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A note on absolutely continuous functions of two variables in the sense of Carathéodory [1]

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Abstract

In this note, the notion of absolute continuity of functions of two variables is discussed. We recall that the set of functions of two variables absolutely continuous in the sense of Carathéodory coincides with the class of functions admitting a certain integral representation. Moreover, we show that absolutely continuous functions in the sense of Carathéodory can be equivalently characterized in terms of their properties with respect to each of the variables.

Key words: absolutely continuous function in the sense of Carathéodory, integral representation, derivative of double integral
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Introduction

The notion of absolute continuity of functions of a single variable has been generalized to more variables in many ways. Each of these constructions observes only some of properties known in one dimensional case like continuity, differentiability almost everywhere or integration by parts, the others are naturally lost. Let us mention absolute continuity due to Schwartz (see [11]), absolutely continuous functions in the sense of Banach or in the sense of Tonelli (see, e.g., [9,10]), and n-absolute continuity introduced by Malý (see [7]). We are interested in the notion of absolute continuity of functions of two variables in the sense of Carathéodory (see [2]) in order to define a solution of a partial differential

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equation of hyperbolic type with noncontinuous right-hand side. More precisely, if we consider the hyperbolic equation

$$\frac{\partial u(t,x)}{\partial t\,\partial x} = f(t,x,u) \tag{$*$}$$

on the rectangle $[a,b]\times[c,d]$, where $f\colon [a,b]\times[c,d]\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, then by a solution of $(*)$ is usually understood a function u absolutely continuous on $[a,b]\times[c,d]$ in the sense of Carathéodory, which satisfies the equality $(*)$ almost everywhere on the rectangle $[a,b]\times[c,d]$ (see, e.g., [1,3,5,14]).

The aim of this note is to recall the definition of an absolutely continuous function of two variables introduced by Carathéodory in 1918 and to show that the set of those functions coincides with the class of functions admitting a certain integral representation. Some basic properties of such functions like the existence of partial derivatives and differentiability almost everywhere are studied, e.g., in [2,12,13,15] (see also survey given in [4]). However, for the investigation of the initial and boundary value problems for the equation $(*)$, we need to show that absolutely continuous functions in the sense of Carathéodory can be *equivalently* characterized in terms of their properties with respect to each of the variables.

1. Notation and definitions

Throughout the paper, $\mathcal{D}=[a,b]\times[c,d]$ denotes the rectangle in $\mathbb{R}^2$ and

$$Q(t,x)=[a,t]\times[c,x] \quad \text{for } (t,x)\in\mathcal{D}. \tag{1.1}$$

As usual, $L\big(\mathcal{D};\mathbb{R}\big)$ stands for the set of Lebesgue integrable functions on $\mathcal{D}$. $AC\big([\alpha,\beta];\mathbb{R}\big)$ and $L\big([\alpha,\beta];\mathbb{R}\big)$ are the sets of absolutely continuous and Lebesgue integrable, respectively, functions on $[\alpha,\beta]\subset\mathbb{R}$. For any measurable set $E\subset\mathbb{R}^n$ $(n=1,2)$, $\operatorname{mes}E$ denotes the Lebesgue measure of E.

We first introduce a definition of an absolutely continuous function of two variables (namely, Definition 1.4), which is based on the definition given by Carathéodory in 1918. Although one of the conditions appearing in Definition 1.4 is slightly different from those stated in Carathéodory's monograph [2], both definitions describe one and the same set of functions (see Remarks 1.5 and 1.7).

Let $\mathfrak{S}(\mathcal{D})$ denote the system of all rectangles $[t_1,t_2]\times[x_1,x_2]$ contained in $\mathcal{D}$. For any rectangle $P\in\mathfrak{S}(\mathcal{D})$, $|P|$ denotes the *volume* of P. We say that rectangles $P_1,P_2\in\mathfrak{S}(\mathcal{D})$ *do not overlap* if the have no interior points in common. Furthermore, rectangles $P_1,P_2\in\mathfrak{S}(\mathcal{D})$ are referred to be *adjoining* if they do not overlap and $P_1\cup P_2\in\mathfrak{S}(\mathcal{D})$.

Definition 1.1 ([6, §5.3])**.** A finite function $F\colon\mathfrak{S}(\mathcal{D})\to\mathbb{R}$ is said to be *additive function of rectangles* if, for any adjoining rectangles $P_1,P_2\in\mathfrak{S}(\mathcal{D})$, the relation

$$F(P_1\cup P_2)=F(P_1)+F(P_2)$$

holds.

