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ON ω-LIMIT SETS OF ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract. We classify ω-limit sets of autonomous ordinary differential equations $x' = f(x), x(0) = x_0$, where f is Lipschitz, in infinite dimensional Banach spaces as being of three types I-III. Let $S \subset X$ be a Polish subset of a Banach space X. S is of type I if there exists a Lipschitz function f and a solution x such that $S = \Omega(x)$ and $x \cap S = \emptyset$. S is of type II if it has non-empty interior and there exists a Lipschitz function f and a solution x such that $S = \Omega(x)$. S is of type III if it has empty interior and $x \subset S$ for every solution x (of an equation where f is Lipschitz) such that $S = \Omega(x)$. Our main results are the following: S is a type I set if and only if there exists an open and connected set $U \subset X$ such that $S \subset \partial U$. Suppose that there exists an open separable and connected set $U \subset X$ such that $S = \overline{U}$. Then S is a type II set. Every separable Banach space with a Schauder basis contains a type III set. Moreover in all these results we show that in addition f may be chosen C^k-smooth whenever the underlying Banach space is C^k-smooth.

1. Introduction and preliminaries

Let X be an infinite-dimensional real Banach space, $f : X \to X$ be a continuous function (geometrically a vector field), and

$$\begin{cases} x'(t) = f(x(t)) \\ x(0) = x_0 \in X \end{cases} \tag{1}$$

be an autonomous ordinary differential equation. The global behavior of the (forward) solution $x : [0, \infty) \to X$ is described by the following notion of ω-limit set.

Definition 1. *Let $x : [0, \infty) \to X$ be a solution of an autonomous differential equation* (1). *We say that $S \subset X$ is an ω-limit set of the solution, if for every $p \in S, \varepsilon > 0, n \in \mathbb{N}$ there is $t_n > n$ such that $\|x(t_n) - p\| < \varepsilon$. We use the notation $S = \Omega(x)$.*

It is easy to see that all ω-limit sets S are closed and separable, so they are Polish topological spaces. We introduce the following types of Polish sets in X.

Definition 2. *Type I. S is of type I if there exists a Lipschitz function f and a solution x of* (1) *such that $S = \Omega(x)$ and $x[0, +\infty) \cap S = \emptyset$.*
Type II. S is of type II if it has non-empty interior and there exists a Lipschitz function f and a solution x of (1) *such that $S = \Omega(x)$.*
Type III. S is of type III if it has empty interior and it contains every solution $x[0, +\infty]$ of (1)*, where f is Lipschitz, such that $S = \Omega(x)$.*

Every ω-limit set S of a Lipschitz equation (1) belongs to precisely one of these types (Proposition 12). Our main results are Theorems 3. 4 and 5 containing a topological characterization of type I sets, a sufficient (but not necessary) condition

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describing type II sets, and a construction of examples of type III in every separable Banach space X with a Schauder basis.

Theorem 3. *Let X be an infinite dimensional Banach space, $S \subset X$ be a Polish subset. Then S is a type I set if and only if there exists an open set $U \subset X$ with the properties:*
1. U is connected.
2. $S \subset \partial U$.
Moreover, if X admits a C^k-smooth bump function then f may be chosen C^k-smooth as well.

If X is a non-separable Banach space, the above theorem is a complete characterization of ω-limit sets for Lipschitz equations. Type II and III sets may exist only in separable spaces.

Theorem 4. *Let X be a separable infinite dimensional Banach space, $S \subset X$ be a Polish subset with non-empty interior. Suppose that there exists an open and connected set $U \subset X$ such that $S = \overline{U}$. Then S is a type II set. Moreover, if X has a C^k-smooth norm then f may be chosen C^k-smooth.*

In the final section 6 we describe how to obtain examples of type II sets that are not a closure of their interior.

