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Integration of parametric measures and the statics of masonry panels

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Abstract In this paper we consider masonry bodies undergoing loads that can be represented by vector valued measures, and prove a result which is an appropriate formulation to this context of the static theorem of the limit analysis. As applications, we study the equilibrium of panels that are subjected both to distributed loads and concentrated forces, and determine equilibrated tensor valued measures. Then, by using an integration procedure for parametric measures, we explicitly calculate equilibrated stress fields that are represented by integrable functions. The obtained solutions are discussed.

Key Words masonry panels, limit analysis, integration of measures

Introduction

The idea that the safety of a masonry arch is guaranteed by the possibility to find a curve of thrust that is in equilibrium with the loads and that is entirely in the interior of the arch is old [H66]. Yet the rigorous proof of the static and kinematic theorems of the limit analysis for solids that do not support the traction is recent [DP98], [LPS10].

In [DP98] the collapse is identified with a deformation process that takes place at constant load (cf. def. 4.1), whereas in [LPS10] an approach is presented which is based on energetic considerations. Let $\Omega \subset \mathbb{R}^n$ be a connected open set with Lipschitz boundary $\partial\Omega$ of outer normal $\boldsymbol{n}$, interpreted as a reference configuration of a body made of no-tension material. The body has a prescribed displacement $\boldsymbol{d}$ on an area measurable subset $\mathcal{D}$ of $\partial\Omega$ and is subjected to body forces $\boldsymbol{b}$ on Ω and surface traction $\boldsymbol{s}$ prescribed on $\mathcal{S} = \partial\Omega \sim \mathcal{D}$. The limit analysis deals with a family of loads $\mathfrak{L}(\lambda)$ that depend linearly on a scalar parameter $\lambda \in \mathbb{R}$, that is $\mathfrak{L}(\lambda) = (\boldsymbol{b}^\lambda, \boldsymbol{s}^\lambda)$, where $\boldsymbol{b}^\lambda = \boldsymbol{b}_0 + \lambda\boldsymbol{b}_1$ and $\boldsymbol{s}^\lambda = \boldsymbol{s}_0 + \lambda\boldsymbol{s}_1$.

The loads $\boldsymbol{b}_0$, $\boldsymbol{s}_0$ and $\boldsymbol{b}_1$, $\boldsymbol{s}_1$ are the permanent and variable parts of the loads, respectively; λ is the loading multiplier. In [LPS10] $\boldsymbol{b}_0$, $\boldsymbol{b}_1$ are supposed to be square integrable functions on Ω with respect to the volume (Lebesgue) measure and $\boldsymbol{s}_0$, $\boldsymbol{s}_1$ square integrable functions on $\mathcal{S}$ with respect to the area (Hausdorff) measure.

Let V be the set of the displacement field that belong to the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^n)$ and vanish on $\mathcal{D}$. Let $\hat{w}$ be the energy density of the no-tension material [DP89], $\hat{\boldsymbol{E}}(\boldsymbol{v})$ the infinitesimal strain tensor, $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and $\mathcal{H}^{n-1}$ the Haudorff measure on $\mathcal{S}$. We define the potential energy of the body

$$\bar{I}(\lambda, \boldsymbol{v}) = \int_\Omega \hat{w}(\hat{\boldsymbol{E}}(\boldsymbol{v}))d\mathcal{L}^n - \int_\Omega \boldsymbol{v} \cdot \boldsymbol{b}^\lambda d\mathcal{L}^n - \int_{\mathcal{S}} \boldsymbol{v} \cdot \boldsymbol{s}^\lambda d\mathcal{H}^{n-1},$$

$(\lambda, \boldsymbol{v}] \in \mathbb{R} \times V$, and the infimum energy $\bar{I}_0(\lambda) = \inf\{\bar{I}(\lambda, \boldsymbol{v}) : \boldsymbol{v} \in V\}$. In [LPS10, Prop. 2.4] it is proved that the function $\bar{I}_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave and upper semicontinuous, so that the set $\Lambda = \{\lambda \in \mathbb{R} : I_0(\lambda) > -\infty\}$ is an interval (possibly empty or degenerate). The elements of Λ can be interpreted as loading multipliers whose corresponding loads $\mathfrak{L}(\lambda)$ are safe, i.e. the body does not collapse, and each finite endpoint λ_c of Λ is called a collapse multiplier. Then, it is proved that $\lambda \in \Lambda$ if and only if there exists a negative semidefinite and square integrable stress field $\boldsymbol{T}$ which equilibrates the loads $\mathfrak{L}(\lambda)$, for each $\boldsymbol{v} \in V$. This result can be interpreted as a formulation of the static theorem of the limit analysis.

In the study of the statics of masonry panels we verified that the problem of finding negative semidefinite stress fields that equilibrate the loads is considerably simplified if instead of stress fields represented by square integrable functions $\boldsymbol{T}$ one admits also stress fields represented by tensor valued measures T, by allowing the presence of curves of concentrated stress [LSZ06]. Several applications of these singular stress fields are presented in [LSZ06], and in [LSZ07], [LSZ09] where the gravity is also taken into account. In all these applications the stress field equilibrating the loads $\mathfrak{L}(\lambda)$ is represented by a measure T^λ which is the sum of an absolutely continuous part with respect to the area measure with density $\boldsymbol{T}_r^\lambda$ and a part which is concentrated on a regular curve, having there a density $\boldsymbol{T}_s^\lambda$ with respect to the length measure. All these stress fields are not represented by square integrable functions and then by themselves cannot guarantee the safety of the loads. In order to overcome this difficulty, in [LSZ08] we present a procedure that in some case allows us to obtain a square integrable stress field once a measure stress field is known. Crucial to the procedure is the fact that both the loads $\mathfrak{L}(\lambda)$ and the equilibrating stress measure T^λ depend on a linear parameter λ. The idea is to take the average of the stress measure over any set $(\mu - \epsilon, \mu + \epsilon)$, where $\epsilon > 0$ is sufficiently small, and μ is any point in the set of parameters. Averaging gives the measure

$$\mathsf{T} = \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} \mathsf{T}^\lambda d\lambda,$$

and it may happen that this measure, in contrast to T^μ, is absolutely continuous with respect to the Lebesgue measure with density $\boldsymbol{T}$, which is square integrable. Because $\mathfrak{L}(\lambda)$ depends linearly on the parameter λ, then it is automatic that $\boldsymbol{T}$ equilibrates the loads $\mathfrak{L}(\mu)$. Although in some case the explicit calculation of the averaged measure $\mathbf{T}$ can be a difficult task, for applications it suffices to know that the averaging procedure leads to the existence of a negative semidefinite square integrable stress field equilibrating the loads.

In the present paper, in order to deal also with concentrations of body forces and surface traction, we allow the loads to be vector valued measures $\mathbf{b}_0$, $\mathbf{b}_1, \mathbf{s}_0$, $\mathbf{s}_1$. Because in this more general case the work of the loads cannot be defined for displacements from V, we consider the set V_0 of all vector valued functions that are continuously differentiable on the closure of Ω and vanish on $\mathcal{D}$, and define the functional of the potential energy

$$I(\lambda, \boldsymbol{v}) = \int_\Omega \hat{w}(\hat{\boldsymbol{E}}(\boldsymbol{v}))d\mathcal{L}^n - \int_\Omega \boldsymbol{v} \cdot d\mathbf{b}^\lambda - \int_\mathcal{S} \boldsymbol{v} \cdot d\mathbf{s}^\lambda,$$

$(\lambda, \boldsymbol{v}) \in \mathbb{R} \times V_0$, and the infimum of energy $I_0(\lambda) = \inf\{I(\lambda, \boldsymbol{v}) : \boldsymbol{v} \in V_0\}$. It can be shown that $\bar{I}(\lambda, \boldsymbol{v})$ is continuous on V and, under the hypothesis that $\mathcal{D}$ is a closed set with Lipschitz boundary, that V_0 is dense in V. Thus we have $\bar{I}_0(\lambda) = I_0(\lambda)$. If there exists a tensor valued measure $\mathbf{T}$ on Ω with values in Sym^- such that

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot d\mathbf{T} = \int_\Omega \boldsymbol{v} \cdot d\mathbf{b}^\lambda + \int_\mathcal{S} \boldsymbol{v} \cdot d\mathbf{s}^\lambda,$$

for every $\boldsymbol{v} \in V_0$, we say that the loads $\mathfrak{L}(\lambda)$ are weakly compatible and that $\mathbf{T}$ weakly equilibrates the loads $\mathfrak{L}(\lambda)$. Moreover, we say that the loads $\mathfrak{L}(\lambda)$ are strongly compatible if it happens that they are equilibrated by a negative semi definite square integrable stress field $\boldsymbol{T}$. In this last case we say also that $\boldsymbol{T}$ strongly equilibrates the loads $\mathfrak{L}(\lambda)$.

In Section 1 we describe the procedure that we use to integrate a parametric family of measures, by collecting several result presented in [LSZ08]. In Section 2, after a brief description of some concepts of the limit analysis, we prove that the general loads $\mathfrak{L}(\lambda)$ are strongly compatible if and only if $I_0(\lambda) > -\infty$ (see Proposition 2.1 below). This result is an appropriate version of the static theorem of the limit analysis when dealing with loads that are represented by vector valued measure. Its counterpart is what was proved in [LPS10], Prop. 2.4, where only loads represented by square integrable functions were considered.