Definition 1.2 ([6, §7.3])**.** We say that an additive function of rectangles $F\colon\mathfrak{S}(\mathcal{D})\to\mathbb{R}$ is *absolutely continuous* if, for every $\varepsilon>0$, there exists $\delta>0$ such that if $P_1,\dots,P_k\in\mathfrak{S}(\mathcal{D})$ are mutually non-overlapping rectangles with the property

$$\sum_{j=1}^{k}|P_j|\le\delta,$$

then the relation

$$\sum_{j=1}^{k} \big|F(P_j)\big| \leq \varepsilon$$

is satisfied.

The following statement gives a sufficient and necessary condition for the function of rectangles to be absolutely continuous.

Theorem 1.3 ([6, §7.3])**.** *The function of rectangles $F\colon \mathfrak{S}(\mathcal{D}) \to \mathbb{R}$ is absolutely continuous if and only if there exists a function $h \in L\big(\mathcal{D};\mathbb{R}\big)$ such that*

$$F(P) = \iint\limits_{P} h(t,x)\mathrm{d}t\mathrm{d}x \quad \textit{for } P \in \mathfrak{S}(\mathcal{D}). \tag{1.2}$$

Now we are in position to define an absolutely continuous function of two variables in the sense of Carathéodory. Having the function $u\colon \mathcal{D} \to \mathbb{R}$, we put

$$F_u\big([t_1,t_2]\times[x_1,x_2]\big) = u(t_1,x_1) - u(t_1,x_2) - u(t_2,x_1) + u(t_2,x_2) \quad \text{for } [t_1,t_2]\times[x_1,x_2] \in \mathfrak{S}(\mathcal{D}). \tag{1.3}$$

The function of rectangles $F_u\colon \mathfrak{S}(\mathcal{D}) \to \mathbb{R}$ defined by the formula (1.3) is said to be *a function of rectangles associated with u.*

Definition 1.4. We say that a function $u\colon \mathcal{D} \to \mathbb{R}$ is *absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory* if the following two conditions hold:

(α) the function of rectangles F_u associated with u is absolutely continuous;

(β) the functions $u(a,\cdot)\colon [a,b] \to \mathbb{R}$ and $u(\cdot,c)\colon [c,d] \to \mathbb{R}$ are absolutely continuous.

Remark 1.5. In Carathéodory's monograph [2], the condition (α) of the previous definition is slightly different. The construction of Carathéodory reads, roughly speaking, as follows.

Let $\mathfrak{T}(\mathcal{D})$ denote the system of all sets σ of the form

$$\sigma = \delta_1 \cup \dots \cup \delta_p\,,$$

where $\delta_1,\dots,\delta_p$ is a finite system of mutually disjoint *open* rectangles contained in $\mathcal{D}$, and let $\mathfrak{C}$ be the system of all measurable subset of $\mathcal{D}$. An interval function $\Phi\colon \mathfrak{T}(\mathcal{D}) \to \mathbb{R}$ is said to be *additive and absolutely continuous* if there exists a set function $G\colon \mathfrak{C} \to \mathbb{R}$ that is additive, absolutely continuous, and such that

$$\Phi(\sigma) = G(\sigma) \quad \text{for } \sigma \in \mathfrak{T}(\mathcal{D}).$$

Having the function $u\colon \mathcal{D} \to \mathbb{R}$, according to Carathéodory, we put

$$\Phi\big(]t_1,t_2[\times]x_1,x_2[\big) = u(t_1,x_1) - u(t_1,x_2) - u(t_2,x_1) + u(t_2,x_2) \tag{1.4}$$

for any $]t_1,t_2[\times]x_1,x_2[\subseteq \mathcal{D}$, and

$$\Phi(\sigma) = \Phi(\delta_1) + \dots + \Phi(\delta_p) \quad \text{for } \sigma = \delta_1 \cup \dots \cup \delta_p \in \mathfrak{T}(\mathcal{D}). \tag{1.5}$$

The interval function $\Phi\colon \mathfrak{T}(\mathcal{D}) \to \mathbb{R}$ defined by the formulae (1.4) and (1.5) is called *interval function associated with u.*

Definition 1.6 (Carathéodory, [2, §567])**.** A function $u\colon \mathcal{D} \to \mathbb{R}$ is said to be *absolutely continuous* if the following two conditions hold:

(α) the interval function Φ associated with u is absolutely continuous;

(β) the functions $u(a,\cdot)\colon [a,b] \to \mathbb{R}$ and $u(\cdot,c)\colon [c,d] \to \mathbb{R}$ are absolutely continuous.