Theorem 5. *Let X be a separable Banach space with a Schauder basis. Then there exists a C^∞-smooth field f, Lipschitz on bounded sets, and a solution x such that $S = \Omega(x)$ is a type III set. Moreover, if X has a C^k-smooth bump then f may be chosen globally Lipschitz and C^k-smooth. Every type III set satisfies the following property. Let $U = \cup_{n=1}^\infty U_n$ be a decomposition of $U = X \setminus S$ into connected components and let x be any solution to a Lipschitz equation, such that $S = \Omega(x)$, then $x \cap \partial U_n = \emptyset$ for every $n \in \mathbb{N}$.*

The statement of the above theorem is slightly imprecise, as we have defined type III sets with respect to (globally) Lipschitz equations. It is surprising that for locally Lipschitz fields no smoothness assumption on X seems to be needed. We see that a necessary condition for S to be of type III is that $R = S \setminus \cup_{n=1}^\infty \overline{U_n}$ is dense in S. However, this condition is not sufficient for a Polish set S to be a type III set, as the set R needs to contain an infinite smooth curve. Containing such a curve is neither sufficient, as the directional derivatives of it may not be extended to a global Lipschitz vector field.
In our note we will be working exclusively with Lipschitz functions f. In this case, the classical finite-dimensional results on the existence and uniqueness of global solutions of (1) and their continuous dependence on the initial conditions still hold.

The structure of ω-limit sets has been studied extensively in finite dimensional spaces not only for differential equations but also in the setting of dynamical systems and ergodic theory. In $\mathbb{R}^2$ their structure is completely described as being either a point or a periodic orbit by the famous Poincaré-Bedixon theorem ([Ha82]). In higher dimensions it is known that they may have non-empty interior as a consequence of the work in [Ka79], [BMK81]. We are not going to discuss here this vast field of research. Recall the basic result in [Ha82] p. 145.

Theorem 6. *Let $X = \mathbb{R}^m$ and $x : [0, \infty) \to X$ be a bounded solution of* (1). *Then an ω-limit set of a x is non-empty, compact and connected.*

If X is infinite-dimensional, then the continuity of f does not guarantee the existence of any solutions to (1) (for a fixed initial condition this is due to Godunov [G75], for any solution it is the result of Shkarin [S03] and [HJ]). However, this is not the only obstacle in generalizing Theorem 6. The first examples of failure of Theorem 6 for C^∞-smooth (thus locally Lipschitz) functions f in infinite dimensional Hilbert space are due to Horst [H86], where noncompact and disconnected ω-limit sets were obtained. A wealth of results on ω-limit sets in infinite dimensional spaces is in the paper of Garay [Ga91], which is dealing mostly with dynamical systems. One of the main results there is the following.

Theorem 7. *(Garay) Let X be a Banach space, $S \subset X$ be a separable and closed subset, such that there exists an open set $U \subset X$ with the properties:*
1. U is connected.
2. $S = \partial U$.
Then there exists a continuous dynamical system with a trajectory $x(t)$, with the property $S = \Omega(x)$.

In fact, the techniques in [Ga91] are very natural, and we have discovered our results below independently of Garay's work, motivated instead by the work of Herzog [H00].

Theorem 8. *(Herzog)*
Let P be any Polish space, $X = \ell_2 \oplus c_0$. Then there exists a locally Lipschitz function $f : X \to X$ such that P is homeomorphic to a ω-limit set of some solution to the autonomous equation $x' = f(x)$.

Our first main result, Theorem 3 is a generalization of both Theorems 8 and 7. One of the key ideas in the proof of our second main result Theorem 4, the use of hypercyclic operators, was again conceived independently, but its relevance was previously observed by Fathi [Fat83] in his work on dynamical systems on the Hilbert space and by Garay in [Ga91].

Our note is organized as follows. In section 2 we collect some background results and notions from the relevant areas of differential equations, topology and Banach space theory. This is followed by technical lemmas needed mainly in the proofs of Theorem 3 and 4. These proofs follow a similar pattern and are given in subsequent sections 3 and 4. In section 5. we prove Theorem 5, which is somewhat different, and turns out to be the most delicate part of the present note. In the final section 6 we discuss some relevant examples.