In the rest of the paper, which is devoted to the applications, we consider rectangular panels that are fixed at its base and undergo several loading conditions. For the panel considered in Section 3, the gravity is the permanent part of the loads and the variable part is an uniformly distributed load of intensity λ, that is applied on the right lateral side of the panel. For this problem, in [LSZ09] a singular stress field which weakly equilibrates the load was found for sufficiently small values of λ, and here we explicitly calculate a corresponding admissible stress field by the integration procedure that is described in Sec. 1. In Section 4, where the gravity is not taken into account, an uniformly distributed load on the top of the panel constitutes the permanent part of the loads. As variable part of the loads we consider, firstly (see Sec. 4.1), a lateral load of intensity λ which is uniformly distributed on a small part of the right lateral side of the panel, starting from the upper corner. For this loads we determine both, a singular stress field which weakly equilibrates the loads and its admissible counterpart, by the integration procedure. Secondly (see Sec. 4.2), we take as variable part of the load a force which is concentrated on the right side of the panel and that corresponds to the resultant of the lateral load that is considered in Sec. 4.1. After determining a singular stress field that weakly equilibrated the loads, by applying the integration procedure we determine a stress field which is represented by an integrable function that is not square integrable. So that, in this case, the integration procedure does not produce an admissible stress field. Nevertheless, as the example shows, this non admissible stress field can be a good approximation of the more complicate admissible stress field that was obtained in Sec. 4.1.

1 Measures and families of measures

Throughout we use the conventions for vectors and second order tensors identical with those in [G81]. Thus Lin denotes the set of all second order tensors on $\mathbb{R}^n$, i.e., linear transformations from $\mathbb{R}^n$ into itself, Sym is the subspace of symmetric tensors, Sym^+ the set of all positive semidefinite elements of Sym; additionally, Sym^- is the set of all negative semidefinite elements of Sym . The scalar product of $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{Lin}$ is defined by $\boldsymbol{A} \cdot \boldsymbol{B} = \mathrm{tr}(\boldsymbol{A}\boldsymbol{B}^{\mathrm{T}})$ and $|\cdot|$ denotes the associated Euclidean norm on Lin.

In this section we introduce the terminology and notation for measures with values in a finite dimensional vectorspace. We refer to [AFP00, Chapter 1] to further details.

Let V be a finite-dimensional vectorspace. By a V valued measure in $\mathbb{R}^n$ we mean a map $\mathbf{m}$ from a system of all Borel sets in $\mathbb{R}^n$ to V which is countably additive in the sense that if B_1, B_2, ... is a disjoint family of Borel sets in $\mathbb{R}^n$ then

$$\mathbf{m}\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \mathbf{m}(B_i).$$

Below we need the choices $V = \mathrm{Sym}$ and $V = \mathbb{R}^n$. We call the Sym valued measures tensor valued measures; these are used to model the stress fields over the body. We call the $\mathbb{R}^n$ valued measures vector valued measures. These are used to model the loads applied to the body.

We say that a function ϕ defined on the system of all Borel sets in $\mathbb{R}^n$ is a nonnegative measure if it takes the values from the set $[0, \infty]$ of nonnegative numbers or ∞ which is countably additive in the sense that if B_1, B_2, ... is a disjoint family of Borel sets in $\mathbb{R}^n$ then

$$\phi\Big(\bigcup_{i=1}^{\infty} B_i\Big) = \sum_{i=1}^{\infty} \phi(B_i)$$

and

$$\phi(\emptyset) = 0.$$

If Ω is a Borel subset of $\mathbb{R}^n$ and $\mathbf{m}$ a V valued measure or nonnegative measure, we say that $\mathbf{m}$ is supported by Ω if $\mathbf{m}(A) = \mathbf{0}$ for any Borel set A such that $A \cap \Omega = \emptyset$. We denote by $\mathcal{M}(\Omega, V)$ the set of all V valued measures supported by Ω. If $\mathbf{m} \in \mathcal{M}(\mathbb{R}^n, V)$, we denote by $|\mathbf{m}|$ the total variation measure of $\mathbf{m}$, i.e., a sacalar valued measure defined for each Borel set $A \subset \mathbb{R}^n$ by

$$|\mathbf{m}|(A) = \sup\{\sum_{i=1}^{\infty} |\mathbf{m}(A_i)|\}$$

where the supremum is taken over all sequences A_i of Borel sets such that

$$\bigcup_{i=1}^{\infty} A_i = A \quad \text{and} \quad A_i \cap A_j = \emptyset \quad \text{if} \quad i \neq j.$$

We further denote by $\mathrm{M}(\mathbf{m})$ the mass of $\mathbf{m}$, defined by $\mathrm{M}(\mathbf{m}) = |\mathbf{m}|(\mathbb{R}^n)$.

We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ [AFP00, Definition 1.52] and if k is an integer, $0 \leq k \leq n$, we denote by $\mathcal{H}^k$ the k-dimensional Hausdorff measure ("k dimensional area") in $\mathbb{R}^n$ [AFP00, Section 2.8]. If ϕ is a nonnegative measure or a V valued measure, we denote by $\phi \llcorner A$ the restriction of ϕ to a Borel set $A \subset \mathbb{R}^n$ defined by

$$\phi \llcorner A(B) = \phi(A \cap B)$$

for any Borel subset B of $\mathbb{R}^n$. Thus if $\mathcal{S}$ is an $n-1$ dimensional surface in $\mathbb{R}^n$ then $\mathcal{H}^{n-1} \llcorner \mathcal{S}$ is the area measure on $\mathcal{S}$.

If ϕ is a nonnegative measure, we denote by $f\phi$ the product of the measure ϕ by a ϕ integrable V valued function f on $\mathbb{R}^n$; one has

$$(f\phi)(A) = \int_A f \, d\phi$$

for any Borel subset A of $\mathbb{R}^n$.

If Ω is an open subset of $\mathbb{R}^n$, we denote by $C_0(\Omega, V)$ the space of all continuous V valued functions on $\mathbb{R}^n$ with compact support that is contained in Ω, and denote by $|\cdot|_{C_0}$ the maximum norm on $C_0(\mathbb{R}^n, V)$.

An ***integrable parametric measure*** is a family $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ of V valued measures on $\mathbb{R}^n$ where $\Lambda \subset \mathbb{R}$ is a $\mathcal{L}^1$ measurable set of parameters such that

(i) for every $f \in C_0(\mathbb{R}^n, V)$ the function $\lambda \mapsto \int_{\mathbb{R}^n} f \cdot d\mathbf{m}^\lambda$ is $\mathcal{L}^1$ measurable on Λ;

(ii) we have

$$c := \int_\Lambda \mathrm{M}(\mathbf{m}^\lambda) \, d\lambda < \infty.$$

We note that parametric measures similar to those defined above occur in the context of disintegration (slicing) of measures [AFP00, Section 2.5] and, what is related, in the context of Young's measures [MU99, Chapter 5].

The following three propositions are taken from [LSZ08].

Proposition 1.1. *If $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ is an integrable parametric measure then there exists a unique V valued measure $\mathbf{m}$ on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} f \cdot d\mathbf{m} = \int_\Lambda \int_{\mathbb{R}^n} f \cdot d\mathbf{m}^\lambda \, d\lambda \tag{1}$$

for each $f \in C_0(\mathbb{R}^n, V)$.

We write

$$\mathbf{m} = \int_\Lambda \mathbf{m}^\lambda \, d\lambda \tag{2}$$

and call $\mathbf{m}$ the ***integral of the family*** $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ with respect to λ.

Next we give two important examples of integrable parametric measures in the subsequent two propositions. In both cases the corresponding integral (2) is absolutely continuous with respect to the Lebesgue measure.

Proposition 1.2. *Let $\{h^\lambda : \lambda \in \Lambda\}$ be a family of V valued functions on $\Omega \subset \mathbb{R}^n$ defined for all λ from a $\mathcal{L}^1$ measurable set $\Lambda \subset \mathbb{R}$ such that the mapping $(\boldsymbol{x}, \lambda) \mapsto h^\lambda(\boldsymbol{x})$ is $\mathcal{L}^{n+1}$ integrable on $\Omega \times \Lambda$, i.e.,*

$$\int_\Lambda \int_\Omega |h^\lambda(\boldsymbol{x})| \, d\boldsymbol{x} d\lambda < \infty. \tag{3}$$

If we define a V valued measure $\mathbf{m}^\lambda$ by

$$\mathbf{m}^\lambda = h^\lambda \mathcal{L}^n \llcorner \Omega$$

then $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ is an integrable parametric measure and we have

$$\int_\Lambda \mathbf{m}^\lambda \, d\lambda = k\mathcal{L}^n \llcorner \Omega$$

where

$$k(\boldsymbol{x}) = \int_\Lambda h^\lambda(\boldsymbol{x}) \, d\lambda$$

for $\mathcal{L}^n$ a.e. $\boldsymbol{x} \in \Omega$.

Proposition 1.3. *Let $\Omega_0 \subset \mathbb{R}^n$ be open, let $\varphi : \Omega_0 \to \mathbb{R}$ be locally Lipschitz continuous and let $g : \Omega_0 \to V$ be $\mathcal{L}^n$ measurable on Ω_0, with*

$$\int_{\Omega_0} |g||\nabla\varphi| \, d\mathcal{L}^n < \infty. \tag{4}$$

Then for $\mathcal{L}^1$ a.e. $\lambda \in \mathbb{R}$ the function g is $\mathcal{H}^{n-1} \llcorner \varphi^{-1}(\lambda)$ integrable; denoting by Λ the set of all such λ we define the measure $\mathbf{m}^\lambda$ by

$$\mathbf{m}^\lambda := g\mathcal{H}^{n-1} \llcorner \varphi^{-1}(\lambda)$$

for each $\lambda \in \Lambda$. Then $\{\mathbf{m}^\lambda : \lambda \in \Lambda\}$ is an integrable parametric measure and we have

$$\int_\Lambda \mathbf{m}^\lambda \, d\lambda = g|\nabla\varphi|\mathcal{L}^n \llcorner \Omega_0. \tag{5}$$

2 Limit analysis

We consider a continuous body represented by a Lipschitz domain [AF03] $\Omega \subset \mathbb{R}^n$ and assume that $\mathcal{D}, \mathcal{S}$ are two disjoint subsets of $\partial\Omega$ such that $\mathcal{D} \cup \mathcal{S} = \partial\Omega$, to be identified below as the set of prescribed boundary displacement and prescribed boundary force. We assume that $\mathcal{D}$ is a closed set with Lipschitz boundary.

