}R_{0i1j} - 4R_{01ij}R_{01ij} \\ &= 4\Phi^2 + C_{ijkl}C_{ijkl} + 8\Phi^S_{ij}\Phi^S_{ij} - 24\Phi^A_{ij}\Phi^A_{ij}. \end{aligned} \tag{5.39}$$

As was pointed out in [9], under the assumption $\Phi^A_{ij}=0$, it reduces to a sum of squares. Thus if any term Φ^0, ${\Phi_{pq}}^0$, C^0_{vpwq} or C^0_{nopq} is non-zero, then there is a scalar curvature singularity at $r=0$.

Note also that for asymptotically flat spacetimes the Kretschmann scalar vanishes for $r \to \infty$ and thus in this case

$$\Phi^0_{wv} = 0 = C^0_{wvyz}. \tag{5.40}$$

5.2. *Type II*

Apart from boost weight zero components of the Weyl tensor, in type II spacetimes boost weight -1 components, Ψ_i, Ψ_{ijk}, and boost weight -2 components, Ψ_{ij}, also appear (see (2.13), (2.14)). However, these negative boost weight components do not enter Bianchi equations (5.1)–(5.3) and thus assuming again $s_{(p)}=1/r$ for all p and $\Phi^A_{ij}=0$ all results obtained in sec. 5.1 for type D spacetimes except of (5.27) are valid for type II spacetimes as well.

In order to determine r-dependence of negative boost weight components of the Weyl tensor, we analyze Bianchi equations (B.1), (B.6), (B.9) and (B.4), which can be rewritten as

$$D\Psi_i - \delta_i\Phi = -2\Psi_i s_{(i)} - \Phi L_{i1} - \Phi^S_{ik}L_{k1}, \tag{5.41}$$

$$-2D\Psi_{ijk} = 2\Psi_{ijk}s_{(k)} + 2\Psi_{[i}\delta_{j]k}s_{(k)} + 2\Phi^S_{k[i}L_{j]1} - C_{klij}L_{l1}, \tag{5.42}$$

$$2D\Psi_{jki} + 2\delta_{[k}\Phi^S_{j]i} = 2\Psi_{kji}s_{(j)} - 2\Psi_{jki}s_{(k)} + 2\Phi^S_{[k|l}\overset{l}{M}_{i|j]} - 2\Phi^S_{li}\overset{l}{M}_{[jk]}, \tag{5.43}$$

$$\begin{aligned} 2D\Psi_{ij} - \Delta\Phi^S_{ij} - \delta_j\Psi_i &= -2\Psi_{ij}s_{(j)} + 2\Psi_{jli}L_{l1} + \Psi_l\overset{l}{M}_{ij} + \Phi N_{ij} \\ &\quad + \Phi^S_{li}N_{lj} + \Phi^S_{jl}\overset{l}{M}_{i1} + \Phi^S_{li}\overset{l}{M}_{j1}\,. \end{aligned} \tag{5.44}$$

Using previous results, from (5.41)

$$\begin{aligned} \Psi_w &= r\left[\tfrac{1}{m+1}(\xi^{\alpha 0}_w\Phi^0_{vv},_\alpha - l^0_{1w}\Phi^0_{vv}) - \Phi^0_{wv}l^0_{1v}\right] + \Psi^0_w \\ &\quad - (l^0_{1w}\Phi^0 m + \xi^{\alpha 0}_w\Phi^0,_\alpha)\tfrac{1}{mr^m} + \tfrac{1}{r^{m+1}}\omega^0_w\Phi^0, \end{aligned} \tag{5.45}$$

$$\Psi_p = \tfrac{1}{2(m+1)}\xi^{\alpha 0}_p\Phi^0_{ww},_\alpha + \tfrac{1}{r^2}\Psi^0_p - \xi^{\alpha 0}_p\Phi^0,_\alpha\tfrac{mF_m(r)}{r^2} \tag{5.46}$$

and from (5.42)

$$\Psi_{pqw} = \Psi^0_{pqw}, \tag{5.47}$$

$$\Psi_{vpw} = \Psi^0_{vpw}, \tag{5.48}$$

$$\Psi_{vzw} = \tfrac{1}{2}(\Phi^0_{wz}l^0_{1v} - \Phi^0_{wv}l^0_{1z} + C^0_{wyvz}l^0_{1y})r + \Psi^0_{vzw}, \tag{5.49}$$

$$\Psi_{vwp} = \tfrac{1}{r}\Psi^0_{vwp}, \tag{5.50}$$

$$\begin{aligned} \Psi_{wpq} &= -\tfrac{r}{4(m+1)}\xi^{\alpha 0}_w\Phi^0_{vv},_\alpha\delta_{pq} + \tfrac{1}{2}(\Phi^0_{pq}l^0_{1w} + C^0_{qzwp}l^0_{1z} - \Psi^0_w\delta_{pq}) + \tfrac{1}{r}\Psi^0_{wpq} \\ &\quad - \tfrac{\delta_{pq}}{2}[l^0_{1w}\Phi^0(m+1) + \xi^{\alpha 0}_w\Phi^0,_\alpha]\tfrac{F_m(r)}{r} + \tfrac{\delta_{pq}}{2mr^{m+1}}\omega^0_w\Phi^0, \end{aligned} \tag{5.51}$$

$$\begin{aligned} \Psi_{oqp} &= \tfrac{1}{4(m+1)}\Phi^0_{ww},_\alpha(\delta_{op}\xi^{\alpha 0}_q - \delta_{pq}\xi^{\alpha 0}_o) + \Psi^0_{oqp}\tfrac{1}{r} + \tfrac{1}{2r^2}(\delta_{pq}\Psi^0_o - \delta_{po}\Psi^0_q) \\ &\quad + \Phi^0,_\alpha\tfrac{F_m(r)}{2r^2}(\delta_{po}\xi^{\alpha 0}_q - \delta_{pq}\xi^{\alpha 0}_o). \end{aligned} \tag{5.52}$$

