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NONNEGATIVE SOLUTIONS OF THE CHARACTERISTIC INITIAL VALUE PROBLEM FOR LINEAR PARTIAL FUNCTIONAL-DIFFERENTIAL EQUATIONS OF HYPERBOLIC TYPE

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Abstract. On the rectangle $\mathcal{D} = [a, b] \times [c, d]$, the problem on the existence and uniqueness of a nonnegative solution of the characteristic initial value problem for the equation

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = \ell(u)(t,x) + q(t,x)$$

is considered, where $\ell : C(\mathcal{D}; \mathbb{R}) \to L(\mathcal{D}; \mathbb{R})$ is a linear bounded operator and $q \in L(\mathcal{D}; \mathbb{R}_+)$.

1. Introduction

On the rectangle $\mathcal{D}$, we consider the linear partial functional–differential equation of hyperbolic type

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = \ell(u)(t,x) + q(t,x), \tag{1.1}$$

where $\ell : C\big(\mathcal{D}; \mathbb{R}\big) \to L\big(\mathcal{D}; \mathbb{R}\big)$ is a linear bounded operator and $q \in L\big(\mathcal{D}; \mathbb{R}\big)$. By a solution of the equation (1.1) is understood a function $u \in C^*\big(\mathcal{D}; \mathbb{R}\big)$[1] satisfying the equality (1.1) almost everywhere on the set $\mathcal{D}$.

Various initial value problems for the equation (1.1) are studied in literature (see, e.g., [2, 4, 7, 8] and references therein). We will consider so–called characteristic initial value problem. In this case, the values of the solution u of (1.1) are prescribed on both characteristics $t = a$ and $x = c$, i.e., the initial conditions are

$$u(t, c) = \varphi(t) \quad \text{for} \quad t \in [a, b], \tag{1.2}$$

$$u(a, x) = \psi(x) \quad \text{for} \quad x \in [c, d], \tag{1.3}$$

where $\varphi : [a, b] \to \mathbb{R}$ and $\psi : [c, d] \to \mathbb{R}$ are absolutely continuous functions such that $\varphi(a) = \psi(c)$.

In this paper, we suggest a new approach to the problem considered which allows us to establish results guaranteeing that the problem (1.1)–(1.3) has a unique solution and this solution is nonnegative whenever the function q is nonnegative and the functions φ, ψ are nonnegative and nondecreasing. In other words, we will give some efficient conditions under which every solution of the problem

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} \geq \ell(u)(t,x), \tag{1.4}$$

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[1] For definition of the set $C^*\big(\mathcal{D}; \mathbb{R}\big)$, see Section 2.

$$u(a,c) \geq 0, \tag{1.5}$$

$$\frac{\partial u(t,c)}{\partial t} \geq 0 \quad \text{for almost all} \quad t \in [a,b], \tag{1.6}$$

$$\frac{\partial u(a,x)}{\partial x} \geq 0 \quad \text{for almost all} \quad x \in [c,d] \tag{1.7}$$

is nonnegative. Recall here that by a solution of the problem (1.4)–(1.7) we understand a function $u \in C^*\big(\mathcal{D};\mathbb{R}\big)$ satisfying the inequality (1.4) almost everywhere on the set $\mathcal{D}$ and verifying also the conditions (1.5)–(1.7). The results obtained in this paper will be further used in the study of the question on the unique solvability of the problem (1.1)–(1.3).

Note also that some analogous results for the first and second order "ordinary" functional–differential equations are established in [1] and [6], respectively.

To simplify the formulation of the main results we introduce the following definition.

Definition 1.1. We will say that an operator $\ell \in \mathcal{L}(\mathcal{D})$ belongs to the set $\mathcal{S}_{ac}(\mathcal{D})$ if every solution of the problem (1.4)–(1.7) is nonnegative.

It is well–known that the problem (1.1)–(1.3) has so–called Fredholm property, i.e., the following theorem is true (see, e.g., [5]).

Theorem 1.1. *The problem* (1.1)–(1.3) *has a unique solution if and only if the corresponding homogeneous problem*

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = \ell(u)(t,x), \tag{1.1$_0$}$$

$$u(t,c) = 0 \quad \textit{for} \quad t \in [a,b], \tag{1.2$_0$}$$

$$u(a,x) = 0 \quad \textit{for} \quad x \in [c,d] \tag{1.3$_0$}$$

has only the trivial solution.

Remark 1.1. Let $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. Then it is clear that the homogeneous problem (1.1$_0$)–(1.3$_0$) has only the trivial solution. Therefore, the problem (1.1)–(1.3) is uniquely solvable for every q, φ, and ψ. Moreover, if the function q is nonnegative and the functions φ, ψ are nonnegative and nondecreasing then the solution of the problem (1.1)–(1.3) is nonnegative.

2. Notation and Definitions

The following notation and definitions are used throughout the paper.

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0,+\infty[$.

N is the set of all natural numbers.

If $x \in \mathbb{R}$ then

$$[x]_+ = \frac{|x|+x}{2}\,, \qquad [x]_- = \frac{|x|-x}{2}\,.$$

$\mathcal{D} = [a,b] \times [c,d]$, where $-\infty < a < b < +\infty$ and $-\infty < c < d < +\infty$.

$C(\mathcal{D};\mathbb{R})$ is the Banach space of continuous functions $u : \mathcal{D} \to \mathbb{R}$ equipped with the norm

$$\|u\|_C = \max\{|u(t,x)| : (t,x) \in \mathcal{D}\}.$$

$C(\mathcal{D};A) = \{u \in C(\mathcal{D};\mathbb{R}) : u(t,x) \in A \text{ for } (t,x) \in \mathcal{D}\}$, where $A \subseteq \mathbb{R}$.

$L(\mathcal{D};\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : \mathcal{D} \to \mathbb{R}$ equipped with the norm

$$\|p\|_L = \iint_{\mathcal{D}} |p(t,x)|dtdx.$$

$L(\mathcal{D};A) = \{p \in L(\mathcal{D};\mathbb{R}) : p(t,x) \in A \text{ for almost all } (t,x) \in \mathcal{D}\}$, where $A \subseteq \mathbb{R}$.

$\mathcal{L}(\mathcal{D})$ is the set of linear bounded operators $\ell : C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$.

$\widetilde{C}([\alpha,\beta];A)$, where $A \subseteq \mathbb{R}$, is the set of absolutely continuous functions $v : [\alpha,\beta] \to A$.

$C^*(\mathcal{D};A)$, where $A \subseteq \mathbb{R}$, is the set of functions $u : \mathcal{D} \to A$ admitting the representation

$$u(t,x) = v_1(t) + v_2(x) + \int_a^t \int_c^x h(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D},$$

where $v_1 \in \widetilde{C}([a,b];\mathbb{R})$, $v_2 \in \widetilde{C}([c,d];\mathbb{R})$, and $h \in L(\mathcal{D};\mathbb{R})$.

$C^*_{loc}([a,b[\times[c,d[\,;A)$, where $A \subseteq \mathbb{R}$, is the set of function $u \in C(\mathcal{D};A)$ such that $u \in C^*([a,b_0] \times [c,d_0];A)$ for every $b_0 \in]a,b[$ and $d_0 \in]c,d[$.

Remark 2.1. One can verify that $v \in C^*(\mathcal{D};\mathbb{R})$ if and only if v satisfies the following conditions:

(a) $v(t,\cdot) \in \widetilde{C}([c,d];\mathbb{R})$ for every $t \in [a,b]$, $v(\cdot,x) \in \widetilde{C}([a,b];\mathbb{R})$ for every $x \in [c,d]$;

(b) $v_t(t,\cdot) \in \widetilde{C}([c,d];\mathbb{R})$ for almost all $t \in [a,b]$, $v_x(\cdot,x) \in \widetilde{C}([a,b];\mathbb{R})$ for almost all $x \in [c,d]$;

(c) $v_{tx} \in L(\mathcal{D};\mathbb{R})$.

We should note here that the set $C^*(\mathcal{D};\mathbb{R})$ coincide with the class of absolutely continuous functions of two variables presented, e.g., in [3, 7].

Definition 2.1. An operator $\ell \in \mathcal{L}(\mathcal{D})$ is said to be nondecreasing if it maps the set $C(\mathcal{D};\mathbb{R}_+)$ into the set $L(\mathcal{D};\mathbb{R}_+)$. The set of nondecreasing operators we denote by $P(\mathcal{D})$. We say that an operator $\ell \in \mathcal{L}(\mathcal{D})$ is nonincreasing if $-\ell \in P(\mathcal{D})$.

Definition 2.2. An operator $\ell \in \mathcal{L}(\mathcal{D})$ is called to be an (a,c)–Volterra operator if, for arbitrary rectangle $[a,t_0] \times [c,x_0] \subseteq \mathcal{D}$ and function $v \in C(\mathcal{D};\mathbb{R})$ such that

$$v(t,x) = 0 \quad \text{for} \quad (t,x) \in [a,t_0] \times [c,x_0],$$

the relation

$$\ell(v)(t,x) = 0 \quad \text{for almost all} \quad (t,x) \in [a,t_0] \times [c,x_0]$$

holds.

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

3. Main Results

In this section, we establish some efficient conditions for the inclusion $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. Theorems formulated below can be reffered to as theorems on functional–differential inequalities. One can say also that $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ if and only if some kind of maximum principle holds for the problem (1.1)–(1.3).

Theorem 3.1. *Let $\ell \in P(\mathcal{D})$. Then $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ if and only if there exists a function $\gamma \in C^*\big(\mathcal{D};\,]0,+\infty[\big)$ such that*

$$\frac{\partial^2\gamma(t,x)}{\partial t\,\partial x} \geq \ell(\gamma)(t,x) \quad \textit{for} \quad (t,x) \in \mathcal{D} \tag{3.1}$$

and either

$$\frac{\partial\gamma(t,c)}{\partial t} \geq 0 \quad \textit{for} \quad t \in [a,b] \tag{3.2}$$

or

$$\frac{\partial\gamma(a,x)}{\partial x} \geq 0 \quad \textit{for} \quad x \in [c,d]. \tag{3.3}$$

By a suitable choice of the function γ in Theorem 3.1 we can derive several sufficient conditions under which the inclusion $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ is true.