2. Auxiliary results

We begin by collecting some well-known results for the convenience of the reader. The first result is classical in the finite dimensional setting, [Ha82], Chapter V.

Theorem 9. *Let $X = \mathbb{R}^n$ be a Banach space, $f : X \to X$ be a Lipschitz and C^k-smooth function. Then for every initial condition $x(0) = x_0$, (1) has a unique solution $x(x_0, t)$ defined on the whole $t \in \mathbb{R}$. Moreover, the function $x(x_0, t) : X \times \mathbb{R} \to X$ is continuous (in other words, the solutions depend continuously on the initial condition), it is C^k-smooth and it is a C^k-smooth diffeomorphism in the first variable.*

We will also need its infinite dimensional analogue. For lack of suitable reference (the existence is mentioned e.g. in [D77], but we have not found an explicit statement on the continuous dependence on the initial condition) we sketch its simple proof.

Theorem 10. *Let X be a Banach space, $f : X \to X$ be a Lipschitz function. Then for every initial condition $x(0) = x_0$, (1) has a unique solution $x(x_0, t)$ defined on the whole $t \in \mathbb{R}$. Moreover, the function $x(x_0, t) : X \times \mathbb{R} \to X$ is continuous (in other words, the solutions depend continuously on the initial condition).*

Proof. Suppose that f is K-Lipschitz, i.e. $\|f(y_1)-f(y_2)\| \le K\|y_1-y_2\|$ for $y_1, y_2 \in X$. We will solve (1) on the interval $J = [0, \delta]$, where $\delta = \frac{1}{2K}$. Let $C(J, X)$ be the Banach space of continuous functions from J into X equipped with the supremum norm. It is easy to see that the continuous operator $T_y : C(J, X) \to C(J, X)$, $y \in X$, defined as

$$T_y(x)(t) = y + \int_0^t f(x(\tau))d\tau \tag{2}$$

is a contraction. More precisely, given $g, h \in C(J)$,

$$\|T_y(g) - T_y(h)\| \le \int_0^\delta \|f(g(\tau)) - f(h(\tau))\|d\tau \le \delta K\|g-h\|_\infty \le \frac{1}{2}\|g-h\|_\infty. \tag{3}$$

By the Banach contraction principle ([F~] Theorem 7.55) T_y has a unique fixed point x. Clearly, x is a solution of (1) with the initial condition $x(0) = y$. It is also clear that the process can be repeated on every interval $[k\delta, (k+1)\delta]$, $k \in \mathbb{Z}$, and the respective solutions can be joined to form a unique solution on the whole $\mathbb{R}$. It suffices to check the continuous dependence on the initial condition y on the interval J. Let x be a fixed point of T_{y_1}. Then $T_{y_2}(x) = (y_2 - y_1) + T_{y_1}(x) = (y_2 - y_1) + x$. So $\|T_{y_2}(x) - x\| = \|y_2 - y_1\|$. By the proof of the contraction principle we know that the fixed point $\tilde{x}$ of T_{y_2} is a limit of $T^n_{y_2}(x)$ and so $\|\tilde{x} - x\| \le \sum_{n=0}^\infty \frac{1}{2^n}\|y_2 - y_1\| = 2\|y_2 - y_1\|$. This finishes the proof. □

Lemma 11. *Let $S = \Omega(x)$ for a solution x of* (1) *with locally Lipschitz function f. Suppose that $x(0) \in S$. Then*

$$S = \overline{\cup_{t\in[0,\infty)} x(t)} \tag{4}$$

In particular, S is connected.