The Weyl components Ψ_i and Ψ_{ijk} as given in (5.45)–(5.52) are subject to (2.14) and therefore

$$\Psi^0_{wvp} = 0, \quad \Psi^0_{pqw} = 0, \quad \Psi^0_{pwv} = \Psi^0_{pvw}, \quad \Psi^0_{wpq} = \Psi^0_{wqp}, \tag{5.53}$$

$$0 = \Psi^0_{vwz} + \Psi^0_{zvw} + \Psi^0_{wzv}, \tag{5.54}$$

$$0 = \Psi^0_{pqo} + \Psi^0_{opq} + \Psi^0_{qop}, \tag{5.55}$$

$$C^0_{vywv}l^0_{1y} = \tfrac{m+2}{2(m+1)}\xi^{\alpha 0}_w\Phi^0_{zz},_\alpha - \tfrac{m+2}{(m+1)}l^0_{1w}\Phi^0_{zz}, \tag{5.56}$$

$$2\Psi^0_{wzz} = l^0_{1z}C^0_{zpwp} + \Psi^0_w(m+1), \tag{5.57}$$

$$\Psi^0_{wpp} = \tfrac{1}{2}l^0_{1w}\Phi^0 \quad \text{and} \quad 2l^0_{1w}\Phi^0 + \xi^{\alpha 0}_w\Phi^0,_\alpha = 0 \quad \text{for} \quad m=1, \tag{5.58}$$

$$\Psi^0_{wpp} = 0 \quad \text{and} \quad 2ml^0_{1w}\Phi^0 + \xi^{\alpha 0}_w\Phi^0,_\alpha = 0 \quad \text{for} \quad m>1, \tag{5.59}$$

$$\Psi^0_{pww} = \tfrac{m}{4(m+1)}\xi^{\alpha 0}_p\Phi^0_{ww},_\alpha\,, \tag{5.60}$$

$$\Psi^0_{pqq} = 0, \tag{5.61}$$

$$\Psi^0_p = 0 \quad \text{and} \quad \xi^{\alpha 0}_p\Phi^0,_\alpha = 0 \quad \text{for} \quad m=1, \tag{5.62}$$

$$\Psi^0_p(m-2) = 0 \quad \text{and} \quad (m-2)\xi^{\alpha 0}_p\Phi^0,_\alpha = 0 \quad \text{for} \quad m>1. \tag{5.63}$$

In order to determine r-dependence of Ψ_{ij} from eqs. (5.44), first we need to find $\overset{i}{M}_{j1}$ and N_{ij}. Note that for $\Phi^A_{ij}=0$, the Ricci eqs. (A.13) reduce to those of the Weyl type III with solution given in (4.10). From Ricci eqs. (A.10),

$$N_{pw} = n^0_{pw}, \quad N_{vw} = -\Phi^0_{wv} r + n^0_{vw}, \quad N_{wp} = \frac{n^0_{wp}}{r}, \tag{5.64}$$

$$N_{pq} \;=\; -\,\Phi^0_{pq} + \frac{n^0_{pq}}{r} + \delta_{pq}\left[r\frac{\Phi^0_{ww}}{2(m+1)} + \Phi^0\frac{F_m(r)}{r}\right]. \tag{5.65}$$

Now r-dependence of Ψ_{ij} can be determined from eqs. (5.44)

$$\begin{aligned}\Psi_{vw} &= r^2\Psi^A_{vw} + r\Psi^B_{vw} + \Psi^C_{vw} - \frac{\overset{p}{m}_{vw}{}^0\Psi^0_p}{2r} - \overset{p}{m}_{vw}{}^0\xi^{\alpha 0}_p\Phi^0{}_{,\alpha}\frac{\ln r+1}{2r}\delta_{1m}\\ &\quad + mF_m(r)\Psi^D_{vw} + \frac{1}{r^m}\Psi^E_{vw} + \frac{1}{r^{m+1}}\Psi^F_{vw},\end{aligned} \tag{5.66}$$

$$\begin{aligned}\Psi_{pw} &= r^2\Psi^A_{pw} + r\Psi^B_{pw} + \Psi^C_{pw}\ln r + \Psi^D_{pw} + \frac{\ln r}{r}\delta_{1m}\Psi^E_{pw} + \frac{1}{r}\Psi^F_{pw}\\ &\quad + \frac{\ln r}{r^2}\delta_{1m}\Psi^G_{pw} + \frac{1}{r^2}\Psi^H_{pw} + \frac{1}{r^{m-1}}\Psi^I_{pw} + \frac{1}{r^m}\Psi^J_{pw} + \frac{1}{r^{m+1}}\Psi^K_{pw},\end{aligned} \tag{5.67}$$

$$\begin{aligned}\Psi_{pq} &= r^2\Psi^A_{pq}\delta_{pq} + r\Psi^B_{pq} + \ln r\,\delta_{1m}\Psi^C_{pq} + \Psi^D_{pq} + \frac{\ln r}{r}\Psi^E_{pq} + \Psi^F_{pw}\frac{1}{r} + \frac{\ln r}{r^2}\delta_{1m}\Psi^G_{pq}\\ &\quad - \frac{1}{r^2}\Psi^H_{pq} + \frac{1}{r^{m-1}}\Psi^I_{pq} + \frac{1}{r^m}\Psi^J_{pq} + \frac{1}{r^{m+1}}\Psi^K_{pq},\end{aligned} \tag{5.68}$$

where Ψ^A_{ij}, Ψ^B_{ij}, … Ψ^K_{ij} do not depend on r. Since in this paper we are mainly interested in the r-dependence of the metric and the Weyl tensor we do not give here quite complicated explicit expressions for Ψ^A_{ij}, Ψ^B_{ij}, … Ψ^K_{ij}.

5.2.1. The case with $L_{1i}=0$ When (5.27) is satisfied (for type D and special cases of other Weyl types considered here) then ω_w can be transformed away by null rotation with fixed $\boldsymbol{\ell}$ (2.3) with $z_w = \omega^0_w$ (4.4) and thus (assuming all $s_{(p)}$ are same) $\omega_i = 0$ for all i. Since now $g^{1\alpha} = X^{\alpha 0}$, we introduce $\tilde{x}^\alpha = \tilde{x}^\alpha(x^\beta, u)$ as in [10], leaving unchanged null hypersurfaces $u=$const and preserving the affine character of the parameter r, to set $\tilde{g}^{1\alpha}=0$, i.e. (omitting the tilde symbol)

$$X^{\alpha 0} = 0. \tag{5.69}$$

Then from (3.10)–(3.13) and (5.37) we get

$$V = -U = -\frac{1}{2(m+1)}\Phi^0_{ww}r^2 + rl^0_{11} - \Phi^0 F_m(r) - U^0, \quad \Omega^{(i)} = 0, \quad Y_\alpha = 0. \tag{5.70}$$

