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# Existence of a Weak Solution to the Navier–Stokes Equation in a General Time–Varying Domain by the Rothe Method

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**Abstract**

We assume that $\Omega^t$ is a domain in $\mathbb{R}^3$, arbitrarily (but continuously) varying for $0 \leq t \leq T$. We impose no conditions on smoothness or shape of $\Omega^t$. We prove the global in time existence of a weak solution of the Navier–Stokes equation with Dirichlet's homogeneous or inhomogeneous boundary condition in $Q_{[0,T)} := \{(\boldsymbol{x}, t);\ 0 \leq t \leq T,\ \boldsymbol{x} \in \Omega^t\}$. The solution satisfies the energy inequality and is weakly continuous in dependence on time in a certain sense. As particular examples, we consider flows around rotating bodies and around a body striking to a rigid wall.

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## 1 Motivation and introduction

The global (in time) existence of a weak solution to the Navier–Stokes equation in a fixed domain $\Omega \subset \mathbb{R}^3$ belongs to classical fundamental results of the qualitative theory of the Navier–Stokes equation. (See e.g. J. Leray 1934 [18], E. Hopf 1952 [15], M. Shinbrot 1970 [20], O. A. Ladyzhenskaya 1969 [17], J. L. Lions 1969 [19], R. Temam 1977 [27] and G. P. Galdi 2000 [9].) The weak solution simulates a flow of a viscous incompressible fluid in domain $\Omega$. Since the fluid can also flow in a domain whose boundaries are not rigid, particularly around moving objects, a natural generalization is to study the weak solution in a time varying domain. The first work on this theme was published in 1970 by H. Fujita and N. Sauer [7]. The authors consider a variable domain $\Omega^t$ whose boundary consists of a finite number of simple closed surfaces of the class $C^3$ at each time $t \in [0, T]$. The surfaces can smoothly vary in dependence on time so that the distance of any two of the surfaces is never less than $\delta_0 > 0$. H. Fujita and N. Sauer proved the existence of a weak solution to the Navier–Stokes equation in the space–time cylinder $\{(\boldsymbol{x}, t);\ 0 < t < T,\ \boldsymbol{x} \in \Omega^t\}$.

There have appeared a series of works studying flows in certain time–varying domains in the last years in literature. The papers which we have in mind in this paragraph concern the motion of one or more bodies, occupying a closed region $B^t$ at time $t$, in a fluid filling a domain $O$. The fluid and the bodies are treated as an interconnected system and the position of the bodies in the fluid is thus not apriori known. While the motion of the fluid is governed by the Navier–Stokes equation and the equation of continuity, the motion of the bodies is described by equations that involve forces and torques with which the fluid acts on the bodies. The weak solvability of such a problem, provided the bodies do not touch each other or they do not strike to the boundary, was proved by B. Desjardins and M. J. Esteban 1999 [4], 2000 [5], K. H. Hoffmann V. N. Starovoitov 1999 [13] (the 2D case), C. Conca, J. San Martín and M. Tucsnak 2000 [3] and M. D. Gunzburger, H. C. Lee, G. Seregin 2000 [12]. The analogous result, without the assumption on the lack of collisions, was proved by J. San Martín, V. N. Starovoitov and M. Tucsnak 2002 [21] (the 2D case), K. H. Hoffmann, V. N. Starovoitov 2000 [14] (motion of a "small" ball in a fluid filling a "large ball") and E. Feireisl 2003 [6]. (The case of a 3D bounded domain $\Omega$. The author explains that there are more possibilities how the solution can be continued after an eventual collision and he uses the simple contact condition which requires that once two bodies touch one another, they remain stuck together forever.) The non–uniqueness of a weak solution in the case of a collision of a body with the boundary was shown by V. N. Starovoitov in 2005 [23] (the 2D case). The strong solvability of the problem was proved, on a time interval up to eventual collisions, by T. Takahashi 2003 [24] and T. Takahashi and M. Tucsnak in 2004 [26] (both papers treat the 2D case). The local (in time) existence of a strong solution in the 3D case was shown by T. Takahashi in 2003 [25]. The author also proved the global existence of a strong solution, as well as an asymptotic stability result, for small data and at the absence of collisions.

Other papers treat the motion of the system bodies–fluid under the assumption that the bodies produce certain velocity profile on the surface and they consequently move in the fluid due to this profile. (The bodies are therefore called the "self–propelled bodies".) The survey of related results is given by G. P. Galdi 2002 in [10].

In this paper, we assume that $\Omega^t$ is a time–varying domain, whose changes and deformations are prescribed, and we study the motion of the fluid in $\Omega^t$ in a given time interval $(0,T)$. $\Omega^t$ can have an arbitrary variable shape and smoothness, we only assume that its changes depend continuously on time. The Dirichlet boundary condition (homogeneous or inhomogeneous) for velocity is modelled by means of a given function $\boldsymbol{a}$, simulating the velocity on the boundary. Function $\boldsymbol{a}$ is required to have certain properties (see conditions (i)–(iii) in Section 2). The conditions on existence of appropriate function $\boldsymbol{a}$ in fact represent the only restriction on the shape and motion of domain $\Omega^t$, nevertheless we show in Sections 6 and 7 that they are satisfied in two concrete examples: a flow around a family of rotating bodies or a flow around a body striking to a wall. Here we also derive conditions on the shape of the body and on the speed of the strike which enable the existence of a weak solution. The paper thus provides a generalization of the existence theorem from [7]. The part concerned with the flow around the rotating bodies (Section 6) generalizes the existence result of W. Borchers 1992 [1]. The sufficient conditions obtained in Section 7 (the collision of the body with the wall) represent a complement to some deductions of V. N. Starovoitov 2003 [22], who derived a series of necessary (however not sufficient) conditions for the existence of a divergence–free flow with properties of a weak solution to the Navier–Stokes equation, in terms of the velocity and shape of a rigid body striking to a fixed boundary.

As to the techniques used in this paper, it is based on the construction of Rothe approximations. This method was already applied to the Navier–Stokes equation e.g. by J. L. Lions in [19] and M. Shinbrot in [20], however in a fixed spatial domain $\Omega$ with a certain smoothness. The difficulties, arising from the fact that our domain is time–variable and of an arbitrary shape, appear especially in the part where we treat the limit transition in a nonlinear term and we therefore need a piece of information on a strong convergence of a sequence of approximations in an appropriate norm. The standard compactness argument based on the Lions–Aubin lemma (see J. L. Lions [19], R. Temam [27]) cannot be used in a usual fashion. A similar problem was solved by D. Bucur,

E. Feireisl, Š. Nečasová and J. Wolf in [2] in connection with a limit of the Navier–Stokes system in a domain with rough boundaries. Here the authors apply a relatively deep information on a "local" pressure developed by J. Wolf in [28]. Our approach uses a different techniques: we prove the strong convergence of local (in space and time) Helmholtz projections of the approximations, which turns out to be sufficient for the limit transition. We show that the weak solution is in some sense weakly continuous in dependence on time and it satisfies an energy–type inequality in Section 5. We point some open problems.

We suppose that $T>0$ and $\Omega^t$ is a time–varying domain in $\mathbb{R}^3$ (for $0\leq t\leq T$), satisfying the following Assumptions 1 and 2:

**Assumption 1 (on domain $\Omega^t$).** $\forall\, t\in[0,T]\ :\ \Omega^t$ *is a non–empty domain in* $\mathbb{R}^3$ *such that* $\mathbb{R}^3=\Omega^t\cup\Gamma^t\cup\Omega^t_c$ *where* $\Omega^t_c$ *is a non–empty open set in* $\mathbb{R}^3$*,* $\Omega^t\cap\Omega^t_c=\emptyset$ *and* $\Gamma^t$ *is a common boundary of* $\Omega^t$ *and* $\Omega^t_c$*.*

We shall also need an assumption on continuity of $\Omega^t$ and $\Gamma^t$ in dependence on $t$. Therefore we define

$$
\begin{aligned}
d_3\big(\Omega^{t_1},\,\Omega^{t_2}\big) &:= \sup_{x\in\Omega^{t_1}} \operatorname{dist}_3\big(\boldsymbol{x};\,\Omega^{t_2}\big), \qquad d_3\big(\Omega^{t_2},\,\Omega^{t_1}\big) := \sup_{x\in\Omega^{t_2}} \operatorname{dist}_3\big(\boldsymbol{x};\,\Omega^{t_1}\big),\\
\widehat{d}_3\big(\Omega^{t_1},\,\Omega^{t_2}\big) &:= \max\big\{d\big(\Omega^{t_1},\,\Omega^{t_2}\big);\,d\big(\Omega^{t_2},\,\Omega^{t_1}\big)\big\}.
\end{aligned}
$$

The subscript 3 indicates that the distances are measured in $\mathbb{R}^3$ and the arguments of $\widehat{d}_3$ are also sets in $\mathbb{R}^3$. If the arguments are sets in $\mathbb{R}^3\times[0,T]$ then we use the subscript $4$. Now the assumption on continuity reads:

**Assumption 2 (on continuity of $\Omega^t$ and $\Gamma^t$).** *We suppose that* $\widehat{d}_3\big(\Omega^{t_1},\,\Omega^{t_2}\big)$ *and* $\widehat{d}_3\big(\Gamma^{t_1},\,\Gamma^{t_2}\big)$ *(as functions of the two variables* $t_1$ *and* $t_2$*) are continuous in* $[0,T]^2$*.*

**Space–time cylinders $Q_I$, $Q_I^c$ and their boundary.** If $I$ is an interval in $[0,T]$ then we denote

$$
\begin{aligned}
Q_I &:= \big\{(\boldsymbol{x},t)\in\mathbb{R}^3\times I;\ \boldsymbol{x}\in\Omega^t\big\}, \qquad Q_I^c := \big\{(\boldsymbol{x},t)\in\mathbb{R}^3\times I;\ \boldsymbol{x}\in\Omega^t_c\big\},\\
\Gamma_I &:= \big\{(\boldsymbol{x},t)\in\mathbb{R}^3\times I;\ \boldsymbol{x}\in\Gamma^t\big\}.
\end{aligned}
$$

Using Assumptions 1 and 2, one can verify that $\mathbb{R}^3\times[0,T)=Q_{[0,T)}\cup\Gamma_{[0,T)}\cup Q^c_{[0,T)}$ where $Q_{[0,T)}$ and $Q^c_{[0,T)}$ are open disjoint sets in $\mathbb{R}^3\times[0,T)$ and $\Gamma_{[0,T)}$ is their common boundary in $\mathbb{R}^3\times[0,T)$.

**The initial–boundary value problem and treatment of the boundary condition.** The purpose of this paper is to prove the existence of a weak solution of the problem

$$\partial_t\boldsymbol{v}-\nu\Delta\boldsymbol{v}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\nabla p\,=\,\boldsymbol{g} \qquad \text{in } Q_{(0,T)}, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{v}\,=\,0 \qquad \text{in } Q_{(0,T)}, \tag{1.2}$$

$$\boldsymbol{v}\,=\,\boldsymbol{a} \qquad \text{in } \Gamma_{(0,T)}, \tag{1.3}$$

$$\boldsymbol{v}(0)\,=\,\boldsymbol{v}_0 \qquad \text{in } \Omega^0. \tag{1.4}$$

Here $\boldsymbol{v}$ is the velocity of the fluid, $p$ is the pressure, $\boldsymbol{g}$ denotes the external specific body force, $\boldsymbol{v}_0$ is the initial velocity and $\nu$ is the kinematic coefficient of viscosity. The density of the fluid is supposed to be equal to one. The Navier–Stokes equation (1.1) expresses the conservation of momentum in a moving fluid. Equation (1.2) is the condition of incompressibility. The boundary condition (1.3) expresses the assumption that the velocity $\boldsymbol{v}$ takes a prescribed value $\boldsymbol{a}$ on the boundary of $\Omega^t$.

Usually, in papers on problems like (1.1)–(1.4) with a non–homogeneous boundary condition of the type (1.3), the authors assume that $\boldsymbol{a}$ is at first given on the boundary, they extend $\boldsymbol{a}$ appropriately to the interior and they search for the solution $\boldsymbol{v}$ in the form $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{u}$ where $\boldsymbol{u}$ is a new unknown function. The advantage of this approach is that $\boldsymbol{u}$ satisfies the homogeneous condition $\boldsymbol{u} = \boldsymbol{0}$ on the boundary. On the other hand, the disadvantage is that in order to construct the extension of $\boldsymbol{a}$, one needs some rate of smoothness of the boundary (for example that it is lipschitzian), which we wish to avoid in this paper. This is why we prefer another approach: we assume from the beginning that $\boldsymbol{a}$ is a given function in $\mathbb{R}^3 \times (0,T)$ such that

$$\boldsymbol{a} \,=\, \boldsymbol{a}^\infty + \boldsymbol{a}^0 \tag{1.5}$$

where $\boldsymbol{a}^\infty$ is a constant vector field in $\mathbb{R}^3$ (playing the role of velocity in infinity) and $\boldsymbol{a}^0$ satisfies certain conditions which will be formulated in Assumption 3 in the next section.

We further look for the velocity $\boldsymbol{v}$ in the form $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{u} \equiv \boldsymbol{a}^\infty + \boldsymbol{a}^0 + \boldsymbol{u}$ where

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{a}^0 + \big((\boldsymbol{a}^\infty + \boldsymbol{a}^0)\cdot\nabla\big)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p \,=\, \boldsymbol{f} \quad \text{in } Q_{(0,T)}, \tag{1.6}$$

$$\operatorname{div}\boldsymbol{u} \,=\, 0 \quad \text{in } Q_{(0,T)}, \tag{1.7}$$

$$\boldsymbol{u} \,=\, \boldsymbol{0} \quad \text{in } \Gamma_{(0,T)}, \tag{1.8}$$

$$\boldsymbol{u}(0) \,=\, \boldsymbol{u}_0 \quad \text{in } \Omega_0, \tag{1.9}$$

$$\boldsymbol{f} \,:=\, \boldsymbol{g} - \partial_t\boldsymbol{a}^0 + \nu\Delta\boldsymbol{a}^0 - \big((\boldsymbol{a}^\infty + \boldsymbol{a}^0)\cdot\nabla\big)\boldsymbol{a}^0. \tag{1.10}$$

## 2 Basic notation and definitions

We shall use the following function spaces and notation:

- $(\,.\,,\,.\,)_{2;\,\mathbb{R}^3}$ is the scalar product in $L^2(\mathbb{R}^3)$ or in $L^2(\mathbb{R}^3)^3$.
- $\|\,.\,\|_{q;\,\mathbb{R}^3}$ denotes the norm in $L^q(\mathbb{R}^3)$ or in $L^q(\mathbb{R}^3)^3$ or in $L^q(\mathbb{R}^3)^9$.
- $\|\,.\,\|_{k,2;\,\mathbb{R}^3}$ is the norm in $W^{k,2}(\mathbb{R}^3)$ or in $W^{k,2}(\mathbb{R}^3)^3$ (for $k = 0, 1, \dots$).
- $C^\infty_{0,\sigma}(\mathbb{R}^3)$ is the linear space of infinitely differentiable divergence–free vector functions in $\mathbb{R}^3$ that have a compact support.
- $L^2_\sigma(\mathbb{R}^3)$ denotes the completion of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ in the norm $\|\,.\,\|_{2;\,\mathbb{R}^3}$.
- $W^{1,2}_{0,\sigma}(\mathbb{R}^3)$ (respectively $\widehat{W}^{1,2}_{0,\sigma}(\mathbb{R}^3)$) denotes the completion of $C^\infty_{0,\sigma}(\mathbb{R}^3)$ in the norm $\|\,.\,\|_{1,2;\,\mathbb{R}^3}$ (respectively $\|\nabla\,.\,\|_{2;\,\mathbb{R}^3}$).
- The spaces $L^2_\sigma(\Omega^t)$, $W^{1,2}_{0,\sigma}(\Omega^t)$ or $\widehat{W}^{1,2}_{0,\sigma}(\Omega^t)$ are defined by analogy, as completions of $C^\infty_{0,\sigma}(\Omega^t)$ in the norms $\|\,.\,\|_{2;\,\Omega^t}$, $\|\,.\,\|_{1,2;\,\Omega^t}$ or $\|\nabla\,.\,\|_{2;\,\Omega^t}$.
- $W^{-1,2}_{0,\sigma}(\mathbb{R}^3)$ denotes the dual to the space $W^{1,2}_{0,\sigma}(\mathbb{R}^3)$. The duality between $W^{-1,2}_{0,\sigma}(\mathbb{R}^3)$ and $W^{1,2}_{0,\sigma}(\mathbb{R}^3)$ is denoted by $\langle\,.\,,\,.\,\rangle_{\mathbb{R}^3}$. The norm in $W^{-1,2}_{0,\sigma}(\mathbb{R}^3)$ is denoted by $\|\,.\,\|_{-1,2;\,\mathbb{R}^3}$.

- If a function is defined in a set $D \subset \mathbb{R}^3$ then the superscript "+" denotes the same function, extended by zero to $\mathbb{R}^3 \smallsetminus D$ so that its domain becomes $\mathbb{R}^3$.