Remark 1.7. Carathéodory also proved (see [2, §453, Satz 1]) that for an interval function $\Phi\colon \mathfrak{I}(\mathcal{D}) \to \mathbb{R}$ to be additive and absolutely continuous it is sufficient and necessary the existence of a function $p \in L\big(\mathcal{D};\mathbb{R}\big)$ such that

$$\Phi(\sigma) = \iint\limits_{\sigma} p(t,x)\mathrm{d}t\mathrm{d}x \quad \text{for } \sigma \in \mathfrak{I}(\mathcal{D}).$$

Consequently, by virtue of Theorem 1.3, *the function $u\colon \mathcal{D} \to \mathbb{R}$ is absolutely continuous in the sense of Definition 1.4 if and only if it is absolutely continuous in the sence of Definition 1.6.*

2. Main result

The main result of this note is the next theorem which is partly proved in [2, 13, 15].

Theorem 2.1. *The following three statements are equivalent:*

(1) the function $u\colon \mathcal{D} \to \mathbb{R}$ is absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory, i.e.,

(α) the function of rectangles F_u associated with u is absolutely continuous;

(β) $u(a,\cdot) \in AC\big([c,d];\mathbb{R}\big)$ and $u(\cdot,c) \in AC\big([a,b];\mathbb{R}\big)$.

(2) the function $u\colon \mathcal{D} \to \mathbb{R}$ admits the integral representation

$$u(t,x) = e + \int\limits_a^t f(s)\mathrm{d}s + \int\limits_c^x g(\eta)\mathrm{d}\eta + \iint\limits_{Q(t,x)} h(s,\eta)\mathrm{d}s\mathrm{d}\eta \quad \textit{for } (t,x) \in \mathcal{D}, \tag{2.1}$$

where $e \in \mathbb{R}$, $f \in L\big([a,b];\mathbb{R}\big)$, $g \in L\big([c,d];\mathbb{R}\big)$, $h \in L\big(\mathcal{D};\mathbb{R}\big)$, and the mapping Q is defined by the formula (1.1)*;*

(3) the function $u\colon \mathcal{D} \to \mathbb{R}$ satisfies:

(a) $u(\cdot,x) \in AC\big([a,b];\mathbb{R}\big)$ for every $x \in [c,d]$, $u(a,\cdot) \in AC\big([c,d];\mathbb{R}\big)$;

(b) $u_t(t,\cdot) \in AC\big([c,d];\mathbb{R}\big)$ for almost every $t \in [a,b]$;

(c) $u_{tx} \in L\big(\mathcal{D};\mathbb{R}\big)$;

Remark 2.2. It is clear that, using the Fubini theorem (see, e.g., [8, XII, §4, Thm. 1]), we get

$$\iint\limits_{Q(t,x)} h(s,\eta)\mathrm{d}s\mathrm{d}\eta = \int\limits_a^t \int\limits_c^x h(s,\eta)\mathrm{d}\eta\mathrm{d}s = \int\limits_c^x \int\limits_a^t h(s,\eta)\mathrm{d}s\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \tag{2.2}$$

and thus the conditions (a)–(c) of Theorem 2.1 can be replaced by the symmetric ones:

(A) $u(t,\cdot) \in AC\big([c,d];\mathbb{R}\big)$ for every $t \in [a,b]$, $u(\cdot,c) \in AC\big([a,b];\mathbb{R}\big)$;

(B) $u_x(\cdot,x) \in AC\big([a,b];\mathbb{R}\big)$ for almost every $x \in [c,d]$;

(C) $u_{xt} \in L\big(\mathcal{D};\mathbb{R}\big)$.

Remark 2.3. It follows from Proposition 2.5 below that an arbitrary function u absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory has both mixed second-order derivatives almost everywhere on the rectangle $\mathcal{D}$ and these derivatives are equivalent, i.e.,

$$u_{tx}(t,x) = u_{xt}(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D}.$$

Moreover, by virtue of Lemma 2.6 below, one can see that u has also integrable on $\mathcal{D}$ both first-order partial derivatives.