Eqs. (3.14)–(3.17) now reduce to

$$g_{11} = 0, \quad g_{01} = 1, \quad g_{1\alpha} = 0, \quad g_{00} = 2V, \quad g_{0\alpha} = 0, \tag{5.71}$$

$$g_{\alpha\beta} = \eta^k_\alpha\eta^k_\beta = \eta^{p0}_\alpha\eta^{p0}_\beta r^2 + \eta^{w0}_\alpha\eta^{w0}_\beta = \overset{2}{\gamma}_{\alpha\beta}r^2 + \overset{0}{\gamma}_{\alpha\beta}, \tag{5.72}$$

and thus *the metric of vacuum spacetimes with a non-twisting multiple WAND (i.e. types II, D, III or N) with $L_{1i}=0$ can be set into the form*

$$\mathrm{d}s^2 = 2V\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + \left(\overset{2}{\gamma}_{\alpha\beta}r^2 + \overset{0}{\gamma}_{\alpha\beta}\right)\mathrm{d}x^\alpha\mathrm{d}x^\beta, \tag{5.73}$$

where functions $\overset{N}{\gamma}_{\alpha\beta}$, $N=0,2$, introduced in (5.72) do not depend on r and V is given in (5.70).

5.2.2. Shearfree case Let us now briefly discuss the shear-free case which occurs for $m=0$ (Kundt spacetimes) and for $m=n-2$ (Robinson-Trautman spacetimes [10]).

Kundt spacetimes in vacuum are necessarily of type II or more special [8] and they thus form $m=0$ subclass of spacetimes studied in the present paper. Note that in contrast with the expanding case, the components of the metric (3.18), including

g_{00}, are at most quadratic polynomials in r. Similarly as in four dimensions boost weight 0, -1 and -2 components of the Weyl tensor are independent on r, linear and quadratic in r, respectively.

In the $m = n-2$ case in four dimensions, eqs. (5.63) are identically satisfied and consequently the corresponding class of Robinson-Trautman spacetimes is very rich and includes e.g. radiative type N and III spacetimes as well as type D C-metric describing uniformly accelerated black holes emitting gravitational radiation. However, in higher dimensions eqs. (5.63) imply $\Psi^0_p = 0$ and using (2.25) $\Phi^0{}_{,\alpha} = 0$. From (5.32) $\Phi^0_{pq} = 0$ and then from (A.12) or (5.43) we get $\Psi^0_{oqp} = 0$. Therefore all components of the Weyl tensor with boost weight -1 vanish. Similarly it can be shown that boost weight -2 components of the Weyl tensor vanish as well. Thus in higher dimensions vacuum shear-free spacetimes admitting non-twisting multiple WAND are necessarily of type D in agreement with [10]. The Weyl tensor is now given by

$$\Phi_{pq} = \delta_{pq}\frac{\Phi^0}{(n-2)r^{n-1}}, \quad C_{sopq} = -2(\delta_{sp}\delta_{oq} - \delta_{op}\delta_{sq})\Phi^0\frac{F_{(n-2)}(r)}{r^2} + \frac{C^0_{sopq}}{r^2}. \qquad (5.74)$$

Note that in four dimensions eq. (5.30) implies $C^0_{sopq} = 0$, while in higher dimensions this term, corresponding essentially to the curvature of the spatial part of the metric $\gamma^2_{\alpha\beta}$ [10], in general does not vanish. Therefore the r-dependence of the Weyl tensor and thus also the asymptotic behaviour of gravitational field in higher dimensions is more complex than in four dimensions¶. This is, however, beyond the scope of the present paper and will be studied elsewhere.

6. Construction of an explicit expanding type N solution in five dimensions with $l^0_{14} = 0$

Apart from usual motivation coming from higher-dimensional general relativity, there is an additional reason for studying type N vacuum spacetimes. For these spacetimes all curvature invariants involving metric, the Riemann tensor and its first covariant derivatives vanish. Such solutions thus belong to VSI$_1$ class of spacetimes [17], which are solutions of various field theories to all orders with a specific effective action containing only certain higher order correction terms (see [17]).

Let us explicitly mention the Einstein-Gauss-Bonnet equations

$$R_{ab} - \tfrac{1}{2}Rg_{ab} = \alpha\left(\tfrac{1}{2}\mathcal{L}_{GB}g_{ab} - 2RR_{ab} + 4R_{ac}R_b{}^c + 4R_{acbd}R^{cd} - 2R_{acde}R_b{}^{cde}\right), \qquad (6.1)$$

where $\mathcal{L}_{GB} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$ and α is the Gauss-Bonnet coupling constant. It can be seen directly that vacuum type N solutions to the Einstein equations solve vacuum Einstein-Gauss-Bonnet equations (6.1) as well since for these spacetimes $R_{acde}R_b{}^{cde} = 0 = R_{abcd}R^{abcd}$.

In this section, we attempt to derive an expanding non-twisting type N vacuum solution and we limit ourselves to a five-dimensional case with an additional assumption $l^0_{14} = 0$. Since resulting metrics we have obtained so far can be obtained by taking a direct product of four-dimensional type N vacuum metrics with an extra dimension, the main purpose of this section is thus to illustrate use of the higher-dimensional NP formalism for constructing exact vacuum solutions. Note that corresponding Bianchi and Ricci equations are quite complex and thus at several points of the calculation we make various assumptions in order to simplify them.

¶ Note that in boost weight zero Weyl components in the $m < n-2$ case terms proportional to r^0 and r^{-1} also appear.

This, however, obviously comes with the price of possibly reducing the resulting class of solutions.

For explicit calculations it turns out to be more convenient to relax the assumption of diagonal Ψ_{ij} from sec. 4.1 and so now there are two independent components of the Weyl tensor $\Psi_{33} = -\Psi_{22}$, $\Psi_{32} = \Psi_{23}$ with the rest of the components vanishing. Therefore we cannot use the form of the Weyl tensor obtained in (4.15) and instead from the Bianchi eqs. (B.4)

$$\Psi_{22} = -\Psi_{33} = \frac{p^0}{r}, \quad \Psi_{23} = \Psi_{32} = \frac{\Pi^0}{r}, \quad \Psi_{pw} = 0 = \Psi_{wv}. \tag{6.2}$$

Assuming $l^0_{14} = 0$, NP equations simplify considerably and are given in Appendix C. In fact the following quantities vanish

$$\overset{2}{m}_{43}{}^0 = \overset{3}{m}_{42}{}^0 = \overset{2}{m}_{44}{}^0 = \overset{3}{m}_{44}{}^0 = 0, \tag{6.3}$$

$$n^0_{24} = n^0_{34} = \overset{2}{m}_{41}{}^0 = \overset{3}{m}_{41}{}^0 = n^0_{23} = n^0_{32} = \omega^0_2 = \omega^0_3 = 0. \tag{6.4}$$