Corollary 3.1. *If $\ell \in P(\mathcal{D})$ then each of the following statements guarantees the inclusion $\ell \in \mathcal{S}_{ac}(\mathcal{D})$:*

a) there exist $k, m \in N$ and $\alpha \in]0,1[$ such that $m > k$ and

$$\rho_m(t,x) \leq \alpha\rho_k(t,x) \quad \textit{for} \quad (t,x) \in \mathcal{D}, \tag{3.4}$$

where

$$\rho_1 \equiv 1, \qquad \rho_{i+1} \equiv \theta(\rho_i) \quad \textit{for} \quad i \in N, \tag{3.5}$$

and

$$\theta(v)(t,x) \stackrel{\text{def}}{=} \int_a^t \int_c^x \ell(v)(s,\eta)d\eta ds \quad \textit{for} \quad (t,x) \in \mathcal{D}; \tag{3.6}$$

b) there exists $\overline{\ell} \in P(\mathcal{D})$ such that

$$\int_a^b \int_c^d \overline{\ell}(1)(s,\eta) \exp\left(\int_s^b \int_\eta^d \ell(1)(\xi_1,\xi_2)d\xi_2 d\xi_1 \right) d\eta ds < 1 \tag{3.7}$$

and the inequality

$$\ell\big(\theta(v)\big)(t,x) - \ell(1)(t,x)\theta(v)(t,x) \leq \overline{\ell}(v)(t,x) \quad \textit{for} \quad (t,x) \in \mathcal{D} \tag{3.8}$$

holds on the set $\big\{v \in C\big(\mathcal{D};\mathbb{R}_+\big) : v(\cdot,c) \equiv 0,\ v(a,\cdot) \equiv 0\big\}$, where θ is defined by (3.6).

Remark 3.1. The assumption $\alpha \in]0,1[$ in Corollary 3.1 a) cannot be replaced by the assumption $\alpha \in]0,1]$ (see Example 7.1).

Remark 3.2. It follows from Corollary 3.1 a) (for $k = 1$ and $m = 2$) that $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ provided that $\ell \in P(\mathcal{D})$ and

$$\int_a^b \int_c^d \ell(1)(s,\eta)d\eta ds < 1.$$

Proposition 3.1. *Let $\ell \in P(\mathcal{D})$ be such that*

$$\int_a^b \int_c^d \ell(1)(s,\eta)d\eta ds = 1. \tag{3.9}$$

Then $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ if and only if the homogeneous problem (1.1_0)–(1.3_0) has only the trivial solution.

Proposition 3.2. *Let $\ell \in P(\mathcal{D})$ be an (a,c)–Volterra operator. Then $\ell \in \mathcal{S}_{ac}(\mathcal{D})$.*

Theorem 3.2. *Let $-\ell \in P(\mathcal{D})$, ℓ be an (a,c)–Volterra operator, and let there exist a function $\gamma \in C^*_{loc}\big([a,b[\times[c,d[\,;\mathbb{R}_+\big)$ satisfying*

$$\frac{\partial^2\gamma(t,x)}{\partial t\,\partial x} \le \ell(\gamma)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}, \tag{3.10}$$

$$\gamma(t,x) > 0 \quad \text{for} \quad (t,x) \in [a,b[\,\times[c,d[\,, \tag{3.11}$$

$$\frac{\partial\gamma(t,c)}{\partial t} \le 0 \quad \text{for} \quad t \in [a,b[\,, \tag{3.12}$$

and

$$\frac{\partial\gamma(a,x)}{\partial x} \le 0 \quad \text{for} \quad x \in [c,d[\,. \tag{3.13}$$

Then the operator ℓ belongs to the set $\mathcal{S}_{ac}(\mathcal{D})$.

Remark 3.3. The assumption (3.11) in Theorem 3.2 is essential and cannot be omitted. Indeed, if there exists a function $\gamma \in C^*_{loc}\big([a,b[\times[c,d[\,;\mathbb{R}_+\big)$ such that the conditions (3.10), (3.12), and (3.13) hold and $\gamma(t_0,x_0) = 0$ for some $(t_0,x_0) \in\,]a,b[\,\times\,]c,d[\,$, then it can happen that $\ell \notin \mathcal{S}_{ac}(\mathcal{D})$ (see Example 7.2).

Corollary 3.2. *Let $-\ell \in P(\mathcal{D})$, ℓ be an (a,c)–Volterra operator, and*

$$\int_a^b \int_c^d |\ell(1)(s,\eta)|d\eta ds \le 1. \tag{3.14}$$

Then $\ell \in \mathcal{S}_{ac}(\mathcal{D})$.

Remark 3.4. The inequality (3.14) in Corollary 3.2 cannot be replaced by the inequality

$$\int_a^b \int_c^d |\ell(1)(s,\eta)|d\eta ds \le 1 + \varepsilon, \tag{3.15}$$

no matter haw small $\varepsilon > 0$ would be (see Example 7.2).

Theorem 3.3. *Let $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P(\mathcal{D})$ and ℓ_1 is an (a, c)–Volterra operator. If*

$$\ell_0 \in \mathcal{S}_{ac}(\mathcal{D}), \qquad -\ell_1 \in \mathcal{S}_{ac}(\mathcal{D}), \tag{3.16}$$

then the operator ℓ belongs to the set $\mathcal{S}_{ac}(\mathcal{D})$.

Remark 3.5. The assumption (3.16) in Theorem 3.3 cannot be replaced neither by the assumption

$$(1-\varepsilon)\ell_0 \in \mathcal{S}_{ac}(\mathcal{D}), \qquad -\ell_1 \in \mathcal{S}_{ac}(\mathcal{D})$$

nor by the assumption

$$\ell_0 \in \mathcal{S}_{ac}(\mathcal{D}), \qquad -(1-\varepsilon)\ell_1 \in \mathcal{S}_{ac}(\mathcal{D}),$$

no matter haw small $\varepsilon > 0$ would be (see Examples 7.3 and 7.4).

Remark 3.6. There is proved in [5] that if a nonincreasing operator belongs to the set $\mathcal{S}_{ac}(\mathcal{D})$ then it is necessarily an (a, c)–Volterra operator. Therefore, in Theorems 3.2 and 3.3, the assumptions on the operators ℓ and ℓ_1, respectively, to be (a, c)–Volterra ones are necessary.

4. Proofs of The Main Results

To prove the statements established in Section 3 we will need the following lemmata.

Lemma 4.1. *Let $v \in C^*(\mathcal{D}; \mathbb{R})$ and $a \le t_1 \le t_2 \le b$, $c \le x_1 \le x_2 \le d$. Then*

$$\begin{aligned} v(t_2, x_2) - v(t_1, x_1) &= \int_{t_1}^{t_2} \frac{\partial v(s, c)}{\partial s}\, ds + \int_{x_1}^{x_2} \frac{\partial v(a, \eta)}{\partial \eta}\, d\eta + \\ &\quad + \int_a^{t_1}\int_{x_1}^{x_2} \frac{\partial^2 v(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds + \int_{t_1}^{t_2}\int_c^{x_2} \frac{\partial^2 v(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds = \\ &= \int_{t_1}^{t_2} \frac{\partial v(s, c)}{\partial s}\, ds + \int_{x_1}^{x_2} \frac{\partial v(a, \eta)}{\partial \eta}\, d\eta + \\ &\quad + \int_a^{t_2}\int_{x_1}^{x_2} \frac{\partial^2 v(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds + \int_{t_1}^{t_2}\int_c^{x_1} \frac{\partial^2 v(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds. \end{aligned} \tag{4.1}$$

Proof. Since $v \in C^*(\mathcal{D}; \mathbb{R})$, the function v admits the representation

$$v(t, x) = v(a, c) + \int_a^t \frac{\partial v(s, c)}{\partial s}\, ds + \int_c^x \frac{\partial v(a, \eta)}{\partial \eta}\, d\eta + \int_a^t\int_c^x \frac{\partial^2 v(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds$$

for $(t, x) \in \mathcal{D}$. Therefore,

$$v(t_2, x_2) - v(t_1, x_2) = \int_{t_1}^{t_2} \frac{\partial v(s, c)}{\partial s}\, ds + \int_{t_1}^{t_2}\int_c^{x_2} \frac{\partial^2 v(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds.$$

On the other hand,

$$v(t_1, x_2) - v(t_1, x_1) = \int_{x_1}^{x_2} \frac{\partial v(a,\eta)}{\partial \eta}\, d\eta + \int_a^{t_1}\int_{x_1}^{x_2} \frac{\partial^2 v(s,\eta)}{\partial s\, \partial \eta}\, d\eta ds.$$

Consequently, the first equality in (4.1) holds. The second equality in (4.1) can be proved analogously. □

Lemma 4.2. *Let* $(t_0, x_0) \in \mathcal{D}$*,* $-\ell \in P(\mathcal{D})$*,* ℓ *be an* (a,c)*–Volterra operator, and let* u *be a solution of the problem* (1.4)–(1.7) *satisfying*

$$u(t_0, x_0) < 0. \tag{4.2}$$

Then

$$\max\big\{u(t,x) : (t,x) \in [a,t_0] \times [c,x_0]\big\} > 0. \tag{4.3}$$

Proof. Obviously, $t_0 \neq a$ and $x_0 \neq c$. Assume that, on the contrary, (4.3) is not true. Then

$$u(t,x) \le 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0,$$

where $\mathcal{D}_0 = [a,t_0] \times [c,x_0]$. Since ℓ is an (a,c)–Volterra operator and $-\ell \in P(\mathcal{D})$, it follows from (1.4) that

$$u_{tx}(t,x) \ge \ell(u)(t,x) \ge 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0.$$

Consequently, according to (1.5)–(1.7) and Lemma 4.1, we get

$$u(t_0,x_0) \ge u(a,c) \ge 0,$$

which contradicts (4.2). □

Lemma 4.3. *Let* $\ell \in P(\mathcal{D})$*. Then* $\ell \in \mathcal{S}_{ac}(\mathcal{D})$ *if and only if the problem*

$$\frac{\partial^2 v(t,x)}{\partial t\, \partial x} \le \ell(v)(t,x), \tag{4.4}$$

$$v(t,c) = 0 \quad \text{for} \quad t \in [a,b], \qquad v(a,x) = 0 \quad \text{for} \quad x \in [c,d] \tag{4.5}$$

has no nontrivial nonnegative solution[2]*.*

Proof. If $\ell \in \mathcal{S}_{ac}(\mathcal{D})$, then it is clear that the problem (4.4), (4.5) has no nontrivial nonnegative solution.

Now suppose that the problem (4.4), (4.5) has no nontrivial nonnegative solution and let u be a solution of the problem (1.4)–(1.7). We will show that the function u is nonnegative. Put

$$\alpha(t,x) = \int_a^t \int_c^x \ell([u]_-)(s,\eta) d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

[2]Recall here that by a solution of the problem (4.4), (4.5) is understood a function $v \in C^*\big(\mathcal{D};\mathbb{R}\big)$ satisfying the inequality (4.4) almost everywhere on the set $\mathcal{D}$ and verifying also the conditions (4.5).