Now we can formulate conditions that we impose on function $\boldsymbol{a}^0$:

**Assumption 3 (on function $\boldsymbol{a}^0$).** *We suppose that*

(i) $\boldsymbol{a}^0 \in L^2(0,T;\, \widehat{W}^{1,2}_{0,\sigma}(\mathbb{R}^3))$,

(ii) $\partial_t \boldsymbol{a}^0 \in L^1(0,T;\, L^2_\sigma(\mathbb{R}^3)) \oplus L^2(0,T;\, W^{-1,2}_{0,\sigma}(\mathbb{R}^3))$,

(iii) $\boldsymbol{a}^0 \in L^r(0,T;\, L^s(\mathbb{R}^3)^3)$ *for some* $r,\ s$ *such that* $3 < s \le r \le +\infty,\quad \dfrac{2}{r} + \dfrac{3}{s} \le 1.$

**More on function $\boldsymbol{f}$.** Let us now return to function $\boldsymbol{f}$ defined by (1.10). It is natural to assume that $\boldsymbol{g} \in L^1(0,T;\, L^2_\sigma(\mathbb{R}^3)) \oplus L^2(0,T;\, W^{-1,2}_{0,\sigma}(\mathbb{R}^3))$. It means that $\boldsymbol{g} = \boldsymbol{g}^I + \boldsymbol{g}^{II}$, where $\boldsymbol{g}^I \in L^1(0,T;\, L^2_\sigma(\mathbb{R}^3))$ and $\boldsymbol{g}^{II} \in L^2(0,T;\, W^{-1,2}_{0,\sigma}(\mathbb{R}^3))$. Thus, if $\boldsymbol{z} \in W^{1,2}_{0,\sigma}(\mathbb{R}^3)$ then

$$\begin{aligned} \big|\langle \boldsymbol{g},\, \boldsymbol{z}\rangle_{\mathbb{R}^3}\big| \ &\le\ \big|(\boldsymbol{g}^I,\, \boldsymbol{z})_{2;\,\mathbb{R}^3}\big| + \big|\langle \boldsymbol{g}^{II},\, \boldsymbol{z}\rangle_{\mathbb{R}^3}\big| \\ &\le\ \|\boldsymbol{g}^I\|_{2;\,\mathbb{R}^3}\, \|\boldsymbol{z}\|_{2;\,\mathbb{R}^3} + \|\boldsymbol{g}^{II}\|_{-1,2;\,\mathbb{R}^3} \left(\|\boldsymbol{z}\|^2_{2;\,\mathbb{R}^3} + \|\nabla \boldsymbol{z}\|^2_{2;\,\mathbb{R}^3}\right)^{1/2} \\ &\le\ \left(\|\boldsymbol{g}^I\|_{2;\,\mathbb{R}^3} + \|\boldsymbol{g}^{II}\|_{-1,2;\,\mathbb{R}^3}\right) \|\boldsymbol{z}\|_{2;\,\mathbb{R}^3} + \|\boldsymbol{g}^{II}\|_{-1,2;\,\mathbb{R}^3}\, \|\nabla \boldsymbol{z}\|_{2;\,\mathbb{R}^3}. \end{aligned}$$

Furthermore, conditions (i) and (ii) guarantee that $\partial_t \boldsymbol{a}^0$ and $\Delta \boldsymbol{a}^0$ belong to the direct sum $L^1(0,T;\, L^2_\sigma(\mathbb{R}^3)) \oplus L^2(0,T;\, W^{-1,2}_{0,\sigma}(\mathbb{R}^3))$ as well. The term $(\boldsymbol{a}^\infty \cdot \nabla)\boldsymbol{a}^0$ satisfies

$$\big|\big((\boldsymbol{a}^\infty \cdot \nabla)\boldsymbol{a}^0,\, \boldsymbol{z}\big)_{2;\,\mathbb{R}^3}\big| \ \le\ |\boldsymbol{a}^\infty|\, \|\nabla \boldsymbol{a}^0\|_{2;\,\mathbb{R}^3}\, \|\boldsymbol{z}\|_{2;\,\mathbb{R}^3}$$

for $\boldsymbol{z} \in L^2(\mathbb{R}^3)^3$. Finally, the term $(\boldsymbol{a}^0 \cdot \nabla)\boldsymbol{a}^0$ satisfies

$$\begin{aligned} \big|\big((\boldsymbol{a}^0 \cdot \nabla)\boldsymbol{a}^0,\, \boldsymbol{z}\big)_{2;\,\mathbb{R}^3}\big| \ &=\ \left|\int_{\mathbb{R}^3} (\boldsymbol{a}^0 \cdot \nabla)\boldsymbol{a}^0 \cdot \boldsymbol{z}\, \mathrm{d}\boldsymbol{x}\right| \ \le\ \|\boldsymbol{a}^0\|_{s;\,\mathbb{R}^3}\, \|\nabla \boldsymbol{a}^0\|_{2;\,\mathbb{R}^3}\, \|\boldsymbol{z}\|_{\frac{2s}{s-2};\,\mathbb{R}^3} \\ &\le\ \|\boldsymbol{a}^0\|_{s;\,\mathbb{R}^3}\, \|\nabla \boldsymbol{a}^0\|_{2;\,\mathbb{R}^3}\, \|\boldsymbol{z}\|^{\frac{3}{s}}_{6;\,\mathbb{R}^3}\, \|\boldsymbol{z}\|^{\frac{s-3}{s}}_{2;\,\mathbb{R}^3} \\ &\le\ \left(\frac{2}{\sqrt{3}}\right)^{\frac{3}{s}} \|\boldsymbol{a}^0\|_{s;\,\mathbb{R}^3}\, \|\nabla \boldsymbol{a}^0\|_{2;\,\mathbb{R}^3}\, \|\nabla \boldsymbol{z}\|^{\frac{3}{s}}_{2;\,\mathbb{R}^3}\, \|\boldsymbol{z}\|^{\frac{s-3}{s}}_{2;\,\mathbb{R}^3} \\ &\le\ \left(\frac{2}{\sqrt{3}}\right)^{\frac{3}{s}} \frac{3}{s}\, \|\nabla \boldsymbol{a}^0\|_{2;\,\mathbb{R}^3}\, \|\nabla \boldsymbol{z}\|_{2;\,\mathbb{R}^3} + \left(\frac{2}{\sqrt{3}}\right)^{\frac{3}{s}} \frac{s-3}{s}\, \|\nabla \boldsymbol{a}^0\|_{2;\,\mathbb{R}^3}\, \|\boldsymbol{a}^0\|^{\frac{s}{s-3}}_{s;\,\mathbb{R}^3}\, \|\boldsymbol{z}\|_{2;\,\mathbb{R}^3} \end{aligned}$$

for $\boldsymbol{z} \in W^{1,2}_0(\mathbb{R}^3)^3$. (We have used the Sobolev inequality – see e.g. [8], p. 31 – and the Young inequality – see e.g. [17], p. 10.) While $\|\nabla \boldsymbol{a}^0\|_{2;\,\mathbb{R}^3} \in L^2(0,T)$ due to condition (i) of Assumption 3, the product $\|\nabla \boldsymbol{a}^0\|_{2;\,\mathbb{R}^3}\, \|\boldsymbol{a}^0\|^{s/(s-3)}_{s;\,\mathbb{R}^3}$ belongs to $L^1(0,T)$ due to conditions (i) and (iii). Summarizing these results, we observe that

(iv) $\boldsymbol{f}(t) \in W^{-1,2}_{0,\sigma}(\mathbb{R}^3)$ for a.a. $t \in (0,T)$ and

$$\big|\big\langle \boldsymbol{f}(t),\, \boldsymbol{z}\big\rangle_{\mathbb{R}^3}\big| \ \le\ \zeta^0(t)\, \|\boldsymbol{z}\|_{2;\,\mathbb{R}^3} + \zeta^1(t)\, \|\nabla \boldsymbol{z}\|_{2;\,\mathbb{R}^3}$$

where $\zeta^1 \in L^2(0,T)$ and $\zeta^0 \in L^1(0,T)$.

**Definition 1 (the weak solution of (1.6)–(1.9)).** *Given functions* $\boldsymbol{u}_0 \in L^2_\sigma(\Omega^0)$ *and* $\boldsymbol{f}$ *satisfying condition (iv). Function* $\boldsymbol{u} \in L^2(0,T;\, \widehat{W}^{1,2}_{0,\sigma}(\mathbb{R}^3)) \cap L^\infty(0,T;\, L^2_\sigma(\mathbb{R}^3))$ *is called a weak solution of the problem (1.6)–(1.9) if*

$$\boldsymbol{u} \;=\; \boldsymbol{0} \qquad \text{a.e. in } Q^c_{[0,T)} \qquad \text{and} \tag{2.1}$$

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} \bigl[-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \nu\,\nabla\boldsymbol{u} : \nabla\boldsymbol{\phi} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{a}^0\cdot\boldsymbol{\phi} + \bigl((\boldsymbol{a}^\infty+\boldsymbol{a}^0)\cdot\nabla\bigr)\boldsymbol{u}\cdot\boldsymbol{\phi} \\ + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}\bigr]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \;=\; \int_0^T \langle\boldsymbol{f},\boldsymbol{\phi}\rangle_{\mathbb{R}^3}\,\mathrm{d}t - \int_{\Omega_0} \boldsymbol{u}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{2.2}$$

*for all* $\boldsymbol{\phi} \in C^\infty(\mathbb{R}^3\times[0,T))^3$ *such that* $\operatorname{div}\boldsymbol{\phi} = 0$ *in* $\mathbb{R}^3\times[0,T)$ *and* $\boldsymbol{\phi}$ *has a compact support in* $Q_{[0,T)}$.

Identity (2.1) simulates the boundary condition (1.8). Indeed, if the common boundary $\Gamma_{[0,T)}$ of $Q_{[0,T)}$ and $Q^c_{[0,T)}$ is so smooth that it enables the existence of a trace then the trace is, due to (2.1), equal to zero.

**The weak solution of (1.1)–(1.4).** If $\boldsymbol{u}$ is a weak solution defined above then the function $\boldsymbol{v} = \boldsymbol{u}+\boldsymbol{a} \equiv \boldsymbol{u}+\boldsymbol{a}^\infty+\boldsymbol{a}^0$ is a weak solution of the problem (1.1)–(1.4) in the sense that $\boldsymbol{v} \in L^2(0,T;\, \widehat{W}^{1,2}_{0,\sigma}(\mathbb{R}^3))$ and

$$\int_0^T \int_{\mathbb{R}^3} \bigl[-\boldsymbol{v}\cdot\partial_t\boldsymbol{\phi} + \nu\,\nabla\boldsymbol{v} : \nabla\boldsymbol{\phi} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{\phi}\bigr]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \;=\; \int_0^T \langle\boldsymbol{g},\boldsymbol{\phi}\rangle_{\mathbb{R}^3}\,\mathrm{d}t - \int_{\Omega_0} \boldsymbol{v}_0\cdot\boldsymbol{\phi}(\,.\,,0)\,\mathrm{d}\boldsymbol{x}$$

for all $\boldsymbol{\phi} \in C^\infty(\mathbb{R}^3\times[0,T))^3$ such that $\operatorname{div}\boldsymbol{\phi} = 0$ in $\mathbb{R}^3\times[0,T)$ and $\boldsymbol{\phi}$ has a compact support in $Q_{[0,T)}$. The function $\boldsymbol{v}_0 \equiv \boldsymbol{u}_0 + \boldsymbol{a}(0)$ represents the initial value of $\boldsymbol{v}$ at time $t=0$.

## 3 Approximations, their estimates and weak convergence

**The time discretization.** We apply Rothe's method. Let $n\in\mathbb{N}$. Put

$$h := \frac{T}{n}, \qquad t_k := kh, \qquad \Omega_k := \Omega^{t_k}, \qquad \boldsymbol{U}_0 := \boldsymbol{u}_0, \qquad \boldsymbol{a}^0_k := \frac{1}{h}\int_{t_{k-1}}^{t_k} \boldsymbol{a}^0(t)\,\mathrm{d}t,$$
$$\boldsymbol{F}_k := \frac{1}{h}\int_{t_{k-1}}^{t_k} \boldsymbol{f}(t)\,\mathrm{d}t, \qquad \zeta^0_k := \frac{1}{h}\int_{t_{k-1}}^{t_k} \zeta^0(t)\,\mathrm{d}t, \qquad \zeta^1_k := \frac{1}{h}\int_{t_{k-1}}^{t_k} \zeta^1(t)\,\mathrm{d}t$$

and we successively solve, for $k=1,\dots,n$, a series of stationary BVP's

$$\begin{aligned} \boldsymbol{U}_k - \boldsymbol{U}^+_{k-1} - \nu h\,\Delta\boldsymbol{U}_k + h\,(\boldsymbol{U}_k\cdot\nabla)\boldsymbol{a}^0_k + h\,\bigl((\boldsymbol{a}^\infty+\boldsymbol{a}^0_k)\cdot\nabla\bigr)\boldsymbol{U}_k \\ + \; h\,(\boldsymbol{U}^+_{k-1}\cdot\nabla)\boldsymbol{U}_k + h\,\nabla P_k \;=\; h\,\boldsymbol{F}_k \qquad \text{in } \Omega_k, \end{aligned} \tag{3.1}$$

$$\operatorname{div}\boldsymbol{U}_k \;=\; 0 \qquad \text{in } \Omega_k, \tag{3.2}$$

$$\boldsymbol{U}_k \;=\; \boldsymbol{0} \qquad \text{in } \partial\Omega_k. \tag{3.3}$$

**Definition 2 (the weak solution of the stationary BVP).** *Given* $\boldsymbol{U}^+_{k-1} \in L^2_\sigma(\mathbb{R}^3)$ *and* $\boldsymbol{F}_k \in W^{-1,2}_{0,\sigma}(\mathbb{R}^3)$. *A function* $\boldsymbol{U}_k \in W^{1,2}_{0,\sigma}(\Omega_k)$ *is called a weak solution of the problem (3.1)–(3.3) if*

$$\int_{\Omega_k} \big[ \boldsymbol{U}_k \cdot \boldsymbol{\phi}_k - \boldsymbol{U}_{k-1}^+ \cdot \boldsymbol{\phi}_k + \nu h\, \nabla \boldsymbol{U}_k : \nabla \boldsymbol{\phi}_k + h\, (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{a}_k^0 \cdot \boldsymbol{\phi}_k + h\, (\boldsymbol{a}^\infty \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi}_k$$
$$+ h\, (\boldsymbol{a}_k^0 \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi}_k + h\, (\boldsymbol{U}_{k-1}^+ \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi}_k \big] \, \mathrm{d}\boldsymbol{x} \ = h\, \langle \boldsymbol{F}_k, \boldsymbol{\phi}_k^+ \rangle_{\mathbb{R}^3} \tag{3.4}$$

*for all* $\boldsymbol{\phi}_k \in W^{1,2}_{0,\sigma}(\Omega_k)$.

**Lemma 1 (the unique weak solvability of the stationary BVP's).** *There exists* $n_0 \in \mathbb{N}$ *such that if* $n \geq n_0$ *then the problems (3.1)–(3.3) (for* $k = 1, \dots, n$*) have unique weak solutions* $\boldsymbol{U}_k$ *which satisfy*

$$\|\boldsymbol{U}_k\|_{2;\,\Omega_k}^2 + \sum_{j=1}^k \|\boldsymbol{U}_j - \boldsymbol{U}_{j-1}^+\|_{2;\,\Omega_j}^2 + 2\nu h \sum_{j=1}^k \|\nabla \boldsymbol{U}_j\|_{2;\,\Omega_j}^2$$
$$\leq \ \|\boldsymbol{U}_0^+\|_{2;\,\mathbb{R}^3}^2 + 2h \sum_{j=1}^k \int_{\Omega_j} (\boldsymbol{U}_j^+ \cdot \nabla) \boldsymbol{U}_j^+ \cdot \boldsymbol{a}_j^0 \, \mathrm{d}\boldsymbol{x} + 2h \sum_{j=1}^k \langle \boldsymbol{F}_j, \boldsymbol{U}_j^+ \rangle_{\mathbb{R}^3}. \tag{3.5}$$

*Proof.* 1) We shall need the estimate of the norm of function $\boldsymbol{a}_k^0$ in $L^s(\Omega_k)^3$:

$$\|\boldsymbol{a}_k^0\|_{s;\,\Omega_k}^{\frac{2s}{s-3}} \ = \ \left[ \int_{\Omega_k} \left| \frac{1}{h} \int_{t_{k-1}}^{t_k} \boldsymbol{a}^0(t) \, \mathrm{d}t \right|^s \mathrm{d}\boldsymbol{x} \right]^{\frac{2}{s-3}} \ \leq \ \left[ \frac{1}{h} \int_{t_{k-1}}^{t_k} \int_{\Omega_k} |\boldsymbol{a}^0|^s \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \right]^{\frac{2}{s-3}}$$
$$\leq \ \frac{1}{h} \int_{t_{k-1}}^{t_k} \left( \int_{\Omega_k} |\boldsymbol{a}^0|^s \, \mathrm{d}\boldsymbol{x} \right)^{\frac{2}{s-3}} \mathrm{d}t \ = \ \frac{1}{h} \int_{t_{k-1}}^{t_k} \|\boldsymbol{a}^0(t)\|_{s;\,\Omega_k}^{\frac{2s}{s-3}} \, \mathrm{d}t. \tag{3.6}$$

2) Let us show that if $n$ is sufficiently large then the bilinear form

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) \ := \ \int_{\Omega_k} \big[ \boldsymbol{u} \cdot \boldsymbol{v} + \nu h\, \nabla \boldsymbol{u} : \nabla \boldsymbol{v} + h\, (\boldsymbol{u} \cdot \nabla) \boldsymbol{a}_k^0 \cdot \boldsymbol{v} + h\, (\boldsymbol{a}^\infty \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{v}$$
$$+ h\, (\boldsymbol{a}_k^0 \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{v} + h\, (\boldsymbol{U}_{k-1}^+ \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{v} \big] \, \mathrm{d}\boldsymbol{x}$$

is $W^{1,2}_{0,\sigma}(\Omega_k)$–elliptic. We have

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{u}) \ = \ \|\boldsymbol{u}\|_{2;\,\Omega_k}^2 + \nu h\, \|\nabla \boldsymbol{u}\|_{2;\,\Omega_k}^2 + h \int_{\Omega_k} (\boldsymbol{u} \cdot \nabla) \boldsymbol{a}_k^0 \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}. \tag{3.7}$$

The last term on the right hand side equals $-h \int_{\Omega_k} (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{a}_k^0 \, \mathrm{d}\boldsymbol{x}$ and it can be therefore estimated as follows:

$$\left| h \int_{\Omega_k} (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \cdot \boldsymbol{a}_k^0 \, \mathrm{d}\boldsymbol{x} \right| \ \leq \ \delta_1 \nu h\, \|\nabla \boldsymbol{u}\|_{2;\,\Omega_k}^2 + \frac{h}{4\delta_1 \nu} \int_{\Omega_k} |\boldsymbol{u}|^2 \, |\boldsymbol{a}_k|^2 \, \mathrm{d}\boldsymbol{x}$$
$$\leq \ \delta_1 \nu h\, \|\nabla \boldsymbol{u}\|_{2;\,\Omega_k}^2 + \frac{h}{4\delta_1 \nu} \, \|\boldsymbol{a}_k\|_{s;\,\Omega_k}^2 \, \|\boldsymbol{u}\|_{\frac{2s}{s-2};\,\Omega_k}^2$$
$$\leq \ \delta_1 \nu h\, \|\nabla \boldsymbol{u}\|_{2;\,\Omega_k}^2 + \frac{h}{4\delta_1 \nu} \, \|\boldsymbol{a}_k^0\|_{s;\,\Omega_k}^2 \, \|\boldsymbol{u}\|_{2;\,\Omega_k}^{2(s-3)/s} \, \|\boldsymbol{u}\|_{6;\,\Omega_k}^{6/s}$$
$$\leq \ \delta_1 \nu h\, \|\nabla \boldsymbol{u}\|_{2;\,\Omega_k}^2 + \frac{h}{4\delta_1 \nu} \, \|\boldsymbol{a}_k^0\|_{s;\,\Omega_k}^2 \, \|\boldsymbol{u}\|_{2;\,\Omega_k}^{2(s-3)/s} \left( \frac{2}{\sqrt{3}} \right)^{6/s} \|\nabla \boldsymbol{u}\|_{2;\,\Omega_k}^{6/s}$$
$$\leq \ \delta_1 \nu h\, \|\nabla \boldsymbol{u}\|_{2;\,\Omega_k}^2 + \delta_2 \nu h\, \|\nabla \boldsymbol{u}\|_{2;\,\Omega_k}^2 + c_1\, h\, \|\boldsymbol{a}_k^0\|_{s;\,\Omega_k}^{2s/(s-3)} \, \|\boldsymbol{u}\|_{2;\,\Omega_k}^2 \tag{3.8}$$

where $\delta_1,\ \delta_2>0$ and

$$c_1 \;=\; c_1(\delta_1,\delta_2,s,\nu) \;=\; \frac{s-3}{s}\,\Big(\frac{4}{\nu\delta_2 s}\Big)^{\frac{3}{s-3}}\,\Big(\frac{1}{4\delta_1\nu}\Big)^{\frac{s}{s-3}}. \tag{3.9}$$