We put

$$V_0 = \{\boldsymbol{v} \in C^1(\operatorname{cl}\Omega, \mathbb{R}^n) : \boldsymbol{v} = \mathbf{0} \text{ on } \mathcal{D}\}$$

and

$$V = \{\boldsymbol{v} \in W^{1,2}(\Omega, \mathbb{R}^n) : \boldsymbol{v} = \mathbf{0} \text{ a.e. on } \mathcal{D}\};$$

here $C^1(\operatorname{cl}\Omega, \mathbb{R}^n)$ is the set of all continuously differentiable mappings $\boldsymbol{v} : \Omega \to \mathbb{R}^n$ such that $\boldsymbol{v}$ and its derivative $\nabla\boldsymbol{v}$ have a continuous extension to the closure $\operatorname{cl}\Omega$ of Ω and $W^{1,2}(\Omega, \mathbb{R}^n)$ is the Sobolev space of all $\mathbb{R}^n$ valued maps such that

$\boldsymbol{v}$ and the distributional derivative $\nabla \boldsymbol{v}$ of $\boldsymbol{v}$ are square integrable on Ω [AF03]. Our assumptions about $\mathcal{D}$ imply that V_0 is a dense subset of V. For any $\boldsymbol{v} \in V$ we define the infinitesimal strain tensor $\hat{\boldsymbol{E}}(\boldsymbol{v})$ of $\boldsymbol{v}$ by

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) = \tfrac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}).$$

We assume that the body is subjected to loads which consist of a body force acting in the interior of Ω and of the surface force acting on $\mathcal{S}$. We represent both the body and surface forces as vector valued measures supported by Ω and $\mathcal{S}$, respectively.

The limit analysis deals with the loads that depend linearly (affinely) on a scalar parameter $\lambda \in \mathbb{R}$. We thus assume that the body and surface forces $\mathbf{b}^\lambda \in \mathcal{M}(\Omega, \mathbb{R}^n)$ and $\mathbf{s}^\lambda \in \mathcal{M}(\mathcal{S}, \mathbb{R}^n)$ corresponding to λ are given by

$$\mathbf{b}^\lambda = \mathbf{b}_0 + \lambda \mathbf{b}_1, \quad \mathbf{s}^\lambda = \mathbf{s}_0 + \lambda \mathbf{s}_1$$

where

$$\mathbf{b}_0, \mathbf{b}_1 \in \mathcal{M}(\Omega, \mathbb{R}^n) \quad \mathbf{s}_0, \mathbf{s}_1 \in \mathcal{M}(\mathcal{S}, \mathbb{R}^n).$$

We call $\mathfrak{L}(\lambda) = (\mathbf{s}^\lambda, \mathbf{b}^\lambda)$ the loads corresponding to λ. If $\boldsymbol{v} \in V_0$ then the work of the loads $\mathfrak{L}(\lambda)$ corresponding to $\boldsymbol{v}$ is

$$\langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle = \int_{\mathcal{S}} \boldsymbol{v} \cdot \, d\mathbf{s}^\lambda + \int_{\Omega} \boldsymbol{v} \cdot \, d\mathbf{b}^\lambda.$$

We note that we define the loads as measures, which allows for the concentration of the body force and more importantly surface tractions. One example to be given below involves a delta type concentrated surface traction. Note also that under general measures $\mathbf{s}$, $\mathbf{b}$ the work of the loads cannot be defined for displacement from the subspace V of the Sobolev space $W^{1,2}(\Omega, \mathbb{R}^n)$. However, if the loads have square integrable densities, i.e., if

$$\mathbf{b}_0 = \boldsymbol{b}_0 \mathcal{L}^n, \quad \mathbf{b}_1 = \boldsymbol{b}_1 \mathcal{L}^n, \quad \mathbf{s}_0 = \boldsymbol{s}_0 \mathcal{H}^{n-1}, \quad \mathbf{s}_1 = \boldsymbol{s}_1 \mathcal{H}^{n-1}, \tag{6}$$

where

$$\boldsymbol{b}_0, \ \boldsymbol{b}_1 \in L^2(\Omega, \mathbb{R}^n), \quad \boldsymbol{s}_0, \ \boldsymbol{s}_1 \in L^2(\mathcal{S}, \mathbb{R}^n), \tag{7}$$

then one can extend the definition of $\boldsymbol{l}(\lambda)$ to elements $\boldsymbol{v}$ of V. If $\boldsymbol{v} \in V$, we define the internal energy $F(\boldsymbol{v})$ of the body corresponding to $\boldsymbol{v}$ by

$$F(\boldsymbol{v}) = \int_{\Omega} \hat{w}(\hat{\boldsymbol{E}}(\boldsymbol{v})) \, d\mathcal{L}^n$$

where $\hat{w}$ is the energy density of the no–tension material [DP89] and

$$\hat{\boldsymbol{E}}(\boldsymbol{v}) = \tfrac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}})$$

is the infinitesimal strain tensor corresponding to the displacement $\boldsymbol{v}$.

In the general context of loads represented by measures we define the potential energy $I(\boldsymbol{v}, \lambda)$ of the body corresponding to the loads $\mathfrak{L}(\lambda)$ and displacement $\boldsymbol{v} \in V_0$ by

$$I(\boldsymbol{v}, \lambda) = F(\boldsymbol{v}) - \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \tag{8}$$

so that $I(\cdot,\lambda) : V_0 \to \mathbb{R}$. Central to our considerations is the infimum energy $I_0(\lambda) \in \mathbb{R} \cup \{-\infty\}$ of the loads $\mathfrak{L}(\lambda)$ defined by

$$I_0(\lambda) = \inf\{I(\boldsymbol{v},\lambda) : \boldsymbol{v} \in V_0\}.$$

The paper [LPS10] works within the special context (6)–(7) and defines the potential energy as a functional $\tilde{I}(\cdot,\lambda) : V \to \mathbb{R}$ given by the right hand side of (8) for each $\boldsymbol{v} \in V$ and the infimum energy $\tilde{I}_0(\lambda)$ by

$$\tilde{I}_0(\lambda) = \inf\{\tilde{I}(\boldsymbol{v},\lambda) : \boldsymbol{v} \in V\}.$$

One can show that $\tilde{I}(\cdot,\lambda)$ is continuous on V and since V_0 is dense in V due to our assumptions on $\mathcal{D}$ and $\mathcal{S}$, one has

$$\tilde{I}_0(\lambda) = I_0(\lambda).$$

The function $I_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is concave and uppersemicontinuous [LPS10] and the set

$$\Lambda := \{\lambda \in \mathbb{R} : I_0(\lambda) > -\infty\}$$

is an interval (possibly empty and open, semiopen or closed). We interpret the elements of Λ as loading multipliers for which the loads $\mathfrak{L}(\lambda)$ are safe, i.e., the body does not collapse. Each finite endpoint λ_c of the interval Λ is called a collapse multiplier with the interpretation that for $\lambda = \lambda_c$ or at least for λ arbitrarily close to λ_c outside Λ the body collapses.

We say that the loads $\mathfrak{L}(\lambda)$ are weakly compatible if there exists a measure $\mathsf{T} \in \mathcal{M}(\Omega, \mathrm{Sym})$ with values in Sym^- such that

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot d\mathsf{T} = \langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle$$

for every $\boldsymbol{v} \in V_0$. If this is the case, we also say that T weakly balances the loads $\mathfrak{L}(\lambda)$. We say that the loads $\mathfrak{L}(\lambda)$ are strongly compatible if there exists a stressfield $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym})$ with values in Sym^- for $\mathcal{L}^n$ a.e. point of Ω such that

$$\int_\Omega \hat{\boldsymbol{E}}(\boldsymbol{v}) \cdot \boldsymbol{T}\, d\mathcal{L}^n = \langle \boldsymbol{l}(\lambda), \boldsymbol{v}\rangle$$

for every $\boldsymbol{v} \in V_0$. If this is the case, we also say that $\boldsymbol{T}$ strongly equilibrates the loads $\mathfrak{L}(\lambda)$.

The strong compatibility is related to $I_0(\lambda)$ as described in the following proposition.

Proposition 2.1. *Consider the general loads $\mathfrak{L}(\cdot)$ and let $\lambda \in \mathbb{R}$. Then the loads $\mathfrak{L}(\lambda)$ are strongly compatible if and only if*

$$I_0(\lambda) > -\infty.$$

The same proposition under the assumptions (6)–(7) and with $I_0(\lambda)$ replaced by $\tilde{I}_0(\lambda)$ is proved in [LPS10]. The proof in the present more general case is similar.

Proof Let $Y := L^2(\Omega, \mathrm{Sym})$ and write $(\boldsymbol{A}, \boldsymbol{B}) = \int_\Omega \boldsymbol{A} \cdot \boldsymbol{B}\, d\mathcal{L}^n$ for the scalar product in Y. Let $X_0 := \{\hat{\boldsymbol{E}}(\boldsymbol{v}) : \boldsymbol{v} \in V_0\}$ so that $X_0 \subset Y$ and let $L_0 : X_0 \to \mathbb{R}$ be defined by

$$L_0(\hat{\boldsymbol{E}}(\boldsymbol{v})) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle \tag{9}$$

for each $\boldsymbol{v} \in V_0$. Let $H : Y \to \mathbb{R}$ be defined by

$$H(\boldsymbol{A}) = \int_\Omega \hat{w}(\boldsymbol{A})\, d\mathcal{L}^n$$

for each $\boldsymbol{A} \in Y$. If $c = I_0(\lambda) \in \mathbb{R}$ then

$$L_0(\boldsymbol{A}) \le H(\boldsymbol{A}) - c \quad \text{for all} \quad \boldsymbol{A} \in X_0.$$

The convexity of $\hat{w}$ implies the convexity of H and hence by the version of the Hahn Banach theorem [FL07, Theorem A.35] there exists a linear extension $L : Y \to \mathbb{R}$ of L_0 such that

$$L(\boldsymbol{A}) \le H(\boldsymbol{A}) - c \quad \text{for all} \quad \boldsymbol{A} \in Y. \tag{10}$$

The continuity of H on Y, which follows from the properties of $\hat{w}$, implies the continuity of L and hence L can be represented by an element $\boldsymbol{T} \in Y$ as a scalar product in Y. Relation (9) then gives

$$(\boldsymbol{T}, \hat{\boldsymbol{E}}(\boldsymbol{v})) = \langle \boldsymbol{l}(\lambda), \boldsymbol{v} \rangle$$

for each $\boldsymbol{v} \in V_0$ and thus $\boldsymbol{T}$ strongly equilibrates the loads $\mathfrak{L}(\lambda)$.