Similarly as in sec. 5.2.1 we transform away ω^0_4. However, here we do not transform away the functions $X^{\alpha 0}$. Then from eqs. (C.9)

$$\omega^0_4 = \overset{4}{m}_{22}{}^0 = \overset{4}{m}_{33}{}^0 = 0, \tag{6.5}$$

and from (C.30)–(C.32) and (C.36)

$$n^0_{42} = n^0_{43} = n^0_{44} = n^0_{41} = 0. \tag{6.6}$$

From (C.4) we get $U^0 = n^0_{22} = n^0_{33}$ and then eqs. (C.45)–(C.47) (now identical with (C.15)–(C.17)) imply

$$U^0 = n^0_{22} = U^0(u). \tag{6.7}$$

Let us assume $U^0 = n^0_{22}$ =const.

Apart from $l^0_{14} = 0$ we make the following simplifying assumptions

$$\overset{2}{m}_{34}{}^0 = 0, \tag{6.8}$$

$$\xi^3_3 \; = \; - \, \xi^2_2 \neq 0, \quad \xi^4_4 \neq 0, \quad \text{all other } \xi^\alpha_k = 0. \tag{6.9}$$

Note that $\overset{2}{m}_{34}{}^0$ always vanishes for diagonal Ψ_{ij}, see (C.51).

Under the assumptions (6.9) from (C.38), (C.39) (C.14), (C.23), (C.26), (C.27), (C.28), (C.35), (C.42), (C.50), (C.51) we obtain that $\xi^{20}_2 = -\xi^{30}_3$, l^0_{11}, n^0_{21}, n^0_{31}, $\overset{3}{m}_{22}{}^0$, $\overset{2}{m}_{33}{}^0$, $\overset{2}{m}_{31}{}^0$, X^{20}, X^{30}, p^0, Π^0 do not depend on $x^3 = z$. From (C.38)–(C.41) it also follows that $\xi^{40}_4 = \xi^{40}_4(u, z)$, $X^{40} = X^{40}(u, z)$ are functions of u, z only.

Eqs. (C.37), (C.29) can be rewritten using (6.9)

$$\xi^{20}_2{}_{,3} \; = \; \overset{3}{m}_{22}{}^0, \tag{6.10}$$

$$\xi^{20}_2{}_{,2} \; = - \, \overset{2}{m}_{33}{}^0, \tag{6.11}$$

$$\xi^{20}_2 (\xi^{20}_2{}_{,22} + \xi^{20}_2{}_{,33}) - (\xi^{20}_2{}_{,2})^2 - (\xi^{20}_2{}_{,3})^2 = 2 n^0_{22}. \tag{6.12}$$

Assuming ξ^{20}_2 to have a form of a polynomial in $x^2 = x$ and $x^3 = y$, after an appropriate translation in x, y, we arrive at

$$\xi^{20}_2 = A_0 P(x, y), \quad P(x, y) = (1 + e x^2 + e y^2), \quad \overset{2}{m}_{33}{}^0 = -2 A_0 e x, \quad \overset{3}{m}_{22}{}^0 = 2 A_0 e y, \tag{6.13}$$

where we set $A_0 = 1/\sqrt{2}$ and $e = n_{22}$ is assumed to be independent on u. From (C.40) and (C.41) it follows

$$X^{20}{}_{,2} = X^{30}{}_{,3}\,, \qquad X^{30}{}_{,2} = -X^{20}{}_{,3}\,, \tag{6.14}$$

with the integrability condition $X^{20}{}_{,22} + X^{20}{}_{,33} = 0$ and from (C.40)

$$l^0_{11} = -\frac{2e(xX^{20} + yX^{30})}{P(x,y)} + X^{20}{}_{,x}\,. \tag{6.15}$$

Then (C.12), (C.13) determine n^0_{21}, n^0_{31}

$$n_{21} = -\tfrac{\sqrt{2}}{2}P(x,y)X^{20}{}_{,xx} + \sqrt{2}e(X^{20} + xX^{20}{}_{,x} - yX^{20}{}_{,y}) - 2\sqrt{2}ex\tfrac{xX^{20}+yX^{30}}{P(x,y)}, \tag{6.16}$$

$$n_{31} = \tfrac{\sqrt{2}}{2}P(x,y)X^{20}{}_{,xy} - \sqrt{2}e(X^{30} + xX^{20}{}_{,y} + yX^{20}{}_{,x}) + 2\sqrt{2}ey\tfrac{xX^{20}+yX^{30}}{P(x,y)}. \tag{6.17}$$

From eq. (C.21) or (C.25) and from (C.22) or (C.24) we get

$$p^0 = -\tfrac{1}{2}X^{20}{}_{,xxx}\,P(x,y)^2, \tag{6.18}$$

$$\Pi^0 = \tfrac{1}{2}X^{20}{}_{,xxy}\,P(x,y)^2. \tag{6.19}$$

Eqs. (3.10)–(3.13) lead to $V = -U$, $Y_j = 0$, $\Omega^i = -\eta^{(i)}_{(i)}X^{(i)0} = -X^{(i)}/\xi^{(i)}_{(i)}$. The contravariant frame vectors now read

$$\ell^a = [0,1,0,0,0], \tag{6.20}$$

$$n^a = \left[1, -\left(-\tfrac{2e(xX^{20}+yX^{30})}{P(x,y)} + X^{20}{}_{,x}\right)r + e, X^{20}, X^{30}, X^{40}\right], \tag{6.21}$$

$$m^a_{(2)} = A_0 P(x,y)\tfrac{1}{r}[0,0,1,0,0], \tag{6.22}$$

$$m^a_{(3)} = -\,A_0 P(x,y)\tfrac{1}{r}[0,0,0,1,0], \tag{6.23}$$

$$m^a_{(4)} = \xi^{40}_4[0,0,0,0,1]; \tag{6.24}$$

and the covariant frame vectors are

$$\ell_a = [1,0,0,0,0], \tag{6.25}$$

$$n_a = \left[\left(-\tfrac{2e(xX^{20}+yX^{30})}{P(x,y)} + X^{20}{}_{,x}\right)r - e, 1, 0, 0, 0\right], \tag{6.26}$$

$$m^{(2)}_a = \tfrac{r}{A_0P(x,y)}[-X^{20},0,1,0,0], \tag{6.27}$$

$$m^{(3)}_a = -\,\tfrac{r}{A_0P(x,y)}[-X^{30},0,0,1,0], \tag{6.28}$$

$$m^{(4)}_a = \left[-\tfrac{X^{40}}{\xi^{40}_4}, 0, 0, 0, \tfrac{1}{\xi^{40}_4}\right], \tag{6.29}$$

where ξ^{40}_4, X^{40} are subject to (C.42), i.e.