It is clear that $\alpha \in C^*\big(\mathcal{D};\mathbb{R}\big)$,

$$\alpha_{tx}(t,x) = \ell([u]_-)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}, \tag{4.6}$$

$$\alpha(t,c) = 0 \quad \text{for} \quad t \in [a,b], \qquad \alpha(a,x) = 0 \quad \text{for} \quad x \in [c,d], \tag{4.7}$$

and

$$\alpha(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

By virtue of (1.4), (1.6), (1.7), (4.6), (4.7), and the assumption $\ell \in P(\mathcal{D})$, we get

$$w_{tx}(t,x) \geq \ell(u + [u]_-)(t,x) = \ell([u]_+)(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D},$$
$$w_t(t,c) \geq 0 \quad \text{for} \quad t \in [a,b], \qquad w_x(a,x) \geq 0 \quad \text{for} \quad x \in [c,d],$$

where

$$w(t,x) = u(t,x) + \alpha(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Consequently, in view of (1.5), Lemma 4.1 yields

$$w(t,x) \geq w(a,c) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}$$

and thus

$$[u(t,x)]_- \leq \alpha(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D} \tag{4.8}$$

because the function α is nonnegative. Now, from (4.6) we get

$$\alpha_{tx}(t,x) \leq \ell(\alpha)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

We have proved that α is a nonnegative solution of the problem (4.4), (4.5). Therefore, $\alpha \equiv 0$ and the condition (4.8) yields $u(t,x) \geq 0$ for $(t,x) \in \mathcal{D}$. Hence $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. □

Lemma 4.4. *Let $f \in L\big(\mathcal{D};\mathbb{R}_+\big)$ be such that*

$$\int_a^b \int_c^d f(s,\eta)\,d\eta\,ds \leq 1. \tag{4.9}$$

Then there exists $(b_0,d_0) \in\,]a,b] \times\,]c,d]$ such that

$$\int_a^t \int_c^x f(s,\eta)\,d\eta\,ds < 1 \quad \textit{for} \quad (t,x) \in \mathcal{D}_0,\ (t,x) \neq (b_0,d_0), \tag{4.10}$$

and

$$f(t,x) = 0 \quad \textit{for} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_0\,, \tag{4.11}$$

where $\mathcal{D}_0 = [a,b_0] \times [c,d_0]$.

Proof. If the inequality (4.9) is strict, then the assertion of lemma holds for $b_0 = b$ and $d_0 = d$. Therefore suppose that

$$\int_a^b \int_c^d f(s,\eta)d\eta ds = 1. \tag{4.12}$$

Put

$$d_0 = \min\left\{ x \in [c,d] : \int_a^b \int_c^x f(s,\eta)d\eta ds = 1 \right\}.$$

It is clear that $d_0 > c$ and

$$\int_a^b \int_c^{d_0} f(s,\eta)d\eta ds = 1, \qquad \int_a^b \int_c^x f(s,\eta)d\eta ds < 1 \quad \text{for} \quad x \in [c, d_0[\,.$$

Further, we put

$$b_0 = \min\left\{ t \in [a,b] : \int_a^t \int_c^{d_0} f(s,\eta)d\eta ds = 1 \right\}.$$

Obviously, $b_0 > a$ and

$$\int_a^{b_0} \int_c^{d_0} f(s,\eta)d\eta ds = 1, \qquad \int_a^t \int_c^{d_0} f(s,\eta)d\eta ds < 1 \quad \text{for} \quad t \in [a, b_0[\,.$$

Let $\mathcal{D}_0 = [a,b_0] \times [c,d_0]$. It is easy to verify that the condition (4.10) holds and

$$\iint_{\mathcal{D} \setminus \mathcal{D}_0} f(t,x)dtdx = 0.$$

Hence (4.11) is also satisfied because the function f is supposed to be nonnegative. □

Now we are in position to prove the main results given in Section 3.

Proof of Theorem 3.1. First suppose that there exists $\gamma \in C^*\big(\mathcal{D};\,]0,+\infty[\big)$ satisfying the conditions (3.1) and (3.2) (resp. (3.1) and (3.3)). Let u be a solution of the problem (1.4)–(1.7). We will show that the function u is nonnegative. Put

$$A = \big\{\lambda \in \mathbb{R}_+ : \lambda\gamma(t,x) + u(t,x) \geq 0 \text{ for } (t,x) \in \mathcal{D}\big\}. \tag{4.13}$$

Since γ is a positive function, we have $A \neq \varnothing$. Let

$$\lambda_0 = \inf A. \tag{4.14}$$

Now we put

$$w(t,x)=\lambda_0\gamma(t,x)+u(t,x) \quad \text{for} \quad (t,x)\in\mathcal{D}. \tag{4.15}$$

It is clear that $\lambda_0\geq 0$, $w\in C^*\big(\mathcal{D};\mathbb{R}\big)$, and

$$w(t,x)\geq 0 \quad \text{for} \quad (t,x)\in\mathcal{D}. \tag{4.16}$$

Therefore, by virtue of (1.4), (3.1), and the assumption $\ell\in P(\mathcal{D})$, we get

$$w_{tx}(t,x)\geq \ell(w)(t,x)\geq 0 \quad \text{for} \quad (t,x)\in\mathcal{D}. \tag{4.17}$$

Assume that

$$\lambda_0>0. \tag{4.18}$$

Then, it follows from (1.5)–(1.7), (3.2) (resp. (3.3)), and (4.18) that

$$w(a,x)>0 \quad \text{for} \quad x\in[c,d], \qquad w_t(t,c)\geq 0 \quad \text{for} \quad t\in[a,b]$$
$$\Big(\text{resp.} \quad w(t,c)>0 \quad \text{for} \quad t\in[a,b], \qquad w_x(a,x)\geq 0 \quad \text{for} \quad x\in[c,d]\Big).$$

Hence, in view of (4.17), Lemma 4.1 yields

$$w(t,x)\geq w(a,x)>0 \quad \text{for} \quad (t,x)\in\mathcal{D}$$
$$\Big(\text{resp.} \quad w(t,x)\geq w(t,c)>0 \quad \text{for} \quad (t,x)\in\mathcal{D}\Big).$$

Consequently, there exists $\varepsilon\in\,]0,\lambda_0]$ such that

$$w(t,x)\geq\varepsilon\gamma(t,x) \quad \text{for} \quad (t,x)\in\mathcal{D},$$

i.e.,

$$(\lambda_0-\varepsilon)\gamma(t,x)+u(t,x)\geq 0 \quad \text{for} \quad (t,x)\in\mathcal{D}.$$

Hence, by virtue of (4.13), we get $\lambda_0-\varepsilon\in A$, which contradicts (4.14).

The contradiction obtained proves that $\lambda_0=0$. Consequently, (4.15) and (4.16) yield

$$u(t,x)=w(t,x)\geq 0 \quad \text{for} \quad (t,x)\in\mathcal{D}$$

and thus $\ell\in\mathcal{S}_{ac}(\mathcal{D})$.

Now suppose that $\ell\in\mathcal{S}_{ac}(\mathcal{D})$. Then, according to Remark 1.1, the problem

$$\frac{\partial^2\gamma(t,x)}{\partial t\,\partial x}=\ell(\gamma)(t,x), \tag{4.19}$$

$$\gamma(t,c)=1 \quad \text{for} \quad t\in[a,b], \qquad \gamma(a,x)=1 \quad \text{for} \quad x\in[c,d] \tag{4.20}$$

has a unique solution γ and

$$\gamma(t,x)\geq 0 \quad \text{for} \quad (t,x)\in\mathcal{D}.$$

By virtue of the assumption $\ell\in P(\mathcal{D})$, the equation (4.19) implies

$$\gamma_{tx}(t,x)\geq 0 \quad \text{for} \quad (t,x)\in\mathcal{D}.$$

Therefore, in view of (4.20) and Lemma 4.1, we get

$$\gamma(t,x) \geq \gamma(a,c) = 1 \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Consequently, $\gamma \in C^*\big(\mathcal{D};\,]0,+\infty[\big)$ and it satisfies the inequalities (3.1), (3.2), and (3.3). □

Proof of Corollary 3.1. a) It is not difficult to verify that the function

$$\gamma(t,x) = \sum_{j=1}^{m} \rho_j(t,x) - \alpha \sum_{j=1}^{k} \rho_j(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}$$

belongs to the set $C^*\big(\mathcal{D};\,]0,+\infty[\big)$ and satisfies (3.1), (3.2), and (3.3). Therefore, Theorem 3.1 guarantees $\ell \in \mathcal{S}_{ac}(\mathcal{D})$.

b) According to (3.7), there exists $\varepsilon > 0$ such that

$$\varepsilon \exp\left(\int_a^b \int_c^d \ell(1)(s,\eta)d\eta ds\right) + \\ + \int_a^b \int_c^d \overline{\ell}(1)(s,\eta) \exp\left(\int_s^b \int_\eta^d \ell(1)(\xi_1,\xi_2)d\xi_2 d\xi_1\right) d\eta ds \leq 1. \quad (4.21)$$

Put

$$\gamma(t,x) = \varepsilon \exp\left(\int_a^t \int_c^x \ell(1)(s,\eta)d\eta ds\right) + \\ + \int_a^t \int_c^x \overline{\ell}(1)(s,\eta) \exp\left(\int_s^t \int_\eta^x \ell(1)(\xi_1,\xi_2)d\xi_2 d\xi_1\right) d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

It is not difficult to verify that $\gamma \in C^*\big(\mathcal{D}; \mathbb{R}_+\big)$ and, in view of the assumption $\ell \in P(\mathcal{D})$, we get

$$\gamma_{tx}(t,x) \geq \ell(1)(t,x)\gamma(t,x) + \overline{\ell}(1)(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}, \quad (4.22)$$

$$\gamma(t,c) = \varepsilon \quad \text{for} \quad t \in [a,b], \qquad \gamma(a,x) = \varepsilon \quad \text{for} \quad x \in [c,d]. \quad (4.23)$$

Hence, by virtue of (4.21)–(4.23), Lemma 4.1 yields

$$0 < \gamma(a,c) \leq \gamma(t,x) \leq \gamma(b,d) \leq 1 \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Now from (4.22) we get

$$\gamma_{tx}(t,x) \geq \ell(1)(t,x)\gamma(t,x) + \overline{\ell}(\gamma)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},$$

and thus, by virtue of Theorem 3.1, we find

$$\widetilde{\ell} \in \mathcal{S}_{ac}(\mathcal{D}), \quad (4.24)$$

where

$$\widetilde{\ell}(w)(t,x) \stackrel{\rm def}{=} \ell(1)(t,x)w(t,x) + \overline{\ell}(w)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{4.25}$$