To prove the converse part of the statement, we let $\boldsymbol{T}$ be a stressfield strongly equilibrating the loads $\mathfrak{L}(\lambda)$. Since $\boldsymbol{T}$ is negative semidefinite and square integrable, we have ([LPS10])

$$H^*(\boldsymbol{T}) = \tfrac{1}{2} \int_\Omega \boldsymbol{T} \cdot \mathsf{C}^{-1} \boldsymbol{T} \mathsf{d}\mathcal{L}^\mathsf{n} < \infty$$

and hence

$$\infty > H^*(\boldsymbol{T}) := \sup\{(\boldsymbol{T}, \boldsymbol{A}) - H(\boldsymbol{A}) : \boldsymbol{A} \in Y\}$$

from which

$$H(\boldsymbol{A}) - (\boldsymbol{T}, \boldsymbol{A}) \ge -H^*(\boldsymbol{T}) \qquad \text{for all} \quad \boldsymbol{A} \in Y;$$

taking $\boldsymbol{A} = \hat{\boldsymbol{E}}(\boldsymbol{v})$ where $\boldsymbol{v} \in V_0$, this is rewritten as

$$I(\boldsymbol{v}, \lambda) \ge c$$

for all $\boldsymbol{v} \in V_0$ [with $c = -H^*(\boldsymbol{T})$]. □

Remark 2.2. Consider the general loads $\mathfrak{L}(\cdot)$ and let $\mu \in \mathbb{R}$ and $\epsilon > 0$. Let $\{\mathbf{U}^\lambda \in \mathcal{M}(\Omega, \mathrm{Sym}), \lambda \in (\mu - \epsilon, \mu + \epsilon)\}$ be a family of measures such that for each $\lambda \in (\mu - \epsilon, \mu + \epsilon)$ the measure $\mathbf{U}^\lambda$ weakly equilibrates the loads $\mathfrak{L}(\lambda)$. It follows from the affine dependence of the loads on the loading multiplier that then the measure $\mathbf{T}$ given by

$$\mathbf{T} = \frac{1}{2\epsilon} \int_{\mu-\epsilon}^{\mu+\epsilon} \mathbf{U}^\lambda\, d\lambda$$

weakly equilibrates the loads $\mathfrak{L}(\lambda)$. It may happen that the stressfield $\mathbf{T}$ is square integrable in the sense that $\mathbf{T} = \boldsymbol{T}\mathcal{L}^n$ for some $\boldsymbol{T} \in L^2(\Omega, \mathrm{Sym})$ even when the individual stressfields $\mathbf{U}^\mu$ are not square integrable. In this way it may happen that λ is a safe multiplier. The analysis of the loads from this point of view is given in the subsequent two sections in case of panels; additional examples are given in [LSZ08].

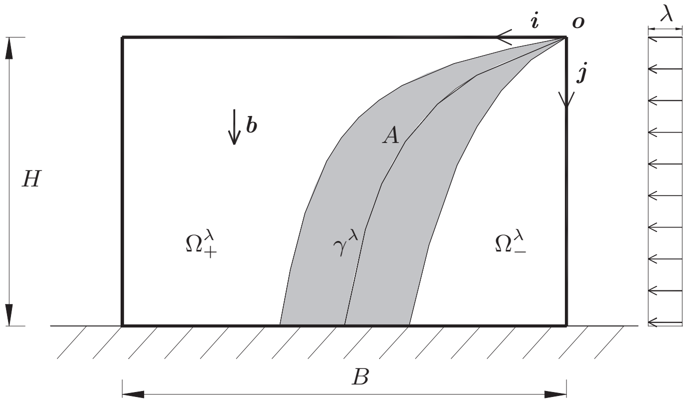

Fig. 1.

3 Panel under gravity and horizontal side loads

In the present section and in Section 4 we consider the rectangular panel

$$\Omega = (0, B) \times (0, H) \subset \mathbb{R}^2;$$

we introduce the coordinate system x, y in $\mathbb{R}^2$ with the origin in the upper right corner of the panel and with the orientation of axes as shown in Figure 1. We denote a general point of Ω by $\boldsymbol{r} = (x, y)$ and the coordinate vectors along the axes x, y by $\boldsymbol{i}$, $\boldsymbol{j}$, respectively. We put

$$\mathcal{D} = (0, B) \times \{H\}, \quad \mathcal{S} = \partial\Omega \sim \mathcal{D}.$$

In the present section we consider the loads $\mathfrak{L}(\lambda) = (\boldsymbol{s}^\lambda, \boldsymbol{b}^\lambda)$ where $\boldsymbol{b}^\lambda = b\boldsymbol{j}$ in Ω, and, for $\boldsymbol{r} = (x, y) \in \mathcal{S}$,

$$\boldsymbol{s}^\lambda(\boldsymbol{r}) = \begin{cases} \lambda\boldsymbol{i} & \text{on} \quad \{0\} \times (0, H), \\ \boldsymbol{0} & \text{elsewhere.} \end{cases}$$

(Here for the sake of simplicity we do not consider distributed loads on the top of the panel). We assume that $b > 0$, $\lambda \geq 0$ and deduce the stressfield in the same way as in [LSZ09]. Ω is divided into the regions Ω_+^λ (on the left) and Ω_-^λ (on the right) by the curve γ^λ, which is the graph of an increasing function $\omega^\lambda : [0, t^\lambda] \to [0, h]$,

$$\omega^\lambda(x) = cbx^2/\lambda, \quad c = 1/2 + \sqrt{3}/6, \tag{11}$$

with unit tangent vector

$$\boldsymbol{t}^\lambda(\boldsymbol{r}) = \frac{(x, 2y)}{\sqrt{x^2 + 4y^2}}. \tag{12}$$

(See [LSZ09, eq. (4.16)], with $\lambda = \beta H$ and $p_0 = 0$). If $\lambda \in (0, \lambda_c)$, with

$$\lambda_c = cbB^2/H,$$

then γ^λ is contained in Ω, except for the endpoints and $t^\lambda = \sqrt{\lambda H/cb}$.

We note that if $\varphi : \Omega \to \mathbb{R}$ is defined by

$$\varphi(\boldsymbol{r}) = cbx^2/y, \quad \boldsymbol{r} = (x, y) \in \Omega, \tag{13}$$

then for any $\lambda \in (0, \lambda_c)$, the curve γ^λ is the level set of φ corresponding to the value of λ,

$$\gamma^\lambda = \{\boldsymbol{r} \in \Omega : \varphi(\boldsymbol{r}) = \lambda\}.$$

Moreover, φ is continuouly differentiable and

$$|\nabla\varphi(\boldsymbol{r})| = \frac{cbx\sqrt{x^2 + 4y^2}}{y^2}. \tag{14}$$

For $\lambda \in (0, \lambda_c)$, the loads $\mathfrak{L}(\lambda)$ are weakly equilibrated by the admissible measure stressfield

$$\mathbf{T}^\lambda = \boldsymbol{T}_r^\lambda \mathcal{L}^2 \llcorner \Omega + \boldsymbol{T}_s^\lambda \mathcal{H}^1 \llcorner \gamma^\lambda$$

where

$$\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = \begin{cases} -by\boldsymbol{j} \otimes \boldsymbol{j} & \text{in} \quad \Omega_+^\lambda \ , \\ -\lambda \boldsymbol{i} \otimes \boldsymbol{i} - bx\boldsymbol{i} \odot \boldsymbol{j} - \dfrac{b^2x^2}{\lambda}\boldsymbol{j} \otimes \boldsymbol{j} & \text{in} \quad \Omega_-^\lambda, \end{cases}$$

with $\boldsymbol{i} \odot \boldsymbol{j} = \boldsymbol{i} \otimes \boldsymbol{j} + \boldsymbol{j} \otimes \boldsymbol{i}$, as can be deduced by [LSZ09, eqs. (4.6), (4.7) and (3.1)], and

$$\boldsymbol{T}_s^\lambda = \sigma^\lambda \boldsymbol{t}^\lambda \otimes \boldsymbol{t}^\lambda, \tag{15}$$

where (see [LSZ09], eq. (2.19), with $s_0 = 0$, and $f(x, y)$ at the end of page 229)

$$\sigma^\lambda(\boldsymbol{r}) = -\frac{\sqrt{3}}{6}bx\sqrt{x^2 + 4y^2} \quad \text{on} \quad \gamma^\lambda \tag{16}$$

and $\boldsymbol{t}^\lambda$ given by (12).