Remark 2.4. It immediately follows from the definition of an absolutely continuous function of a single variable that if $v \in AC\big([a,b];\mathbb{R}\big)$ then also $|v| \in AC\big([a,b];\mathbb{R}\big)$. Unfortunately, functions of two variables *do not have* this property. Indeed, the function u defined by the formula

$$u(t,x) = t - x \quad \text{for } (t,x) \in [0,1] \times [0,1]$$

is absolutely continuous on $[0,1] \times [0,1]$ in the sense of Carathéodory but $|u|$ is not, because the condition (b) of Theorem 2.1 does not hold for $|u|$.

To prove Theorem 2.1 we need the following proposition.

Proposition 2.5. *Let $h \in L\big(\mathcal{D};\mathbb{R}\big)$ and*

$$u(t,x) = \iint\limits_{Q(t,x)} h(s,\eta)\mathrm{d}s\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \tag{2.3}$$

where the mapping Q is defined by the formula (1.1)*. Then:*

(i) there exists a set $E_1 \subseteq [a,b]$ such that $\operatorname{mes} E_1 = b - a$ *and*

$$u_t(t,x) = \int_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } t \in E_1 \text{ and } x \in [c,d]; \tag{2.4}$$

(ii) there exists a set $E_2 \subseteq [c,d]$ such that $\operatorname{mes} E_2 = d - c$ *and*

$$u_x(t,x) = \int_a^t h(s,x)\mathrm{d}s \quad \text{for } t \in [a,b] \text{ and } x \in E_2;$$

(iii) there exists a set $E_3 \subseteq \mathcal{D}$ such that $\operatorname{mes} E_3 = (b-a)(d-c)$ *and*

$$u_{tx}(t,x) = h(t,x) \quad \text{for } (t,x) \in E_3; \tag{2.5}$$

(iv) there exists a set $E_4 \subseteq \mathcal{D}$ such that $\operatorname{mes} E_4 = (b-a)(d-c)$ *and*

$$u_{xt}(t,x) = h(t,x) \quad \text{for } (t,x) \in E_4.$$

Parts (i), (ii) are proved in [13, §7], the proofs of parts (iii), (iv) are based on the proof of Lemma 1 stated in [15]. For the sake of completeness we prove Proposition 2.5 in detail.

Lemma 2.6. *Let $h \in L\big(\mathcal{D};\mathbb{R}\big)$ and*

$$\varphi(t,x) = \int_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } t \in E,\ x \in [c,d], \tag{2.6}$$

where $E \subseteq [a,b]$ is such that $\operatorname{mes} E = b - a$*. Then the function φ is measurable on the rectangle $\mathcal{D}$.*

This lemma is stated in [2, §569]; we give here a slightly simpler proof.

Proof of Lemma 2.6. It is clear that $E \times [c,d]$ is measurable subset of the rectangle $\mathcal{D}$ with the measure equal to $(b-a)(d-c)$, and thus the function φ is defined almost everywhere on $\mathcal{D}$.

First suppose that the function h is bounded, i.e.,

$$|h(t,x)| \le M \quad \text{for a. e. } (t,x) \in \mathcal{D}.$$

It is well-known (see, e.g., [8, XII, §1]) that there exists a sequence $\{f_n\}_{n=1}^{+\infty}$ of functions continuous on $\mathcal{D}$ such that

$$\lim_{n\to+\infty} f_n(t,x) = h(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D}. \tag{2.7}$$

Without loss of generality we can assume that

$$|f_n(t,x)| \le M \quad \text{for } (t,x) \in \mathcal{D},\ n \in \mathbb{N}. \tag{2.8}$$

According to (2.7) and the Fubini theorem (see, e.g., [8, XI, §5, Cor. 1]), there exists a set $F \subseteq [a,b]$ such that $\operatorname{mes} F = b-a$ and the condition

$$\lim_{n\to+\infty} f_n(t_0,x) = h(t_0,x) \quad \text{for a. e. } x \in [c,d]. \tag{2.9}$$

holds for every $t_0 \in F$. Now we put

$$\psi_n(t,x) = \int_c^x f_n(t,\eta)\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D},\ n \in \mathbb{N}. \tag{2.10}$$