According to Lemma 4.3, to prove corollary it is sufficient to show that the problem (4.4), (4.5) has no nontrivial nonnegative solution. Let v be a nonnegative solution of the problem (4.4), (4.5). We will show that $v \equiv 0$. Put

$$u(t,x) = \theta(v)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}, \tag{4.26}$$

where θ is defined by (3.6). Obviously,

$$\begin{gathered} u_{tx}(t,x) = \ell(v)(t,x) \geq v_{tx}(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}, \\ u(t,c) = 0 \quad \text{for} \quad t \in [a,b], \qquad u(a,x) = 0 \quad \text{for} \quad x \in [c,d]. \end{gathered} \tag{4.27}$$

Consequently, in view of (4.5), Lemma 4.1 yields

$$u(t,x) \geq v(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{4.28}$$

On the other hand, by virtue of (3.8), (4.25)–(4.28), and the assumptions $\ell, \overline{\ell} \in P(\mathcal{D})$, we get

$$\begin{aligned} u_{tx}(t,x) = \ell(v)(t,x) \leq \ell(1)(t,x)u(t,x) + \ell(u)(t,x) - \ell(1)(t,x)u(t,x) = \\ = \ell(1)(t,x)u(t,x) + \ell\big(\theta(v)\big)(t,x) - \ell(1)(t,x)\theta(v)(t,x) \leq \\ \leq \ell(1)(t,x)u(t,x) + \overline{\ell}(v)(t,x) \leq \ell(1)(t,x)u(t,x) + \overline{\ell}(u)(t,x) = \\ = \widetilde{\ell}(u)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}. \end{aligned}$$

Now, by (4.24), (4.27), (4.28), and Lemma 4.3, we obtain $u \equiv 0$. Consequently, (4.28) implies $v \equiv 0$, i.e., the problem (4.4), (4.5) has no nontrivial nonnegative solution. □

Proof of Proposition 3.1. Suppose that (3.9) holds and the homogeneous problem (1.1_0)–(1.3_0) has only the trivial solution. We will show that $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. According to Theorem 1.1, the problem (4.19), (4.20) has a unique solution γ. Put

$$\gamma_0 = \min\{\gamma(t,x) : (t,x) \in \mathcal{D}\} \tag{4.29}$$

and choose $(t_0, x_0) \in \mathcal{D}$ such that $\gamma(t_0,x_0) = \gamma_0$.

Assume that

$$\gamma_0 \leq 0. \tag{4.30}$$

Then, in view of (4.20), Lemma 4.1 yields

$$\gamma(t_0,x_0) = 1 + \int_a^{t_0}\int_c^{x_0} \ell(\gamma)(s,\eta)d\eta ds.$$

Therefore, on account of (3.9), (4.29), (4.30), and the assumption $\ell \in P(\mathcal{D})$, we get

$$\gamma_0 \geq 1 + \gamma_0 \int_a^b\int_c^d \ell(1)(s,\eta)d\eta ds = 1 + \gamma_0,$$

a contradiction.

The contradiction obtained proves that $\gamma_0 > 0$. Consequently, Theorem 3.1 guarantees the inclusion $\ell \in \mathcal{S}_{ac}(\mathcal{D})$.

The converse implication is trivial. $\square$

Proof of Proposition 3.2. It is not difficult to verify that the assumptions of Corollary 3.1 b) are satisfied with $\overline{\ell} \equiv 0$ because the operator ℓ is supposed to be an (a,c)–Volterra one. $\square$

Proof of Theorem 3.2. Let u be a solution of the problem (1.4)–(1.7). We will show that the function u is nonnegative. Assume that, on the contrary,

$$\min\{u(t,x) : (t,x) \in \mathcal{D}\} < 0. \tag{4.31}$$

Then there exists $(t_0, x_0) \in]a,b[\times]c,d[$ such that

$$u(t_0, x_0) < 0. \tag{4.32}$$

Put $\mathcal{D}_0 = [a, t_0] \times [c, x_0]$ and

$$A = \big\{\lambda \in \mathbb{R}_+ : \lambda\gamma(t,x) - u(t,x) \geq 0 \text{ for } (t,x) \in \mathcal{D}_0\big\}. \tag{4.33}$$

Since the function γ is positive on $\mathcal{D}_0$, we have $A \neq \varnothing$. Let

$$\lambda_0 = \inf A. \tag{4.34}$$

Now we put

$$w(t,x) = \lambda_0\gamma(t,x) - u(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{4.35}$$

It is clear that $w \in C^*\big(\mathcal{D}_0; \mathbb{R}\big)$ and

$$w(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0. \tag{4.36}$$

Moreover, according to (4.32)–(4.34) and Lemma 4.2, we get

$$\lambda_0 > 0. \tag{4.37}$$

From (1.4), (3.10), (4.35), and (4.37) we obtain

$$w_{tx}(t,x) \leq \ell(w)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Since ℓ is an (a,c)–Volterra operator, $-\ell \in P(\mathcal{D})$, and (4.36) holds, the last inequality implies

$$w_{tx}(t,x) \leq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0. \tag{4.38}$$

Further, from (1.6), (1.7), (3.12), (3.13), (4.35), and (4.37) we get

$$w_t(t,c) \leq 0 \quad \text{for} \quad t \in [a, t_0], \qquad w_x(a,x) \leq 0 \quad \text{for} \quad x \in [c, x_0]. \tag{4.39}$$

Hence, by virtue of (4.32), Lemma 4.1 yields

$$w(t,x) \geq w(t_0, x_0) > 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0.$$

Consequently, there exists $\varepsilon \in]0, \lambda_0]$ such that

$$w(t,x) \geq \varepsilon\gamma(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}_0,$$

i.e.,

$$(\lambda_0 - \varepsilon)\gamma(t,x) - u(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0.$$

Hence, in view of (4.33), we get $\lambda_0 - \varepsilon \in A$, which contradicts (4.34). □

Proof of Corollary 3.2. According to Lemma 4.4, there exists a point $(b_0, d_0) \in]a,b] \times]c,d]$ such that

$$\int_a^t \int_c^x |\ell(1)(s,\eta)| d\eta ds < 1 \quad \text{for} \quad (t,x) \in \mathcal{D}_0, \ (t,x) \neq (b_0, d_0),$$

and

$$\ell(1)(t,x) = 0 \quad \text{for} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_0\,, \tag{4.40}$$

where $\mathcal{D}_0 = [a, b_0] \times [c, d_0]$. Put

$$\gamma(t,x) = 1 - \int_a^t \int_c^x |\ell(1)(s,\eta)| d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}_0.$$

Since ℓ is a nonincreasing (a,c)–Volterra operator, by Theorem 3.2 we get

$$\ell_0 \in \mathcal{S}_{ac}(\mathcal{D}_0), \tag{4.41}$$

where ℓ_0 is the restriction of ℓ to the space $C(\mathcal{D}_0; \mathbb{R})$.

Now let u be a solution of the problem (1.4)–(1.7). We will show that the function u is nonnegative. In view of (4.41), we find

$$u(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D}_0\,. \tag{4.42}$$

On the other hand, the assumption $-\ell \in P(\mathcal{D})$ guarantees that the relations

$$\ell(1)(t,x)\max\{u(s,\eta) : (s,\eta) \in \mathcal{D}\} \leq \ell(u)(t,x) \leq \\ \leq \ell(1)(t,x)\min\{u(s,\eta) : (s,\eta) \in \mathcal{D}\}$$

hold for $(t,x) \in \mathcal{D}$ and thus, by virtue of (4.40), we get

$$\ell(u)(t,x) = 0 \quad \text{for} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_0\,.$$

Consequently, (1.4) implies

$$u_{tx}(t,x) \geq 0 \quad \text{for} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_0\,. \tag{4.43}$$

Let $(t_0, x_0) \in \mathcal{D} \setminus \mathcal{D}_0$ be an arbitrary point. Put

$$t_1 = \min\{t_0, b_0\}, \qquad x_1 = \min\{x_0, d_0\},$$

and

$$D^* = [a, t_0] \times [c, x_0] \setminus [a, t_1] \times [c, x_1].$$

Clearly, $(t_1, x_1) \in \mathcal{D}_0$ and $\mathcal{D}^* \subseteq \mathcal{D} \setminus \mathcal{D}_0$. Then, in view of (1.6), (1.7), (4.42), (4.43), and Lemma 4.1, we get

$$u(t_0, x_0) = u(t_1, x_1) + \int_{t_1}^{t_0} \frac{\partial u(s, c)}{\partial s}\, ds + \int_{x_1}^{x_0} \frac{u(a, \eta)}{d\eta}\, d\eta + \iint_{\mathcal{D}^*} \frac{\partial^2 u(s, \eta)}{\partial s\, \partial \eta}\, dsd\eta \geq 0.$$

Therefore, we have proved that $u(t, x) \geq 0$ for $(t, x) \in \mathcal{D} \setminus \mathcal{D}_0$, which together with (4.42) ensures that the function u is nonnegative on the set $\mathcal{D}$. Consequently, $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. □

Proof of Theorem 3.3. Let u be a solution of the problem (1.4)–(1.7). We will show that the function u is nonnegative. According to the inclusion $-\ell_1 \in \mathcal{S}_{ac}(\mathcal{D})$ and Remark 1.1, the problem

$$\frac{\partial^2 w(t, x)}{\partial t\, \partial x} = -\ell_1(w)(t, x) - \ell_0([u]_-)(t, x), \tag{4.44}$$

$$w(t, c) = 0 \quad \text{for} \quad t \in [a, b], \qquad w(a, x) = 0 \quad \text{for} \quad x \in [c, d] \tag{4.45}$$

has a unique solution w and

$$w(t, x) \leq 0 \quad \text{for} \quad (t, x) \in \mathcal{D}. \tag{4.46}$$

In view of (1.4)–(1.7), (4.44), (4.45), and the assumption $\ell_0 \in P(\mathcal{D})$ we get

$$\frac{\partial^2}{\partial t\, \partial x} \Big(u(t, x) - w(t, x) \Big) \geq -\ell_1(u - w)(t, x) + \ell_0([u]_+)(t, x) \geq$$
$$\geq -\ell_1(u - w)(t, x) \quad \text{for} \quad (t, x) \in \mathcal{D},$$
$$\frac{\partial}{\partial t} \Big(u(t, c) - w(t, c) \Big) \geq 0 \quad \text{for} \quad t \in [a, b],$$
$$\frac{\partial}{\partial x} \Big(u(a, x) - w(a, x) \Big) \geq 0 \quad \text{for} \quad x \in [c, d],$$

and

$$u(a, c) - w(a, c) \geq 0.$$

Consequently, the inclusion $-\ell_1 \in \mathcal{S}_{ac}(\mathcal{D})$ yields

$$u(t, x) \geq w(t, x) \quad \text{for} \quad (t, x) \in \mathcal{D}. \tag{4.47}$$