$$-\xi^{40}_4{}_{,u} - X^{40}\xi^{40}_4{}_{,4} + \xi^{40}_4 X^{40}{}_{,4} = 0. \tag{6.30}$$

The metric thus reads

$$\begin{aligned}\mathrm{d}s^2 = {}&\left[2l^0_{11}r - 2e + \left(\tfrac{r}{\xi^{20}_2}\right)^2((X^{20})^2 + (X^{30})^2) + \left(\tfrac{X^{40}}{\xi^{40}_4}\right)^2\right]\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r \\ &-2\mathrm{d}u\left[\left(\tfrac{r}{\xi^{20}_2}\right)^2(X^{20}\mathrm{d}x + X^{30}\mathrm{d}y) + \left(\tfrac{1}{\xi^{40}_4}\right)^2X^{40}\mathrm{d}z\right] + \left(\tfrac{r}{\xi^{20}_2}\right)^2(\mathrm{d}x^2 + \mathrm{d}y^2) + \left(\tfrac{1}{\xi^{40}_4}\right)^2\mathrm{d}z^2.\end{aligned} \tag{6.31}$$

Introducing $\tilde{z} = \int 1/\xi^{40}_4\mathrm{d}z$ and using (6.30) the metric (6.31) reduces to

$$\begin{aligned}\mathrm{d}s^2 = {}&\left[2l^0_{11}r - 2e + \left(\tfrac{r}{\xi^{20}_2}\right)^2((X^{20})^2 + (X^{30})^2)\right]\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r \\ &- 2\mathrm{d}u\left(\tfrac{r}{\xi^{20}_2}\right)^2(X^{20}\mathrm{d}x + X^{30}\mathrm{d}y) + \left(\tfrac{r}{\xi^{20}_2}\right)^2(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}\tilde{z}^2,\end{aligned} \tag{6.32}$$

where

$$\mathrm{d}\tilde{z} = \tfrac{1}{\xi_4^{40}}\mathrm{d}z - \tfrac{X^{40}}{\xi_4^{40}}\mathrm{d}u. \tag{6.33}$$

So the metric (6.32) represents a direct product of a four-dimensional Robinson-Trautman type N vacuum solution with an extra dimension.

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Appendix A. Ricci identities

The Ricci equations, i.e. contractions of the Ricci identities $v_{a;bc} - v_{a;cb} = R_{sabc}v^s$ with the frame vectors (2.1), in higher dimensions, given in full generality in eqs. (11a)–(11p) in [8], are rewritten here for vacuum spacetimes with a geodetic multiple WAND (Weyl types II, D, III or N) in a parallelly propagated frame

$$DL_{11} = -L_{1i}L_{i1} - C_{0101}\,, \tag{A.1}$$

$$DL_{1i} = -L_{1j}L_{ji}\,, \tag{A.2}$$

$$\triangle L_{1i} - \delta_i L_{11} = L_{11}(L_{1i} - L_{i1}) - 2L_{j[1|}N_{j|i]} - L_{1j}(N_{ji} + \overset{j}{M}_{i1}) + C_{101i}\,, \tag{A.3}$$

$$\delta_{[j|}L_{1|i]} = -L_{11}L_{[ij]} - L_{1k}\overset{k}{M}_{[ij]} - L_{k[j|}N_{k|i]} + \tfrac{1}{2}C_{01ij}\,, \tag{A.4}$$

$$DL_{i1} = -L_{ij}L_{j1}\,, \tag{A.5}$$

$$-DN_{i1} = N_{ij}L_{j1} - C_{101i}\,, \tag{A.6}$$

$$DL_{ij} = -L_{ik}L_{kj}\,, \tag{A.7}$$

$$\begin{aligned}\triangle N_{ij} - \delta_j N_{i1} &= -L_{11}N_{ij} - N_{i1}(-2L_{1j} + L_{j1})\\ &\quad + 2N_{k[1|}\overset{k}{M}_{i|j]} - N_{ik}(N_{kj} + \overset{k}{M}_{j1}) - C_{1i1j}\,,\end{aligned} \tag{A.8}$$

$$\triangle L_{ij} - \delta_j L_{i1} = L_{11}L_{ij} - L_{i1}L_{j1} + 2L_{k[1|}\overset{k}{M}_{i|j]} - L_{ik}(N_{kj} + \overset{k}{M}_{j1}) - C_{0i1j}\,, \tag{A.9}$$

$$DN_{ij} = -N_{ik}L_{kj} - C_{0j1i}\,, \tag{A.10}$$

$$\delta_{[j|}L_{i|k]} = L_{1[j|}L_{i|k]} + L_{i1}L_{[jk]} + L_{il}\overset{l}{M}_{[jk]} + L_{l[j|}\overset{l}{M}_{i|k]}\,, \tag{A.11}$$

$$\delta_{[j|}N_{i|k]} = -L_{1[j|}N_{i|k]} + N_{i1}L_{[jk]} + N_{il}\overset{l}{M}_{[jk]} + N_{l[j|}\overset{l}{M}_{i|k]} - \tfrac{1}{2}C_{1ijk}\,, \tag{A.12}$$

$$D\overset{i}{M}_{j1} = -\overset{i}{M}_{jk}L_{k1} - C_{01ij}\,, \tag{A.13}$$

$$D\overset{i}{M}_{jk} = -\overset{i}{M}_{jl}L_{lk}\,, \tag{A.14}$$

$$\begin{aligned}\triangle\overset{i}{M}_{jk} - \delta_k\overset{i}{M}_{j1} &= 2N_{j[1|}L_{i|k]} + 2L_{j[1|}N_{i|k]} + \overset{i}{M}_{j1}(L_{1k} - L_{k1}) + 2\overset{i}{M}_{l[1|}\overset{l}{M}_{j|k]}\\ &\quad - \overset{i}{M}_{jl}(N_{lk} + \overset{l}{M}_{k1}) - C_{1kij}\,,\end{aligned} \tag{A.15}$$

$$\begin{aligned}\delta_{[k|}\overset{i}{M}_{j|l]} &= N_{i[l|}L_{j|k]} + L_{i[l|}N_{j|k]} + L_{[kl]}\overset{i}{M}_{j1}\\ &\quad + \overset{i}{M}_{p[k|}\overset{p}{M}_{j|l]} + \overset{i}{M}_{jp}\overset{p}{M}_{[kl]} - \tfrac{1}{2}C_{ijkl}\,.\end{aligned} \tag{A.16}$$