It is easy to verify that the functions ψ_n are continuous on $\mathcal{D}$. Indeed, by virtue of (2.8), for any $(t_1,x_1),(t_2,x_2) \in \mathcal{D}$ and $n \in \mathbb{N}$ we get

$$\begin{aligned}|\psi_n(t_2,x_2) - \psi_n(t_1,x_1)| &= \left| \int_c^{x_2} f_n(t_2,\eta)\mathrm{d}\eta - \int_c^{x_1} f_n(t_1,\eta)\mathrm{d}\eta \right| \le \\ &\le \int_c^d |f_n(t_2,\eta) - f_n(t_1,\eta)|\mathrm{d}\eta + M|x_2 - x_1|,\end{aligned}$$

and thus continuity of ψ_n on $\mathcal{D}$ follows from continuity of f_n. Consequently, the functions ψ_n are measurable on $\mathcal{D}$.

On the other hand, in view of (2.8)–(2.10), the Lebesgue convergence theorem yields

$$\varphi(t,x) = \int_c^x h(t,\eta)\mathrm{d}\eta = \lim_{n\to+\infty} \int_c^x f_n(t,\eta)\mathrm{d}\eta = \lim_{n\to+\infty} \psi_n(t,x) \quad \text{for } t \in E \cap F,\ x \in [c,d],$$

i.e.,

$$\varphi(t,x) = \lim_{n\to+\infty} \psi_n(t,x) \quad \text{for a. e. } (t,x) \in \mathcal{D}.$$

Therefore, the function φ is measurable on $\mathcal{D}$ (see, e.g., [8, XII, §1]).

Now suppose that the function h is unbounded. For any $k \in \mathbb{N}$, we put

$$h_k(t,x) = \begin{cases} -k & \text{for } (t,x) \in \mathcal{D},\ h(t,x) < -k, \\ h(t,x) & \text{for } (t,x) \in \mathcal{D},\ |h(t,x)| \le k, \\ k & \text{for } (t,x) \in \mathcal{D},\ h(t,x) > k. \end{cases}$$

It is clear that the functions h_k ($k \in \mathbb{N}$) are measurable and bounded on $\mathcal{D}$ and, for any $t_0 \in E$, we have

$$|h_k(t_0,x)| \le |h(t_0,x)| \quad \text{for a. e. } x \in [c,d],\ k \in \mathbb{N} \tag{2.11}$$

and

$$h(t_0,x)=\lim_{k\to+\infty} h_k(t_0,x) \quad \text{for a. e. } x\in[c,d]. \tag{2.12}$$

Moreover, according to the above-proved, the functions φ_k ($k\in\mathbb{N}$) defined by the formula

$$\varphi_k(t,x)=\int_c^x h_k(t,\eta)\mathrm{d}\eta \quad \text{for } t\in E,\ x\in[c,d],\ k\in\mathbb{N}$$

are measurable on $\mathcal{D}$. On the other hand, in view of (2.11) and (2.12), it follows from the Lebesgue convergence theorem that

$$\lim_{k\to+\infty}\int_c^x h_k(t,\eta)\mathrm{d}\eta=\int_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } t\in E,\ x\in[c,d],$$

i.e.,

$$\varphi(t,x)=\lim_{k\to+\infty}\varphi_k(t,x) \quad \text{for a. e. } (t,x)\in\mathcal{D}.$$

Consequently, the function φ is measurable on $\mathcal{D}$ (see, e.g., [8, XII, §1]). $\square$

Proof of Proposition 2.5. We first extent the function h outside of $\mathcal{D}$ by setting

$$h(t,x)=0 \quad \text{for } (t,x)\in\mathbb{R}^2\setminus\mathcal{D}.$$

(i) Without loss of generality we can assume that

$$h(t,x)\geq 0 \quad \text{for a. e. } (t,x)\in\mathcal{D}$$

(in contrary case we represent the function h as the difference of its positive and negative parts). It follows from (2.2) and (2.3) that, for any $x\in[c,d]$, there exists a set $A(x)\subseteq\,]a,b[$ such that $\operatorname{mes}A(x)=b-a$ and

$$u_t(t,x)=\int_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } x\in[c,d] \text{ and } t\in A(x). \tag{2.13}$$

Put $E_1=\cap_{x\in B}A(x)$, where $B=\big([c,d]\cap\mathbb{Q}\big)\cup\{c,d\}$. Since the set B is countable, the set E_1 is measurable and $\operatorname{mes}E_1=b-a$. Moreover, the condition (2.13) yields

$$u_t(t,x)=\int_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } t\in E_1 \text{ and } x\in B. \tag{2.14}$$

We will show that the last relation holds for every $x\in[c,d]$.