We now write

$$\mathbf{T}^\lambda = \mathbf{T}_r^\lambda + \mathbf{T}_s^\lambda$$

where

$$\mathbf{T}_r^\lambda = \boldsymbol{T}_r^\lambda \mathcal{L}^2 \llcorner \Omega, \quad \mathbf{T}_s^\lambda = \boldsymbol{T}_s^\lambda \mathcal{H}^1 \llcorner \gamma^\lambda.$$

We note that $\mathbf{T}_r^\lambda$ is of the form considered in Prop. 1.2 and that the integrability condition (3) is satisfied. Hence for $0 < \mu < \lambda_c$ and $\epsilon > 0$ such that

$$\Lambda = (\mu - \epsilon, \mu + \epsilon) \subset (0, \lambda_c) \tag{17}$$

the measure

$$\mathbf{T}_r = \frac{1}{2\epsilon} \int_\Lambda \mathbf{T}_r^\lambda$$

is an absolutely continuous measure with respect to $\mathcal{L}^2 \llcorner \Omega$,

$$\mathbf{T}_r = \boldsymbol{T}_r \mathcal{L}^2 \llcorner \Omega.$$

In order to compute the density $\boldsymbol{T}_r$ we put

$$A = \{\boldsymbol{r} = (x, y) : bcx^2/y \in \Lambda\}. \tag{18}$$

Following the same procedure as in [LSZ08, Sec. 5] we obtain

$$\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = \begin{cases} -by\boldsymbol{j}\otimes\boldsymbol{j} & \text{in} \quad \Omega_+^\lambda \sim A\ , \\ -\mu\boldsymbol{i}\otimes\boldsymbol{i} - bx\boldsymbol{i}\odot\boldsymbol{j} - b^2x^2/\lambda\boldsymbol{j}\otimes\boldsymbol{j} & \text{in} \quad \Omega_-^\lambda \sim A, \\ (2\varepsilon)^{-1}\left(\xi_1(\boldsymbol{r})\boldsymbol{i}\otimes\boldsymbol{i} + \xi_2(\boldsymbol{r})\boldsymbol{i}\odot\boldsymbol{j} + \xi_3(\boldsymbol{r})\boldsymbol{j}\otimes\boldsymbol{j}\right. & \text{in} \quad A, \end{cases} \tag{19}$$

where

$$\xi_1(\boldsymbol{r}) = \frac{b^2c^2x^4}{2y^2} - \frac{1}{2}(\mu+\epsilon)^2,$$

$$\xi_2(\boldsymbol{r}) = bx(\frac{bcx^2}{y} - \mu - \epsilon),$$

$$\xi_3(\boldsymbol{r}) = by(\mu-\varepsilon) - b^2x^2\left(c + ln\frac{y(\mu+\varepsilon)}{bcx^2}\right).$$

Let us derive the third regime of (19). We have

$$\begin{aligned} \boldsymbol{T}_r(\boldsymbol{r}) \quad &= (2\varepsilon)^{-1}\int_{\mu-\epsilon}^{\mu+\epsilon} \boldsymbol{T}_r^\lambda d\lambda = (2\varepsilon)^{-1}\Big\{\int_{\mu-\epsilon}^{bcx^2/y} -by\boldsymbol{j}\otimes\boldsymbol{j}d\lambda \\ &+ \int_{bcx^2/y}^{\mu+\epsilon}(-\lambda\boldsymbol{i}\otimes\boldsymbol{i} - bx\boldsymbol{i}\odot\boldsymbol{j} - \frac{b^2x^2}{\lambda}\boldsymbol{j}\otimes\boldsymbol{j})d\lambda\Big\} \\ &= (2\varepsilon)^{-1}\Big\{\Big(\frac{b^2c^2x^4}{2y^2} - \frac{1}{2}(\mu+\epsilon)^2\Big)\boldsymbol{i}\otimes\boldsymbol{i} + bx\Big(\frac{bcx^2}{y} - \mu - \varepsilon\Big)\boldsymbol{i}\odot\boldsymbol{j} \\ &+ \Big[by(\mu-\epsilon) - b^2x^2\Big(c + ln\frac{y(\mu+\varepsilon)}{bcx^2}\Big)\Big]\boldsymbol{j}\otimes\boldsymbol{j}; \end{aligned}$$

this reduces to the third regime in (19). We note that the density $\boldsymbol{T}_r$ is bounded in Ω.

We now consider measures $\mathbf{T}_s^\lambda$. Firstly we note that $|\nabla\varphi||\boldsymbol{T}_s^\lambda|$ is bounded in

$$\Omega_0 = \{\boldsymbol{r} = (x,y) : cbx^2/y \in (0,\lambda_c)\} = \varphi^{-1}(0,\lambda_c),$$

in view of (14), (15) and (16). Then Prop. 1.3 says that for any interval Λ as in (17) the measure

$$\mathbf{T}_s = \frac{1}{2\epsilon}\int_\Lambda \mathbf{T}_s^\lambda\, d\lambda, \tag{20}$$

is $\mathcal{L}^2$ absolutely continuous over Ω with density given by (5), i.e.,

$$\mathbf{T}_s = \boldsymbol{T}_s(\boldsymbol{r})\mathcal{L}^2 \llcorner \Omega_0,$$

where

$$\boldsymbol{T}_s(\boldsymbol{r}) = \begin{cases} (2\varepsilon)^{-1}\,\boldsymbol{T}_s^\lambda(\boldsymbol{r})|\nabla\varphi(\boldsymbol{r})| & \text{if} \quad \boldsymbol{r}\in A, \\ \boldsymbol{0} & \text{otherwise.} \end{cases}$$

(Note that $\varphi(\boldsymbol{r}) \in \Lambda$ if and only if $\boldsymbol{r} \in A$, by (13) and (18)). In our case we have

$$\begin{aligned} \boldsymbol{T}_s^\lambda(\boldsymbol{r})|\nabla\varphi(\boldsymbol{r})| \quad &= \sigma^\lambda(\boldsymbol{r})\,\boldsymbol{t}^\lambda(\boldsymbol{r})\otimes\boldsymbol{t}^\lambda(\boldsymbol{r})|\nabla\varphi(\boldsymbol{r})| \\ &= \frac{\sqrt{3}cb^2x^2}{6y^2}(x,2y)\otimes(x,2y) \\ &= \frac{\sqrt{3}cb^2x^2}{6y^2}\Big(x^2\boldsymbol{i}\otimes\boldsymbol{i} + 2xy\boldsymbol{i}\odot\boldsymbol{j} + 4y^2\boldsymbol{j}\otimes\boldsymbol{j}\Big) \end{aligned}$$

by (14), (15) and (16).

Finally, we obtain the admissible and equilibrated stressfield T which is $\mathcal{L}^2$ absolutely continuous over Ω,

$$\mathsf{T} = \boldsymbol{T}(\boldsymbol{r})\mathcal{L}^2 \llcorner \Omega, \tag{21}$$

with $\boldsymbol{T}(\boldsymbol{r}) = \boldsymbol{T}_r(\boldsymbol{r}) + \boldsymbol{T}_s(\boldsymbol{r})$,

$$\boldsymbol{T}(\boldsymbol{r}) = \begin{cases} -by\boldsymbol{j}\otimes\boldsymbol{j} & \text{in} \quad \Omega_+^\lambda \sim A, \\ -\mu\boldsymbol{i}\otimes\boldsymbol{i} - bx\boldsymbol{i}\odot\boldsymbol{j} - b^2x^2/\lambda\boldsymbol{j}\otimes\boldsymbol{j} & \text{in} \quad \Omega_-^\lambda \sim A, \\ \boldsymbol{S}(\boldsymbol{r}) & \text{in} \quad A, \end{cases} \tag{22}$$

where

$$\begin{aligned} \boldsymbol{S}(\boldsymbol{r}) \quad &= -(2\varepsilon)^{-1}\Big\{\Big[-\frac{b^2x^4}{12y^2} + \frac{1}{2}(\mu+\epsilon)^2\Big]\boldsymbol{i}\otimes\boldsymbol{i} \\ &+\Big[-\frac{b^2x^3}{3y} + bx(\mu+\varepsilon)\Big]\boldsymbol{i}\odot\boldsymbol{j} \\ &+\Big[\Big(\frac{\sqrt{3}}{2} + \frac{5}{6} + ln\frac{y(\mu+\varepsilon)}{bcx^2}\Big)b^2x^2 - by(\mu-\epsilon)\Big]\boldsymbol{j}\otimes\boldsymbol{j}\Big\}. \end{aligned} \tag{23}$$

It is an easy matter to verify that div $\boldsymbol{T} + b\boldsymbol{j} = \boldsymbol{0}$.

4 Panel under singular side loads

We consider the panel Ω, the coordinate system, and the sets $\mathcal{S}$, $\mathcal{D}$ as in the first paragraph of Sect. 3.

4.1 Regular case

Firstly, we study the problem when the panel undergoes an uniformly distributed load on the top and on an upper part of the right lateral side, i.e. we consider the loads $\mathfrak{L}(\lambda) = (\boldsymbol{s}^\lambda, \boldsymbol{0})$ where, for $\boldsymbol{r} = (x, y) \in \mathcal{S}$,

$$\boldsymbol{s}^\lambda(\boldsymbol{r}) = \begin{cases} p\boldsymbol{j} & \text{on} \quad \{0\}\times(0,B), \\ \lambda\boldsymbol{i} & \text{on} \quad \{0\}\times(0,\alpha), \\ \mathbf{0} & \text{elsewhere}, \end{cases}$$

with $0 < \alpha \leq H$ and $\lambda \geq 0$ (Figure 2).