(We have used the Sobolev inequality and the Young inequality, see e.g. [8], pp. 22, 31. Number $s$ is the exponent from Assumption 3.) Thus,

$$\mathcal{A}(\boldsymbol{u},\boldsymbol{u}) \;\geq\; \big[1-c_1\,h\,\|\boldsymbol{a}_k^0\|_{s;\,\Omega_k}^{2s/(s-3)}\big]\,\|\boldsymbol{u}\|_{2:\,\Omega_k}^2+\nu h\,[1-(\delta_1+\delta_2)]\,\|\nabla\boldsymbol{u}\|_{2:\,\Omega_k}^2\,.$$

The numbers $\delta_1$ and $\delta_2$ can be chosen so small that $\delta_1+\delta_2=\frac{1}{2}$. Due to estimate (3.6) and the inequality $2s/(s-3)\leq r,\ h\,\|\boldsymbol{a}_k^0\|_{s;\,\Omega_k}^{2s/(s-3)}$ can be made arbitrarily small (uniformly for $k=1,\dots,n$) by choosing $h$ sufficiently small (which corresponds to $n$ sufficiently large). Thus, if $n$ is sufficiently large, we have

$$\mathcal{A}(\boldsymbol{u},\boldsymbol{u}) \;\geq\; \frac{1}{2}\,\|\boldsymbol{u}\|_{2\;\Omega_k}^2+\frac{\nu h}{2}\,\|\nabla\boldsymbol{u}\|_{2:\,\Omega_k}^2. \tag{3.10}$$

3) $\boldsymbol{F}_k$ is an element of $W_{0,\sigma}^{-1,2}(\mathbb{R}^3)$. We can also consider $\boldsymbol{F}_k$ to a bounded linear functional acting on $W_{0,\sigma}^{1,2}(\Omega_k)$, putting $\langle\boldsymbol{F}_k,\boldsymbol{\phi}_k\rangle_{\Omega_k}:=\langle\boldsymbol{F}_k,\boldsymbol{\phi}_k^+\rangle_{\mathbb{R}^3}$ for all $\boldsymbol{\phi}_k\in W_{0,\sigma}^{1,2}(\Omega_k)$. Due to the Lax–Millgram lemma, there exists a unique $\boldsymbol{U}_k\in W_{0,\sigma}^{1,2}(\Omega_k)$ such that the identity

$$\mathcal{A}(\boldsymbol{U}_k,\boldsymbol{\phi}_k) \;=\; h\,\big\langle\boldsymbol{F}_k,\,\boldsymbol{\phi}_k\big\rangle_{\Omega_k} \tag{3.11}$$

holds for all $\boldsymbol{\phi}_k\in W_{0,\sigma}^{1,2}(\Omega_k)$. Consequently, $\boldsymbol{U}_k$ satisfies (3.4) for all $\boldsymbol{\phi}_k\in W_{0,\sigma}^{1,2}(\Omega_k)$.

4) Let us derive (3.5). Substituting $\boldsymbol{\phi}_k=\boldsymbol{U}_k$ to (3.4), we obtain

$$\begin{aligned}
&\big(\boldsymbol{U}_k-\boldsymbol{U}_{k-1}^+,\boldsymbol{U}_k\big)_{2:\,\Omega_k}+\nu h\,\|\nabla\boldsymbol{U}_k\|_{2:\,\Omega_k}^2\\
&\quad=\;\; -h\int_{\Omega_k}(\boldsymbol{U}_k\cdot\nabla)\boldsymbol{a}_k^0\cdot\boldsymbol{U}_k\,\mathrm{d}\boldsymbol{x}+h\,\langle\boldsymbol{F}_k,\boldsymbol{U}_k^+\rangle_{\mathbb{R}^3},
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&\|\boldsymbol{U}_k-\boldsymbol{U}_{k-1}^+\|_{2:\,\Omega_k}^2+\nu h\,\|\nabla\boldsymbol{U}_k\|_{2:\,\Omega_k}^2\\
&\quad=\;\; -\big(\boldsymbol{U}_k-\boldsymbol{U}_{k-1}^+,\boldsymbol{U}_{k-1}^+\big)_{2:\,\Omega_k}-h\int_{\Omega_k}(\boldsymbol{U}_k\cdot\nabla)\boldsymbol{a}_k^0\cdot\boldsymbol{U}_k\,\mathrm{d}\boldsymbol{x}+h\,\langle\boldsymbol{F}_k,\boldsymbol{U}_k^+\rangle_{\mathbb{R}^3}\\
&\quad=\;\; \|\boldsymbol{U}_{k-1}^+\|_{2:\,\Omega_k}^2-h\int_{\Omega_k}(\boldsymbol{U}_k\cdot\nabla)\boldsymbol{a}_k\cdot\boldsymbol{U}_k\,\mathrm{d}\boldsymbol{x}-(\boldsymbol{U}_k,\boldsymbol{U}_{k-1}^+)_{2:\,\Omega_k}+h\,\langle\boldsymbol{F}_k,\boldsymbol{U}_k^+\rangle_{\mathbb{R}^3}.
\end{aligned}$$

Substituting here for $(\boldsymbol{U}_k,\boldsymbol{U}_{k-1}^+)_{2:\,\Omega_k}$ again from (3.12), we obtain

$$\begin{aligned}
&\|\boldsymbol{U}_k\|_{2:\,\Omega_k}^2+\|\boldsymbol{U}_k-\boldsymbol{U}_{k-1}^+\|_{2:\,\Omega_k}^2+\nu h\,\|\nabla\boldsymbol{U}_k\|_{2:\,\Omega_k}^2 \;=\; \|\boldsymbol{U}_{k-1}^+\|_{2:\,\Omega_k}^2\\
&\qquad -2h\int_{\Omega_k}(\boldsymbol{U}_k\cdot\nabla)\boldsymbol{a}_k^0\cdot\boldsymbol{U}_k\,\mathrm{d}\boldsymbol{x}-\nu h\,\|\nabla\boldsymbol{U}_k\|_{2:\,\Omega_k}^2+2h\,\langle\boldsymbol{F}_k,\boldsymbol{U}_k^+\rangle_{\mathbb{R}^3},\\
&\|\boldsymbol{U}_k^+\|_{2:\,\mathbb{R}^3}^2+\|\boldsymbol{U}_k^+-\boldsymbol{U}_{k-1}^+\|_{2:\,\Omega_k}^2+2\nu h\,\|\nabla\boldsymbol{U}_k^+\|_{2:\,\Omega_k}^2 \;\leq\; \|\boldsymbol{U}_{k-1}^+\|_{2:\,\mathbb{R}^3}^2\\
&\qquad -2h\int_{\Omega_k}(\boldsymbol{U}_k^+\cdot\nabla)\boldsymbol{a}_k^0\cdot\boldsymbol{U}_k^+\,\mathrm{d}\boldsymbol{x}-\nu h\,\|\nabla\boldsymbol{U}_k\|_{2:\,\Omega_k}^2+2h\,\langle\boldsymbol{F}_k,\boldsymbol{U}_k^+\rangle_{\mathbb{R}^3}.
\end{aligned}$$

Integrating by parts in the term which contains $\boldsymbol{a}_k^0$, writing the inequality with $j$ instead of $k$ and summing for $j=1,\dots,k$, we arrive at (3.5). $\square$

**Approximate solutions and their estimates.** Now we define

$$\boldsymbol{u}_n(t) := \boldsymbol{U}_k^+ \qquad \text{for } t_{k-1} < t \le t_k; \quad k = 1,\dots,n.$$

Inequality (3.5) enables us to estimate $\boldsymbol{u}_n$. If $t_{k-1} < t \le t_k$ then

$$\|\boldsymbol{u}_n(t)\|_{2;\,\mathbb{R}^3}^2 + 2\nu \int_0^{t_k} \|\nabla \boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2 \, \mathrm{d}\tau \;\le\; \|\boldsymbol{u}_0\|_{2;\,\Omega_0}^2$$
$$+ 2\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \int_{\Omega_j} \big(\boldsymbol{u}_n(\tau)\cdot\nabla\big)\boldsymbol{u}_n(\tau)\cdot\boldsymbol{a}^0(\tau)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau + 2\int_0^{t_k} \langle \boldsymbol{f}(\tau), \boldsymbol{u}_n(\tau)\rangle_{\mathbb{R}^3}\,\mathrm{d}\tau. \tag{3.13}$$

Applying (3.8) and estimating the norm of $\boldsymbol{a}_j^0$ by means of (3.6), we obtain

$$\|\boldsymbol{u}_n(t)\|_{2;\,\mathbb{R}^3}^2 + 2\nu\big[1-(\delta_1+\delta_2)\big] \int_0^{t_k} \|\nabla \boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2 \, \mathrm{d}\tau \;\le\; \|\boldsymbol{u}_0\|_{2;\,\Omega_0}^2$$
$$+ 2c_1 \int_0^{t_k} \|\boldsymbol{a}^0(\tau)\|_{s;\,\mathbb{R}^3}^{\frac{2s}{s-3}}, \|\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2\,\mathrm{d}\tau + 2\int_0^{t_k} \langle \boldsymbol{f}(\tau), \boldsymbol{u}_n(\tau)\rangle_{\mathbb{R}^3}\,\mathrm{d}\tau. \tag{3.14}$$

The integrals on the right hand side can be split to the integrals from $0$ to $t$ and from $t$ to $t_k$. The integrals from $t$ to $t_k$ can be estimated:

$$2c_1 \int_t^{t_k} \|\boldsymbol{a}^0(\tau)\|_{s;\,\mathbb{R}^3}^{\frac{2s}{s-3}} \|\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2\,\mathrm{d}\tau + 2\int_t^{t_k} \langle \boldsymbol{f}(\tau), \boldsymbol{u}_n(\tau)\rangle_{\mathbb{R}^3}\,\mathrm{d}\tau$$
$$\le\; c_2(h)\,\|\boldsymbol{u}_n(t)\|_{2;\,\mathbb{R}^3}^2 + 2\int_t^{t_k} \big[\zeta^1(\tau)\,\|\nabla\boldsymbol{u}(\tau)\|_{2;\,\mathbb{R}^3} + \zeta^0(\tau)\,\|\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}\big]\,\mathrm{d}\tau$$
$$\le\; c_2(h)\,\|\boldsymbol{u}_n(t)\|_{2;\,\mathbb{R}^3}^2 + \int_t^{t_k} \Big[2\nu\,(1-\delta_1-\delta_2)\,\|\nabla\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2$$
$$+ \frac{\zeta^1(\tau)^2}{2\nu\,(1-\delta_1-\delta_2)} + \zeta^0(\tau) + \zeta^0(\tau)\,\|\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2\Big]\,\mathrm{d}\tau$$
$$\le\; \big[c_2(h)+c_3(h)\big]\,\|\boldsymbol{u}_n(t)\|_{2;\,\mathbb{R}^3}^2 + 2\nu\,(1-\delta_1-\delta_2)\int_t^{t_k} \|\nabla\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2\,\mathrm{d}\tau + c_4(h)$$

where $c_2(h)\to 0$, $c_3(h)\to 0$ and $c_4(h)\to 0$ as $h\to 0+$. If we further apply the estimate from condition (iv) to the integral from $0$ to $t$ of $\langle \boldsymbol{f}(\tau), \boldsymbol{u}_n(\tau)\rangle_{\mathbb{R}^3}$ in (3.14), we obtain

$$\big[1-c_2(h)-c_3(h)\big]\,\|\boldsymbol{u}_n(t)\|_{2;\,\mathbb{R}^3}^2 + 2\nu\big[1-(\delta_1+\delta_2)\big]\int_0^t \|\nabla\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2\,\mathrm{d}\tau$$
$$\le\; \|\boldsymbol{u}_0\|_{2;\,\Omega_0}^2 + 2c_1\int_0^t \|\boldsymbol{a}^0(\tau)\|_{s;\,\mathbb{R}^3}^{\frac{2s}{s-3}} \|\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2\,\mathrm{d}\tau$$
$$+ 2\int_0^t \big[\zeta^1(\tau)\,\|\nabla\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3} + \zeta^0(\tau)\,\|\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}\big]\,\mathrm{d}\tau + c_4(h)$$
$$\le\; \|\boldsymbol{u}_0\|_{2;\,\Omega_0}^2 + 2c_1\int_0^t \|\boldsymbol{a}^0(\tau)\|_{s;\,\mathbb{R}^3}^{\frac{2s}{s-3}} \|\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2\,\mathrm{d}\tau + 2\nu\delta_3\int_0^t \|\nabla\boldsymbol{u}_n(\tau)\|_{2;\,\mathbb{R}^3}^2\,\mathrm{d}\tau$$

$$+\int_0^t \big[2c_1\,\|\boldsymbol{a}^0(\tau)\|_{s;\,\mathbb{R}^3}^{\frac{2s}{s-3}}+\zeta^0(\tau)\big]\,\|\boldsymbol{u}_n(\tau)\|_{2:\,\mathbb{R}^3}^2\,\mathrm{d}\tau+\int_0^t \Big[\frac{\zeta^1(\tau)^2}{2\nu\delta_3}+\zeta^0(\tau)\Big]\,\mathrm{d}\tau$$
$$+\,c_4(h). \tag{3.15}$$

Estimate (3.15) holds for all $t\in(0,T)$. Suppose further that $h$ and $\delta_3$ are so small that $c_2(h)+c_3(h)<1$ and $\delta_1+\delta_2+\delta_3<1$. Denote

$$\begin{aligned}
y_h(t) &:= \big[1-c_2(h)-c_3(h)\big]\,\|\boldsymbol{u}_n(t)\|_{2:\,\mathbb{R}^3}^2,\\
\psi_h(t) &:= \|\boldsymbol{u}_0\|_{2:\,\Omega_0}^2+\int_0^t \Big[\frac{\zeta^1(\tau)^2}{4\nu\delta_3}+\frac{\zeta^0(\tau)}{4}\Big]\,\mathrm{d}\tau+c_4(h),\\
\vartheta_h(t) &:= \frac{2c_1\,\|\boldsymbol{a}^0(t)\|_{s;\,\mathbb{R}^3}^{\frac{2s}{s-3}}+\zeta^0(t)}{1-c_2(h)-c_3(h)}, \qquad z_h(t)\;:=\;\int_0^t \vartheta_h(\tau)\,y_h(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

The inequality (3.15) can now be shortly written as

$$y_h(t)+2\nu\,\big[1-(\delta_1+\delta_2+\delta_3)\big]\int_0^t \|\nabla\boldsymbol{u}_n(\tau)\|_{2:\,\mathbb{R}^3}^2\,\mathrm{d}\tau\;\le\;\psi_h(t)+z_h(t). \tag{3.16}$$

Elementary calculations show that $z'(t)-\vartheta(t)\,z(t)\le\vartheta(t)\,\psi(t)$ and therefore

$$z_h(t)\;\le\;\int_0^t \psi_h(\tau)\,\vartheta_h(\tau)\,\exp\biggl(\int_\tau^t \vartheta_h(\sigma)\,\mathrm{d}\sigma\biggr)\,\mathrm{d}\tau.$$

Using this estimate on the right hand side of (3.16) and expressing the functions $y_h(t)$, $\psi_h(t)$, $\vartheta_h$ from of their definition, we obtain

$$\begin{aligned}
&\big[1-c_2(h)-c_3(h)\big]\,\|\boldsymbol{u}_n(t)\|_{2:\,\mathbb{R}^3}^2+2\nu\,\big[1-(\delta_1+\delta_2+\delta_3)\big]\int_0^t \|\nabla\boldsymbol{u}_n(\tau)\|_{2:\,\mathbb{R}^3}^2\,\mathrm{d}\tau\\
&\le\;\psi_h(t)+\int_0^t \psi_h(\tau)\,\vartheta_h(\tau)\,\exp\biggl(\int_\tau^t \vartheta_h(\sigma)\,\mathrm{d}\sigma\biggr)\,\mathrm{d}\tau.
\end{aligned} \tag{3.17}$$

We observe that there exist upper bounds $c_5=c_5(\delta_1,\delta_2,\delta_3,r,s,\boldsymbol{a},\zeta^1,\zeta^0,\nu)$ and $c_6=c_6(\delta_1,\delta_2,\delta_3,r,s,\boldsymbol{a},\zeta^1,\zeta^0,\nu)$, independent of $n$ such that

$$\|\boldsymbol{u}_n(t)\|_{2:\,\mathbb{R}^3}\;\le\;c_5 \qquad \text{for a.a. } t\in(0,T), \tag{3.18}$$
$$\int_0^T \|\nabla\boldsymbol{u}_n(\tau)\|_{2:\,\mathbb{R}^3}^2\,\mathrm{d}\tau\;\le\;c_6 \tag{3.19}$$

for $n$ sufficiently large. Obviously, due to the relation between $\boldsymbol{U}_k^+$ and $\boldsymbol{u}_n$, the same inequalities also hold for $\boldsymbol{U}_k^+$ and $\nabla\boldsymbol{U}_k$:

$$\|\boldsymbol{U}_k^+\|_{2:\,\mathbb{R}^3}\;\le\;c_5 \qquad \text{for } k=1,\dots,n, \tag{3.20}$$
$$h\sum_{k=1}^n \|\nabla\boldsymbol{U}_k^+\|_{2:\,\mathbb{R}^3}^2\;\le\;c_6. \tag{3.21}$$

**Weak convergence of the approximate solutions.** Inequalities (3.18) and (3.19) provide the estimates of the sequence $\{\boldsymbol{u}_n\}$ in the spaces $L^\infty(0,T;\,L^2_\sigma(\mathbb{R}^3))$ and $L^2(0,T;\,W^{1,2}_{0,\sigma}(\mathbb{R}^3))$. Thus, there exists $\boldsymbol{u} \in L^\infty(0,T;\, L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;\, W^{1,2}_{0,\sigma}(\mathbb{R}^3))$ and a subsequence of $\{\boldsymbol{u}_n\}$ (denoted again by $\{\boldsymbol{u}_n\}$) such that

$$\boldsymbol{u}_n \longrightarrow \boldsymbol{u} \qquad \text{as } n \to +\infty \qquad \text{weakly in } L^2(0,T;\, W^{1,2}_{0,\sigma}(\mathbb{R}^3)), \tag{3.22}$$

$$\boldsymbol{u}_n \longrightarrow \boldsymbol{u} \qquad \text{as } n \to +\infty \qquad \text{weakly–}* \text{ in } L^\infty(0,T;\, L^2_\sigma(\mathbb{R}^3)). \tag{3.23}$$

## 4 Verification that the limit function $u$ is a weak solution

We will show that $\boldsymbol{u}$ is a weak solution of (1.5)–(1.8) in this section.