Let $t\in E_1$ be arbitrary but fixed. Let, moreover, $\{l_n\}_{n=1}^{+\infty}$ be an arbitrary sequence of nonzero numbers such that

$$\lim_{n\to+\infty} l_n=0. \tag{2.15}$$

Put

$$\varphi_n(x)=\frac{1}{l_n}\int_t^{t+l_n}\int_c^x h(s,\eta)\mathrm{d}\eta\mathrm{d}s \quad \text{for } x\in[c,d],\ n\in\mathbb{N}. \tag{2.16}$$

According to (2.14)–(2.16), we get

$$\lim_{n\to+\infty}\varphi_n(x)=\int_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } x\in B. \tag{2.17}$$

Since the function h is supposed to be nonnegative, it is clear that the functions φ_n $(n\in\mathbb{N})$ are nondecreasing on $[c,d]$.

We will show that the relation (2.17) holds for every $x\in[c,d]$. Let $x_0\in[c,d]$ and $\varepsilon>0$ be arbitrary but fixed. Then there exist $x_1,x_2\in B$ such that $x_1\le x_0\le x_2$ and

$$\int_{x_1}^{x_0} h(t,\eta)\mathrm{d}\eta<\frac{\varepsilon}{2}\,, \qquad \int_{x_0}^{x_2} h(t,\eta)\mathrm{d}\eta<\frac{\varepsilon}{2}\,. \tag{2.18}$$

Moreover, by virtue of (2.17), there exists $n_0\in\mathbb{N}$ such that

$$\left|\varphi_n(x_k)-\int_c^{x_k} h(t,\eta)\mathrm{d}\eta\right|<\frac{\varepsilon}{2} \quad \text{for } n\ge n_0\,,\ k=1,2. \tag{2.19}$$

Using (2.18), (2.19), and the monotonicity of φ_n, we get

$$\varphi_n(x_0)-\int_c^{x_0} h(t,\eta)\mathrm{d}\eta\le\varphi_n(x_2)-\int_c^{x_2} h(t,\eta)\mathrm{d}\eta+\int_{x_0}^{x_2} h(t,\eta)\mathrm{d}\eta<\varepsilon \quad \text{for } n\ge n_0$$

and

$$\int_c^{x_0} h(t,\eta)\mathrm{d}\eta-\varphi_n(x_0)\le\int_{x_1}^{x_0} h(t,\eta)\mathrm{d}\eta+\int_c^{x_1} h(t,\eta)\mathrm{d}\eta-\varphi_n(x_1)<\varepsilon \quad \text{for } n\ge n_0\,,$$

and thus

$$\left|\varphi_n(x_0)-\int_c^{x_0} h(t,\eta)\mathrm{d}\eta\right|<\varepsilon \quad \text{for } n\ge n_0.$$

Consequently, in view of arbitrariness of x_0 and ε, the relation

$$\lim_{n\to+\infty}\varphi_n(x)=\int_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } x\in[c,d]$$

holds. Since the sequence $\{l_n\}_{n=1}^{+\infty}$ was also arbitrary, we have proved that

$$u_t(t,x)=\int_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } x\in[c,d].$$

Mention on arbitrariness of $t\in E_1$ completes the proof of the part (i).

(ii) The proof is analogous to the previous case and thus we omit it.

(iii) According to the above-proved part (i), there exists a set $E_1\subseteq[a,b]$ such that $\operatorname{mes}E_1=b-a$ and the condition (2.4) holds. Let

$$E_3=\left\{(t,x)\in E_1\times\,]c,d[\,:\ \lim_{l\to0}\frac{u_t(t,x+l)-u_t(t,x)}{l}=h(t,x)\right\}. \tag{2.20}$$

Then it is clear that, for any $(t,x)\in E_3$, there exists $u_{tx}(t,x)$ and the condition (2.5) is satisfed. We will show that the set E_3 is measurable with the measure equal to $(b-a)(d-c)$.