Proof. This follows immediately from Theorem 10. Indeed, the solution keeps returning arbitrarily close to the point $p = x(0)$, and so it does the same for every $x(t)$, $t \in [0, \infty)$. □

Proposition 12. *Let X be a Banach space. Then every ω-limit set of a Lipschitz equation* (1) *belongs to precisely one of the types I-III.*

Proof. We only need to prove that if S is an ω-limit set such that $S \cap x \ne \emptyset$ for every solution x with $S = \Omega(x)$, then S is of type III. If $S \cap x \ne \emptyset$, there exists $t_0 \in [0, \infty)$ with $x(t_0) \in S$. So there exists a sequence $t_n \nearrow \infty$ with $x(t_n) \to x(t_0)$. By Theorem 10 we immediately see that $x(t_n + s - t_0) \to x(s)$, so every $x(s)$, $s > t$ belongs again to S. The case $s < t$ follows similarly considering the backward solution. □

The solution x to (1) is usually called a forward solution (meaning the parameter t tends to $+\infty$). A solution for $t \in (-\infty, 0]$ is called backward solution. For equations with a Lipschitz f both solutions exist and are unique. A set $S \subset X$ is said to be f-invariant if every (forward and backward) solution with initial condition $x(0) \in S$ stays in S. If f is Lipschitz, it follows from Theorem 10 that S is f-invariant iff $X \setminus S$ is f-invariant.

We will need some facts from general topology. We say that a topological space U is arcwise connected if for any $p, q \in U$ there exists a continuous mapping $\gamma : [0,1] \to U$, $\gamma(0) = p, \gamma(1) = q$. It is well-known and easy to show that an open subset of a Banach space is arcwise connected if and only if it is connected ([HY61], Theorem 3-17). Recall the definition of absolute neighborhood retract. A compact metrizable topological space V is said to be an absolute neighborhood retract (ANR) if for

every homeomorphic image of $V \subset V'$, where V' is a metric space, there exists some open neighborhood U, $V \subset U \subset V'$, and a continuous mapping $g : U \to V$, $g \restriction_V = Id$. The classical examples of ANR are convex sets in $\mathbb{R}^n$. We refer to [Bo67] for a thorough treatment of the subject.
Let us recall some properties of higher smoothness. We will work exclusively with the Fréchet smoothness (see [DGZ93]). Let X, Y be Banach spaces, $M : X^n \to Y$ be a mapping with the property

$$M(a_1^1 h_1^1 + a_1^2 h_1^2, \dots, a_n^1 h_n^1 + a_n^2 h_n^2) = \sum_{i_1,\dots,i_n \in \{1,2\}} (\prod_{j=1}^{n} a_{i_j}) M(h_1^{i_1}, \dots, h_n^{i_n}). \tag{5}$$

Then M is called n-linear. The norm of multilinear forms is

$$\|M\| = \sup\{M(h_1, \dots, h_i) : \|h_i\| \leq 1\}. \tag{6}$$

The algebra of symmetric multilinear is equipped with the symmetric product $\odot$ among the forms. We have that $\|M \odot N\| \leq \binom{i+j}{j} \|M\|\|N\|$, whenever M is an i-linear form on X and N is a j-linear form on X. Moreover, $\|\phi^i\| = i!\|\phi\|$, for $\phi \in X^*$ ([Fed69] p.47). By definition, higher derivatives are symmetric multilinear forms. More precisely, given Banach spaces X, Y, an open $U \subset X$ a function $f : U \to Y$ is C^k-Frechet smooth if at every $a \in U$, there exists a symmetric multilinear form $M \in \mathcal{L}(^i X, Y), 0 \leq i \leq k$, so that $D^i f(a)(h_1, \dots, h_i) = M(h_1, \dots, h_i)$ (we are using the convention that $D^0 f = f$), and the mappings $a \to D^i f(a)$ are continuous. Let U be an open subset of a Banach space X, $h : U \to \mathbb{R}$ and $f : U \to X$ be C^k-smooth. The Leibnitz formula for the derivative of a product ([Fed69], p.222) can be formulated in the following way.

$$D^i(h \cdot f) = \sum_{j=0}^{i} D^{i-j} h(z) \odot D^j f(z) \text{ for } i = 0, \dots, k \tag{7}$$

Indeed, we can verify that $D^{i-j} h(z) \in \mathcal{L}(^{i-j} X, \mathbb{R})$, $D^j f(a) \in \mathcal{L}(^j X, X)$ and so their symmetric product belongs to $\mathcal{L}(^i X, X)$ as required. If $f : X \to Y$, $g : Y \to Z$ are C^k-smooth, $b = f(a)$, the the chain rule formula holds.