**Supports of $\boldsymbol{u}_n$.** Let us denote

$$Q_n := \bigcup_{k=1}^{n} \Omega_k \times [t_{k-1}, t_k), \qquad \Gamma_n := \bigcup_{k=1}^{n} \partial\Omega_k \times [t_{k-1}, t_k).$$

The support of $\boldsymbol{u}_n$ is a subset of $\overline{Q}_n$.

**Lemma 2.** $\lim_{n\to+\infty} \widehat{d}(Q_n,\, Q_{[0,T)}) = 0$ *and* $\lim_{n\to+\infty} \widehat{d}(\Gamma_n,\, \Gamma_{[0,T)}) = 0.$

*Proof.* Let $\epsilon > 0$ be given. Then due to Assumption 2, there exists $\delta > 0$ such that if $t_1,\, t_2 \in [0,T]$, $|t_1 - t_2| < \delta$ then $\widehat{d}(\Omega^{t_1}, \Omega^{t_2}) < \epsilon$. (The symbol $\widehat{d}$ was defined in Section 1.)

Let us choose $n \in \mathbb{N}$ so large that $h = T/n < \delta$. Each point $(\boldsymbol{x}, t) \in Q_n$ belongs to $\Omega_k \times [t_{k-1}, t_k]$ for some $k \in \{1; \dots; n\}$. Then

$$\mathrm{dist}_4\big((\boldsymbol{x}, t);\, Q_{[0,T)}\big) \le \mathrm{dist}_3\big(\boldsymbol{x}, \Omega^t\big) \le \widehat{d}(\Omega_k, \Omega^t) < \epsilon$$

because $|t - t_k| < \delta$. Hence $d(Q_n, Q_{[0,T)}) < \epsilon$. The inequality $d(Q_{[0,T)}, Q_n) < \epsilon$ (for $n$ sufficiently large) can be proved similarly. This implies the the first equality in Lemma 2. The second equality can be proved in the same way, we only use the continuity of $\widehat{d}(\Gamma^{t_1}, \Gamma^{t_2})$ instead of $\widehat{d}(\Omega^{t_1}, \Omega^{t_2})$. □

**Lemma 3 (on condition (2.1)).** *The identity $\boldsymbol{u} = \boldsymbol{0}$ holds a.e. in $Q^c_{[0,T)}$.*

*Proof.* Let $m \in \mathbb{N}$. Denote by $\mathcal{U}_m(Q_{[0,T)})$ the $\frac{1}{m}$–neighborhood of $Q_{(0,T)}$ in $\mathbb{R}^3 \times [0,T)$. The set $Q^c_{[0,T)}$ can be expressed in the form $Q^c_{[0,T)} = K_{1m} \cup K_{2m}$ where $K_{1m} = Q^c_{[0,T)} \cap \mathcal{U}_m(Q_{[0,T)})$ and $K_{2m} = Q^c_{[0,T)} \smallsetminus K_{1m}$. Put $K_1 := \cap_{m\in\mathbb{N}} K_{1m}$ and $K_2 := Q^c_{[0,T)} \smallsetminus K_1 = \cup_{m\in\mathbb{N}} K_{2m}$. Set $K_1$ contains only points whose distance from $Q_{[0,T)}$ is zero. Hence $K_1 \subset Q_{[0,T)} \cup \Gamma_{[0,T)}$. However, $K_1 \subset Q^c_{[0,T)}$. Thus, $K_1 = \emptyset$ and $Q^c_{[0,T)} = K_2$.

Consider a fixed $m \in \mathbb{N}$ for a while. Due to Lemma 2, the support of $\boldsymbol{u}_n$ is disjoint with $K_{2m}$ for $n$ large enough. We claim that the weak limit $\boldsymbol{u}$ of the sequence $\{\boldsymbol{u}_n\}$ is equal to zero a.e. in $K_{2m}$. Due to (3.22), we have

$$0 = \int_{K_{2m}} \boldsymbol{u}_n \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \longrightarrow \int_{K_{2m}} \boldsymbol{u} \cdot \boldsymbol{u} \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t = \int_{K_{2m}} |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t.$$

This shows that $\boldsymbol{u}$ equals zero a.e. in $K_{2m}$. Since $Q^c_{[0,T)}$ is the union of $K_{2m}$ for $m \in \mathbb{N}$, $\boldsymbol{u}$ equals zero a.e. in $Q^c_{[0,T)}$. □

**Integral identity (2.2).** We will further show that $\boldsymbol{u}$ satisfies (2.2) for each infinitely differentiable and divergence–free test function $\boldsymbol{\phi}$ in $\mathbb{R}^3 \times [0,T)$ that has a compact support in $Q_{[0,T)}$. Thus, let $\boldsymbol{\phi}$ be such a function. We shall consider function $\boldsymbol{\phi}$ to be fixed from now. As it is usual, we denote by $\boldsymbol{\phi}(t)$ the function of spatial variables, which arises from $\boldsymbol{\phi}$ if $t \in [0,T]$ is fixed.

We denote by $\mathcal{I}_n$ the left–hand side of (2.2) with $\boldsymbol{u}_n$ instead of $\boldsymbol{u}$. It can be expressed as follows:

$$\begin{aligned} \mathcal{I}_n \; := \; & \int_0^T \int_{\mathbb{R}^3} \big[ -\boldsymbol{u}_n \cdot \partial_t \boldsymbol{\phi} + \nu \, \nabla \boldsymbol{u}_n : \nabla \boldsymbol{\phi} + (\boldsymbol{u}_n \cdot \nabla) \boldsymbol{a}^0 \cdot \boldsymbol{\phi} + \big( (\boldsymbol{a}^\infty + \boldsymbol{a}^0) \cdot \nabla \big) \boldsymbol{u}_n \cdot \boldsymbol{\phi} \\ & + (\boldsymbol{u}_n \cdot \nabla) \boldsymbol{u}_n \cdot \boldsymbol{\phi} \big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ = \; & \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{\Omega_k} \big[ -\boldsymbol{U}_k \cdot \partial_t \boldsymbol{\phi} + \nu \, \nabla \boldsymbol{U}_k : \nabla \boldsymbol{\phi} + (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{a}^0 \cdot \boldsymbol{\phi} \\ & + \big( (\boldsymbol{a}^\infty + \boldsymbol{a}^0) \cdot \nabla \big) \boldsymbol{U}_k \cdot \boldsymbol{\phi} + (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi} \big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ = \; & \int_{\Omega_1} \boldsymbol{U}_0 \cdot \boldsymbol{\phi}(t_0) \, \mathrm{d}\boldsymbol{x} + \sum_{k=1}^n \int_{\Omega_k} \big[ \boldsymbol{U}_k - \boldsymbol{U}_{k-1} \big] \cdot \boldsymbol{\phi}(t_{k-1}) \, \mathrm{d}\boldsymbol{x} \\ & + \sum_{k=1}^n \int_{\Omega_k} \big[ \nu h \, \nabla \boldsymbol{U}_k : \nabla \boldsymbol{\phi}(t_{k-1}) + h \, (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{a}_k^0 \cdot \boldsymbol{\phi}(t_{k-1}) \\ & + h \, \big( (\boldsymbol{a}^\infty + \boldsymbol{a}_k^0) \cdot \nabla \big) \boldsymbol{U}_k \cdot \boldsymbol{\phi}(t_{k-1}) + h \, (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi}(t_{k-1}) \big] \, \mathrm{d}\boldsymbol{x} \\ & + \mathcal{I}_{n1} + \mathcal{I}_{n2} + \mathcal{I}_{n3} + \mathcal{I}_{n4} + \mathcal{I}_{n5} \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} \mathcal{I}_{n1} &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{\Omega_k} \nu \, \nabla \boldsymbol{U}_k : \nabla [\boldsymbol{\phi}(t) - \boldsymbol{\phi}(t_{k-1})] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \\ \mathcal{I}_{n2} &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{\Omega_k} \Big[ (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{a}^0(t) \cdot \boldsymbol{\phi}(t) - (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{a}^0(t) \cdot \boldsymbol{\phi}(t_{k-1}) \Big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \\ \mathcal{I}_{n3} &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{\Omega_k} \Big[ (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{a}^0(t) \cdot \boldsymbol{\phi}(t_{k-1}) - (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{a}_k^0 \cdot \boldsymbol{\phi}(t_{k-1}) \Big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \\ \mathcal{I}_{n4} &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{\Omega_k} \Big[ (\boldsymbol{a}^0(t) \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi}(t) - (\boldsymbol{a}_k^0 \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi}(t_{k-1}) \Big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t, \\ \mathcal{I}_{n5} &= \sum_{k=1}^n \int_{t_{k-1}}^{t_k} \int_{\Omega_k} \Big[ (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi}(t) - (\boldsymbol{U}_k \cdot \nabla) \boldsymbol{U}_k \cdot \boldsymbol{\phi}(t_{k-1}) \Big] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t. \end{aligned}$$

Due to the smoothness of function $\boldsymbol{\phi}$ and Assumption 2, $\operatorname{supp} \boldsymbol{\phi}(t_{k-1})$ is a subset of $\Omega_k \cap \Omega_{k-1}$ for $n$ large enough. Hence we can replace $\boldsymbol{U}_{k-1}$ by $\boldsymbol{U}_{k-1}^+$ in the integral over $\Omega_k$ on the right hand side of (4.1) and we can use (3.4) with $\boldsymbol{\phi}_k = \boldsymbol{\phi}(t_{k-1})$. We get

$$\mathcal{I}_n = \int_{\Omega_0} \boldsymbol{u}_0 \cdot \boldsymbol{\phi}(0) \, \mathrm{d}\boldsymbol{x} + \int_0^T \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle_{\mathbb{R}^3} \, \mathrm{d}t + \mathcal{I}_{n1} + \mathcal{I}_{n2} + \mathcal{I}_{n3} + \mathcal{I}_{n4} + \mathcal{I}_{n5} + \mathcal{I}_{n6} \tag{4.2}$$

where

$$\mathcal{I}_{n6} \;:=\; \sum_{k=1}^{n} \big\langle \boldsymbol{F}_k,\, \boldsymbol{\phi}(t_{k-1})^+ \big\rangle_{\mathbb{R}^3} - \int_0^T \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle_{\mathbb{R}^3}\, \mathrm{d}t.$$

Let us denote $\Omega_k' := \Omega_k \cap \big[\cup_{0\leq t<T} \operatorname{supp} \boldsymbol{\phi}(t)\big]$. Then the 3D Lebesgue measure of $\Omega_k'$ is bounded above by a constant, depending on $\boldsymbol{\phi}$, but independent of $k$. Now we can estimate $\mathcal{I}_{n1}$ and $\mathcal{I}_{n2}$ by means of (3.20) and (3.21):

$$\begin{aligned}
|\mathcal{I}_{n1}| \;&\leq\; \nu \sum_{k=1}^{n} \bigg| \int_{\Omega_k} \nabla \boldsymbol{U}_k : \bigg( \int_{t_{k-1}}^{t_k} \big[ \nabla\boldsymbol{\phi}(t) - \nabla\boldsymbol{\phi}(t_{k-1}) \big]\, \mathrm{d}t \bigg)\, \mathrm{d}\boldsymbol{x} \bigg| \\
&\leq\; \nu \sum_{k=1}^{n} \int_{\Omega_k} |\nabla \boldsymbol{U}_k|\, \bigg| \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t} \nabla \partial_\tau \boldsymbol{\phi}(\tau)\, \mathrm{d}\tau\, \mathrm{d}t \bigg|\, \mathrm{d}\boldsymbol{x} \\
&\leq\; \nu\, C(\boldsymbol{\phi})\, h^2 \sum_{k=1}^{n} \|\nabla \boldsymbol{U}_k\|_{2:\,\Omega_k} \;\leq\; \sqrt{\nu}\, C(\boldsymbol{\phi})\, \sqrt{h^3}\, \sqrt{n}\, \bigg( \nu h \sum_{k=1}^{n} \|\nabla \boldsymbol{U}_k\|^2_{2:\,\Omega_k} \bigg)^{1/2} \\
&\leq\; \sqrt{\nu}\, C(\boldsymbol{\phi})\, h\, \sqrt{T}\, c_6^{1/2}, \\
|\mathcal{I}_{n2}| \;&\leq\; C(\boldsymbol{\phi})\, h \sum_{k=1}^{n} \|\boldsymbol{U}_k^+\|_{2;\,\Omega_k} \int_{t_{k-1}}^{t_k} \|\nabla \boldsymbol{a}^0(t)\|_{2;\,\Omega_k}\, \mathrm{d}t \\
&\leq\; C(\boldsymbol{\phi})\, h \bigg( h \sum_{k=1}^{n} \|\boldsymbol{U}_k^+\|^2_{2;\,\Omega_k} \bigg)^{1/2} \bigg( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \|\nabla \boldsymbol{a}^0(t)\|^2_{2;\,\Omega_k}\, \mathrm{d}t \bigg)^{1/2} \\
&\leq\; C(\boldsymbol{\phi})\, h\, c_5^{1/2} \bigg( \int_0^T \|\nabla \boldsymbol{a}^0(t)\|^2_{2;\,\Omega_k}\, \mathrm{d}t \bigg)^{1/2}.
\end{aligned}$$

These estimates and identities show that $\mathcal{I}_{n1} \to 0$ and $\mathcal{I}_{n2} \to 0$ if $n \to +\infty$ (and so $h \to 0$). Similar estimates of $\mathcal{I}_{n4}$–$\mathcal{I}_{n6}$ show that all these terms also tend to zero if $n \to +\infty$. The integral $\mathcal{I}_{n3}$ equals zero:

$$\begin{aligned}
\mathcal{I}_{n3} \;&=\; \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \int_{\Omega_k} (\boldsymbol{U}_k^+ \cdot \nabla)\boldsymbol{\phi}(t_{k-1}) \cdot \big[ \boldsymbol{a}^0(t) - \boldsymbol{a}_k^0 \big]\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \\
&=\; \sum_{k=1}^{n} \int_{\Omega_k} (\boldsymbol{U}_k^+ \cdot \nabla)\boldsymbol{\phi}(t_{k-1}) \cdot \bigg( \int_{t_{k-1}}^{t_k} \boldsymbol{a}^0(t)\, \mathrm{d}t - \int_{t_{k-1}}^{t_k} \boldsymbol{a}^0(\tau)\, \mathrm{d}\tau \bigg)\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \;=\; 0.
\end{aligned}$$

Thus, we have proved that

$$\lim_{n\to+\infty} \mathcal{I}_n \;=\; \int_{\Omega_0} \boldsymbol{u}_0 \cdot \boldsymbol{\phi}(0)\, \mathrm{d}\boldsymbol{x} + \int_0^T \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle_{\mathbb{R}^3}\, \mathrm{d}t. \tag{4.3}$$

In order to verify that $\boldsymbol{u}$ is a weak solution of the problem (1.5)–(1.8), we still need to show that

$$\begin{aligned}
\lim_{n\to+\infty} \mathcal{I}_n \;\;&=\;\; \int_0^T \int_{\mathbb{R}^3} \big[ -\boldsymbol{u} \cdot \partial_t \boldsymbol{\phi} + \nu\, \nabla \boldsymbol{u} : \nabla \boldsymbol{\phi} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{a}^0 \cdot \boldsymbol{\phi} \\
&\qquad + \big( (\boldsymbol{a}^\infty + \boldsymbol{a}^0) \cdot \nabla \big) \boldsymbol{u} \cdot \boldsymbol{\phi} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \cdot \boldsymbol{\phi} \big]\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t.
\end{aligned} \tag{4.4}$$

The identity

$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}^3} \big[-\boldsymbol{u}_n\cdot\partial_t\boldsymbol{\phi} + \nu\,\nabla\boldsymbol{u}_n : \nabla\boldsymbol{\phi} + (\boldsymbol{u}_n\cdot\nabla)\boldsymbol{a}^0\cdot\boldsymbol{\phi} + \big((\boldsymbol{a}^\infty+\boldsymbol{a}^0)\cdot\nabla\big)\boldsymbol{u}_n\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$
$$= \int_0^T \int_{\mathbb{R}^3} \big[-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \nu\,\nabla\boldsymbol{u} : \nabla\boldsymbol{\phi} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{a}^0\cdot\boldsymbol{\phi} + \big((\boldsymbol{a}^\infty+\boldsymbol{a}^0)\cdot\nabla\big)\boldsymbol{u}\cdot\boldsymbol{\phi}\big]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \tag{4.5}$$

follows from (3.22) and (3.23). Thus, it remains to show that

$$\lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}^3} (\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n\cdot\boldsymbol{\phi}\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \;=\; \int_0^T \int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.6}$$

It follows from the definition of $\mathcal{I}_n$, (4.3) and (4.5) that the limit on the left hand side exists. Thus, to show that it equals the right hand side of (4.6), it is sufficient to check the value of the limit for an arbitrary subsequence of $\{\boldsymbol{u}_n\}$. It is an objective of next paragraphs.

Let us denote by $d$ the positive distance $\mathrm{dist}_4(\mathrm{supp}\,\boldsymbol{\phi}, \Gamma_{[0,T]})$. Due to Assumption 2, there exists $m\in\mathbb{N}$ such that if $t_1,\, t_2\in[0,T]$, $|t_1-t_2|\le 2T/m$ then $\widehat{d}_3(\Gamma^{t_1},\Gamma^{t_2}) < \frac{1}{10}d$.

Further, we denote $\tau_j = jT/m$ (for $j=1,\dots,m$). There exist $m+1$ infinitely differentiable functions $\theta_0,\dots,\theta_m$ on $[0,T]$ such that $0\le\theta_j\le 1$ and

$$\begin{aligned}
&\mathrm{supp}\,\theta_0 \subset J_0 := (\tau_0,\tau_1),\\
&\mathrm{supp}\,\theta_j \subset J_j := (\tau_{j-1},\tau_{j+1}) \qquad \text{for } j=1,\dots,m-1,\\
&\mathrm{supp}\,\theta_m \subset J_m := (\tau_{m-1},\tau_m),\\
&\sum_{j=0}^{m} \theta_j(t) = 1 \quad \text{for } 0\le t\le T.
\end{aligned}$$

Now we put $\boldsymbol{\phi}_j := \theta_j\boldsymbol{\phi}$. Let $K_j$ be the orthogonal projection of $\mathrm{supp}\,\boldsymbol{\phi}_j$ onto $\mathbb{R}^3$. If $t,\, s\in J_j$ then

$$\mathrm{dist}_3(\mathrm{supp}\,\boldsymbol{\phi}(t),\Gamma^s) \;\ge\; \mathrm{dist}_3(\mathrm{supp}\,\boldsymbol{\phi}(t),\Gamma^t) - \widehat{d}_3(\Gamma^t,\Gamma^s) \;>\; d-\tfrac{1}{10}d \;=\; \tfrac{9}{10}d.$$

Hence $\forall\, s\in\overline{J_j}\;:\;\mathrm{dist}_3(K_j,\Gamma^s) > \frac{9}{10}d$. There exists a bounded open set $\Omega'_j$ in $\mathbb{R}^3$ with the boundary of the class $C^{1,1}$ that has a finite number of components and $K_j\subset\Omega'_j\subset\overline{\Omega'_j}\subset\Omega^s$ (for all $s\in\overline{J_j}$).