Put $W=E_1\times\,]c,d[$ and

$$\varphi_k(t,x)=\inf_{0<|l|<1/k}\left\{\frac{1}{l}\int\limits_x^{x+l}h(t,\eta)\mathrm{d}\eta\right\}\quad\text{for }(t,x)\in W,\ k\in\mathbb{N},$$

$$\psi_k(t,x)=\sup_{0<|l|<1/k}\left\{\frac{1}{l}\int\limits_x^{x+l}h(t,\eta)\mathrm{d}\eta\right\}\quad\text{for }(t,x)\in W,\ k\in\mathbb{N}.$$

For any $(t,x)\in W$ fixed, the function of variable l in braces is continuous, and thus the functions φ_k and ψ_k can be defined as follows

$$\varphi_k(t,x)=\inf_{\substack{0<|l|<1/k\\ l\in\mathbb{Q}}}\left\{\frac{1}{l}\int\limits_x^{x+l}h(t,\eta)\mathrm{d}\eta\right\}\quad\text{for }(t,x)\in W,\ k\in\mathbb{N},\tag{2.21}$$

$$\psi_k(t,x)=\sup_{\substack{0<|l|<1/k\\ l\in\mathbb{Q}}}\left\{\frac{1}{l}\int\limits_x^{x+l}h(t,\eta)\mathrm{d}\eta\right\}\quad\text{for }(t,x)\in W,\ k\in\mathbb{N}.\tag{2.22}$$

It is clear that the set W is measurable with the measure equal to $(b-a)(d-c)$ and, for any $l\in\mathbb{Q}$ fixed, the function of (t,x) in braces is measurable on $\mathcal{D}$ (see Lemma 2.6), and thus it is measurable on W. Consequently, the functions φ_k and ψ_k, which are defined as the point-wise infimum and supremum of countable system of measurable functions, are measurable on W (see [8, XII, §1]). Moreover, in view of (2.21) and (2.22), the relations

$$\varphi_1(t,x)\le\varphi_2(t,x)\le\cdots\quad\text{for }(t,x)\in W,\qquad \psi_1(t,x)\ge\psi_2(t,x)\ge\cdots\quad\text{for }(t,x)\in W$$

holds. Consequently, there exist point-wise (finite or infinite) limits

$$\varphi(t,x)=\lim_{k\to+\infty}\varphi_k(t,x),\qquad \psi(t,x)=\lim_{k\to+\infty}\psi_k(t,x)\quad\text{for }(t,x)\in W.$$

Now it follows from measurability of the functions φ_k and ψ_k that the limit functions φ and ψ are also measurable. By virtue of (2.4), for any $(t,x)\in\mathcal{D}$ we get

$$\varphi(t,x)=\lim_{k\to+\infty}\left(\inf_{0<|l|<1/k}\left\{\frac{1}{l}\int\limits_x^{x+l}h(t,\eta)\mathrm{d}\eta\right\}\right)=$$

$$=\lim_{\delta\to0+}\left(\inf_{0<|l|<\delta}\left\{\frac{1}{l}\int\limits_x^{x+l}h(t,\eta)\mathrm{d}\eta\right\}\right)=\liminf_{l\to0}\frac{u_t(t,x+l)-u_t(t,x)}{l}\,.$$

One can show in a similar manner that

$$\psi(t,x)=\limsup_{l\to0}\frac{u_t(t,x+l)-u_t(t,x)}{l}\quad\text{for }(t,x)\in W.$$

Now it is clear that the set E_3 defined by the formula (2.20) satisfies

$$E_3=\{(t,x)\in W\colon \varphi(t,x)=\psi(t,x)=h(t,x)\}\,.$$

This set is however measurable, because the function φ, ψ, and h are measurable on W. Moreover, the condition (2.4) guarantees that, for any $t \in E_1$, the section $E_3(t)$ of the set E_3 has the measure equal to $b-a$. Therefore, using the Fubini theorem (see, e.g., [8, XI, §5, Thm. 1]) we get

$$\operatorname{mes} E_3 = \int\limits_{E_1} \operatorname{mes} E_3(t)\mathrm{d}t = (d-c)\operatorname{mes} E_1 = (b-a)(d-c).$$

(iv) The proof is analogous to the previous case, but the part (ii) has to be used instead of (i). □

Now we are in position to prove Theorem 2.1.