Now, (4.46) and (4.47) imply

$$-[u(t, x)]_- \geq w(t, x) \quad \text{for} \quad (t, x) \in \mathcal{D}. \tag{4.48}$$

On the other hand, by virtue of (4.44), (4.46), (4.48), and the assumptions $\ell_0, \ell_1 \in P(\mathcal{D})$, we obtain

$$w_{tx}(t, x) \geq \ell_0(w)(t, x) - \ell_1(w)(t, x) \geq \ell_0(w)(t, x) \quad \text{for} \quad (t, x) \in \mathcal{D}.$$

Hence, the inclusion $\ell_0 \in \mathcal{S}_{ac}(\mathcal{D})$, on account of (4.45), implies

$$w(t, x) \geq 0 \quad \text{for} \quad (t, x) \in \mathcal{D},$$

which, together with (4.47), guarantees $u(t, x) \geq 0$ for $(t, x) \in \mathcal{D}$. □

5. Operators With Deviating Arguments

In this section, we will establish some consequences of the main results for the operators with deviating arguments, i.e., for the case when the operator ℓ is given by one of the following formulae:

$$\ell(v)(t,x) \stackrel{\text{def}}{=} p(t,x)v\big(\tau_0(t,x),\mu_0(t,x)\big) \quad \text{for} \quad (t,x)\in\mathcal{D}, \tag{5.1}$$

$$\ell(v)(t,x) \stackrel{\text{def}}{=} -\,g(t,x)v\big(\tau_1(t,x),\mu_1(t,x)\big) \quad \text{for} \quad (t,x)\in\mathcal{D}, \tag{5.2}$$

$$\begin{aligned}\ell(v)(t,x) \stackrel{\text{def}}{=} & p(t,x)v\big(\tau_0(t,x),\mu_0(t,x)\big)- \\ & -g(t,x)v\big(\tau_1(t,x),\mu_1(t,x)\big) \quad \text{for} \quad (t,x)\in\mathcal{D}.\end{aligned} \tag{5.3}$$

Here we suppose that $p, g \in L(\mathcal{D},\mathbb{R}_+)$ and $\tau_i : \mathcal{D} \to [a,b]$, $\mu_i : \mathcal{D} \to [c,d]$ are measurable functions $(i = 0,1)$.

Throughout this section, the following notation will be used:

$$\tau_0^* = \text{ess sup}\,\big\{\tau_0(t,x) : (t,x)\in\mathcal{D}\big\}, \quad \mu_0^* = \text{ess sup}\,\big\{\mu_0(t,x) : (t,x)\in\mathcal{D}\big\}.$$

At first we formulate all the statements, the proofs are given later.

Theorem 5.1. *Let at least one of the following items be fulfilled:*

a) there exists $\alpha \in]0,1[$ *such that*

$$\begin{aligned}\int_a^t\int_c^x p(s,\eta)\left(\int_a^{\tau_0(s,\eta)}\int_c^{\mu_0(s,\eta)} p(\xi_1,\xi_2)d\xi_2 d\xi_1\right)d\eta ds \le \\ \le \alpha\int_a^t\int_c^x p(s,\eta)d\eta ds \quad \textit{for} \quad (t,x)\in\mathcal{D};\end{aligned} \tag{5.4}$$

b)

$$\begin{aligned}\int_a^b\int_c^d p(s,\eta)\Big(f_1\big(s,\mu_0(s,\eta)\big)+f_2\big(s,\eta\big)\Big)\times \\ \times\exp\left(\int_s^b\int_\eta^d p(\xi_1,\xi_2)d\xi_2 d\xi_1\right)d\eta ds < 1,\end{aligned} \tag{5.5}$$

where, for $(t,x)\in\mathcal{D}$,

$$f_1(t,x) \stackrel{\text{def}}{=} \frac{1}{2}\Big(1+\text{sgn}(\tau_0(t,x)-t)\Big)\int_t^{\tau_0(t,x)}\int_c^x p(s,\eta)d\eta ds, \tag{5.6}$$

$$f_2(t,x) \stackrel{\text{def}}{=} \frac{1}{2}\Big(1+\text{sgn}(\mu_0(t,x)-x)\Big)\int_a^t\int_x^{\mu_0(t,x)} p(s,\eta)d\eta ds; \tag{5.7}$$

c)

$$\int_a^b \int_c^d p(s,\eta)\Big(f_1(s,\eta) + f_2\big(\tau_0(s,\eta),\eta\big)\Big)\times$$
$$\times \exp\left(\int_s^b \int_\eta^d p(\xi_1,\xi_2)d\xi_2 d\xi_1\right) d\eta ds < 1, \quad (5.8)$$

where the functions f_1 *and* f_2 *are defined by* (5.6) *and* (5.7)*, respectively.*

Then the operator ℓ *given by* (5.1) *belongs to the set* $\mathcal{S}_{ac}(\mathcal{D})$*.*

Remark 5.1. The assumption $\alpha \in]0,1[$ in Theorem 5.1 a) cannot be replaced by the assumption $\alpha \in]0,1]$ (see Example 7.1).

Theorem 5.2. *Let one of the following item be fulfilled:*

a)

$$\int_a^{\tau_0^*} \int_c^{\mu_0^*} p(s,\eta)d\eta ds < 1; \quad (5.9)$$

b)

$$\int_a^{\tau_0^*} \int_c^{\mu_0^*} p(s,\eta)d\eta ds > 1 \quad (5.10)$$

and

$$\operatorname{ess\,sup}\left\{ \int_t^{\tau_0(t,x)} \int_c^{x} p(s,\eta)d\eta ds + \int_a^{\tau_0(t,x)} \int_x^{\mu_0(t,x)} p(s,\eta)d\eta ds : (t,x)\in\mathcal{D}\right\} < \omega^*, \quad (5.11)$$

where

$$\omega^* = \sup\left\{ \frac{1}{y}\ln\left(y + \frac{y}{\exp\left(y\int_a^{\tau_0^*}\int_c^{\mu_0^*} p(s,\eta)d\eta ds\right) - 1}\right) : y>0\right\}. \quad (5.12)$$

Then the operator ℓ *given by* (5.1) *belongs to the set* $\mathcal{S}_{ac}(\mathcal{D})$*.*

Theorem 5.3. *Let*

$$\int_a^{\tau_0^*} \int_c^{\mu_0^*} p(s,\eta)d\eta ds = 1. \quad (5.13)$$

Then the operator ℓ *given by* (5.1) *belongs to the set* $\mathcal{S}_{ac}(\mathcal{D})$ *if and only if*

$$\int_a^{\tau_0^*}\int_c^{\mu_0^*} p(s,\eta)\left(\int_a^{\tau_0(s,\eta)}\int_c^{\mu_0(s,\eta)} p(\xi_1,\xi_2)d\xi_2 d\xi_1\right)d\eta ds \neq 1. \tag{5.14}$$

Theorems 5.1–5.3 contain some integral conditions for the operator ℓ defined by (5.1) to belong to the set $\mathcal{S}_{ac}(\mathcal{D})$. The following theorem gives a different kind of conditions.

Theorem 5.4. *Let the function* p *be essentially bounded and*

$$\text{ess sup}\Big\{p(t,x)\big(\tau_0(t,x)-a\big)\big(\mu_0(t,x)-c\big) : (t,x)\in\mathcal{D}\Big\} < 1. \tag{5.15}$$

Then the operator ℓ *given by* (5.1) *belongs to the set* $\mathcal{S}_{ac}(\mathcal{D})$.

Remark 5.2. The strict inequality (5.15) in the previous theorem cannot be replaced by the nonstrict one (see Example 7.5).

Theorem 5.5. *Let*

$$g(t,x)\big(\tau_1(t,x)-t\big) \le 0 \quad \textit{for} \quad (t,x)\in\mathcal{D}, \tag{5.16}$$

$$g(t,x)\big(\mu_1(t,x)-x\big) \le 0 \quad \textit{for} \quad (t,x)\in\mathcal{D}, \tag{5.17}$$

and

$$\int_a^b\int_c^d g(s,\eta)d\eta ds \le 1. \tag{5.18}$$

Then the operator ℓ *given by* (5.2) *belongs to the set* $\mathcal{S}_{ac}(\mathcal{D})$.

Remark 5.3. The constant 1 on the right–hand side of the inequality (5.18) cannot be replaced by the constant $1+\varepsilon$, no matter how small $\varepsilon > 0$ would be (see Example 7.2).

Theorem 5.6. *Let the conditions* (5.16) *and* (5.17) *be satisfied and let*

$$\text{ess sup}\Big\{g(t,x)\gamma\big(\tau_1(t,x),\mu_1(t,x)\big) : (t,x)\in\mathcal{D}\Big\} \le 1, \tag{5.19}$$

where

$$\gamma(t,x) = (b-a)(d-c)-(t-a)(x-c) \quad \textit{for} \quad (t,x)\in\mathcal{D}. \tag{5.20}$$

Then the operator ℓ *given by* (5.2) *belongs to the set* $\mathcal{S}_{ac}(\mathcal{D})$.

Remark 5.4. The inequality (5.19) in the previous theorem cannot be replaced by the inequality

$$\text{ess sup}\Big\{g(t,x)\gamma\big(\tau_1(t,x),\mu_1(t,x)\big) : (t,x)\in\mathcal{D}\Big\} \le 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be (see Example 7.6).