In order to prove (4.6), it is sufficient to show that

$$\lim_{n\to+\infty} \int_{J_j} \int_{\Omega'_j} (\boldsymbol{u}_n\cdot\nabla)\boldsymbol{u}_n\cdot\boldsymbol{\phi}_j\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \;=\; \int_{J_j} \int_{\Omega'_j} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}_j\;\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \tag{4.7}$$

for all $j=0,1,\dots,m$. We denote by $P_\sigma^j$ the Helmholtz projection in $L^2(\Omega'_j)^3$. Put $\boldsymbol{w}_n^j := P_\sigma^j\boldsymbol{u}_n$. The function $[I-P_\sigma^j]\boldsymbol{u}_n$ has the form $\nabla\varphi_n^j$ for an appropriate scalar function $\varphi_n^j$. (4.7) can now be written as

$$\lim_{n\to+\infty} \int_{J_j} \int_{\Omega'_j} \Big[(\boldsymbol{w}_n^j\cdot\nabla)\boldsymbol{w}_n^j\cdot\boldsymbol{\phi}_j + (\boldsymbol{w}_n^j\cdot\nabla)\nabla\varphi_n^j\cdot\boldsymbol{\phi}_j + (\nabla\varphi_n^j\cdot\nabla)\boldsymbol{w}_n^j\cdot\boldsymbol{\phi}_j$$

$$+ (\nabla\varphi_n^j \cdot \nabla)\nabla\varphi_n^j \cdot \boldsymbol{\phi}_j \Big]\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}t \;=\; \int_{J_j} \int_{\Omega'_j} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}_j\; \mathrm{d}\boldsymbol{x}\, \mathrm{d}t. \tag{4.8}$$

Since $(\nabla\varphi_n^j \cdot \nabla)\nabla\varphi_n^j = \nabla\big(\frac{1}{2}|\nabla\varphi_n^j|^2\big)$ and $\boldsymbol{\phi}(t) \in L^2_\sigma(\Omega'_j)$, the integral of $(\nabla\varphi_n^j \cdot \nabla)\nabla\varphi_n^j \cdot \boldsymbol{\phi}_j$ on $\Omega'_j$ equals zero.

The convergence (3.22) and (3.23) and the boundedness of operator $P_\sigma^j$ in $L^2(\Omega'_j)^3$ and in $W^{1,2}(\Omega'_j)^3$ imply that

$$\boldsymbol{w}_n^j \;\longrightarrow\; \boldsymbol{w}^j = P_\sigma^j \boldsymbol{u}, \quad \text{and} \quad \nabla\varphi_n^j \longrightarrow \nabla\varphi^j = [I - P_\sigma^j]\boldsymbol{u} \quad \text{for } \; n \to +\infty \tag{4.9}$$

weakly in $L^2(J_j;\, W^{1,2}(\Omega'_j)^3)$ and weakly–$*$ in $L^\infty(J_j;\, L^2_\sigma(\Omega'_j))$.

**Strong convergence of a subsequence of $\{\boldsymbol{w}_n^j\}$.** We are going to show that there exists a subsequence of $\{\boldsymbol{w}_n^j\}$ that tends to $\boldsymbol{w}^j$ strongly in $L^2(J_j;\, L^2_\sigma(\Omega'_j))$ as $n \to +\infty$. We shall therefore use the next lemma, see Theorem 5.2 in J. L. Lions [19].

**Lemma 4.** *Let $0 < \gamma < \frac{1}{2}$ and let $H_0$, $H$ and $H_1$ be Hilbert spaces such that $H_0 \hookrightarrow\hookrightarrow H \hookrightarrow H_1$. Let $\mathcal{H}^\gamma(\mathbb{R};\, H_0,\, H_1)$ denote the Banach space $\big\{w \in L^2(\mathbb{R};\, H_0);\; |\vartheta|^\gamma\, \hat{w}(\vartheta) \in L^2(\mathbb{R};\, H_1)\big\}$ with the norm*

$$\|w\|_{\mathcal{H}^\gamma(\mathbb{R};\, H_0,\, H_1)} \;:=\; \big(\|w\|^2_{L^2(\mathbb{R};\, H_0)} + \|\, |\vartheta|^\gamma\, \hat{w}(\vartheta)\|^2_{L^2(\mathbb{R};\, H_1)}\big)^{1/2}.$$

*(Here $\hat{w}(\vartheta)$ is the Fourier transform of $w(t)$.) Let $\mathcal{H}^\gamma(a,b;\, H_0,\, H_1)$ further denote the Banach space of restrictions of functions from $\mathcal{H}^\gamma(\mathbb{R};\, H_0,\, H_1)$ onto the interval $(a,b)$, with the norm*

$$\|w\|_{\mathcal{H}^\gamma(a,b;\, H_0,\, H_1)} \;:=\; \inf\; \|z\|_{\mathcal{H}^\gamma(\mathbb{R};\, H_0,\, H_1)}$$

*where the infimum is taken over all $z \in \mathcal{H}^\gamma(\mathbb{R};\, H_0,\, H_1)$ such that $z = w$ a.e. in $(a,b)$. Then $\mathcal{H}^\gamma(0,T;\, H_0,\, H_1) \hookrightarrow\hookrightarrow L^2(a,b;\, H)$.*

Consider $j \in \{1;\, \dots;\, m\}$ fixed. We shall use Lemma 4 with $(a,b) = J_j$, $H_0 = W^{1,2}(\Omega'_j)^3 \cap L^2_\sigma(\Omega'_j)$, $H = L^2_\sigma(\Omega'_j)$ and $H_1 = W^{-1,2}_{0,\sigma}(\Omega'_j)$. We claim that $\{\boldsymbol{w}_n^j\}$ is bounded in the space $\mathcal{H}^\gamma(J_j;\, H_0,\, H_1)$. The boundedness of $\{\boldsymbol{w}_n^j\}$ in $L^2(J_j;\, H_0)$ follows from (3.18), (3.19) and from the boundedness of operator $P_\sigma^j$ in $L^2(\Omega'_j)^3$ and in $W^{1,2}(\Omega'_j)^3$. Thus, we only need to verify that $\{|\vartheta|^\gamma\, \hat{\boldsymbol{w}^j}_n\}$ is bounded in the space $L^2(J_j;\, H_1)$, i.e. in $L^2(J_j;\, W^{-1,2}_{0,\sigma}(\Omega'_j))$. Let $\boldsymbol{z}_n^j$ be an extension by zero of $\boldsymbol{w}_n^j$ from the time interval $J_j$ onto $\mathbb{R}$. Then

$$\hat{\boldsymbol{z}^j}_n(\vartheta) \;=\; \int_{J_j} \mathrm{e}^{-2\pi\mathrm{i}t\vartheta}\, \boldsymbol{w}_n^j(t)\, \mathrm{d}t \;=\; \sum_{k\in\Lambda_j^n} \int_{t_{k-1}}^{t_k} \mathrm{e}^{-2\pi\mathrm{i}t\vartheta}\, P_\sigma^j \boldsymbol{U}_k\, \mathrm{d}t \tag{4.10}$$

where $\Lambda_j^n$ is the set of such indices $k \in \{1;\dots;n\}$ that $[\mathbb{R}^3 \times (t_{k-1}, t_k)] \cap \operatorname{supp} \boldsymbol{\phi}_j \neq \emptyset$. $\Lambda_j^n$ has the form $\Lambda_j^n = \{l; l+1; \dots; q\}$ where $1 \le l \le q \le n$. Calculating the integrals in (4.10), we obtain

$$\hat{\boldsymbol{z}}_n^j(\vartheta) \;=\; \sum_{k=l}^{q} \frac{1}{2\pi\mathrm{i}\vartheta} \left[\mathrm{e}^{-2\pi\mathrm{i}\, t_{k-1}\vartheta} - \mathrm{e}^{-2\pi\mathrm{i}\, t_k\vartheta}\right] P_\sigma^j \boldsymbol{U}_k$$

$$= \frac{1}{2\pi i\vartheta}\left[e^{-2\pi i\, t_{l-1}}\, P_\sigma^j \boldsymbol{U}_l - e^{-2\pi i\, t_q}\, P_\sigma^j \boldsymbol{U}_q\right] + \frac{1}{2\pi i\vartheta}\sum_{k=l+1}^{q} e^{-2\pi i\, t_{k-1}\vartheta}\,[P_\sigma^j \boldsymbol{U}_k - P_\sigma^j \boldsymbol{U}_{k-1}]. \quad (4.11)$$

Since $\Omega'_j \subset \Omega^s$ for all $s \in J_j$, we also have $\Omega'_j \subset \Omega_k$ for all $k \in \Lambda_j^n$ (if $n$ is large enough). If $|\vartheta| \leq 1$ then, using (4.10) and (3.20), we can estimate $\|\, |\vartheta|^\gamma\, \hat{\boldsymbol{z}}_n^j(\vartheta)\|_{-1,2;\,\Omega'_j}$ as follows:

$$\|\, |\vartheta|^\gamma\, \hat{\boldsymbol{z}}_n^j(\vartheta)\|_{-1,2;\,\Omega'_j} \;\leq\; C(\Omega'_j)\, |\vartheta|^\gamma \sum_{k=l}^{q} h\, \|\boldsymbol{U}_k\|_{2:\,\Omega'_j} \;\leq\; C(\Omega'_j)\, \sqrt{c_5}\, |\vartheta|^\gamma. \quad (4.12)$$

If $|\vartheta| > 1$ then, using (4.11), we get

$$\begin{aligned}
\|\, |\vartheta|^\gamma\, \hat{\boldsymbol{z}}_n(\vartheta)\|_{-1,2;\,\Omega'_j} \;\leq\;& \frac{|\vartheta|^{\gamma-1}}{2\pi}\left(\|P_\sigma^j \boldsymbol{U}_l\|_{-1,2;\,\Omega'_j} + \|P_\sigma^j \boldsymbol{U}_q\|_{-1,2;\,\Omega'_j}\right) \\
&+ \frac{|\vartheta|^{\gamma-1}}{2\pi}\sum_{k=l+1}^{q} \|P_\sigma^j \boldsymbol{U}_k - P_\sigma^j \boldsymbol{U}_{k-1}\|_{-1,2;\,\Omega'_j} \\
\leq\;& C(\Omega'_j)\, |\vartheta|^{\gamma-1}\left(\|\boldsymbol{U}_l\|_{2:\,\Omega'_j} + \|\boldsymbol{U}_q\|_{2:\,\Omega'_j}\right) \\
&+ \frac{|\vartheta|^{\gamma-1}}{2\pi}\sum_{k=l+1}^{q} \sup_{\psi} \frac{1}{\|\boldsymbol{\psi}\|_{1,2;\,\Omega'_j}}\left|\int_{\Omega'_j} (\boldsymbol{U}_k - \boldsymbol{U}_{k-1})\cdot\boldsymbol{\psi}\, d\boldsymbol{x}\right|.
\end{aligned}$$

The supremum is taken over all $\boldsymbol{\psi} \in W_{0,\sigma}^{1,2}(\Omega'_j)$ such that $\|\boldsymbol{\psi}\|_{1,2;\,\Omega'_j} > 0$. The function $\boldsymbol{\psi}^+$ (i.e. $\boldsymbol{\psi}$ extended by zero to $\mathbb{R}^3 \smallsetminus \Omega'_j$) belongs to $W_{0,\sigma}^{1,2}(\Omega_k)$ for all $k = l, \dots, q$. Moreover, $\boldsymbol{U}_{k-1}$ coincides with $\boldsymbol{U}_{k-1}^+$ in $\Omega'_j$. Hence the integral of $(\boldsymbol{U}_k - \boldsymbol{U}_{k-1})\cdot\boldsymbol{\psi}$ on $\Omega'_j$ equals the integral of the same function on $\Omega_k$ and it can be therefore expressed by means of (3.4). Thus, using also (3.20), (3.21), (3.6) and condition (ii), we obtain

$$\begin{aligned}
\|\, |\vartheta|^\gamma\, \hat{\boldsymbol{z}}_n(\vartheta)\|_{-1,2;\,\Omega'_j} \;\leq\;& C(\Omega'_j)\, |\vartheta|^{\gamma-1}\sqrt{c_5} \\
&+ \frac{|\vartheta|^{\gamma-1}}{2\pi}\sum_{k=l+1}^{q} \sup_{\psi} \frac{1}{\|\boldsymbol{\psi}\|_{1,2;\,\Omega'_j}}\left|\int_{\Omega'_j}\left[-\nu h\, \nabla\boldsymbol{U}_k : \nabla\boldsymbol{\psi} + h\,(\boldsymbol{U}_k\cdot\nabla)\boldsymbol{\psi}\cdot\boldsymbol{a}_k^0\right.\right. \\
&\left.\left. - h\,(\boldsymbol{a}^\infty\cdot\nabla)\boldsymbol{U}_k\cdot\boldsymbol{\psi} + (\boldsymbol{a}_k^0\cdot\nabla)\boldsymbol{\psi}\cdot\boldsymbol{U}_k - h\,(\boldsymbol{U}_k\cdot\nabla)\boldsymbol{U}_k\cdot\boldsymbol{\psi}\right] d\boldsymbol{x} + h\,\langle \boldsymbol{F}_k, \boldsymbol{\psi}^+\rangle_{\mathbb{R}^3}\right| \\
\leq\;& C\sqrt{c_5}\, |\vartheta|^{\gamma-1} + \frac{|\vartheta|^{\gamma-1}}{2\pi}\sum_{k=l+1}^{q} \sup_{\psi} \frac{1}{\|\boldsymbol{\psi}\|_{1,2;\,\Omega'_j}}\Big\{\nu h\, \|\nabla\boldsymbol{U}_k\|_{2:\,\Omega'_j}\, \|\nabla\boldsymbol{\psi}\|_{2:\,\Omega'_j} \\
&+ 2h\, \|\boldsymbol{U}_k\|_{\frac{2s}{s-2};\,\Omega'_j}\, \|\nabla\boldsymbol{\psi}\|_{2:\,\Omega'_j}\, \|\boldsymbol{a}_k^0\|_{s;\,\Omega'_j} + h\, |\boldsymbol{a}^\infty|\, \|\nabla\boldsymbol{U}_k\|_{2:\,\Omega'_j}\, \|\boldsymbol{\psi}\|_{2;\,\Omega'_j} \\
&+ h\, \|\boldsymbol{U}_k\|_{2:\,\Omega'_j}^{1/2}\, \|\nabla\boldsymbol{U}_k\|_{2:\,\Omega'_j}^{3/2}\, \|\boldsymbol{\psi}\|_{6;\,\Omega'_j} \\
&+ \int_{t_{k-1}}^{t_k}\left[\zeta^0(t)\, \|\boldsymbol{\psi}\|_{2:\,\Omega'_j} + \zeta^1(t)\, \|\nabla\boldsymbol{\psi}\|_{2:\,\Omega'_j}\right] dt\Big\} \\
\leq\;& C\, |\vartheta|^{\gamma-1} + h\, C\, |\vartheta|^{\gamma-1}\sum_{k=2}^{n}\Big\{\nu\, \|\nabla\boldsymbol{U}_k\|_{2:\,\Omega_k} \\
&+ \|\boldsymbol{U}_k\|_{2:\,\Omega'_j}^{\frac{s-3}{s}}\, \|\boldsymbol{U}_k\|_{6;\,\Omega'_j}^{\frac{3}{s}}\, \|\boldsymbol{a}_k^0\|_{s;\,\Omega'_j} + \|\nabla\boldsymbol{U}_k\|_{2:\,\Omega_k} + \|\boldsymbol{U}_k\|_{2:\,\Omega_k}^{1/2}\, \|\nabla\boldsymbol{U}_k\|_{2:\,\Omega_k}^{3/2}\Big\}
\end{aligned}$$

$$\leq \; C\,|\vartheta|^{\gamma-1} + C\,|\vartheta|^{\gamma-1}\,(\nu+1)\left(\sum_{k=2}^{n} h\,\|\nabla \boldsymbol{U}_k\|_{2;\,\Omega_k}^2\right)^{1/2}$$
$$+\, C\,|\vartheta|^{\gamma-1}\left(\sum_{k=2}^{n} h\,\|\boldsymbol{U}_k\|_{2;\,\Omega_k}^{2\,\frac{s-3}{s}}\right)^{\frac{1}{2}}\left(\sum_{k=2}^{n} h\,\|\nabla \boldsymbol{U}_k\|_{2;\,\Omega_k}^2\right)^{\frac{3}{2s}}\left(\sum_{k=2}^{n} h\,\|\nabla \boldsymbol{a}_k^0\|_{s;\,\mathbb{R}^3}^r\right)^{\frac{1}{r}}$$
$$+\, C\,|\vartheta|^{\gamma-1}\left(\sum_{k=2}^{n} h\,\|\nabla \boldsymbol{U}_k\|_{2;\,\Omega_k}^2\right)^{3/4}$$
$$\leq \; C\,|\vartheta|^{\gamma-1} + C\,|\vartheta|^{\gamma-1}\left(\sum_{k=2}^{n} h\,\|\boldsymbol{a}_k^0\|_{s;\,\mathbb{R}^3}^r\right)^{1/r} \;\leq\; C\,|\vartheta|^{\gamma-1}. \tag{4.13}$$

(We have used the interpolation inequality

$$\|\boldsymbol{U}_k\|_{\frac{2s}{s-2};\,\Omega'_j} \;\leq\; \|\boldsymbol{U}_k\|_{2;\,\Omega'_j}^{(s-3)/s}\,\|\boldsymbol{U}_k\|_{6;\,\Omega'_j}^{3/s} \;\leq\; \|\boldsymbol{U}_k\|_{2;\,\Omega_k}^{(s-3)/s}\,\|\boldsymbol{U}_k\|_{6;\,\Omega_k}^{3/s}$$

and the Sobolev inequality; see [8], p. 31. The generic constant $C$ may depend on $\Omega'_j$, $\boldsymbol{a}^\infty$, $T$, $\nu$, $c_5$ and $c_6$.) These estimates hold for $|\vartheta| > 1$ and the constant $C$ is independent of $n$. Since $0 < \gamma < \frac{1}{2}$, we observe that the sequence $\{|\vartheta|^\gamma\,\hat{\boldsymbol{z}}_n^j(\vartheta)\}$ is bounded in $L^2(\mathbb{R};\,W_{0,\sigma}^{-1,2}(\Omega'_j))$. Consequently, the sequence $\{\boldsymbol{w}_n^j\}$ is bounded in $\mathcal{H}^\gamma(J_j;\,W^{1,2}(\Omega'_j)^3,\,W_{0,\sigma}^{-1,2}(\Omega'_j))$. This space is reflexive, hence there exists a subsequence (we denote it again by $\{\boldsymbol{w}_n^j\}$) which converges weakly in $\mathcal{H}^\gamma(J_j;\,W^{1,2}(\Omega'_j)^3,\,W_{0,\sigma}^{-1,2}(\Omega'_j))$. Due to (4.9), the limit must be $\boldsymbol{w}^j$. Applying now Lemma 4, we have: $\boldsymbol{w}_n^j \longrightarrow \boldsymbol{w}^j = P_\sigma^j \boldsymbol{u}$ strongly in $L^2(J_j;\,L^2(\Omega'_j)^3)$. This strong convergence, together with the weak convergence (4.9), enables us to pass to the limit in the first three terms on the left hand side of (4.7). This procedure is standard (see e.g. J. L. Lions [19] or R. Temam [27]), therefore we omit the details. Using also the equality

$$\int_{\Omega'_j} (\nabla\varphi\cdot\nabla)\nabla\varphi\cdot\boldsymbol{\phi}_j\,\mathrm{d}\boldsymbol{x} \;=\; 0,$$

following from the inclusion $\boldsymbol{\phi}_j \in L^2_\sigma(\Omega'_j)$ and from the identity $(\nabla\varphi\cdot\nabla)\nabla\varphi = \nabla\big(\frac{1}{2}|\nabla\varphi|^2\big)$, we can prove the validity of (4.8). We have thus completed the proof of the theorem:

**Theorem 1 (existence of a weak solution).** *Suppose that Assumptions 1 and 2 hold. Then the problem (1.5)–(1.8) has a weak solution that coincides with function $\boldsymbol{u}$ from (3.22) and (3.23).*

## 5 The local weak continuity of the weak solution and the energy–type inequality

**Theorem 2 (the local weak continuity).** *Suppose that Assumptions 1 and 2 hold. The weak solution $\boldsymbol{u}$ of the problem (1.5)–(1.8), given by (3.22) and (3.23), can be modified on a set of measure zero so that if $t_0 \in [0,T)$ and $\boldsymbol{\psi} \in L^2_\sigma(\Omega^{t_0})$ then*

$$\lim_{t\to t_0,\;0\leq t<T} \big(\boldsymbol{u}(t),\boldsymbol{\psi}\big)_{2;\,\Omega^{t_0}} \;=\; \big(\boldsymbol{u}(t_0),\boldsymbol{\psi}\big)_{2;\,\Omega^{t_0}}. \tag{5.1}$$

We call this type of weak continuity of $\boldsymbol{u}$ the “local weak continuity” because the space from which we can choose function $\boldsymbol{\psi}$ (i.e. $L^2_\sigma(\Omega^{t_0})$) depends on $t_0$.