In order to define a measure stress field which is in equilibrium with the loads applied to the panel, we consider a curve γ^λ which divides Ω into the regions Ω_+^λ (on the left) and Ω_-^λ (on the right). The curve γ^λ is the graph of an increasing function $\omega^\lambda : [0, t^\lambda] \to [0, H]$,

$$\omega^\lambda(x) = \begin{cases} \sqrt{\dfrac{p}{\lambda}}x & \text{if} \quad 0 \leq x \leq \alpha\sqrt{\lambda/p}, \\ \dfrac{px^2}{2\alpha\lambda} + \dfrac{1}{2}\alpha & \text{if} \quad x > \alpha\sqrt{\lambda/p}, \end{cases}$$

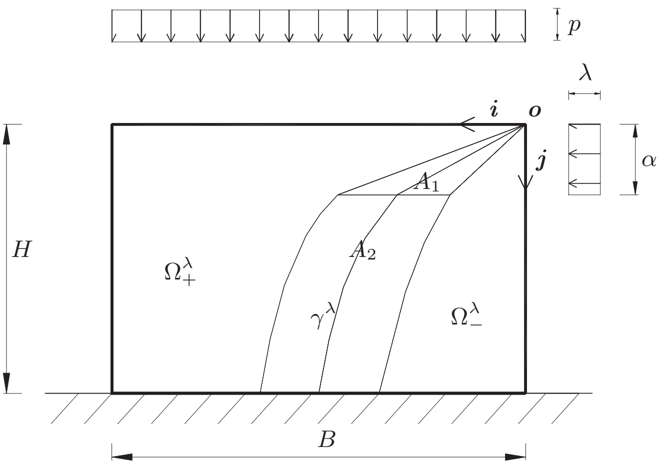

Fig. 2.

with unit tangent vector

$$\boldsymbol{t}^\lambda(\boldsymbol{r}) = \begin{cases} \dfrac{(x,y)}{\sqrt{x^2+y^2}} & \text{for} \quad \boldsymbol{r} \in \gamma^\lambda \cap \Omega_1, \\ \dfrac{(x, 2y-\alpha)}{\sqrt{x^2+(2y-\alpha)^2}} & \text{for} \quad \boldsymbol{r} \in \gamma^\lambda \cap \Omega_2, \end{cases} \tag{24}$$

where

$$\Omega_1 = \{\boldsymbol{r} = (x,y) \in \Omega : 0 < y \leq \alpha\}, \tag{25}$$

$$\Omega_2 = \{\boldsymbol{r} = (x,y) \in \Omega : \alpha < y < H\}. \tag{26}$$

If $\lambda \in (0, \lambda_{\mathrm{c}})$, with

$$\lambda_{\mathrm{c}} = \frac{pB^2}{\alpha(2H-\alpha)}, \tag{27}$$

then $t^\lambda = \sqrt{\alpha\lambda(2H-\alpha)/p}$ and γ^λ is contained in Ω, except for the endpoints.

We note that if $\varphi : \Omega \to \mathbb{R}$ is defined by

$$\varphi(\boldsymbol{r}) = \begin{cases} \dfrac{px^2}{y^2} & \text{for} \quad \boldsymbol{r} \in \Omega_1, \\ \dfrac{px^2}{\alpha(2y-\alpha)} & \text{for} \quad \boldsymbol{r} \in \Omega_2, \end{cases} \tag{28}$$

then, for any $\lambda \in (0, \lambda_{\mathrm{c}})$, the curve γ^λ is the level set of φ corresponding to the value of λ. Moreover, φ is continuously differentiable in Ω and we have

$$|\nabla\varphi(\boldsymbol{r})| = \begin{cases} \dfrac{2px}{y^3}\sqrt{x^2+y^2} & \text{for } \boldsymbol{r} \ \in \Omega_1, \\ \dfrac{2px}{\alpha(2y-\alpha)^2}\sqrt{x^2+(2y-\alpha)^2} & \text{for } \boldsymbol{r} \ \in \Omega_2. \end{cases} \tag{29}$$

For $\lambda \in (0, \lambda_{\mathrm{c}})$, the loads $\mathfrak{L}(\lambda)$ are weakly equilibrated by the admissible measure stressfield

$$\mathbf{T}^\lambda = \boldsymbol{T}_r^\lambda \mathcal{L}^2 \,\llcorner\, \Omega + \boldsymbol{T}_s^\lambda \mathcal{H}^1 \,\llcorner\, \gamma^\lambda$$

where

$$\boldsymbol{T}_r^\lambda(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j} & \text{in} \quad \Omega_+^\lambda \ , \\ -\lambda\boldsymbol{i}\otimes\boldsymbol{i} & \text{in} \quad \Omega_-^\lambda \cap \Omega_1, \\ \boldsymbol{0} & \text{elsewhere,} \end{cases}$$

with Ω_1 defined by (25), and

$$\boldsymbol{T}_s^\lambda = \sigma^\lambda \boldsymbol{t}^\lambda \otimes \boldsymbol{t}^\lambda, \tag{30}$$

with

$$\sigma^\lambda(\boldsymbol{r}) = \begin{cases} -\dfrac{px}{y}\sqrt{x^2+y^2} & \text{for} \quad \boldsymbol{r}\in\gamma^\lambda\cap\Omega_1, \\ \dfrac{-px}{(2y-\alpha)}\sqrt{x^2+(2y-\alpha)^2} & \text{for} \quad \boldsymbol{r}\in\gamma^\lambda\cap\Omega_2, \end{cases} \tag{31}$$

and $\boldsymbol{t}^\lambda$ given by (24). We now write

$$\mathbf{T}^\lambda = \mathbf{T}_r^\lambda + \mathbf{T}_s^\lambda$$

where

$$\mathbf{T}_r^\lambda = \boldsymbol{T}_r^\lambda \mathcal{L}^2 \llcorner \Omega, \quad \mathbf{T}_s^\lambda = \boldsymbol{T}_s^\lambda \mathcal{H}^1 \llcorner \gamma^\lambda.$$

We note that $\mathbf{T}_r^\lambda$ is of the form considered in Prop. 1.2 and that the integrability condition (3) is satisfied. Hence, for $0<\mu<\lambda_c$ and $\varepsilon>0$ as in (17) the measure

$$\mathbf{T}_r = \frac{1}{2\varepsilon}\int_\Lambda \mathbf{T}_r^\lambda d\lambda$$

is an absolutely continuous measure with respect to $\mathcal{L}^2 \llcorner \Omega$,

$$\mathbf{T}_r = \boldsymbol{T}_r \mathcal{L}^2 \llcorner \Omega.$$

In order to compute the density $\boldsymbol{T}_r$, we put

$$A_1 = \{\boldsymbol{r}=(x,y)\in\Omega_1 : \lambda = \frac{px^2}{y^2}\in\Lambda\} \tag{32}$$

and

$$A_2 = \{\boldsymbol{r}=(x,y)\in\Omega_2 : \lambda = \frac{px^2}{\alpha(2y-\alpha)}\in\Lambda\}. \tag{33}$$

Following the procedure in [LSZ08, Sec. 5], we obtain

$$\boldsymbol{T}_r(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j} & \text{for} \quad \boldsymbol{r}\in\Omega_+^\lambda \sim (A_1\cup A_2)\ , \\ -\mu\boldsymbol{i}\otimes\boldsymbol{i} & \text{for} \quad \boldsymbol{r}\in(\Omega_-^\lambda\cap\Omega_1)\sim A_1, \\ \boldsymbol{0} & \text{for} \quad \boldsymbol{r}\in(\Omega_-^\lambda\cap\Omega_2)\sim A_2, \\ -(2\varepsilon)^{-1}\left(\xi_1(\boldsymbol{r})\boldsymbol{i}\otimes\boldsymbol{i}+\xi_2(\boldsymbol{r})\boldsymbol{j}\otimes\boldsymbol{j}\right) & \text{for} \quad \boldsymbol{r}\in A_1, \\ -(2\varepsilon)^{-1}p\left(\dfrac{px^2}{\alpha(2y-\alpha)}-\mu+\epsilon\right)\boldsymbol{j}\otimes\boldsymbol{j} & \text{for} \quad \boldsymbol{r}\in A_2, \end{cases}$$

where

$$\xi_1(\boldsymbol{r}) = \frac{1}{2}\left((\mu+\epsilon)^2 - \frac{p^2x^4}{y^4}\right), \tag{34}$$

$$\xi_2(\boldsymbol{r}) = p\left(\frac{px^2}{y^2} - \mu + \epsilon\right). \tag{35}$$

We note that the density $\boldsymbol{T}_r$ is bounded in Ω.

We now consider the measures T_s^λ. Firstly we note that $|\nabla\varphi||\boldsymbol{T}_s^\lambda|$ is bounded in

$$\Omega_0 = \varphi^{-1}(0,\lambda_c) = \{\boldsymbol{r} = (x,y) \in \Omega_1 : \lambda = \frac{px^2}{y^2} \in (0,\lambda_c)\} \cup$$
$$\{\boldsymbol{r} = (x,y) \in \Omega_2 : \lambda = \frac{px^2}{\alpha(2y-\alpha)} \in (0,\lambda_c)\},$$

in view of (29), (30) and (31). Then Prop. 1.3 says that for any interval Λ as in (17), the measure

$$\mathsf{T}_s = \frac{1}{2\varepsilon}\int_\Lambda \mathsf{T}_s^\lambda d\lambda$$

is $\mathcal{L}^2$ absolutely continuous over Ω_0 with density given by (5), i.e.,

$$\mathsf{T}_s = \boldsymbol{T}_s(\boldsymbol{r})\mathcal{L}^2 \llcorner \Omega_0,$$

where

$$\boldsymbol{T}_s(\boldsymbol{r}) = \begin{cases} (2\varepsilon)^{-1}\,\boldsymbol{T}_s^\lambda(\boldsymbol{r})|\nabla\varphi(\boldsymbol{r})| & \text{for} \quad \boldsymbol{r} \in A_1 \cup A_2, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

(Note that $\varphi(\boldsymbol{r}) \in \Lambda$ if and only if $\boldsymbol{r} \in A_1 \cup A_2$, by (28), (32) and (33)).