Theorem 5.7. *Let the functions p, τ_0, μ_0 satisfy one of the items a)–c) in Theorem 5.1 or the assumptions of Theorems 5.2 or 5.4 or the conditions* (5.13) *and* (5.14)*, while the functions g, τ_1, μ_1 satisfy the conditions* (5.16)*,* (5.17)*, and either the inequality* (5.18) *or* (5.19) *is fulfilled. Then the operator ℓ given by* (5.3) *belongs to the set $\mathcal{S}_{ac}(\mathcal{D})$.*

Proof of Theorem 5.1. Let the operator ℓ be defined by (5.1). Obviously, $\ell \in P(\mathcal{D})$.

a) According to (5.4), we have

$$\rho_3(t,x) \le \alpha\rho_2(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},$$

where ρ_2 and ρ_3 are given by (3.5). Therefore, the assumptions of Corollary 3.1 a) are satisfied.

b) For $(t,x) \in \mathcal{D}$, we put

$$\begin{aligned}
&\overline{\ell}(v)(t,x) \stackrel{\text{def}}{=} \\
&= p(t,x)\Bigg[\frac{1}{2}\big(1+\operatorname{sgn}(\tau_0(t,x)-t)\big)\int\limits_t^{\tau_0(t,x)}\int\limits_c^{\mu_0(t,x)} p(s,\eta)v\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds + \\
&\qquad + \frac{1}{2}\big(1+\operatorname{sgn}(\mu_0(t,x)-x)\big)\int\limits_a^{t}\int\limits_x^{\mu_0(t,x)} p(s,\eta)v\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds\Bigg].
\end{aligned}$$

It is clear that $\overline{\ell} \in P(\mathcal{D})$ and

$$\begin{aligned}
&\ell\big(\theta(v)\big)(t,x) - \ell(1)(t,x)\theta(v)(t,x) = \\
&\qquad = p(t,x)\int\limits_a^{\tau_0(t,x)}\int\limits_c^{\mu_0(t,x)} p(s,\eta)v\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds - \\
&\qquad - p(t,x)\int\limits_a^{t}\int\limits_c^{x} p(s,\eta)v\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds = \\
&\qquad = p(t,x)\Bigg[\int\limits_t^{\tau_0(t,x)}\int\limits_c^{\mu_0(t,x)} p(s,\eta)v\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds + \\
&\qquad + \int\limits_a^{t}\int\limits_x^{\mu_0(t,x)} p(s,\eta)v\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds\Bigg] \le \\
&\qquad \le \overline{\ell}(v)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},\ v \in C\big(\mathcal{D};\mathbb{R}_+\big),
\end{aligned}$$

where θ is given by (3.6). On the other hand, by virtue of (5.5), the inequality (3.7) holds. Hence, the assumptions of Corollary 3.1 b) are satisfied.

c) The proof is similar to the previous case but the operator $\overline{\ell}$ should be defined by

$$\overline{\ell}(v)(t,x) \stackrel{\text{def}}{=}$$
$$= p(t,x)\Bigg[\frac{1}{2}\big(1+\operatorname{sgn}(\tau_0(t,x)-t)\big)\int\limits_t^{\tau_0(t,x)}\int\limits_c^{x} p(s,\eta)v\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds+$$
$$+\frac{1}{2}\big(1+\operatorname{sgn}(\mu_0(t,x)-x)\big)\int\limits_a^{\tau_0(t,x)}\int\limits_x^{\mu_0(t,x)} p(s,\eta)v\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds\Bigg]$$

for $(t,x)\in\mathcal{D}$. □

Proof of Theorem 5.2. Let the operator ℓ be defined by (5.1). Obviously, $\ell\in P(\mathcal{D})$.

First suppose that (5.9) holds. Let

$$\ell^*(v)(t,x) \stackrel{\text{def}}{=} p(t,x)v\big(\tau_0(t,x),\mu_0(t,x)\big) \quad \text{for} \quad (t,x)\in\mathcal{D}^*, \tag{5.21}$$

where $\mathcal{D}^*=[a,\tau_0^*]\times[c,\mu_0^*]$. In other words, ℓ^* is the restriction of ℓ to the space $C\big(\mathcal{D}^*,\mathbb{R}\big)$. According to (5.9) and Remark 3.2, it is clear that $\ell^*\in\mathcal{S}_{ac}(\mathcal{D}^*)$. However, by Lemma 4.1, it can be easily verified that $\ell\in\mathcal{S}_{ac}(\mathcal{D})$, as well.

Now suppose that (5.10) and (5.11) are satisfied, where the number ω^* is given by (5.12). Then there exist $y_0>0$ and $\varepsilon\in[0,1[$ such that

$$\int\limits_t^{\tau_0(t,x)}\int\limits_c^{x} p(s,\eta)d\eta ds+\int\limits_a^{\tau_0(t,x)}\int\limits_x^{\mu_0(t,x)} p(s,\eta)d\eta ds\le$$
$$\le\frac{1}{y_0}\ln\left(y_0+\frac{y_0\varepsilon}{\exp\left(y_0\int\limits_a^{\tau_0^*}\int\limits_c^{\mu_0^*}p(s,\eta)d\eta ds\right)-\varepsilon}\right) \quad \text{for} \quad (t,x)\in\mathcal{D}.$$

Consequently, the inequality

$$\int\limits_a^{\tau_0(t,x)}\int\limits_c^{\mu_0(t,x)} p(s,\eta)d\eta ds-\int\limits_a^{t}\int\limits_c^{x} p(s,\eta)d\eta ds\le$$
$$\le\frac{1}{y_0}\ln\left(\frac{y_0\exp\left(y_0\int\limits_a^{\tau_0(t,x)}\int\limits_c^{\mu_0(t,x)}p(s,\eta)d\eta ds\right)}{\exp\left(y_0\int\limits_a^{\tau_0(t,x)}\int\limits_c^{\mu_0(t,x)}p(s,\eta)d\eta ds\right)-\varepsilon}\right) \tag{5.22}$$

holds for $(t,x) \in \mathcal{D}$. Put

$$\gamma(t,x) = \exp\left(y_0 \int_a^t \int_c^x p(s,\eta)\,d\eta\,ds \right) - \varepsilon \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Obviously, $\gamma \in C^*\big(\mathcal{D};\,]0,+\infty[\big)$ and, in view of (5.22), γ satisfies the inequalities (3.1), (3.2), and (3.3). Therefore, by virtue of Theorem 3.1, we get $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. $\square$

To prove Theorem 5.3 we need the following lemma.

Lemma 5.1. *Let $\mathcal{D}^* = [a,\tau_0^*] \times [c,\mu_0^*]$, $p \in L\big(\mathcal{D}^*;\mathbb{R}_+\big)$ be such that* (5.13) *holds, and let $u \in C^*\big(\mathcal{D}^*;\mathbb{R}\big)$ be a function satisfying*

$$u_{tx}(t,x) = p(t,x)u\big(\tau_0(t,x),\mu_0(t,x)\big) \quad \textit{for} \quad (t,x) \in \mathcal{D}^*, \tag{5.23}$$

$$u(t,c) = 0 \quad \textit{for} \quad t \in [a,\tau_0^*], \qquad u(a,x) = 0 \quad \textit{for} \quad x \in [c,\mu_0^*]. \tag{5.24}$$

Then the function u does not change its sign.

Proof. Assume that, on the contrary, u changes its sign. Put

$$M = \max\{u(t,x) : (t,x) \in \mathcal{D}^*\}, \quad m = -\min\{u(t,x) : (t,x) \in \mathcal{D}^*\}, \tag{5.25}$$

and choose $(t_M,x_M), (t_m,x_m) \in \mathcal{D}^*$ such that

$$u(t_M,x_M) = M, \qquad u(t_m,x_m) = -m. \tag{5.26}$$

Obviously,

$$M > 0, \qquad m > 0, \tag{5.27}$$

and without loss of generality we can assume that $t_m \le t_M$. It is also clear that either

$$x_m < x_M \tag{5.28}$$

or

$$x_m \ge x_M. \tag{5.29}$$

First suppose that (5.28) holds. According to (5.23) and (5.24), Lemma 4.1 yields

$$u(t_M,x_M) - u(t_m,x_m) = \int_a^{t_m} \int_{x_m}^{x_M} p(s,\eta)u\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)\,d\eta\,ds + \\ + \int_{t_m}^{t_M} \int_c^{x_M} p(s,\eta)u\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)\,d\eta\,ds.$$

Hence, in view of (5.25)–(5.27), we get

$$M + m \le M \int_a^{t_m} \int_{x_m}^{x_M} p(s,\eta)\,d\eta\,ds + M \int_{t_m}^{t_M} \int_c^{x_M} p(s,\eta)\,d\eta\,ds \le M \int_a^{\tau_0^*} \int_c^{\mu_0^*} p(s,\eta)\,d\eta\,ds,$$

which, on account of (5.13), contradicts (5.27).

Now suppose that (5.29) is satisfied. According to (5.23) and (5.24), Lemma 4.1 implies

$$u(t_M,x_M)-u(t_m,x_M)=\int\limits_{t_m}^{t_M}\int\limits_{c}^{x_M}p(s,\eta)u\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds,$$

$$u(t_m,x_m)-u(t_m,x_M)=\int\limits_{a}^{t_m}\int\limits_{x_M}^{x_m}p(s,\eta)u\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds.$$

Hence, in view of (5.25)–(5.27), we get

$$M-u(t_m,x_M)\le M\int\limits_{t_m}^{t_M}\int\limits_{c}^{x_M}p(s,\eta)d\eta ds,$$

$$u(t_m,x_M)+m\le m\int\limits_{a}^{t_m}\int\limits_{x_M}^{x_m}p(s,\eta)d\eta ds.$$

Therefore

$$M+m\le\max\{M,m\}\left(\int\limits_{a}^{t_m}\int\limits_{x_M}^{x_m}p(s,\eta)d\eta ds+\int\limits_{t_m}^{t_M}\int\limits_{c}^{x_M}p(s,\eta)d\eta ds\right)\le$$

$$\le\max\{M,m\}\int\limits_{a}^{\tau_0^*}\int\limits_{c}^{\mu_0^*}p(s,\eta)d\eta ds,$$

which, on account of (5.13), contradicts (5.27). □

Proof of Theorem 5.3. Let $\mathcal{D}^*=[a,\tau_0^*]\times[c,\mu_0^*]$ and the operator $\ell\in\mathcal{L}(\mathcal{D})$ be defined by (5.1). Let, moreover, ℓ^* be the restriction of ℓ to the space $C\big(\mathcal{D}^*,\mathbb{R}\big)$, i.e., ℓ^* is given by (5.21). Since

$$\Big(\tau_0(t,x),\mu_0(t,x)\Big)\in\mathcal{D}^*\quad\text{for almost all}\quad(t,x)\in\mathcal{D},$$

it is easy to verify that $\ell\in\mathcal{S}_{ac}(\mathcal{D})$ if and only if $\ell^*\in\mathcal{S}_{ac}(\mathcal{D}^*)$. However, according to Proposition 3.1, $\ell^*\in\mathcal{S}_{ac}(\mathcal{D}^*)$ if and only if the homogeneous problem (5.23), (5.24) has only the trivial solution. Consequently, to prove Theorem 5.3 it is sufficient to show that the problem (5.23), (5.24) has only the trivial solution if and only if the condition (5.14) is satisfied.