Appendix B. Bianchi equations

We present here Bianchi identities projected onto a parallelly propagated null frame (2.1) for vacuum spacetimes with a geodetic multiple WAND. General form of these identities can be found in Appendix B in [7].

$$DC_{101i} - \delta_i C_{0101} = -C_{0101}L_{i1} - C_{01is}L_{s1} - 2C_{101s}L_{si} - C_{0i1s}L_{s1}, \tag{B.1}$$

$$- \triangle C_{01ij} + 2\delta_{[j|}C_{101|i]} = 2C_{101[j|}L_{1|i]} + 2C_{101[i|}L_{|j]1} + 2C_{1[i|1s}L_{s|j]} + C_{1sij}L_{s1}$$
$$+2C_{0101}N_{[ji]} + 2C_{01[i|s}N_{s|j]} + 2C_{0s1[j|}N_{s|i]} + 2C_{01[i|s} \overset{s}{M}_{|j]1} +2C_{101s} \overset{s}{M}_{[ji]}, \tag{B.2}$$

$$-DC_{01ij} = 2C_{0101}L_{[ij]} + 2C_{01[i|s}L_{s|j]} + 2C_{0[i|1s}L_{s|j]}, \tag{B.3}$$

$$DC_{1i1j} - \triangle C_{0j1i} - \delta_j C_{101i} = 2C_{101i}L_{[1j]} + 2C_{1i[j|s}L_{s|1]} + C_{0101}N_{ij}$$
$$-C_{01is}N_{sj} + C_{0s1i}N_{sj} + C_{0j1s} \overset{s}{M}_{i1} +2C_{0s1i} \overset{s}{M}_{j1} +C_{101s} \overset{s}{M}_{ij}, \tag{B.4}$$

$$DC_{0i1j} = -C_{0101}L_{ij} - C_{0i1s}L_{sj} - C_{01is}L_{sj}, \tag{B.5}$$

$$-DC_{1kij} - \delta_k C_{01ij} = -C_{01ij}L_{k1} + 2C_{0k1[i}L_{j]1} + 2C_{101[i}L_{j]k}$$
$$+2C_{[1|sij}L_{s|k]} + 2C_{01[i|s} \overset{s}{M}_{|j]k}, \tag{B.6}$$

$$2\delta_{[k|}C_{0i1|j]} = 2C_{0i1[j|}L_{|k]1} - C_{01jk}L_{i1} + 2C_{101[j|}L_{i|k]} + 2C_{1[k|is}L_{s|j]}$$
$$-C_{isjk}L_{s1} + 2C_{0i1s} \overset{s}{M}_{[kj]} +2C_{0s1[k|} \overset{s}{M}_{i|j]}, \tag{B.7}$$

$$0 = 0, \tag{B.8}$$

$$DC_{1ijk} + 2\delta_{[k|}C_{0|j]1i} = 2C_{101i}L_{[jk]} + 2C_{1i[k|s}L_{s|j]} + 2C_{0[k|1s} \overset{s}{M}_{i|j]} -2C_{0s1i} \overset{s}{M}_{[jk]}, \tag{B.9}$$

$$\triangle C_{1ijk} + 2\delta_{[k|}C_{1i1|j]} = 2C_{1i1[j}L_{k]1} + 4C_{1i1[k|}L_{1|j]} - C_{1ijk}L_{11} + C_{01jk}N_{i1}$$
$$+2C_{0[j|1i}N_{|k]1} + 2C_{101[k|}N_{i|j]} + 2C_{101i}N_{[kj]} + 2C_{1i[k|s}N_{s|j]} + 2C_{1[k|is}N_{s|j]}$$
$$-C_{isjk}N_{s1} - 2C_{1i1s} \overset{s}{M}_{[jk]} +2C_{1[k|1s} \overset{s}{M}_{i|j]} +2C_{1i[k|s} \overset{s}{M}_{|j]1} -C_{1sjk} \overset{s}{M}_{i1}, \tag{B.10}$$

$$-\delta_{\{i|}C_{01|jk\}} = C_{101\{i}L_{jk\}} - C_{101\{i}L_{kj\}} + C_{1s\{ij|}L_{s|k\}}$$
$$+C_{01\{i|s} \overset{s}{M}_{|jk\}} -C_{01\{i|s} \overset{s}{M}_{|kj\}}, \tag{B.11}$$

$$-DC_{ijkm} = 2C_{01ij}L_{[km]} + 2C_{0[k|1j}L_{i|m]} + 2C_{0[m|1i}L_{j|k]} + 2C_{ij[k|s}L_{s|m]}, \tag{B.12}$$

$$- \triangle C_{ijkm} + 2\delta_{[k|}C_{1|m]ij} = 2C_{1i[1|m}L_{j|k]} + 2C_{1[j|1k}L_{|i]m} + 2C_{1j[k|m}L_{i|1]} + 2C_{1kij}L_{[1m]}$$
$$+2C_{1mij}L_{[k1]} - 2C_{01ij}N_{[km]} + 2C_{0[i|1m}N_{|j]k} + 2C_{0[j|1k}N_{|i]m} + 2C_{ij[k|s}N_{s|m]}$$
$$+2C_{1k[i|s} \overset{s}{M}_{|j]m} +2C_{1m[j|s} \overset{s}{M}_{|i]k} +2C_{1sij} \overset{s}{M}_{[km]} +2C_{ij[k|s} \overset{s}{M}_{|m]1} +2C_{[i|skm} \overset{s}{M}_{|j]1}, \tag{B.13}$$

$$\delta_{\{j|}C_{1i|mk\}} = C_{1i1\{j}L_{mk\}} - C_{1i1\{j}L_{km\}} + C_{1i\{jk|}L_{1|m\}} + C_{01\{jm|}N_{i|k\}} - C_{0\{j|1i}N_{|km\}}$$
$$+C_{0\{j|1i}N_{|mk\}} + C_{is\{jk|}N_{s|m\}} - C_{1i\{j|s} \overset{s}{M}_{|mk\}} +C_{1i\{j|s} \overset{s}{M}_{|km\}} +C_{1s\{jk|} \overset{s}{M}_{i|m\}}, \tag{B.14}$$