*Proof.* Suppose for simplicity that $t_0 > 0$. There exists a sequence $\{\boldsymbol{\psi}_n\}$ in $C^\infty_{0,\sigma}(\Omega^{t_0})$ such that $\boldsymbol{\psi}_n \to \boldsymbol{\psi}$ in $L^2_\sigma(\Omega^{t_0})$. Due to Assumptions 1 and 2, there exist $\xi_n > 0$ such that $\operatorname{supp}\boldsymbol{\psi}_n \subset \Omega^t$ for $t_0 - \xi_n < t < t_0 + \xi_n$. Since

$$\big|\big(\boldsymbol{u}(t)-\boldsymbol{u}(t_0),\boldsymbol{\psi}\big)_{2;\,\Omega^{t_0}}\big| \;\leq\; \big|\big(\boldsymbol{u}(t)-\boldsymbol{u}(t_0),\boldsymbol{\psi}-\boldsymbol{\psi}_n\big)_{2;\,\Omega^{t_0}}\big| + \big|\big(\boldsymbol{u}(t)-\boldsymbol{u}(t_0),\boldsymbol{\psi}_n\big)_{2;\,\Omega^{t_0}}\big|$$

and the first term on the right hand side can be made arbitrarily small by choosing $n$ sufficiently large, it is sufficient to prove (5.1) with a fixed function $\boldsymbol{\psi}_n$ instead of $\boldsymbol{\psi}$. Denote by $\Omega'$ a bounded sub–domain of $\Omega^t$ (for all $t \in (t_0-\xi_n, t_0+\xi_n)$), containing $\operatorname{supp}\boldsymbol{\psi}_n$.

Let us now return to the integral identity (2.2) in Definition 1. Considering $\boldsymbol{\phi}$ with a compact support in $\Omega' \times (t_0-\xi_n, t_0+\xi_n)$, we have

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^3} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \;=\; \left|\int_{t_0-\xi_n}^{t_0+\xi_n} \int_{\Omega'} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\right| \\
&\quad\leq\; \int_{t_0-\xi_n}^{t_0+\xi_n} \|\nabla\boldsymbol{u}\|_{2;\,\Omega'}\,\|\boldsymbol{u}\|_{3;\,\Omega'}\,\|\boldsymbol{\phi}\|_{6;\,\Omega'}\,\mathrm{d}t \;\leq\; \int_{t_0-\xi_n}^{t_0+\xi_n} \|\nabla\boldsymbol{u}\|^{3/2}_{2;\,\Omega'}\,\|\boldsymbol{u}\|^{1/2}_{2;\,\Omega'}\,\|\boldsymbol{\phi}\|_{6;\,\Omega'}\,\mathrm{d}t \\
&\quad\leq\; \sqrt{c_5}\,C\left(\int_{t_0-\xi_n}^{t_0+\xi_n} \|\nabla\boldsymbol{u}\|^2_{2;\,\Omega'}\,\mathrm{d}t\right)^{3/4}\left(\int_{t_0-\xi_n}^{t_0+\xi_n} \|\nabla\boldsymbol{\phi}\|^4_{2;\,\Omega'}\,\mathrm{d}t\right)^{1/4}, \\
&\left|\int_0^T \langle \boldsymbol{f},\boldsymbol{\phi}\rangle_{\mathbb{R}^3}\,\mathrm{d}t\right| \;=\; \left|\int_{t_0-\xi_n}^{t_0+\xi_n} \big[\zeta^1(t)\,\|\nabla\boldsymbol{\phi}(t)\|_{2;\,\Omega'} + \zeta^0(t)\,\|\boldsymbol{\phi}(t)\|_{2;\,\Omega'}\big]\,\mathrm{d}t\right| \\
&\quad\leq\; C \sup_{t_0-\xi_n<t<t_0+\xi_n} \mathrm{ess}\, \|\boldsymbol{\phi}(t)\|_{1,2;\,\Omega'}.
\end{aligned}$$

We can similarly estimate all other terms in (2.2), except for the first term which equals the integral $\int_{t_0-\xi_n}^{t_0+\xi_n}\int_{\Omega'} \boldsymbol{u}\cdot\partial_t\boldsymbol{\phi}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$. Thus, we can deduce that the time derivative (in the sense of distributions) of $\boldsymbol{u}$ belongs to $L^1(t_0-\xi_n, t_0+\xi_n;\, W^{-1,2}_{0,\sigma}(\Omega'))$. Hence $\boldsymbol{u}$ can be modified on a set of measure zero so that it becomes an element of $W^{-1,2}_{0,\sigma}(\Omega')$, continuously depending on $t$ in $(t_0-\xi_n, t_0+\xi_n)$. This property of $\boldsymbol{u}$ implies that

$$\begin{aligned}
&\big|\big(\boldsymbol{u}(t)-\boldsymbol{u}(t_0),\boldsymbol{\psi}_n\big)_{2;\,\Omega^{t_0}}\big| \;=\; \big|\big(\boldsymbol{u}(t)-\boldsymbol{u}(t_0),\,\boldsymbol{\psi}_n\big)_{2;\,\Omega'}\big| \\
&\quad\leq\; \big\|\boldsymbol{u}(t)-\boldsymbol{u}(t_0)\big\|_{-1,2;\,\Omega'}\,\|\boldsymbol{\psi}_n\|_{1,2;\,\Omega'} \;\longrightarrow\; 0 \quad \text{for } t \to t_0.
\end{aligned}$$

Thus, (5.1) holds with $\boldsymbol{\psi} = \boldsymbol{\psi}_n \in C^\infty_{0,\sigma}(\Omega^{t_0})$. Hence it also holds with $\boldsymbol{\psi} \in L^2_\sigma(\Omega^{t_0})$.

Since $Q_{[0,T)}$ can be expressed as a countable union of cylinders of the type $\Omega' \times (t-\xi, t+\xi)$ (or $\Omega' \times [0,\xi)$), where $\Omega'$ is a bounded domain in $\mathbb{R}^3$, $\xi > 0$ and the closure of each cylinder is a subset of $Q_{[0,T)}$, the modification of function $\boldsymbol{u}$ on a set of measure zero in $Q_{(0,T)}$ can be made independently of a concrete choice of $t_0 \in [0,T)$ and function $\boldsymbol{\psi}$. □

**The energy–type inequality.** Let $t \in (0,T)$. Inequality (3.18) provides the uniform estimate of $\boldsymbol{u}_n(t)$ in $L^2_\sigma(\mathbb{R}^3)$. Thus, there exists a subsequence $\{\boldsymbol{u}^t_n\}$ of $\{\boldsymbol{u}_n\}$ and $\boldsymbol{u}^t \in L^2_\sigma(\mathbb{R}^3)$ such that $\boldsymbol{u}^t_n(t) \to \boldsymbol{u}^t$ weakly in $L^2_\sigma(\mathbb{R}^3)$. The norm of $\boldsymbol{u}^t$ satisfies

$$\|\boldsymbol{u}^t\|_{2;\,\mathbb{R}^3} \;\leq\; \lim_{n\to+\infty}\ \inf\ \|\boldsymbol{u}^t_n(t)\|_{2;\,\mathbb{R}^3}.$$

Due to (3.23), $\boldsymbol{u}^t = \boldsymbol{u}(t)$, with a possible exception of a set of $t \in (0,T)$ of measure zero. We can obviously modify $\boldsymbol{u}(t)$ at these exceptional times $t$ so that $\|\boldsymbol{u}(t)\|_{2;\,\mathbb{R}^3} = \|\boldsymbol{u}^t\|_{2;\,\mathbb{R}^3}$ for all

$t \in (0,T)$. The modification can be made only in the complementary set $\Omega^t_c$ to $\Omega^t$, so that it does not disturb the modification of $\boldsymbol{u}$ in $Q_{(0,T)}$, considered in the proof of the Theorem 2. Passing now to the limit (for $n \to +\infty$) in the inequality (3.17) (where we consider $\boldsymbol{u}^t_n$ instead of $\boldsymbol{u}_n$), and also using the weak convergence (3.22) and the limits $c_2(h) \to 0$, $c_3(h) \to 0$, $c_4(h) \to 0$ as $h \to 0+$, we obtain the inequality

$$\begin{aligned} &\|\boldsymbol{u}(t)\|^2_{2;\,\mathbb{R}^3} + 2\nu\left[1-(\delta_1+\delta_2+\delta_3)\right] \int_0^t \|\nabla\boldsymbol{u}(\tau)\|^2_{2;\,\mathbb{R}^3}\,\mathrm{d}\tau \\ &\quad\leq\ \psi_0(t) + \int_0^t \psi_0(\tau)\,\vartheta_0(\tau)\,\exp\biggl(\int_\tau^t \vartheta_0(\sigma)\,\mathrm{d}\sigma\biggr)\,\mathrm{d}\tau. \end{aligned} \tag{5.2}$$

Recall that $\delta_1$, $\delta_2$ and $\delta_3$ are arbitrary positive numbers (the interesting case is $\delta_1+\delta_2+\delta_3 < 1$) and $c_1 = c_1(\delta_1,\delta_2,r,s,\nu)$ is given by (3.9). This inequality provides the upper bound for the kinetic energy, associated with the flow field $\boldsymbol{u}$, at time $t$ and for the dissipation of mechanical energy in the time interval $(0,t)$. We can thus formulate the following theorem:

**Theorem 3 (the energy–type inequality).** *Suppose that Assumptions 1 and 2 hold. The weak solution $\boldsymbol{u}$ of the problem (1.5)–(1.8), given by (3.22) and (3.23), can be modified on a set of measure zero so that it satisfies the inequality (5.2) for all $t \in (0,T)$. The modification does not influence the local weak continuity of $\boldsymbol{u}$, stated in Theorem 2.*

We call the inequality (5.2) the "energy–type inequality" and not merely the "energy inequality", because it contains artificial parameters $\delta_1$, $\delta_2$ and $\delta_3$, constant $c_1$ and functions $\zeta^1$ and $\zeta^0$, which rather overshadow the affect of the viscosity, the boundary conditions and the specific body force on the development of the kinetic energy associated with the velocity field $\boldsymbol{u}$ and its dissipation. We discuss the validity of another form of the energy inequality, which seems to be in some sense more natural than (5.2), in the next paragraph.

**An open question.** Since $\boldsymbol{u}_n(\tau)$ (for $t_{j-1} < \tau < t_j$) equals zero a.e. in $\mathbb{R}^3 - \Omega_j$, we can write the inequality (3.13) in the form

$$\begin{aligned} &\|\boldsymbol{u}_n(t)\|^2_{2;\,\mathbb{R}^3} + 2\nu \int_0^t \|\nabla\boldsymbol{u}_n(\tau)\|^2_{2;\,\mathbb{R}^3}\,\mathrm{d}\tau\ \leq\ \|\boldsymbol{u}_0\|^2_{2;\,\Omega_0} \\ &\quad + 2\int_0^t\int_{\mathbb{R}^3} \bigl(\boldsymbol{u}_n(\tau)\cdot\nabla\bigr)\boldsymbol{u}_n(\tau)\cdot\boldsymbol{a}^0(\tau)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau + 2\int_0^t \langle\boldsymbol{f}(\tau),\boldsymbol{u}_n(\tau)\rangle_{\mathbb{R}^3}\,\mathrm{d}\tau + c_7(h) \end{aligned} \tag{5.3}$$

for $0 < t < T$, where $c_7(h) \to 0$ as $h \to 0$. There arises a question whether we can also pass to the limit in this inequality (with the sequence $\{\boldsymbol{u}_n\}$ or at least with a subsequence of $\{\boldsymbol{u}_n\}$) and derive the inequality

$$\begin{aligned} &\|\boldsymbol{u}(t)\|^2_{2;\,\mathbb{R}^3} + 2\nu \int_0^t \|\nabla\boldsymbol{u}_n(\tau)\|^2_{2;\,\mathbb{R}^3}\,\mathrm{d}\tau\ \leq\ \|\boldsymbol{u}_0\|^2_{2;\,\Omega^0} \\ &\quad + 2\int_0^t\int_{\mathbb{R}^3} \bigl(\boldsymbol{u}(\tau)\cdot\nabla\bigr)\boldsymbol{u}(\tau)\cdot\boldsymbol{a}^0(\tau)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau + 2\int_0^t \langle\boldsymbol{f}(\tau),\boldsymbol{u}(\tau)\rangle_{\mathbb{R}^3}\,\mathrm{d}\tau. \end{aligned} \tag{5.4}$$

Using similar arguments as in the previous paragraph, one can show that the limit inferior of the left hand side of (5.3) is greater than or equal to the left hand side of (5.4). However, it is an open question if we can also pass to the limit on the right hand side of (5.3), namely, whether

$$\lim_{n\to+\infty}\int_0^t\int_{\mathbb{R}^3} \bigl(\boldsymbol{u}_n(\tau)\cdot\nabla\bigr)\boldsymbol{u}_n(\tau)\cdot\boldsymbol{a}^0(\tau)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau\ \leq\ \int_0^t\int_{\mathbb{R}^3} \bigl(\boldsymbol{u}(\tau)\cdot\nabla\bigr)\boldsymbol{u}(\tau)\cdot\boldsymbol{a}^0(\tau)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\tau \tag{5.5}$$

holds. The limit on the left hand side is of the same type as the limit in (4.6). However, the test function $\boldsymbol{\phi}$ in (4.6) had a compact support in $Q_{[0,T)}$, which strongly helped us to calculate the limit. Now, (5.5) contains the function $\boldsymbol{a}^0$ instead of $\boldsymbol{\phi}$ and $\boldsymbol{a}^0$ does not generally have a compact support in $Q_{[0,T)}$. (Interesting applications even require $\boldsymbol{a}^0$ to be non–zero on the boundary of $Q_{[0,T)}$.) Thus, the procedure used in Section 4 in the evaluation of the limit in (4.6), cannot be applied to the limit in (5.5). We therefore leave the question whether (5.5) is true (at least for some subsequence of $\{\boldsymbol{u}_n\}$) as open.

## 6 Example 1: The flow around rotating bodies

**Definition of function $\boldsymbol{a}$.** Let us consider $N$ compact bodies, rotating around the axes given by the parametric equations $\boldsymbol{x} = \boldsymbol{q}_i + s\,\boldsymbol{e}_i;\ s \in \mathbb{R}$ with constant angular velocities $\omega_1, \ldots, \omega_N$ ($i = 1, \ldots, N$). We put $\boldsymbol{\omega}_i := \omega_i\,\boldsymbol{e}_i$. We suppose that the distance between any of the bodies at an arbitrary time $t \in [0,T]$ does nor exceed $d > 0$. We denote by $K_i^t$ the closed region occupied by the $i$–th body at time $t$. The domain, filled by the fluid at time $t$, is $\Omega^t := \mathbb{R}^3 \smallsetminus \big(\cup_{i=1}^N K_i^t\big)$.

The assumption on adherence of the fluid to the body on its surface leads to the boundary condition for velocity:

$$\boldsymbol{v}(\boldsymbol{x},t) \;=\; \boldsymbol{\omega}_i \times (\boldsymbol{x}-\boldsymbol{q}_i) \;\equiv\; -\,\mathbf{curl}\,\big(\tfrac{1}{2}\,|\boldsymbol{x}-\boldsymbol{q}_i|^2\boldsymbol{\omega}_i\big) \quad \text{for } \boldsymbol{x} \in \partial K_i^t;\;\; i = 1,\ldots,N. \tag{6.1}$$

Let $\eta_i$ be a $C^\infty$ cut–off function in $\mathbb{R}^3 \times [0,T]$ such that

$$\eta_i(\boldsymbol{x},t) \begin{cases} = 1 & \text{if} \quad \mathrm{dist}(\boldsymbol{x},K_i^t) < \frac{1}{4}d, \\ \in (0,1) & \text{if} \quad \frac{1}{4}d \le \mathrm{dist}(\boldsymbol{x},K_i^t) < \frac{1}{2}d, \\ = 0 & \text{if} \quad \mathrm{dist}(\boldsymbol{x},K_i^t) \ge \frac{1}{2}d. \end{cases}$$

Now we put $\boldsymbol{a} := \boldsymbol{a}^\infty + \boldsymbol{a}^0$ where $\boldsymbol{a}^\infty = \frac{1}{2}\,\mathbf{curl}\,\big(\boldsymbol{a}^\infty \times \boldsymbol{x}\big)$ represents the constant velocity in infinity and

$$\boldsymbol{a}^0(\boldsymbol{x},t) \;:=\; -\frac{1}{2}\,\mathbf{curl}\,\left(\sum_{i=1}^N \eta_i(\boldsymbol{x},t)\,\big[|\boldsymbol{x}-\boldsymbol{q}_i|^2\boldsymbol{\omega}_i + (\boldsymbol{a}^\infty \times \boldsymbol{x})\big]\right).$$

Function $\boldsymbol{a}$ now satisfies the boundary condition (6.1) and $\boldsymbol{a}^0$ also satisfies other conditions named in Assumption 3. Consequently, Theorems 1, 2 and 3 are applicable and they provide the existence of a weak solution of the problem (1.5)–(1.8), as well as the information on its local weak continuity and the energy–type inequality.