Proof of Theorem 2.1. To prove the theorem it is sufficient to show that that the implications (1) ⇒ (2), (2) ⇒ (3), and (3) ⇒ (1) are satitisfied.

(1) ⇒ (2): Suppose that the function $u\colon \mathcal{D} \to \mathbb{R}$ is absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory, i.e., the conditions (α) and (β) are fulfilled. According to the condition (β) and Theorem 1.3, there exists a function $h \in L\big(\mathcal{D};\mathbb{R}\big)$ such that

$$F_u(P) = \iint\limits_{P} h(t,x)\mathrm{d}t\mathrm{d}x \quad \text{for } P \in \mathfrak{S}(\mathcal{D}).$$

In particular, we have

$$F_u\big(Q(t,x)\big) = \iint\limits_{Q(t,x)} h(s,\eta)\mathrm{d}s\mathrm{d}\eta \quad \text{for } (t,x) \in \mathcal{D}, \tag{2.23}$$

where the mapping Q is defined by the formula (1.1). Furthermore, the relation (1.3) yields

$$F_u\big(Q(t,x)\big) = u(a,c) - u(t,c) - u(a,x) + u(t,x) \quad \text{for } (t,x) \in \mathcal{D}. \tag{2.24}$$

On the other hand, the condition (α) guarantees that there exist $f \in L\big([a,b];\mathbb{R}\big)$ and $g \in L\big([c,d];\mathbb{R}\big)$ such that

$$u(t,c) = u(a,c) + \int\limits_a^t f(s)\mathrm{d}s \quad \text{for } t \in [a,b]$$

and

$$u(a,x) = u(a,c) + \int\limits_c^x g(\eta)\mathrm{d}\eta \quad \text{for } x \in [c,d].$$

Now, using the last two relations, it follows from (2.23) and (2.24) that the function u admits the integral representation (2.1) with $e = u(a,c)$.

(2) ⇒ (3): Suppose that the function $u\colon \mathcal{D} \to \mathbb{R}$ admits the integral representation (2.1), where $e \in \mathbb{R}$, $f \in L\big([a,b];\mathbb{R}\big)$, $g \in L\big([c,d];\mathbb{R}\big)$, and $h \in L\big(\mathcal{D};\mathbb{R}\big)$. By virtue

of the relations (2.2), it is clear that the condition (a) holds. Moreover, according to Proposition 2.5(i), there exists a set $E_1 \subseteq [a,b]$ such that $\operatorname{mes} E_1 = b-a$ and

$$u_t(t,x) = f(t) + \int\limits_c^x h(t,\eta)\mathrm{d}\eta \quad \text{for } t \in E_1 \text{ and } x \in [c,d].$$

Therefore, the condition (b) is fulfilled. Finally, Proposition 2.5(iii) guarantees the validity of the condition (2.5), where $E_3 \subseteq \mathcal{D}$ is a measurable set with the measure equal to $(b-a)(d-c)$. Consequently, the condition (c) also holds.

(3) $\Rightarrow$ (1): Suppose that the function $u\colon \mathcal{D} \to \mathbb{R}$ satisfies the conditions (a)–(c). Obviously, the condition (β) holds.

Let $P = [t_1,t_2] \times [x_1,x_2] \in \mathfrak{S}(\mathcal{D})$ be an arbitrary rectangle. By virtue of (1.3), we have

$$F_u(P) = u(t_1,x_1) - u(t_1,x_2) - u(t_2,x_1) + u(t_2,x_2)$$

and thus, using the conditions (a)–(c), we get

$$F_u(P) = \int\limits_{t_1}^{t_2} u_s(s,x_2)\mathrm{d}s - \int\limits_{t_1}^{t_2} u_s(s,x_1)\mathrm{d}s = \int\limits_{t_1}^{t_2} \big[u_s(s,x_2) - u_s(s,x_1)\big]\mathrm{d}s =$$

$$= \int\limits_{t_1}^{t_2} \left(\int\limits_{x_1}^{x_2} u_{s\eta}(s,\eta)\mathrm{d}\eta \right) \mathrm{d}s = \iint\limits_P u_{s\eta}(s,\eta)\mathrm{d}s\mathrm{d}\eta.$$

Therefore, in view of the condition (c) and Theorem 1.3, it is clear that the function of rectangles F_u associated with u is absolutely continuous, i.e., the condition (α) holds. Consequently, the function u is absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory. $\square$

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