Let u be a solution of the problem (5.23), (5.24). By virtue of Lemma 5.1, we can assume that

$$u(t,x)\ge0\quad\text{for}\quad(t,x)\in\mathcal{D}^*.\tag{5.30}$$

Put

$$f(t,x) \stackrel{\text{def}}{=} \int\limits_a^t \int\limits_c^x p(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}^*.$$

Since u satisfies (5.23) and (5.24), Lemma 4.1 yields

$$u(\tau_0^*,\mu_0^*) - u(t,x) = \int\limits_a^{\tau_0^*} \int\limits_x^{\mu_0^*} p(s,\eta)u\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds +$$

$$+ \int\limits_t^{\tau_0^*} \int\limits_c^x p(s,\eta)u\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}^*.$$

Therefore, in view of (5.30), we get

$$u(t,x) \le u(\tau_0^*,\mu_0^*) \quad \text{for} \quad (t,x) \in \mathcal{D}^* \tag{5.31}$$

and

$$u(\tau_0^*,\mu_0^*) - u(t,x) \le u(\tau_0^*,\mu_0^*) \left(\int\limits_a^{\tau_0^*} \int\limits_x^{\mu_0^*} p(s,\eta)d\eta ds + \int\limits_t^{\tau_0^*} \int\limits_c^x p(s,\eta)d\eta ds \right) =$$

$$= u(\tau_0^*,\mu_0^*)\Big(f(\tau_0^*,\mu_0^*) - f(t,x)\Big) \quad \text{for} \quad (t,x) \in \mathcal{D}^*. \tag{5.32}$$

From (5.13) and (5.32) we obtain

$$u(\tau_0^*,\mu_0^*)f(t,x) \le u(\tau_0^*,\mu_0^*)\Big(f(\tau_0^*,\mu_0^*) - 1\Big) + u(t,x) =$$

$$= u(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}^*. \tag{5.33}$$

On the other hand, on account of (5.23), (5.24), and (5.31), we get

$$u(t,x) = \int\limits_a^t \int\limits_c^x p(s,\eta)u\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds \le$$

$$\le u(\tau_0^*,\mu_0^*)f(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}^*. \tag{5.34}$$

Now, it follows from (5.33) and (5.34) that

$$u(t,x) = u(\tau_0^*,\mu_0^*) \int\limits_a^t \int\limits_c^x p(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}^*. \tag{5.35}$$

Finally, on account of the relation (5.35), we obtain

$$u(t,x)=\int_a^t\int_c^x p(s,\eta)u\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds=$$

$$=\int_a^t\int_c^x p(s,\eta)\left(u(\tau_0^*,\mu_0^*)\int_a^{\tau_0(s,\eta)}\int_c^{\mu_0(s,\eta)} p(\xi_1,\xi_2)d\xi_2 d\xi_1\right)d\eta ds$$

for $(t,x)\in\mathcal{D}^*$ and thus,

$$u(\tau_0^*,\mu_0^*)\left[1-\int_a^{\tau_0^*}\int_c^{\mu_0^*} p(s,\eta)\left(\int_a^{\tau_0(s,\eta)}\int_c^{\mu_0(s,\eta)} p(\xi_1,\xi_2)d\xi_2 d\xi_1\right)d\eta ds\right]=0. \tag{5.36}$$

We have proved that every solution u of the problem (5.23), (5.24) admits the representation (5.35) and, moreover, $u(\tau_0^*,\mu_0^*)$ satisfies (5.36). Therefore, if (5.14) holds, then the problem (5.23), (5.24) has only the trivial solution.

It remains to show that if (5.14) is not satisfied, i.e.,

$$\int_a^{\tau_0^*}\int_c^{\mu_0^*} p(s,\eta)f\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds=1, \tag{5.37}$$

then the problem (5.23), (5.24) has a nontrivial solution. Indeed, since

$$f\big(\tau_0(t,x),\mu_0(t,x)\big)\le f(\tau_0^*,\mu_0^*)\quad\text{for}\quad (t,x)\in\mathcal{D}^*,$$

in view of (5.13) and (5.37), we get

$$0\le\int_a^t\int_c^x p(s,\eta)\Big(f(\tau_0^*,\mu_0^*)-f\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)\Big)d\eta ds\le$$

$$\le\int_a^{\tau_0^*}\int_c^{\mu_0^*} p(s,\eta)\Big(f(\tau_0^*,\mu_0^*)-f\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)\Big)d\eta ds=$$

$$=1-\int_a^{\tau_0^*}\int_c^{\mu_0^*} p(s,\eta)f\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds=0\quad\text{for}\quad (t,x)\in\mathcal{D}^*.$$

Consequently,

$$\int_a^t\int_c^x p(s,\eta)\Big(f(\tau_0^*,\mu_0^*)-f\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)\Big)d\eta ds=0\quad\text{for}\quad (t,x)\in\mathcal{D}^*,$$

i.e.,

$$f(t,x)=\int_a^t\int_c^x p(s,\eta)f\big(\tau_0(s,\eta),\mu_0(s,\eta)\big)d\eta ds\quad\text{for}\quad (t,x)\in\mathcal{D}^*,$$

Thus f is a nontrivial solution of the problem (5.23), (5.24). □

Proof of Theorem 5.4. Let the operator ℓ be defined by (5.1). Obviously, $\ell \in P(\mathcal{D})$.

According to (5.15), there exists $\varepsilon > 0$ such that

$$p(t,x)\Big(\big(\tau_0(t,x)-a\big)\big(\mu_0(t,x)-c\big)+\varepsilon\Big) \leq 1 \quad \text{for} \quad (t,x) \in \mathcal{D}. \tag{5.38}$$

Put

$$\gamma(t,x) = (t-a)(x-c)+\varepsilon \quad \text{for} \quad (t,x) \in \mathcal{D}.$$

Obviously, $\gamma \in C^*\big(\mathcal{D};\,]0,+\infty[\big)$ and, in view of (5.38), γ satisfies the inequalities (3.1)–(3.3). Therefore, by virtue of Theorem 3.1, we get $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. □

Proof of Theorem 5.5. Let the operator ℓ be defined by (5.2). It is clear that, in view of the assumptions (5.16) and (5.17), the operator ℓ is an (a,c)–Volterra one. Therefore, the validity of theorem follows immediately from Corollary 3.2. □

Proof of Theorem 5.6. Let the operator ℓ be defined by (5.2). It is clear that, in view of the assumptions (5.16) and (5.17), the operator ℓ is an (a,c)–Volterra one. Moreover, by virtue of the assumption (5.19), the function γ given by (5.20) satisfies the inequalities (3.10)–(3.13). Hence, Theorem 3.2 guarantees the inclusion $\ell \in \mathcal{S}_{ac}(\mathcal{D})$. □

Proof of Theorem 5.7. The validity of theorem follows from Theorem 3.3 and Theorems 5.1–5.6. □

6. Further Remarks

We have investigated the characteristic initial value problem for the equation (1.1) when the values of the solution u of (1.1) are prescribed on both characteristics $t = a$ and $x = c$. However it is clear that the values of the solution can be prescribed on characteristics $t = a$ and $x = d$, $t = b$ and $x = c$ or $t = b$ and $x = d$. Then we obtain the other three problems, which have the same properties as the problem (1.1)–(1.3). Let us introduce the following definition.

Definition 6.1. We say that an operator $\ell \in \mathcal{L}(\mathcal{D})$ belongs to the set $\mathcal{S}_{ad}(\mathcal{D})$, if every solution of the problem

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} \leq \ell(u)(t,x),$$

$$u(a,d) \geq 0,$$

$$\frac{\partial u(t,d)}{\partial t} \geq 0 \quad \text{for} \quad t \in [a,b], \qquad \frac{\partial u(a,x)}{\partial x} \leq 0 \quad \text{for} \quad x \in [c,d]$$

is nonnegative.

An operator $\ell \in \mathcal{L}(\mathcal{D})$ is said to belong to the set $\mathcal{S}_{bc}(\mathcal{D})$, if every solution of the problem

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} \leq \ell(u)(t,x),$$

$$u(b,c) \geq 0,$$

$$\frac{\partial u(t,c)}{\partial t} \leq 0 \quad \text{for} \quad t \in [a,b], \qquad \frac{\partial u(b,x)}{\partial x} \geq 0 \quad \text{for} \quad x \in [c,d]$$

is nonnegative.

We say that an operator $\ell \in \mathcal{L}(\mathcal{D})$ belongs to the set $\mathcal{S}_{bd}(\mathcal{D})$, if every solution of the problem

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} \geq \ell(u)(t,x),$$

$$u(b,d) \geq 0,$$

$$\frac{\partial u(t,d)}{\partial t} \leq 0 \quad \text{for} \quad t \in [a,b], \qquad \frac{\partial u(b,x)}{\partial x} \leq 0 \quad \text{for} \quad x \in [c,d]$$

is nonnegative.

Now we can follow the same ideas and steps presented above and we can obtain results concerning the sets $\mathcal{S}_{ad}(\mathcal{D})$, $\mathcal{S}_{bc}(\mathcal{D})$, and $\mathcal{S}_{bd}(\mathcal{D})$. However, these results can be immediately derived from the above–proved ones using the following transformations.