## 7 Example 2: The flow around a body perpendicularly striking to a plane

**The geometry of the flow field.** We consider a flow of a viscous incompressible fluid in the half–space $\mathbb{R}^3_+ := \{\boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3;\ x_3 > 0\}$, around the body which occupies the closed region $B^0$ at time $t = 0$. The body moves perpendicularly towards the $x_1$, $x_2$–plane in the time interval $[0,T]$ so that its motion is purely translational and its distance $\delta^t$ (at time $t$) from the $x_1, x_2$–plane satisfies the condition

(vi) $\delta^t$ is a differentiable non–increasing function of $t$ on the interval $[0, T]$ such that $\delta^0 > 0$ and $\delta^T = 0$.

Hence the body takes the region

$$B^t \; = \; \bigl\{ (x_1, x_2, x_3) \in \mathbb{R}^3;\ (x_1,\, x_2,\, x_3 + \delta^0 - \delta^t) \in B^0 \bigr\}$$

at a general time $t$ and it strikes to the $x_1,\, x_2$–plane at time $T$. The domain, filled by the fluid, is $\Omega^t = \mathbb{R}^3_+ \smallsetminus B^t$.

The system of coordinates can be chosen so that the nearest point $S^t$ of the to the $x_1,\, x_2$–plane has the coordinates $S^t = (0, 0, \delta^t)$ and the body thus strikes to the $x_1,\, x_2$–plane at the point $O = (0, 0, 0)$. (See Fig. 1.)

It will be further advantageous to work in the cylindrical coordinates $\rho$, $\varphi$, $x_3$, whose origin is point $O$. We denote the corresponding unit vectors by $\boldsymbol{e}_\rho$, $\boldsymbol{e}_\varphi$ and $\boldsymbol{e}_3$.

We choose sufficiently large numbers $l$ and $h$ such that $B^t$ is a subset of the cylinder $\{\rho < l,\ 0 < x_3 < h\}$ for all $t \in (0, T)$. We assume that there exists $\rho_0 \in (0, l)$ and positive numbers $a$, $b$ such that

$$\forall\, (\rho, \varphi, x_3) \in B^t \ :\ \rho \le \rho_0 \Longrightarrow x_3 \ge \delta^t + a\rho^b. \tag{7.1}$$

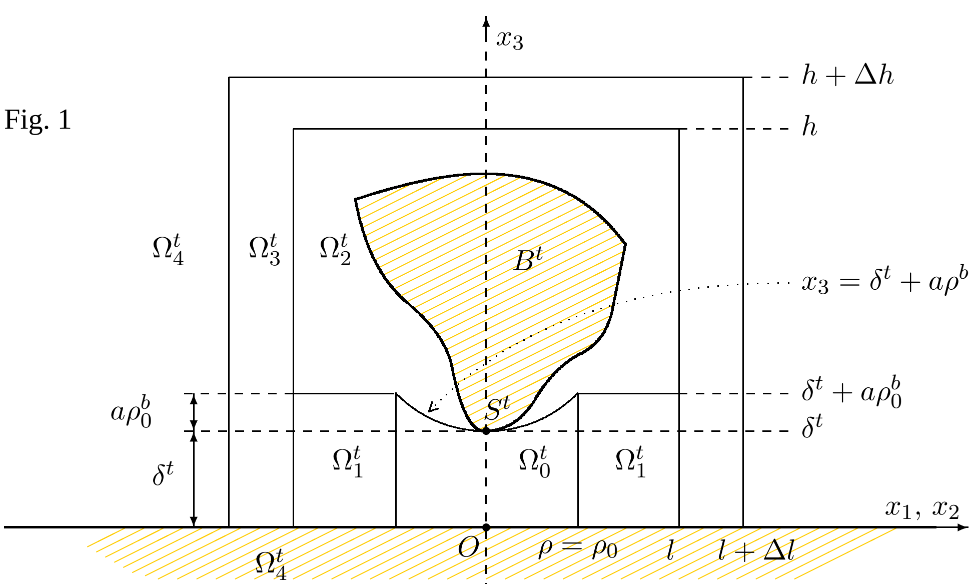


Fig. 1

**Definition of function $\boldsymbol{a}$.** In this geometrical configuration, we will construct function $\boldsymbol{a}$ of the form (1.4) in $\mathbb{R}^3 \times (0, T)$, satisfying all the conditions from Assumption 3. We put $\boldsymbol{a}^\infty = \boldsymbol{0}$.

The crucial part of $\Omega^t$, where the stroke occurs, is

$$\Omega_0^t \;:=\; \bigl\{ \boldsymbol{x} = (\rho, \varphi, x_3;\ 0 \le \rho < \rho_0,\ 0 < x_3 < \delta^t + a\rho^b \bigr\}.$$

We also denote

$$\Omega_1^t \quad := \quad \bigl\{ (\rho, \varphi, x_3) \in \mathbb{R}^3;\ \ \rho_0 \le \rho < l,\ \ 0 < x_3 < \delta^t + a\rho_0^b \bigr\},$$

$$\begin{aligned}
\Omega_2^t &:= \big\{(\rho,\varphi,x_3)\in\mathbb{R}^3;\ \ \rho<l,\ \ 0<x_3<h\big\}\smallsetminus(\Omega_0^t\cup\Omega_1^t),\\
\Omega_3^t &:= \big\{(\rho,\varphi,x_3)\in\mathbb{R}^3;\ \ \rho<l+\Delta l,\ \ 0<x_3<h+\Delta h\big\}\smallsetminus\big(\Omega_0^t\cup\Omega_1^t\cup\Omega_2^t\big),\\
\Omega_4^t &:= \mathbb{R}^3\smallsetminus\big(\Omega_0^t\cup\Omega_1^t\cup\Omega_2^t\cup\Omega_3^t\big)
\end{aligned}$$

where $\Delta l$ and $\Delta h$ are chosen positive numbers. The sets $\Omega_0^t$–$\Omega_4^t$ are mutually disjoint and their union is $\Omega^t$. Set $\Omega_2^t$ contains $B^t$. (See Fig. 1.)

Let $\eta$ be an infinitely differentiable cut–off function of one variable such that $\eta(s)=0$ for $s\le 0$, $0\le\eta(s)\le 1$ for $0<s<1$ and $\eta(s)=1$ for $s\ge 1$. We put

$$\boldsymbol{a}^0(\rho,x_3,t) \;:=\; \mathbf{curl}\ \boldsymbol{w}(\rho,x_3,t)\,\dot{\delta}^t \tag{7.2}$$

where

$$\boldsymbol{w}(\rho,x_3,t) \;:=\; \begin{cases} \tfrac{1}{2}\rho\,\eta\Big(\dfrac{x_3}{\delta^t+a\rho^b}\Big)\,\boldsymbol{e}_\varphi & \text{for}\quad (\boldsymbol{x},\varphi,x_3)\in\Omega_0^t,\\[2mm] \tfrac{1}{2}\rho\,\eta\Big(\dfrac{x_3}{\delta^t+a\rho_0^b}\Big)\,\boldsymbol{e}_\varphi & \text{for}\quad (\boldsymbol{x},\varphi,x_3)\in\Omega_1^t,\\[2mm] \tfrac{1}{2}\rho\,\boldsymbol{e}_\varphi & \text{for}\quad (\boldsymbol{x},\varphi,x_3)\in\Omega_2^t,\\[2mm] \mathbf{0} & \text{for}\quad (\boldsymbol{x},\varphi,x_3)\in\Omega_4^t. \end{cases}$$

In order to avoid complicated formulas, we simply assume that function $\boldsymbol{w}$ in set $\Omega_3^t$ is a smooth extension of $\boldsymbol{w}$ from $\Omega_1^t\cup\Omega_2^t$, which is zero near the boundary with $\Omega_4^t$. Obviously,

$$\boldsymbol{a}^0 \;=\; a_\rho^0\,\boldsymbol{e}_\rho+a_3^0\,\boldsymbol{e}_3 \tag{7.3}$$

where $a_\rho^0=a_3^0=0$ in $\mathbb{R}^3_-\cup\Omega_4^t$ and $a_\rho^0=0$, $a_3^0=\dot{\delta}^t$ in $\Omega_2^t$.

We shall further examine what properties of $\delta^t$ imply that the function $\boldsymbol{a}^0$ satisfies conditions (i)–(iii) from Assumption 3. We will naturally focus on the behavior of $\boldsymbol{a}^0$ in the most interesting part $\Omega_0^t$ of $\Omega^t$. We can calculate that $a_\varphi^0=0$ and

$$a_\rho^0(\rho,x_3,t) \;=\; -\frac{\rho}{2}\,\partial_3\eta\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\dot{\delta}^t \;=\; -\frac{\rho}{2}\,\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{\dot{\delta}^t}{\delta^t+a\rho^b}, \tag{7.4}$$

$$\begin{aligned}
a_3^0(\rho,x_3,t) &= \frac{1}{\rho}\,\partial_\rho\Big[\frac{\rho^2}{2}\,\eta\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\Big]\,\dot{\delta}^t\\
&= \Big[\eta\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)-\frac{\rho}{2}\,\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{x_3\,ab\rho^{b-1}}{[\delta^t+a\rho^b]^2}\Big]\,\dot{\delta}^t
\end{aligned} \tag{7.5}$$

in domain $\Omega_0^t$. In order to verify the conditions (i)–(iii), we shall estimate $\|\nabla\boldsymbol{a}^0\|^2_{2;\,\Omega_0^t}$, $\|\partial_t\boldsymbol{a}^0\|_{2;\,\Omega_0^t}$ and $\|\boldsymbol{a}^0\|^r_{s;\,\Omega_0^t}$ (for $s$ and $r$ satisfying the assumptions from condition (iii)).

**The estimate of** $\|\nabla\boldsymbol{a}^0\|^2_{2;\,\Omega_0^t}$**.** The component $\partial_\rho a_\rho^0$ of $\nabla\boldsymbol{a}^0$ can be estimated on $\Omega_0^t$ as follows:

$$\begin{aligned}
\int_{\Omega_0^t}|\partial_\rho a_\rho^0|^2\,\mathrm{d}\boldsymbol{x} \;&=\; 2\pi\int_0^{\rho_0}\rho\,\mathrm{d}\rho\,\Big[\int_0^{\delta^t+a\rho^b}|\partial_\rho a_\rho^0|^2\,\mathrm{d}x_3\Big]\\
= \;& \int_0^{\rho_0}\rho\,\mathrm{d}\rho\int_0^{\delta^t+a\rho^b}\Big[-\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{\dot{\delta}^t}{2\,[\delta^t+a\rho^b]}+\eta''\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{ab\rho^b\,x_3\,\dot{\delta}^t}{2\,[\delta^t+a\rho^b]^3}
\end{aligned}$$

$$
+\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{ab\rho^b\,\dot\delta^t}{2\,[\delta^t+a\rho^b]^2}\bigg]^2\,\mathrm{d}x_3
$$

$$
\begin{aligned}
&\le\ C\int_0^{\rho_0}\rho\,\mathrm{d}\rho\int_0^{\delta^t+a\rho^b}\bigg[\frac{(\dot\delta^t)^2}{[\delta^t+a\rho^b]^2}+\frac{\rho^{2b}\,x_3^2\,(\dot\delta^t)^2}{[\delta^t+a\rho^b]^6}+\frac{\rho^{2b}\,(\dot\delta^t)^2}{[\delta^t+a\rho^b]^4}\bigg]\,\mathrm{d}x_3\\
&\le\ C\int_0^{\rho_0}\bigg[\frac{\rho\,(\dot\delta^t)^2}{\delta^t+a\rho^b}+\frac{\rho^{2b+1}\,(\dot\delta^t)^2}{[\delta^t+a\rho^b]^3}\bigg]\,\mathrm{d}\rho\ \le\ C\int_0^{\rho_0}\frac{\rho\,(\dot\delta^t)^2}{\delta^t+a\rho^b}\,\mathrm{d}\rho\\
&=\ C\int_0^{\sigma_0}\frac{\sigma^{\frac{2-b}{b}}}{\delta^t+\sigma}\,\mathrm{d}\sigma\,(\dot\delta^t)^2\ \le\ \left\{\begin{array}{ll} C\,(\dot\delta^t)^2 & \text{for } 0<b<2,\\ C\,\ln\Big(1+\dfrac{\sigma_0}{\delta^t}\Big)\,(\dot\delta^t)^2 & \text{for } b=2,\\ C\,(\delta^t)^{\frac{2-b}{b}}\,(\dot\delta^t)^2 & \text{for } 2<b. \end{array}\right\}
\end{aligned}
\tag{7.6}
$$

(We have used the substitution $a\rho^b=\sigma$, i.e. $ab\rho^{b-1}\,\mathrm{d}\rho=\mathrm{d}\sigma$, and the notation $\sigma_0=a\rho_0^b$.) The generic constant $C$ may depend on the numbers $\rho_0$, $l$, $h$, $\Delta l$, $\Delta h$, $a$, $b$ and on function $\eta$ in this section.

The estimate of $\partial_3 a_\rho^0$ yields:

$$
\begin{aligned}
&\int_{\Omega_0^t}|\partial_3 a_\rho^0|^2\,\mathrm{d}\boldsymbol{x}\ =\ 2\pi\int_0^{\rho_0}\rho\,\mathrm{d}\rho\,\bigg[\int_0^{\delta^t+a\rho^b}|\partial_3 a_\rho^0|^2\,\mathrm{d}x_3\bigg]\\
&=\ \int_0^{\rho_0}\rho\,\mathrm{d}\rho\,\bigg(\int_0^{\delta^t+a\rho^b}\frac{\rho^2}{4}\,\eta''\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)^2\,\frac{(\dot\delta^t)^2}{[\delta^t+a\rho^b]^4}\,\mathrm{d}x_3\bigg)\ \le\ C\int_0^{\rho_0}\frac{\rho^3\,(\dot\delta^t)^2}{[\delta^t+a\rho^b]^3}\,\mathrm{d}\rho\\
&=\ C\int_0^{\sigma_0}\frac{\sigma^{\frac{4-b}{b}}}{[\delta^t+\sigma]^3}\,\mathrm{d}\sigma\,(\dot\delta^t)^2\ \le\ \left\{\begin{array}{ll} C\,(\dot\delta^t)^2 & \text{for } 0<b<\frac{4}{3},\\ C\,\ln\Big(1+\dfrac{\sigma_0}{\delta^t}\Big)\,(\dot\delta^t)^2 & \text{for } b=\frac{4}{3},\\ C\,(\delta^t)^{\frac{4-3b}{b}}\,(\dot\delta^t)^2 & \text{for } \frac{4}{3}<b. \end{array}\right\}
\end{aligned}
\tag{7.7}
$$

We can similarly derive that

$$
\int_{\Omega_0^t}|\partial_\rho a_3^0|^2\,\mathrm{d}\boldsymbol{x}\ \le\ C\,(\dot\delta^t)^2.
$$

The estimate of $\partial_3 a_3^0$ is the same as (7.6). Hence we arrive at the inequality

$$
\|\nabla\boldsymbol{a}^0\|_{2;\,\Omega_0^t}^2\ \le\ \left\{\begin{array}{ll} C\,(\dot\delta^t)^2 & \text{for } 0<b<\frac{4}{3},\\ C\,\ln\Big(1+\dfrac{\sigma_0}{\delta^t}\Big)\,(\dot\delta^t)^2 & \text{for } b=\frac{4}{3},\\ C\,(\delta^t)^{\frac{4-3b}{b}}\,(\dot\delta^t)^2 & \text{for } \frac{4}{3}<b. \end{array}\right\}
\tag{7.8}
$$

**The estimate of $\|\partial_t\boldsymbol{a}^0\|_{2;\,\Omega_0^t}$.** Calculating the derivative $\partial_t\boldsymbol{a}^0$ from (7.4) and (7.5), we obtain

$$
\begin{aligned}
\|\partial_t\boldsymbol{a}^0\|_{2;\,\Omega_0^t}^2\ =\ 2\pi\int_0^{\rho_0}\rho\,\mathrm{d}\rho\int_0^{\delta^t+a\rho^b}\bigg\{\bigg[&\frac{\rho}{2}\,\eta''\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{x_3\,(\dot\delta^t)^2}{[\delta^t+a\rho^b]^3}\\
&-\frac{\rho}{2}\,\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{\ddot\delta^t}{\delta^t+a\rho^b}+\frac{\rho}{2}\,\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{(\dot\delta^t)^2}{[\delta^t+a\rho^b]^2}\bigg]^2
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[-\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{x_3\,(\dot\delta^t)^2}{[\delta^t+a\rho^b]^2}+\frac{\rho}{2}\,\eta''\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{x_3^2\,ab\rho^{b-1}\,(\dot\delta^t)^2}{[\delta^t+a\rho^b]^4}\\
&\qquad+\frac{\rho}{2}\,\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{x_3\,ab\rho^{b-1}\,(\dot\delta^t)^2}{[\delta^t+a\rho^b]^3}+\eta\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\ddot\delta^t\\
&\qquad-\frac{\rho}{2}\,\eta'\Big(\frac{x_3}{\delta^t+a\rho^b}\Big)\,\frac{x_3\,ab\rho^{b-1}}{[\delta^t+a\rho^b]^2}\,\ddot\delta^t\Bigg]^2\Bigg\}\,\mathrm{d}x_3\\
\leq\;& C\int_0^{\rho_0}\Bigg[\frac{\rho^3\,(\dot\delta^t)^4}{[\delta^t+a\rho^b]^3}+\frac{\rho^3\,(\ddot\delta^t)^2}{\delta^t+a\rho^b}+\frac{\rho\,(\dot\delta^t)^4}{\delta^t+a\rho^b}+\frac{\rho^{2b+1}\,(\dot\delta^t)^4}{[\delta^t+a\rho^b]^3}+(\ddot\delta^t)^2+\frac{\rho^{2b+1}\,(\ddot\delta^t)^2}{\delta^t+a\rho^b}\Bigg]\,\mathrm{d}\rho\\
=\;& C\int_0^{\sigma_0}\Bigg[\frac{\sigma^{\frac{4-b}{b}}\,(\dot\delta^t)^4}{[\delta^t+\sigma]^3}+\frac{\sigma^{\frac{4-b}{b}}\,(\ddot\delta^t)^2}{\delta^t+\sigma}+\frac{\sigma^{\frac{2-b}{b}}\,(\dot\delta^t)^4}{\delta^t+\sigma}+\frac{\sigma^{\frac{2+b}{b}}\,(\dot\delta^t)^4}{[\delta^t+\sigma]^3}+(\ddot\delta^t)^2\Bigg]\,\mathrm{d}\sigma. \qquad (7.9)
\end{aligned}
$$