$$0 = C_{01\{jk|}L_{i|m\}} - C_{0i1\{j}L_{km\}} + C_{0i1\{j}L_{mk\}} + C_{is\{jk|}L_{s|m\}}, \tag{B.15}$$

$$\delta_{\{k|}C_{ij|nm\}} = C_{1j\{km|}L_{i|n\}} - C_{1i\{km|}L_{j|n\}} - C_{1\{k|ij}L_{|mn\}} + C_{1\{k|ij}L_{|nm\}}$$
$$+C_{ij\{k|s} \overset{s}{M}_{|mn\}} -C_{ij\{k|s} \overset{s}{M}_{|nm\}} +C_{is\{km|} \overset{s}{M}_{j|n\}} -C_{js\{km|} \overset{s}{M}_{i|n\}} . \tag{B.16}$$

Appendix C. The Ricci and Bianchi equations for five-dimensional type N spacetimes with all $L_{1i} = 0$

Ricci eqs. (A.4)

$$n^0_{32} = n^0_{23}, \tag{C.1}$$
$$n^0_{24} = 0, \tag{C.2}$$
$$n^0_{34} = 0, \tag{C.3}$$

Ricci eqs. (A.9)

$$U^0 = n^0_{22} = n^0_{33}, \tag{C.4}$$
$$n^0_{23} = 0 = n^0_{32}, \tag{C.5}$$
$$0 = n^0_{24} + \overset{2}{m}_{41}{}^0, \tag{C.6}$$
$$0 = n^0_{34} + \overset{3}{m}_{41}{}^0. \tag{C.7}$$

Ricci eqs. (A.11)

$$\omega^0_2 = 0 = \omega^0_3, \tag{C.8}$$
$$\omega^0_4 = \overset{4}{m}_{22}{}^0 = \overset{4}{m}_{33}{}^0, \tag{C.9}$$
$$\overset{2}{m}_{43}{}^0 = 0 = \overset{3}{m}_{42}{}^0, \tag{C.10}$$
$$\overset{2}{m}_{44}{}^0 = 0 = \overset{3}{m}_{44}{}^0. \tag{C.11}$$

Ricci eqs. (A.3)

$$\xi^{\alpha 0}_2 l^0_{11},_\alpha = -n^0_{21}, \tag{C.12}$$
$$\xi^{\alpha 0}_3 l^0_{11},_\alpha = -n^0_{31}, \tag{C.13}$$
$$\xi^{\alpha 0}_4 l^0_{11},_\alpha = 0. \tag{C.14}$$

Ricci eqs. (A.12) with $n_{24} = n_{34} = 0$ from (C.2), (C.3), with (6.8)

$$\xi^{\alpha 0}_2 n^0_{22},_\alpha = n^0_{42} \overset{4}{m}_{22}{}^0, \tag{C.15}$$
$$\xi^{\alpha 0}_3 n^0_{22},_\alpha = n^0_{43} \overset{4}{m}_{22}{}^0, \tag{C.16}$$
$$\xi^{\alpha 0}_4 n^0_{22},_\alpha = n^0_{44} \overset{4}{m}_{22}{}^0, \tag{C.17}$$
$$\xi^{\alpha 0}_3 n^0_{42},_\alpha - \xi^{\alpha 0}_2 n^0_{43},_\alpha = n^0_{42} \overset{2}{m}_{32}{}^0 - n^0_{43} \overset{3}{m}_{23}{}^0, \tag{C.18}$$
$$\xi^{\alpha 0}_2 n^0_{44},_\alpha - \xi^{\alpha 0}_4 n^0_{42},_\alpha = 0, \tag{C.19}$$
$$\xi^{\alpha 0}_3 n^0_{44},_\alpha - \xi^{\alpha 0}_4 n^0_{43},_\alpha = 0. \tag{C.20}$$

Ricci eqs. (A.8), with $n_{24} = n_{34} = 0$,

$$n^0_{22},_u + X^{\alpha 0} n^0_{22},_\alpha - \xi^{\alpha 0}_2 n^0_{21},_\alpha = -2 l^0_{11} n^0_{22} + n^0_{31} \overset{3}{m}_{22}{}^0 - p^0, \tag{C.21}$$
$$\xi^{\alpha 0}_3 n^0_{21},_\alpha = n^0_{31} \overset{2}{m}_{33}{}^0 + \Pi^0, \tag{C.22}$$
$$\xi^{\alpha 0}_4 n^0_{21},_\alpha = 0, \tag{C.23}$$
$$\xi^{\alpha 0}_2 n^0_{31},_\alpha = n^0_{21} \overset{3}{m}_{22}{}^0 + \Pi^0, \tag{C.24}$$
$$n^0_{22},_u + X^{\alpha 0} n^0_{22},_\alpha - \xi^{\alpha 0}_3 n^0_{31},_\alpha = -2 l^0_{11} n^0_{22} + n^0_{41} \overset{4}{m}_{22}{}^0 + n^0_{21} \overset{2}{m}_{33}{}^0 + p^0, \tag{C.25}$$
$$\xi^{\alpha 0}_4 n^0_{31},_\alpha = 0. \tag{C.26}$$

Ricci eqs. (A.16)

$$\xi_4^{\alpha 0}\, \overset{2}{m}_{32}{}^0{}_{,\alpha} = 0, \tag{C.27}$$

$$\xi_4^{\alpha 0}\, \overset{2}{m}_{33}{}^0{}_{,\alpha} = 0, \tag{C.28}$$

$$\xi_3^{\alpha 0}\, \overset{2}{m}_{32}{}^0{}_{,\alpha} - \xi_2^{\alpha 0}\, \overset{2}{m}_{33}{}^0{}_{,\alpha} = 2n_{22}^0 + (\overset{4}{m}_{22}{}^0)^2 + (\overset{2}{m}_{32}{}^0)^2 + (\overset{2}{m}_{33}{}^0)^2, \tag{C.29}$$

$$\xi_2^{\alpha 0}\, \overset{4}{m}_{22}{}^0{}_{,\alpha} = -n_{42}^0, \tag{C.30}$$

$$\xi_3^{\alpha 0}\, \overset{4}{m}_{22}{}^0{}_{,\alpha} = -n_{43}^0, \tag{C.31}$$

$$\xi_4^{\alpha 0}\, \overset{4}{m}_{22}{}^0{}_{,\alpha} = -n_{44}^0. \tag{C.32}$$