Let the operators $\Phi, \Psi, \Omega : L\big(\mathcal{D};\mathbb{R}\big) \to L\big(\mathcal{D};\mathbb{R}\big)$ be defined by

$$\begin{aligned}
\Phi(w)(t,x) &\stackrel{\text{def}}{=} w(t, c+d-x) \quad \text{for} \quad (t,x) \in \mathcal{D},\\
\Psi(w)(t,x) &\stackrel{\text{def}}{=} w(a+b-t, x) \quad \text{for} \quad (t,x) \in \mathcal{D},\\
\Omega(w)(t,x) &\stackrel{\text{def}}{=} w(a+b-t, c+d-x) \quad \text{for} \quad (t,x) \in \mathcal{D}.
\end{aligned}$$

Let, moreover, $\Phi_0, \Psi_0, \Omega_0 : C\big(\mathcal{D};\mathbb{R}\big) \to C\big(\mathcal{D};\mathbb{R}\big)$ be the restrictions of Φ, Ψ, Ω to the space $C\big(\mathcal{D};\mathbb{R}\big)$. Now, for any $\ell \in \mathcal{L}(\mathcal{D})$, we put

$$\begin{aligned}
\ell_\Phi(w)(t,x) &\stackrel{\text{def}}{=} -\Phi\big(\ell(\Phi_0(w))\big)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},\\
\ell_\Psi(w)(t,x) &\stackrel{\text{def}}{=} -\Psi\big(\ell(\Psi_0(w))\big)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D},\\
\ell_\Omega(w)(t,x) &\stackrel{\text{def}}{=} \Omega\big(\ell(\Omega_0(w))\big)(t,x) \quad \text{for} \quad (t,x) \in \mathcal{D}.
\end{aligned}$$

Then it is not difficult to verify that the following implications are true:

$$\ell \in \mathcal{S}_{ac}(\mathcal{D}) \iff \ell_\Phi \in \mathcal{S}_{ad}(\mathcal{D}) \iff \ell_\Psi \in \mathcal{S}_{bc}(\mathcal{D}) \iff \ell_\Omega \in \mathcal{S}_{bd}(\mathcal{D})$$

7. Counter–examples

Example 7.1. Let the operator ℓ be defined by (5.1), where $\tau_0 \equiv b$, $\mu_0 \equiv d$, and $p \in L\big(\mathcal{D};\mathbb{R}_+\big)$ is such that

$$\int_a^b \int_c^d p(s,\eta)\,d\eta\,ds = 1.$$

Obviously, $\ell \in P(\mathcal{D})$ and, for any $m > k$ $(m, k \in N)$, the condition (3.4) holds with $\alpha = 1$, where the functions ρ_i $(i \in N)$ are defined by (3.5) and (3.6). Moreover, the condition (5.4) is satisfied with $\alpha = 1$.

On the other hand, the function

$$u(t,x) = \int_a^t \int_c^x p(s,\eta)d\eta ds \quad \text{for} \quad (t,x) \in \mathcal{D}$$

is a nontrivial solution of the problem (1.1_0)–(1.3_0). Therefore, by virtue of Remark 1.1, we find $\ell \notin \mathcal{S}_{ac}(\mathcal{D})$.

Example 7.2. Let $(t_0, x_0) \in]a,b[\times]c,d[$ and $\varepsilon > 0$. Put $\mathcal{D}_1 = [t_0, b] \times [x_0, d]$,

$$\tau_1(t,x) = \begin{cases} a & \text{for} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_1 \\ t_0 & \text{for} \quad (t,x) \in \mathcal{D}_1 \end{cases},$$

and

$$\mu_1(t,x) = \begin{cases} c & \text{for} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_1 \\ x_0 & \text{for} \quad (t,x) \in \mathcal{D}_1 \end{cases}.$$

Let the operator ℓ be defined by (5.2), where $g \in L\big(\mathcal{D}; \mathbb{R}_+\big)$ is such that

$$\int_a^{t_0} \int_c^{x_0} g(s,\eta)d\eta ds = \frac{\varepsilon}{1+\varepsilon}, \qquad \int_{t_0}^{b} \int_{x_0}^{d} g(s,\eta)d\eta ds = 1 + \frac{\varepsilon^2}{1+\varepsilon},$$

$$g(t,x) = 0 \quad \text{for} \quad (t,x) \in [a, t_0] \times [x_0, d] \cup [t_0, b] \times [c, x_0].$$

Obviously, ℓ is an (a,c)–Volterra operator and the condition (3.15) holds. Further, it is not difficult to verify that the function $\gamma \in C^*\big(\mathcal{D}; \mathbb{R}_+\big)$, defined by

$$\gamma(t,x) = \begin{cases} \frac{\varepsilon}{1+\varepsilon} - \int_a^t \int_c^x g(s,\eta)d\eta ds & \text{for} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_1 \\ 0 & \text{for} \quad (t,x) \in \mathcal{D}_1 \end{cases},$$

satisfies the conditions (3.10), (3.12), (3.13), and $\gamma(t_0, x_0) = 0$.

On the other hand, the function

$$u(t,x) = \begin{cases} 1 - \int_a^t \int_c^x g(s,\eta)d\eta ds & \text{for} \quad (t,x) \in \mathcal{D} \setminus \mathcal{D}_1 \\ \left(1 - \frac{\varepsilon}{1+\varepsilon}\right) \left(1 - \int_{t_0}^t \int_{x_0}^x g(s,\eta)d\eta ds\right) & \text{for} \quad (t,x) \in \mathcal{D}_1 \end{cases}$$

is a solution of the problem (1.4)–(1.7) with $u(b,d) = -\frac{\varepsilon^2}{(1+\varepsilon)^2} < 0$, and thus $\ell \notin \mathcal{S}_{ac}(\mathcal{D})$.

Example 7.3. Let $\varepsilon \in]0,1[$ and let $p, g \in L(\mathcal{D};\mathbb{R}_+)$ be such that

$$\int_a^b \int_c^d p(s,\eta)d\eta ds = 1+\varepsilon, \qquad \int_a^b \int_c^d g(s,\eta)d\eta ds < 1. \tag{7.1}$$

Let $\ell = \ell_0 - \ell_1$, where

$$\ell_0(v)(t,x) \stackrel{\text{def}}{=} p(t,x)v(b,d), \qquad \ell_1(v)(t,x) \stackrel{\text{def}}{=} g(t,x)v(a,c). \tag{7.2}$$

According to Remark 3.2 and Corollary 3.2, we find

$$(1-\varepsilon)\ell_0 \in \mathcal{S}_{ac}(\mathcal{D}), \qquad -\ell_1 \in \mathcal{S}_{ac}(\mathcal{D}).$$

Note also that the homogeneous problem (1.1_0)–(1.3_0) has only the trivial solution. Indeed, if u_0 is a solution of the problem (1.1_0)–(1.3_0) then Lemma 4.1 yields

$$u_0(b,d) - u_0(a,c) = \\ = u_0(b,d) \int_a^b \int_c^d p(s,\eta)d\eta ds - u_0(a,c) \int_a^b \int_c^d g(s,\eta)d\eta ds. \tag{7.3}$$

Consequently, in view of (1.2_0) and (7.1), we get $u_0(b,d) = 0$. Now, (1.1_0) implies $\frac{\partial^2}{\partial t\,\partial x} u_0(t,x) = 0$ for $(t,x) \in \mathcal{D}$ and thus, $u_0 \equiv 0$. Therefore, the problem (1.1_0), (1.2), (1.3) with $\varphi \equiv 1$ and $\psi \equiv 1$ has a unique solution u.

On the other hand, by virtue of (7.1), Lemma 4.1 yields

$$u(b,d) - u(a,c) = (1+\varepsilon)u(b,d) - u(a,c) \int_a^b \int_c^d g(s,\eta)d\eta ds,$$

i.e.,

$$\varepsilon u(b,d) = \int_a^b \int_c^d g(s,\eta)d\eta ds - 1.$$

Hence, u is a solution of the problem (1.4)–(1.7) with $u(b,d) < 0$, and thus $\ell \notin \mathcal{S}_{ac}(\mathcal{D})$.

Example 7.4. Let $\varepsilon \in]0,1[$ and let $p, g \in L(\mathcal{D};\mathbb{R}_+)$ be such that

$$\int_a^b \int_c^d p(s,\eta)d\eta ds < 1, \qquad \int_a^b \int_c^d g(s,\eta)d\eta ds = 1+\varepsilon. \tag{7.4}$$

Let $\ell = \ell_0 - \ell_1$, where ℓ_0 and ℓ_1 are defined by (7.2). According to Remark 3.2 and Corollary 3.2, we find

$$\ell_0 \in \mathcal{S}_{ac}(\mathcal{D}), \qquad -(1-\varepsilon)\ell_1 \in \mathcal{S}_{ac}(\mathcal{D}).$$

Note also that the homogeneous problem (1.1_0)–(1.3_0) has only the trivial solution. Indeed, if u_0 is a solution of the problem (1.1_0)–(1.3_0) then Lemma 4.1 yields (7.3). Consequently, in view of (1.2_0) and (7.4), we get $u_0(b,d) = 0$. Now, (1.1_0) implies $\frac{\partial^2}{\partial t\,\partial x} u_0(t,x) = 0$ for $(t,x) \in \mathcal{D}$ and thus, $u_0 \equiv 0$. Therefore, the problem (1.1_0), (1.2), (1.3) with $\varphi \equiv 1$ and $\psi \equiv 1$ has a unique solution u.

On the other hand, by virtue of (7.4), Lemma 4.1 yields

$$u(b,d) - u(a,c) = u(b,d) \int_a^b \int_c^d p(s,\eta)d\eta ds - (1+\varepsilon)u(a,c),$$

i.e.,

$$u(b,d)\left(1 - \int_a^b \int_c^d p(s,\eta)d\eta ds\right) = -\varepsilon.$$

Hence, u is a solution of the problem (1.4)–(1.7) with $u(b,d) < 0$, and thus $\ell \notin \mathcal{S}_{ac}(\mathcal{D})$.

Example 7.5. Let the operator ℓ be defined by (5.1), where $\tau_0 \equiv b$, $\mu_0 \equiv d$, and $p \equiv \big[(b-a)(d-s)\big]^{-1}$. It is clear that

$$\text{ess sup}\Big\{p(t,x)\big(\tau_0(t,x) - a\big)\big(\mu_0(t,x) - c\big) : (t,x) \in \mathcal{D}\Big\} = 1.$$

However the function

$$u(t,x) = (t-a)(x-c) for (t,x) \in \mathcal{D}$$

is a nontrivial solution of the problem (1.1_0)–(1.3_0) and thus $\ell \notin \mathcal{S}_{ac}(\mathcal{D})$.

Example 7.6. Let $\varepsilon > 0$ and let the operator ℓ be defined by (5.2), where $\tau_1 \equiv a$, $\mu_1 \equiv c$, and $g \equiv \big(1+\varepsilon\big)[(b-a)(d-s)\big]^{-1}$. It is clear that the conditions (5.16) and (5.17) are satisfied, and

$$\text{ess sup}\Big\{g(t,x)\Big[(b-a)(d-c) - \big(\tau_1(t,x) - a\big)\big(\mu_1(t,x) - c\big)\Big] : (t,x) \in \mathcal{D}\Big\} = 1+\varepsilon.$$

On the other hand, the function

$$u(t,x) = (b-a)(d-c) - (1+\varepsilon)(t-a)(x-c) \quad \text{for} \quad (t,x) \in \mathcal{D}$$

is a solution of the problem (1.1_0), (1.2), (1.3) with $\psi \equiv (b-a)(d-c)$ and $\varphi \equiv (b-a)(d-c)$. Since $u(b,d) = -\varepsilon(b-a)(d-c) < 0$ we get $\ell \notin \mathcal{S}_{ac}(\mathcal{D})$.

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