$$D^i(g \circ f)(a) = \sum_{\alpha \in S(i)} D^{\sum \alpha} g(b) \circ [D^1 f(a)^{\alpha_1} \odot \cdots \odot D^k f(a)^{\alpha_k}]/\alpha! \tag{8}$$

for $i \leq k$, where $S(i)$ is the set of all k-termed sequences of nonnegative integers such that

$$\sum_{j=1}^{k} j\alpha_j = i. \tag{9}$$

In the particular case when f is a linear mapping, $D^i f(a) = 0$ for all $a \in X, i \geq 2$, so we obtain a special case:

$$D^i(g \circ f)(a) = D^i g(b) \circ f^i / i! \tag{10}$$

This implies (using the definition of $\odot$ in [Fed69]) the obvious fact that composing with linear mappings of norm at most one from the left preserves the upper estimates for the higher derivatives.

$$\|D^i(g \circ f)(a)\| \leq \|D^i(g)(f(a))\| \|f\|^i. \tag{11}$$

Standard operations among functions preserve higher smoothness as in the finite dimensional situation. We refer to [Fed69] for more details and proofs.

We will use finite partitions of unity on Banach spaces. We say that a collection $\{\psi_i\}_{i=1}^n$ of real and non-negative C^k-smooth functions is a partition of unity if $\sum_{i=1}^m \psi_i(z) = 1$ for every $z \in X$. We will often use the well-known fact that given a bounded and closed set $V \subset \mathbb{R}^n$ and an open set $U \supset V$, there is a C^∞-smooth partition of unity $\{\psi_1, \psi_2\}$ with all derivatives uniformly bounded, such that $\psi_1(V) = 1$ and $\operatorname{supp}(\psi_1) \subset U$.
Recall that a norm $\|\cdot\|$ on a Banach space X is said to be C^k-smooth if it is C^k-smooth as a function, away from the origin. A bump function on X is a real nonzero function with a bounded support set. By composing a C^k-smooth norm $\|\cdot\|$ with a suitable real C^∞-smooth function we obtain a C^k-smooth bump. Let X be a Banach space, $A, B \subset X$ be bounded closed and convex sets with nonempty interior, and let $L : A \to B$ be a topological homeomorphism such that both L, L^{-1} are C^k-smooth mappings in the interior of the sets A, B. We call such L a C^k-smooth diffeomorphism between A, B.
We are going to work with curves in a Banach space. A curve γ comes with a specific parametrization, i.e. a one-to-one function $\gamma : I \to X$ where $I = [a, b] \subset \mathbb{R}$ is some interval. A curve is C^k-smooth if its derivatives $\gamma^{(i)}$, $i \le k$, exist and are continuous in the interior of I, they have one-sided limits at endpoints, and also $\gamma'(t) \ne 0$ on I. We will occasionally use also the notation $\widehat{\gamma}$ for the image of the curve γ, i.e. $\widehat{\gamma} = \{\gamma(t) : t \in I\}$. Given $p, q \in X$ we denote by pq the segment joining this pair of points, i.e. $pq = \{r : r = tp + (1-t)q, t \in [0, 1]\}$.
It is clear that a solution x of (1) is also a curve provided $x' \ne 0$. In some places, we will be using the term solution curve (instead of just solution), in order to emphasize the geometrical position of the solution x. This will be useful in describing various deformations of the mapping f and its solutions into some prescribed forms.
Let $S \subset X$ be a subset. We use the notation $[S] = \overline{\operatorname{span}} S$ for the closed linear span of S. Given $A, B \subset X$, we use the notation $A + B = \{z \in X : z = a + b, a \in A, b \in B\}$.
We are going to use Schauder bases (c.f. [F~] Chap. 6). Recall that a normalized Schauder basis of a separable Banach space X is a biorthogonal system $\{e_n; \phi_n\}_{n=1}^\infty$, $e_n \in S_X$, $\phi_n \in X^*$ with properties:

1. $\phi_m(x_n) = \delta_m^n$ the Kronecker delta.
2. $[x_n : n \in \mathbb{N}] = X$

The sums $z = \sum_{n=1}^\infty \phi_n(z)e_n$ are convergent, and the projections $P_k(X) \to X$, $P_k(z) = \sum_{n=1}^k \phi_n(z)e_n$ are 1-bounded onto the finite dimensional subspace $X_k = [e_n : n \le k]$, and $X = X_k \oplus \operatorname{Ker} P_k$ is a topological sum. Also, $P_k \circ P_n = P_{\min\{k,n\}}$. By $X_n = [e_i : 1 \le i \le n]$ we denote the n-dimensional subspaces. We have $X = \overline{\cup_{i=1}^\infty X_n}$. All classical separable Banach spaces (such as $L_p[0, 1], \ell_p, C[0, 1]$ etc.) admit a (normalized) Schauder basis, although this is not true for every separable Banach space due to a famous counterexample by Enflo.

The following notion is of central importance for our work. Later, we are going to construct smooth vector fields with additional properties by using suitable transports.

Definition 13. *Let $X = Y \oplus [y]$, $\|y\| = 1$, $J = [a, b] \subset \mathbb{R}$. Let $A, B \subset Y$ be bounded closed and convex neighborhoods of the origin, and let $L : A \to B, L(0) = 0$ be a C^k-smooth diffeomorphism. Let U be an open set in X, $p, q \in U$. We say that $p + A$ can be C^k-smoothly L-transported to $q + B$ in U if there exists a closed interval $J = [a, b]$, $\varepsilon > 0$ and a one-to-one Lipschitz C^k-smooth mapping $F : J \times (p + A) \to U$ such that:*
1. $F(b, p + z) = q + L(z)$, and for every $z \in A$, $t \to F(t, z)$ is a C^k-smooth curve,
2. if $t < \varepsilon$, then $F(a + t, p') = p' + ty$, $F(b - t, p') = q + L(p' - p) - ty$.

3. for a fixed $t \in J$, $p \to F(t,p)$ is an C^k-smooth diffeomorphism onto its range $F(t,A)$ that is a convex set.
In this case we say that F is a C^k-smooth L-transport of $p+A$ to $q+B$ inside U. We also say that F is a transport of $p+A$ onto $q+B$ inside U if it is an L-transport for a suitable C^k-smooth diffeomorphism L, and we may also omit the specification of the degree of smoothness. The set $T_F = \cup_{t,z} F(t,z)$ is called tubus of transport F.

$T_F \subset U$ is a closed set. Most of the time we will work with a very special type of transports, for which the mapping $p+z \to F(t,p+z)$ is an affine mapping for every $t \in J$. In this case, the following simple fact holds.

Fact 14. *Suppose that the mapping $p+z \to F(t,p+z)$ is an affine mapping for every $t \in J$. If $A' \subset A$ is a convex subset, then the relativization of the transport $F : J \times A$ to the subset $J \times A'$ again gives rise to a transport, that will be called a restriction of F.*

The role of the particular decomposition $X = Y \oplus [y]$ is not important for us, as in our applications it serves together with condition 2 to compose subsequent transports. In particular, it is clear that the following holds.

Lemma 15. *Let $X = Y \oplus [y]$, $\|y\| = 1$. $I_1 = [a,b], I_2 = [b,c]$. Let $A, B, C \subset Y$ be bounded convex neigborhoods of zero, $B = L_1(A)$, $C = L_2(B)$, where L_1, L_2 are C^k-smooth diffeomorphisms. Let $p, q, r \in X$, and U be an open set in X. Let $F_1 : I_1 \times (p+A) \to U$ be an L_1-transport of $p+A$ to $q+B$, $F_2 : I_2 \times (q+B) \to U$ be a L_2-transport of $q+B$ to $r+C$. Assume that $T_{F_1} \cap T_{F_2} = B$. Then $F = F_1 ^\frown F_2 : I_1 \cup I_2 \times (p+A) \to U$,*

$$F(t,p+z) = \begin{cases} F_1(t,p+z) & \text{for } t \in [a,b] \\ F_2(t-b, q+L_1(z)) & \text{for } t \in [b,c] \end{cases} \tag{12}$$

is a $L_2 \circ L_1$-transport of $p+A$ to $r+C$.