For $\boldsymbol{r} \in A_1$ we have

$$(2\varepsilon)^{-1}\,\boldsymbol{T}_s^\lambda(\boldsymbol{r})|\nabla\varphi(\boldsymbol{r})| = -(2\varepsilon)^{-1}\frac{2p^2x^2}{y^4}(x,y)\otimes(x,y) =$$
$$-(2\varepsilon)^{-1}\frac{2p^2x^2}{y^4}\Big(x^2\boldsymbol{i}\otimes\boldsymbol{i} + xy\boldsymbol{i}\odot\boldsymbol{j} + y^2\boldsymbol{j}\otimes\boldsymbol{j}\Big).$$

For $\boldsymbol{r} \in A_2$ we have

$$(2\varepsilon)^{-1}\,\boldsymbol{T}_s^\lambda(\boldsymbol{r})|\nabla\varphi(\boldsymbol{r})| = -(2\varepsilon)^{-1}\frac{2p^2x^2}{\alpha(2y-\alpha)^3}(x,2y-\alpha)\otimes(x,2y-\alpha) =$$
$$-(2\varepsilon)^{-1}\frac{2p^2x^2}{\alpha(2y-\alpha)^3}\Big(x^2\boldsymbol{i}\otimes\boldsymbol{i} + x(2y-\alpha)\boldsymbol{i}\odot\boldsymbol{j} + (2y-\alpha)^2\boldsymbol{j}\otimes\boldsymbol{j}\Big).$$

Finally, we can write the admissible stressfield $\mathsf{T} = \boldsymbol{T}(\boldsymbol{r})\mathcal{L}^2 \llcorner \Omega$ which strongly equilibrates the loads $\mathfrak{L}(\mu)$, with

$$\boldsymbol{T}(\boldsymbol{r}) = \boldsymbol{T}_r(\boldsymbol{r}) + \boldsymbol{T}_s(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j} & \text{for } \boldsymbol{r} \in \Omega_+^\lambda \sim (A_1 \cup A_2) \\ -\mu\boldsymbol{i}\otimes\boldsymbol{i} & \text{for } \boldsymbol{r} \in \left(\Omega_-^\lambda \cap \Omega_1\right) \sim A_1, \\ \boldsymbol{S_1}(\boldsymbol{r}) & \text{for } \boldsymbol{r} \in A_1, \\ \boldsymbol{S_2}(\boldsymbol{r}) & \text{for } \boldsymbol{r} \in A_2, \\ \mathbf{0} & \text{elsewhere,} \end{cases} \tag{36}$$

where, for ξ_1 and ξ_2 given by (34) and (35),

$$\boldsymbol{S_1}(\boldsymbol{r}) = -(2\varepsilon)^{-1}\left\{\left(\xi_1(\boldsymbol{r}) + \frac{2p^2x^4}{y^4}\right)\boldsymbol{i}\otimes\boldsymbol{i} + \frac{2p^2x^3}{y^3}\boldsymbol{i}\odot\boldsymbol{j}\right.$$
$$\left. + \left(\xi_2(\boldsymbol{r}) + \tfrac{2p^2x^2}{y^2}\right)\boldsymbol{j}\otimes\boldsymbol{j}\right\},$$

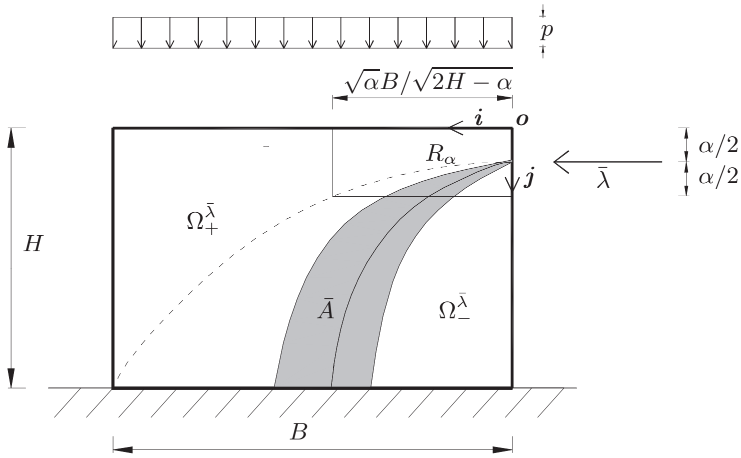

Fig. 3.

$$\boldsymbol{S_2}(\boldsymbol{r}) = -(2\varepsilon)^{-1}\Big\{\frac{2p^2x^4}{\alpha(2y-\alpha)^3}\boldsymbol{i}\otimes\boldsymbol{i}+\frac{2p^2x^3}{\alpha(2y-\alpha)^3}\boldsymbol{i}\odot\boldsymbol{j} + \left(\frac{3p^2x^2}{\alpha(2y-\alpha)} - p\mu + p\epsilon\right)\boldsymbol{j}\otimes\boldsymbol{j}\Big\}. \tag{37}$$

4.2 Singular case

Now we consider the situation when the distributed lateral load is substituted by its resultant of intensity $\bar{\lambda} = \alpha\lambda$ applied at the point $(0, \alpha/2)$ (Figure 3). That is, we consider the loads $\mathfrak{L}^{\bar{\lambda}} = (\mathbf{s}^{\bar{\lambda}}, \mathbf{0})$ with $\mathbf{s}^{\bar{\lambda}}$ given by the measure,

$$\mathbf{s}^{\bar{\lambda}} = \mathrm{s}_0\mathcal{H}^1 \llcorner \mathcal{S} + \bar{\lambda}\delta_\alpha\boldsymbol{i},$$

where δ_α is the Dirac measure at the point $(0, \alpha/2)$, and

$$\mathrm{s}_0(\boldsymbol{r}) = \begin{cases} p\boldsymbol{j} & \text{if} \quad \boldsymbol{r}\in(0,B)\times\{0\}, \\ \mathbf{0} & \text{if} \quad \boldsymbol{r}\in\mathcal{S}\sim(0,B)\times\{0\}. \end{cases}$$

In this case Ω is divided into the regions $\Omega_+^{\bar{\lambda}}$ and $\Omega_-^{\bar{\lambda}}$ by the curve $\gamma^{\bar{\lambda}}$, which is the graph of an increasing function $\omega^{\bar{\lambda}} : [0, t^{\bar{\lambda}}] \to [0, H]$,

$$\omega^{\bar{\lambda}}(x) = \frac{px^2}{2\bar{\lambda}} + \frac{\alpha}{2},$$

with unit tangent vector

$$\boldsymbol{t}^{\bar{\lambda}}(\boldsymbol{r}) = \frac{(x, 2y-\alpha)}{\sqrt{x^2+(2y-\alpha)^2}} \quad \text{for } \boldsymbol{r}\in\gamma^{\bar{\lambda}}. \tag{38}$$

If $\bar{\lambda}\in(0,\bar{\lambda}_{\mathrm{c}})$, with

$$\bar{\lambda}_{\mathrm{c}} = \frac{pB^2}{2H-\alpha},$$

then $\gamma^{\bar\lambda}$ is contained in Ω, except for the endpoints [LSZ06, Example 2], and $t^{\bar\lambda} = \sqrt{\bar\lambda(2H-\alpha)/p}$.

We note that if $\bar\varphi : \Omega \to \mathbb{R}$ is defined by

$$\bar\varphi(\boldsymbol{r}) = \frac{px^2}{2y-\alpha} \tag{39}$$

then, for any $\bar\lambda \in (0, \bar\lambda_{\rm c})$, the curve $\gamma^{\bar\lambda}$ is the level set of $\bar\varphi$ corresponding to the value of $\bar\lambda$. Moreover, $\bar\varphi$ is continuously differentiable in Ω and we have

$$|\nabla\bar\varphi(\boldsymbol{r})| = \frac{2px}{(2y-\alpha)^2}\sqrt{x^2+(2y-\alpha)^2}. \tag{40}$$

For $\bar\lambda \in (0, \bar\lambda_{\rm c})$, the loads $\mathfrak{L}^{\bar\lambda}$ are weakly equilibrated by the admissible measure stressfield

$$\mathsf{T}^{\bar\lambda} = \boldsymbol{T}_r^{\bar\lambda}\mathcal{L}^2 \llcorner \Omega + \boldsymbol{T}_s^{\bar\lambda}\mathcal{H}^1 \llcorner \gamma^{\bar\lambda}$$

where

$$\boldsymbol{T}_r^{\bar\lambda}(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j} & \text{in} \quad \Omega_+^{\bar\lambda}, \\ \mathbf{0} & \text{in} \quad \Omega_-^{\bar\lambda}, \end{cases}$$

and

$$\boldsymbol{T}_s^{\bar\lambda}(\boldsymbol{r}) = \sigma^{\bar\lambda}(\boldsymbol{r})\boldsymbol{t}^{\bar\lambda}(\boldsymbol{r}) \otimes \boldsymbol{t}^{\bar\lambda}(\boldsymbol{r}), \tag{41}$$

with

$$\sigma^{\bar\lambda}(\boldsymbol{r}) = -\frac{px}{2y-\alpha}\sqrt{x^2+(2y-\alpha)^2}, \quad \boldsymbol{r} \in \gamma^{\bar\lambda}, \tag{42}$$

and $\boldsymbol{t}^{\bar\lambda}$ given by (38). We now write

$$\mathsf{T}^{\bar\lambda} = \mathsf{T}_r^{\bar\lambda} + \mathsf{T}_s^{\bar\lambda}$$

where

$$\mathsf{T}_r^{\bar\lambda} = \boldsymbol{T}_r^{\bar\lambda}\mathcal{L}^2 \llcorner \Omega, \quad \mathsf{T}_s^{\bar\lambda} = \boldsymbol{T}_s^{\bar\lambda}\mathcal{H}^1 \llcorner \gamma^{\bar\lambda}$$