Ricci eqs. (A.15)

$$\overset{2}{m}_{32}{}^0{}_{,u} + X^{\alpha 0}\, \overset{2}{m}_{32}{}^0{}_{,\alpha} - \xi_2^{\alpha 0}\, \overset{2}{m}_{31}{}^0{}_{,\alpha} = -\overset{2}{m}_{32}{}^0 l_{11}^0 + n_{31}^0 + \overset{2}{m}_{33}{}^0\, \overset{2}{m}_{31}{}^0, \tag{C.33}$$

$$\overset{2}{m}_{33}{}^0{}_{,u} + X^{\alpha 0}\, \overset{2}{m}_{33}{}^0{}_{,\alpha} - \xi_3^{\alpha 0}\, \overset{2}{m}_{31}{}^0{}_{,\alpha} = -\overset{2}{m}_{33}{}^0 l_{11}^0 - n_{21}^0 - \overset{2}{m}_{32}{}^0\, \overset{2}{m}_{31}{}^0, \tag{C.34}$$

$$\xi_4^{\alpha 0}\, \overset{2}{m}_{31}{}^0{}_{,\alpha} = 0, \tag{C.35}$$

$$\overset{2}{m}_{42}{}^0{}_{,u} + X^{\alpha 0}\, \overset{2}{m}_{42}{}^0{}_{,\alpha} = -\overset{2}{m}_{42}{}^0 l_{11}^0 + n_{41}^0. \tag{C.36}$$

Commutators (2.41)

$$\xi_2^{\beta 0} \xi_3^{\alpha 0}{}_{,\beta} - \xi_3^{\beta 0} \xi_2^{\alpha 0}{}_{,\beta} = \overset{3}{m}_{22}{}^0 \xi_2^{\alpha 0} - \overset{2}{m}_{33}{}^0 \xi_3^{\alpha 0}, \tag{C.37}$$

$$\xi_2^{\beta 0} \xi_4^{\alpha 0}{}_{,\beta} - \xi_4^{\beta 0} \xi_2^{\alpha 0}{}_{,\beta} = \xi_2^{\alpha 0}\, \overset{4}{m}_{22}{}^0, \tag{C.38}$$

$$\xi_3^{\beta 0} \xi_4^{\alpha 0}{}_{,\beta} - \xi_4^{\beta 0} \xi_3^{\alpha 0}{}_{,\beta} = \xi_3^{\alpha 0}\, \overset{4}{m}_{22}{}^0, \tag{C.39}$$

commutators (2.40)

$$-\xi_2^{\alpha 0}{}_{,u} - X^{\beta 0} \xi_2^{\alpha 0}{}_{,\beta} + \xi_2^{\beta 0} X^{\alpha 0}{}_{,\beta} = \xi_2^{\alpha 0} l_{11}^0 + n_{42}^0 \xi_4^{\alpha 0} - \overset{2}{m}_{31}{}^0 \xi_3^{\alpha 0}, \tag{C.40}$$

$$-\xi_3^{\alpha 0}{}_{,u} - X^{\beta 0} \xi_3^{\alpha 0}{}_{,\beta} + \xi_3^{\beta 0} X^{\alpha 0}{}_{,\beta} = \xi_3^{\alpha 0} l_{11}^0 + n_{43}^0 \xi_4^{\alpha 0} + \overset{2}{m}_{31}{}^0 \xi_2^{\alpha 0}, \tag{C.41}$$

$$-\xi_4^{\alpha 0}{}_{,u} - X^{\beta 0} \xi_4^{\alpha 0}{}_{,\beta} + \xi_4^{\beta 0} X^{\alpha 0}{}_{,\beta} = \xi_4^{\alpha 0} n_{44}^0, \tag{C.42}$$

commutators (2.37)

$$\xi_2^{\alpha 0} \omega_4^0{}_{,\alpha} = -n_{42}^0, \tag{C.43}$$

$$\xi_3^{\alpha 0} \omega_4^0{}_{,\alpha} = -n_{43}^0, \tag{C.44}$$

commutators (2.36)

$$\xi_2^{\alpha 0} U^0{}_{,\alpha} = n_{42}^0 \omega_4^0, \tag{C.45}$$

$$\xi_3^{\alpha 0} U^0{}_{,\alpha} = n_{43}^0 \omega_4^0, \tag{C.46}$$

$$-\omega_4^0{}_{,u} - X^{\alpha 0} \omega_4^0{}_{,\alpha} + \xi_4^{\alpha 0} U^0{}_{,\alpha} = n_{44}^0 \omega_4^0 + \overset{4}{m}_{22}{}^0 l_{11}^0 + n_{41}^0. \tag{C.47}$$

Bianchi eqs. (B.10)

$$-\xi_2^{\alpha 0} p^0{}_{,\alpha} - \xi_3^{\alpha 0} \Pi^0{}_{,\alpha} = 2p^0\, \overset{2}{m}_{33}{}^0 + 2\Pi^0\, \overset{3}{m}_{22}{}^0, \tag{C.48}$$

$$\xi_3^{\alpha 0} p^0{}_{,\alpha} - \xi_2^{\alpha 0} \Pi^0{}_{,\alpha} = 2p^0\, \overset{2}{m}_{32}{}^0 + 2\Pi^0\, \overset{2}{m}_{33}{}^0, \tag{C.49}$$

$$\xi_4^{\alpha 0} p^0{}_{,\alpha} = 2\Pi^0\, \overset{2}{m}_{34}{}^0, \tag{C.50}$$

$$\xi_4^{\alpha 0} \Pi^0{}_{,\alpha} = -p^0\, \overset{2}{m}_{34}{}^0, \tag{C.51}$$

$$p^0\, \overset{4}{m}_{23}{}^0 = p^0\, \overset{4}{m}_{32}{}^0 = 0, \tag{C.52}$$

$$\Pi^0 \overset{4}{m}_{32}{}^0 = \Pi^0 \overset{4}{m}_{23}{}^0 = 0, \tag{C.53}$$

$$p^0 \overset{2}{m}_{44}{}^0 + \Pi^0 \overset{3}{m}_{44}{}^0 \;=\; - p^0 \overset{3}{m}_{44}{}^0 + \Pi^0 \overset{2}{m}_{44}{}^0 = 0. \tag{C.54}$$

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