Applying the integration by parts to the integral of the first term on the right hand side, we obtain

$$
\begin{aligned}
\int_0^{\sigma_0}\frac{\sigma^{\frac{4-b}{b}}\,(\dot\delta^t)^4}{[\delta^t+\sigma]^3}\,\mathrm{d}\sigma \;&=\; \Bigg[\frac{b}{4}\,\frac{\sigma^{\frac{4}{b}}\,(\dot\delta^t)^4}{[\delta^t+\sigma]^3}\Bigg]_{\sigma=0}^{\sigma_0}+\int_0^{\sigma_0}\frac{3b}{4}\,\frac{\sigma^{\frac{4}{b}}\,(\dot\delta^t)^4}{[\delta^t+\sigma]^4}\,\mathrm{d}\sigma\\
&\leq\; C\,(\dot\delta^t)^4+C\int_0^{\sigma_0}[\delta^t+\sigma]^{\frac{4-4b}{b}}\,(\dot\delta^t)^4\,\mathrm{d}\sigma \;\leq\; \begin{cases} C\,(\dot\delta^t)^4 & \text{for } b<\frac{4}{3},\\ C\,\ln\Big(1+\dfrac{\sigma_0}{\delta^t}\Big)\,(\dot\delta^t)^4 & \text{for } b=\frac{4}{3},\\ C\,(\delta^t)^{\frac{4-3b}{b}}\,(\dot\delta^t)^4 & \text{for } \frac{4}{3}<b.\end{cases}
\end{aligned}
$$

The integrals of the third and the fourth term on the right hand side of (7.9) satisfy the same estimates. The integral of the second term can be estimated as follows:

$$
\begin{aligned}
\int_0^{\sigma_0}\frac{\sigma^{\frac{4-b}{b}}\,(\ddot\delta^t)^2}{\delta^t+\sigma}\,\mathrm{d}\sigma \;&=\; \Bigg[\frac{b}{4}\,\frac{\sigma^{\frac{4}{b}}\,(\ddot\delta^t)^2}{(\delta^t+\sigma)}\Bigg]_{\sigma=0}^{\sigma_0}+\int_0^{\sigma_0}\frac{b}{4}\,\frac{\sigma^{\frac{4}{b}}\,(\ddot\delta^t)^2}{[\delta^t+\sigma]^2}\,\mathrm{d}\sigma\\
&\leq\; C\,(\ddot\delta^t)^2+C\int_0^{\sigma_0}[\delta^t+\sigma]^{\frac{4-2b}{b}}\,(\ddot\delta^t)^2\,\mathrm{d}\sigma \;\leq\; \begin{cases} C\,(\ddot\delta^t)^2 & \text{for } b<4,\\ C\,\ln\Big(1+\dfrac{\sigma_0}{\delta^t}\Big)\,(\ddot\delta^t)^2 & \text{for } b=4,\\ C\,(\delta^t)^{\frac{4-b}{b}}\,(\dot\delta^t)^4 & \text{for } 4<b.\end{cases}
\end{aligned}
$$

Thus, we obtain the inequality

$$
\|\partial_t\boldsymbol{a}^0\|_{2;\,\Omega_0^t} \;\leq\; \left\{\begin{array}{ll} C\,(\dot\delta^t)^2+C\,\ddot\delta^t & \text{for } b<\frac{4}{3},\\ C\,\Big[\ln\Big(1+\dfrac{\sigma_0}{\delta^t}\Big)\Big]^{1/2}\,(\dot\delta^t)^2+C\,\ddot\delta^t & \text{for } b=\frac{4}{3},\\ C\,(\delta^t)^{\frac{4-3b}{2b}}\,(\dot\delta^t)^2+C\,\ddot\delta^t & \text{for } \frac{4}{3}<b<4,\\ C\,(\delta^t)^{\frac{4-3b}{2b}}\,(\dot\delta^t)^2+C\,\Big[\ln\Big(1+\dfrac{\sigma_0}{\delta^t}\Big)\Big]^{1/2}\,\ddot\delta^t & \text{for } b=4,\\ C\,(\delta^t)^{\frac{4-3b}{2b}}\,(\dot\delta^t)^2+C\,(\delta^t)^{\frac{4-b}{2b}}\,\ddot\delta^t & \text{for } b>4.\end{array}\right\} \qquad (7.10)
$$

**The estimate of $\|\boldsymbol{a}^0\|_{s;\,\Omega_0^t}^r$.** Let $s>3$ and $r=2s/(s-3)$. Then we have

$$
\|\boldsymbol{a}^0\|_{s;\,\Omega_0^t}^r \;=\; \Big\{\int_{\Omega_0^t}|\boldsymbol{a}^0|^s\,\mathrm{d}\boldsymbol{x}\Big\}^{r/s} \;\leq\; C\,\Big\{\int_0^{\rho_0}\rho\,\mathrm{d}\rho\int_0^{\delta^t+a\rho^b}\big(|a_\rho^0|^s+|a_3^0|^s\big)\,\mathrm{d}x_3\Big\}^{\frac{2}{s-3}}
$$

$$\begin{aligned}
&\leq \; C\left\{\int_0^{\rho_0}\rho\,\mathrm{d}\rho\int_0^{\delta^t+a\rho^b}\left[\frac{\rho^s}{[\delta^t+a\rho^b]^s}+1+\frac{\rho^s\,x_3^s\,\rho^{sb-s}}{[\delta^t+a\rho^b]^{2s}}\right]\mathrm{d}x_3\right\}^{\frac{2}{s-3}}(\dot{\delta}^t)^{\frac{2s}{s-3}} \\
&\leq \; C\left\{\int_0^{\rho_0}\left[\frac{\rho^{s+1}}{[\delta^t+a\rho^b]^{s-1}}+1+\frac{\rho^{sb+1}}{[\delta^t+a\rho^b]^{s-1}}\right]\mathrm{d}\rho\right\}^{\frac{2}{s-3}}(\dot{\delta}^t)^{\frac{2s}{s-3}} \\
&= \; C\left\{\int_0^{\sigma_0}\left[\frac{\sigma^{\frac{s+2}{b}-1}}{[\delta^t+\sigma]^{s-1}}+1+\frac{\sigma^{s-1+\frac{2}{b}}}{[\delta^t+\sigma]^{s-1}}\right]\mathrm{d}\sigma\right\}^{\frac{2}{s-3}}(\dot{\delta}^t)^{\frac{2s}{s-3}} \\
&\leq \; C\left\{\int_0^{\sigma_0}\frac{\sigma^{\frac{s+2}{b}-1}}{[\delta^t+\sigma]^{s-1}}\,\mathrm{d}\sigma\right\}^{\frac{2}{s-3}}(\dot{\delta}^t)^{\frac{2s}{s-3}} \\
&= \; C\left\{\left[\frac{b}{s+2}\,\frac{\sigma^{\frac{s+2}{b}}}{[\delta^t+\sigma]^{s-1}}\right]_{\sigma=0}^{\sigma_0}+\frac{(s+2)(s-1)}{b}\int_0^{\sigma_0}\frac{\sigma^{\frac{s+2}{b}}}{[\delta^t+\sigma]^s}\,\mathrm{d}\sigma\right\}^{\frac{2}{s-3}}(\dot{\delta}^t)^{\frac{2s}{s-3}} \\
&\leq \; C\left\{\int_0^{\sigma_0}\frac{\sigma^{\frac{s+2}{b}}}{[\delta^t+\sigma]^s}\,\mathrm{d}\sigma\right\}^{\frac{2}{s-3}}(\dot{\delta}^t)^{\frac{2s}{s-3}} \\
&\leq \; \begin{cases} C\,(\dot{\delta}^t)^{\frac{2s}{s-3}} & \text{for } b<1+\frac{3}{s-1}, \\ C\left[\ln\!\left(1+\dfrac{\sigma_0}{\delta^t}\right)\right]^{\frac{2}{s-3}}(\dot{\delta}^t)^{\frac{2s}{s-3}} & \text{for } b=1+\frac{3}{s-1}, \\ C\,(\delta^t)^{(\frac{s+2}{b}-s+1)\frac{2}{s-3}}\,(\dot{\delta}^t)^{\frac{2s}{s-3}} & \text{for } b>1+\frac{3}{s-1}. \end{cases}
\end{aligned}$$

Now, if $0<b\leq 1$ then $b<1+\frac{3}{s-1}$ and we have

$$\|\boldsymbol{a}^0\|^r_{s;\,\Omega_0^t} \;\leq\; C\,(\dot{\delta}^t)^{\frac{2s}{s-3}} \tag{7.11}$$

for all $s>3$. If $1<b<\frac{5}{2}$ then we can choose $s=(b+2)/(b-1)$, which implies that $s>3$ and $b=1+\frac{3}{s-1}$. Hence

$$\|\boldsymbol{a}^0\|^r_{s;\,\Omega_0^t} \;\leq\; C\left[\ln\!\left(1+\frac{\sigma_0}{\delta^t}\right)\right]^{\frac{2}{s-3}}(\dot{\delta}^t)^{\frac{2s}{s-3}}. \tag{7.12}$$

Finally, if $\frac{5}{2}\leq b$ then $b>1+\frac{3}{s-1}$ for all $s>3$ and therefore we have

$$\|\boldsymbol{a}^0\|^r_{s;\,\Omega_0^t} \;\leq\; C\,(\delta^t)^{(\frac{s+2}{b}-s+1)\frac{2}{s-3}}\,(\dot{\delta}^t)^{\frac{2s}{s-3}}. \tag{7.13}$$

**Integrability of the right hand sides of (7.8), (7.10)–(7.13).** It can be easily checked that the same estimates as (7.8), (7.10)–(7.13) hold not only on $\Omega_0^t$, but also on all other parts of $\mathbb{R}^3$.

The conditions (i)–(iii) of Assumption 3 are satisfied if the right hand sides of (7.8), (7.10)–(7.13) are integrable functions of $t$ on the interval $(0,T)$. This requirement represents five additional conditions on function $\delta^t$ (following from the five inequalities (7.8), (7.10)–(7.13)). Suppose further, for simplicity, that $\delta^t$ is a power function of the form

$$\delta^t \;=\; c_8\,(T-t)^\gamma \tag{7.14}$$

where $c_8>0$ and $\gamma>0$. Then $\dot{\delta}^t=-\gamma c_8(T-t)^{\gamma-1}$.

The right hand sides in (7.8) have the forms

$$
\begin{aligned}
&C\,(T-t)^{2\gamma-2} && \text{for } 0<b<\tfrac{4}{3},\\
&C\,\ln\Big(1+\frac{\sigma_0}{c_8(T-t)^{\gamma}}\Big)\,(T-t)^{2\gamma-2} && \text{for } b=\tfrac{4}{3},\\
&C\,(T-t)^{\frac{4-3b}{b}\,\gamma+2\gamma-2} && \text{for } \tfrac{4}{3}<b.
\end{aligned}
$$

The condition of integrability on the interval $(0,T)$ requires that $\gamma>\frac{1}{2}$ for $b\leq\frac{4}{3}$ and $\gamma>\frac{b}{4-b}$ for $\frac{4}{3}<b<4$. If $b\geq 4$ then $\frac{4-3b}{b}\gamma+2\gamma-2\leq -1$ independently of $\gamma>0$ and the function $C\,(T-t)^{\frac{4-3b}{b}\,\gamma+2\gamma-2}$ can therefore not be integrable on $(0,T)$.

The analysis of the right hand sides in (7.10) shows that we need $\gamma\geq 1$ for $0<b<4$.

Finally, we can verify that the right hand sides of (7.12)–(7.13) are integrable on $(0,T)$ if $\gamma>1-\frac{s-3}{2s}$ for $0<b<\frac{5}{2}$ and $\gamma>g(b,s):=(s+3)b/(2s+4+2b)$ for $b\geq\frac{5}{2}$. Since $g(b,\,.)$ is an increasing function of $s$ on $[3,+\infty)$ for each $b\geq\frac{5}{2}$, one can always find $s>3$ such that $\gamma>g(b,s)$ if $\gamma>g(b,3)=\frac{3b}{5+b}$. However, this condition need not be taken into account because we already have the condition $\gamma>\frac{b}{4-b}$ from the previous paragraph and since $\frac{b}{4-b}>\frac{3b}{5+b}$ for $b\geq\frac{5}{2}$, we do not need to take the condition $\gamma>\frac{3b}{5+b}$ into account.

The next theorem summarizes the results of this section and it also directly applies Theorems 1, 2 and 3.

**Theorem 4.** *Suppose that domain $\Omega^t$ has the form described at the beginning of this section. (See also Fig. 1.) Suppose that $\boldsymbol{a}^{\infty}=\mathbf{0}$ and function $\boldsymbol{a}^0$ has the form (7.2). Suppose that $\delta^t$ has the form (7.14), where*

$$
\left.\begin{aligned}
&\gamma\geq 1 && \textit{for }\ 0<b<2,\\
&\gamma>\tfrac{b}{4-b} && \textit{for }\ 2\leq b<4.
\end{aligned}\right\}
\tag{7.15}
$$

*Then $\boldsymbol{a}^0$ satisfies all the conditions (i)–(iii) of Assumption 3.*

*Consequently, Theorems 1 and 2 are applicable. It means that given $\boldsymbol{u}_0\in L^2_{\sigma}(\Omega^0)$ and $\boldsymbol{f}$ satisfying condition (iv), there exists a solution $\boldsymbol{u}$ of the problem (1.6)–(1.9). The solution is, after a possible modification on a set of measure zero, locally weakly continuous as an element of $L^2_{\sigma}(\Omega^t)$ in dependence on $t$ (which means that it satisfies (5.1)) and it satisfies the energy–type inequality (5.2) for all $t\in(0,T)$.*

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## References

[1] W. BORCHERS: Zur Stabilität und Faktorisierungsmethode für die Navier–Stokes–Gleichungen inkompressibler viskoser Flüssigkeiten. Habilitation Thesis, Univ. of Paderborn 1992.

[2] D. BUCUR, E. FEIREISL, Š. NEČASOVÁ, J. WOLF: On the asymptotic limit of the Navier-Stokes system on domains with rough boundaries. Preprint 2006.

[3] C. CONCA, J. SAN MARTÍN, M. TUCSNAK: Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid. *Comm. in Partial Diff. Equations* **25** (5&6), 2000, 1019–1042.

[4] B. DESJARDINS, M. J. ESTEBAN: Existence of weak solutions for the motion of rigid bodies in a viscous fluid. *Arch. Rat. Mech. Anal.* **146**, 1999, 59–71.

[5] B. DESJARDINS, M. J. ESTEBAN: On weak solutions for fluid–rigid structure interaction: compressible and incompressible models. *Comm. in Partial Diff. Equations* **25** (7&8), 2000, 1399–1413.

[6] E. FEIREISL: On the motion of rigid bodies in a viscous incompressible fluid. *J. of Evolution Equations* **3**, 2003, 419–441.

[7] H. FUJITA, N. SAUER: On existence of weak solutions of the Navier–Stokes equations in regions with moving boundaries. *J. Fac. Sci. Univ. Tokyo,* Sec. 1A, **17**, 1970, 403–420.

[8] G. P. GALDI: *An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Vol. I, Linear Steady Problems.* Springer Tracts in Natural Philosophy 38, 1998.

[9] G. P. GALDI: An Introduction to the Navier–Stokes initial–boundary value problem. In *Fundamental Directions in Mathematical Fluid Mechanics*, ed. G.P.Galdi, J.Heywood, R.Rannacher, series "Advances in Mathematical Fluid Mechanics", Vol. 1, Birkhauser–Verlag, Basel 2000, 1–98.

[10] G. P. GALDI: *On the motion of a rigid body in a viscous fluid: a mathematical analysis with applications.* In *Handbook of Mathematical Fluid Dynamics I*, Ed. S. Friedlander and D. Serre, Elsevier 2002.

[11] C. GRANDMONT, Y. MADAY: Existence de solutions d'un problème de couplage fluide–structure bidimensionel instationnaire. *C. R. Acad. Sci. Paris,* Ser. I, **326**, 1998, 525–530.

[12] M. D. GUNZBURGER, H. C. LEE, G. SEREGIN: Global existence of weak solutions for viscous incompressible flows around a moving rigid body in three dimensions. *J. Math. Fluid Mech.* **2**, 2000, 219–266.

[13] K. H. HOFFMANN, V. N. STAROVOITOV: On a motion of a solid body in a viscous fluid. Two–dimensional case. *Adv. Math. Sci. Appl.* **9**, 1999, 633–648.

[14] K. H. HOFFMANN, V. N. STAROVOITOV: Zur Bewegung einer Kugel in einen zähen Flüssigkeit. *Documenta Mathematica* **5**, 2000, 15–21.

[15] E. HOPF: Über die Anfangswertaufgabe für die Hydrodynamischen Grundgleichungen. *Math. Nachr.* **4**, 1950, 213–231.

[16] N. V. JUDAKOV: The solvability of the problem of the motion of a rigid body in a viscous incompressible fluid. *Dinamika Sploshnoj Sredy,* **18**, 1974, 249–253.

[17] O. A. LADYZHENSKAYA: *The Mathematical Theory of Viscous Incompressible Flow.* Gordon and Breach, New York 1969.

[18] J. LERAY: Sur le mouvements d'un liquide visqueux emplissant l'espace. *Acta Mathematica* **63**, 1934, 193–248.

[19] J. L. LIONS: *Quelques méthodes de résolution des problémes aux limites non linéaires.* Dunod, Gauthier–Villars, Paris 1969.

[20] M. SHINBROT: Fractional derivatives of solutions of the Navier–Stokes equations. *Arch. Rat. Mech. Anal.* **40**, 1970–1971 139–154.

[21] J. SAN MARTÍN, V. N. STAROVOITOV, M. TUCSNAK: Global weak solutions for the two–dimensional motion of several rigid bodies in an incompressible viscous fluid. *Arch. Rat. Mech. Anal.* **161**, 2002, 113–147.

[22] V. N. STAROVOITOV: Behavior of a rigid body in an incompressible viscous fluid near a boundary. *Int. Series of Num. Math.* **147**, 2003, 313–327.

[23] V. N. STAROVOITOV: Nonuniqueness of a solution to the problem on motion of a rigid body in a viscous incompressible fluid. *J. of Math. Sci.* **130** (4), 2005, 4893–4898.

[24] T. TAKAHASHI: Existence of strong solutions for the problem of a rigid–fluid system. *C. R. Acad. Sci. Paris,* Ser. I, **336**, 2003, 453–458.

[25] T. TAKAHASHI: Analysis of strong solutions for the equations modeling the motion of a rigid–fluid system in a bounded domain. *Adv. Differential Equations* **8**, No. 12, 2003, 1499–1532

[26] T. TAKAHASHI, M. TUCSNAK: Global strong solutions for the two–dimensional motion of an infinite cylinder in a viscous fluid. *J. Math. Fluid Mech.*, **6**, 2004, 53–77.

[27] R. TEMAM: *Navier–Stokes Equations.* North–Holland, Amsterdam–New York–Oxford 1977.

[28] J. WOLF: Existence of weak solutions to the equations of non-stationary motion of non-newtonian fluids with shear rate dependent viscosity. *J. Math. Fluid Mech.*, **8**, 2006, 1–35.

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