We note that $\mathsf{T}_r^{\bar\lambda}$ is of the form considered in Prop. 1.2 and that the integrability condition (3) is satisfied. Hence, for $0 < \bar\mu < \bar\lambda_{\rm c}$ and $\bar\epsilon$ such that

$$\bar\Lambda = (\bar\mu - \bar\epsilon, \bar\mu + \bar\epsilon) \subset (0, \bar\lambda_{\rm c}) \tag{43}$$

the measure

$$\bar{\mathsf{T}}_r = \frac{1}{2\bar\epsilon}\int_{\bar\Lambda} \mathsf{T}_r^{\bar\lambda} d\bar\lambda$$

is an absolutely continuous measure with respect to $\mathcal{L}^2 \llcorner \Omega$,

$$\bar{\mathsf{T}}_r = \bar{\boldsymbol{T}}_r\mathcal{L}^2 \llcorner \Omega.$$

In order to compute the density $\bar{\boldsymbol{T}}_r$, we put

$$\bar A = \{\boldsymbol{r} = (x,y) \in \Omega : \bar\lambda = \frac{px^2}{2y-\alpha} \in \bar\Lambda\}. \tag{44}$$

With the same procedure followed in [LSZ08, Sec. 5], we obtain

$$\bar{\boldsymbol{T}}_r = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j} & \text{for} \quad \boldsymbol{r}\in\Omega_+^{\bar\lambda}\sim\bar{A}, \\ \boldsymbol{0} & \text{for} \quad \boldsymbol{r}\in\Omega_-^{\bar\lambda}\sim\bar{A}, \\ -(2\bar\varepsilon)^{-1}p\left(\dfrac{px^2}{2y-\alpha}-\bar\mu+\bar\varepsilon\right)\boldsymbol{j}\otimes\boldsymbol{j} & \text{for} \quad \boldsymbol{r}\in\bar{A}. \end{cases}$$

Consider measures $\mathsf{T}_s^{\bar\lambda}$. Firstly we note that $|\nabla\bar\varphi||\boldsymbol{T}_s^{\bar\lambda}|$ is (not bounded but) integrable in

$$\bar\Omega_0 = \bar\varphi^{-1}(0,\bar\lambda_c) = \{\boldsymbol{r}=(x,y)\in\Omega : \bar\lambda = \frac{px^2}{2y-\alpha}\in(0,\bar\lambda_c)\},$$

in view of (40), (41) and (42). (See Remark 4.1 at the end of this Section). Then Prop. 1.3 says that for any interval $\bar\Lambda$ as in (43) the measure

$$\bar{\mathsf{T}}_s = \frac{1}{2\bar\varepsilon}\int_\Lambda \mathsf{T}_s^{\bar\lambda}d\bar\lambda$$

is $\mathcal{L}^2$ absolutely continuous over $\bar\Omega_0$ with density given by (5), i.e.,

$$\mathsf{T}_s = \bar{\boldsymbol{T}}_s(\boldsymbol{r})\mathcal{L}^2 \llcorner \bar\Omega_0,$$

where

$$\bar{\boldsymbol{T}}_s = \begin{cases} (2\bar\varepsilon)^{-1}\,\bar{\boldsymbol{T}}^s_{\bar\lambda}(\boldsymbol{r})|\nabla\bar\varphi(\boldsymbol{r})| & \text{for} \quad \boldsymbol{r}\in\bar{A}, \\ \boldsymbol{0} & \text{otherwise.} \end{cases}$$

(Note that $\bar\varphi(\boldsymbol{r})\in\bar\Lambda$ if and only if $\boldsymbol{r}\in\bar{A}$, by (39) and (44)).

For $\boldsymbol{r}\in\bar{A}$ we have

$$\begin{aligned}(2\bar\varepsilon)^{-1}\quad \bar{\boldsymbol{T}}_s(\boldsymbol{r})|\nabla\bar\varphi(\boldsymbol{r})| &= -(2\bar\varepsilon)^{-1}\frac{2p^2x^2}{(2y-\alpha)^3}(x,2y-\alpha)\otimes(x,2y-\alpha) = \\ &-(2\bar\varepsilon)^{-1}\frac{2p^2x^2}{(2y-\alpha)^3}\Big(x^2\boldsymbol{i}\otimes\boldsymbol{i}+x(2y-\alpha)\,\boldsymbol{i}\odot\boldsymbol{j}+(2y-\alpha)^2\boldsymbol{j}\otimes\boldsymbol{j}\Big).\end{aligned}$$

Finally, we obtain the admissible and equilibrated stressfield $\bar{\mathsf{T}} = \bar{\boldsymbol{T}}(\boldsymbol{r})\mathcal{L}^2\llcorner\Omega$ which equilibrates the loads $\mathfrak{L}(\mu)$,

$$\bar{\boldsymbol{T}}(\boldsymbol{r}) = \bar{\boldsymbol{T}}_r(\boldsymbol{r}) + \bar{\boldsymbol{T}}_s(\boldsymbol{r}) = \begin{cases} -p\boldsymbol{j}\otimes\boldsymbol{j} & \text{for} \quad \boldsymbol{r}\in\Omega_+^{\lambda}\sim\bar{A}, \\ \boldsymbol{0} & \text{for} \quad \boldsymbol{r}\in\Omega_-^{\lambda}\sim\bar{A}, \\ \bar{\boldsymbol{S}}(\boldsymbol{r}) & \text{for} \quad \boldsymbol{r}\in\bar{A}, \end{cases} \tag{45}$$

where

$$\bar{\boldsymbol{S}}(\boldsymbol{r}) = -(2\bar\varepsilon)^{-1}\Big\{\frac{2p^2x^4}{(2y-\alpha)^3}\boldsymbol{i}\otimes\boldsymbol{i}+\frac{2p^2x^3}{(2y-\alpha)^2}\boldsymbol{i}\odot\boldsymbol{j}+(\frac{3p^2x^2}{2y-\alpha}-p\bar\mu+p\bar\varepsilon)\boldsymbol{j}\otimes\boldsymbol{j}\Big\}. \tag{46}$$

We note that $\bar{\boldsymbol{T}}(\boldsymbol{r})$, defined in (45), belongs to $L^1(\Omega,\mathrm{Sym})$ but not to $L^2(\Omega,\mathrm{Sym})$ (see Remark 4.2 below) and hence it cannot be used to apply the static theorem of limit analysis (see Prop. 2.1). Nevertheless, for $\lambda = \bar\lambda/\alpha$, in the region Ω_2 defined in (26) γ^λ coincides with $\gamma^{\bar\lambda}$ (see $(28)_2$, (39)) and σ^λ coincides with $\sigma^{\bar\lambda}$

(see $(31)_2$, (42)). Moreover, for $\mu=\bar{\mu}/\alpha$ and $\epsilon=\bar{\epsilon}/\alpha$, $\boldsymbol{S}_2$ defined in A_2 by (37) coincides with $\bar{\boldsymbol{S}}$ defined in $\bar{A}$ by (46). Therefore the densities $\boldsymbol{T}$ and $\bar{\boldsymbol{T}}$ are different from one another only in a region that is contained into the rectangle $R_\alpha=(0,\sqrt{\alpha\bar{\lambda}_{\rm c}/p})\times(0,\alpha)$ (see (36) and (45)), whose area is

$$\frac{\alpha\sqrt{\alpha}B}{\sqrt{2H-\alpha}}.$$

Then, for every $\alpha\in(0,H)$, we have

$$\int_\Omega|\boldsymbol{T}-\bar{\boldsymbol{T}}|d\mathcal{L}^2\leq\int_{R\alpha}|\boldsymbol{T}-\bar{\boldsymbol{T}}|d\mathcal{L}^2,$$

and, because $|\boldsymbol{T}-\bar{\boldsymbol{T}}|\in L^1(\Omega,\mathbb{R})$, for each $\nu>0$, there exists $\alpha>0$ such that

$$\int_\Omega|\boldsymbol{T}-\bar{\boldsymbol{T}}|d\mathcal{L}^2\leq\nu.$$

In applications where α/H is small, sometimes one can use the stress field defined in (45) to approximate that defined in (36).

Remark 4.1. In view of (40), (41) and (42) we have

$$|\nabla\bar{\varphi}(\boldsymbol{r})||\boldsymbol{T}_s^{\bar{\lambda}}(\boldsymbol{r})|=\frac{2p^2x^2}{(2y-\alpha)^3}(x^2+(2y-\alpha)$$

For sake of simplicity we put $p=1$, and by the change of variables

$$x=\sqrt{tu},\ y=\frac{1}{2}(u+\alpha) \tag{47}$$

we get $J=\frac{1}{4}\sqrt{tu}/t$ and then

$$\int_{\bar{\Omega}_0}|\nabla\bar{\varphi}(\boldsymbol{r})||\boldsymbol{T}_s^{\bar{\lambda}}(\boldsymbol{r})|d\mathcal{L}^2=$$

$$\frac{1}{2}\int_0^{B^2/(2H-\alpha)}\int_0^{2H-\alpha}\left(\frac{t\sqrt{t}}{\sqrt{u}}+\sqrt{tu}\right)dt\,du<\infty.$$

Remark 4.2. In order to prove that $\bar{\boldsymbol{S}}(\boldsymbol{r})$ defined in (46) belongs to $L^1(\Omega,\mathrm{Sym})$ but not to $L^2(\Omega,\mathrm{Sym})$, it is enough to consider the first term, $\frac{2p^2x^4}{(2y-\alpha)^3}$. By the change of variables (47) we obtain, for $p=1$,

$$\int_{\bar{A}}\frac{2x^4}{(2y-\alpha)^3}d\mathcal{L}^2=\frac{1}{2}\left(\int_{\mu-\epsilon}^{\mu+\epsilon}t\sqrt{t}\,dt\right)\left(\int_0^{2H-\alpha}\frac{du}{\sqrt{u}}\right)<\infty,$$

and

$$\int_{\bar{A}}\left(\frac{2x^4}{(2y-\alpha)^3}\right)^2d\mathcal{L}^2=\left(\int_{\mu-\epsilon}^{\mu+\epsilon}t^3\sqrt{t}\,dt\right)\left(\int_0^{2H-\alpha}\frac{du}{u\sqrt{u}}\right)=\infty.$$

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