Proof. The main point of the proof is to show that the mapping F behaves well around the value b of the first variable. First, note that $F_1(t,p+z) = q + L_1(z) + (t-b)y$ for $t \nearrow b$. On the other hand, $F_2(t, q+z') = q + z' + ty$ for $t \searrow b$, so $q + F_2(t, q+L_1(z)) = q + L_1(z) + (t-b)y$. This shows that the concatenation of the transports is well-defined and indeed C^k-smooth. The remaining conditions are easy to verify. □

The convexity condition contained in 3 of Definition 13 is chosen in order to obtain the following important property.

Lemma 16. *Let $F : J \times A \to U$ be a transport of A to B in U, $\varepsilon > 0$. Then there exists a finite collection of open convex sets C_i, $\overline{C}_i \subset U$, so that $T_F \subset \cup_{i=1}^n C_i \subset T_F + \varepsilon B_X$.*

Proof. This follows easily by compactness of J, the convexity of the mappings $p \to F(t,p)$ for each $t \in J$ and the Lipschitz condition for F. □

In order to construct transport mappings we use the following notion.

Definition 17. *Let $X = Y \oplus [y]$. Let $I = [a,b]$ be an interval and $\varepsilon > 0$. We say that a curve $\gamma : I \to X$ is ε-planar if:*
0. γ is C^∞-smooth
1. for every $t \in I$ it holds that $B(\gamma(t), 3\varepsilon) \cap \widehat{\gamma}$ lies in some two-dimensional affine subspace of X,
2. $\{s : \gamma(s) \in B(\gamma(t), 3\varepsilon)\}$ is an interval.
3. if $t < 3\varepsilon$ then $\gamma(a+t) = \gamma(a) + ty$, $\gamma(b-t) = \gamma(b) - ty$.

The following lemma is clear.

Lemma 18. *Let U be an open and arcwise connected subset of a Banach space X, $p, q \in U$. Then there exist $\varepsilon > 0$ and a ε-planar curve γ from p to q.*

Proof. Choose a finite sequence of points $\{p_i\}_{i=0}^n$, $p = p_0$, $q = p_n$ such that the line segments $p_i p_{i+1}$ lie in U. Moreover, $p_1 - p_0$ is a positive multiple of $p_n - p_{n-1}$, $\mathrm{dist}(p_i p_{i+1}, X \setminus U) > \varepsilon$ and $\mathrm{dist}(p_i p_{i+1}, p_j p_{j+1}) > \varepsilon$ whenever $j > i + 1$. It remains to "smoothen the corners" of this broken line in small enough neighbourhoods of p_i by a standard procedure. □

Lemma 19. *Let $\gamma : I \to X$, $X = \mathbb{R}^n$ be a ε-planar curve with curvature $\kappa(t) < \frac{1}{2\varepsilon}$. Then for every point $h \in X$, $\mathrm{dist}(h, \widehat{\gamma}) < \varepsilon$ there exists a unique nearest point from $\widehat{\gamma}$.*

Proof. The set R of nearest points is a closed subset of $R_\delta(h) = \{y : \|h - y\| = \delta\}$, $\delta = \mathrm{dist}(h, \hat{\gamma}) < \varepsilon$. If R is not a singleton, then there exist $t < s$ such that $p = \gamma(t)$ and $q = \gamma(s)$ both lie in R. By condition 3 in the Definition 17 none of p, q can be an endpoint of γ. By condition 1 in the Definition 17 there exists a two dimensional affine subspace $H \subset X$ such that $\gamma(r) \in H$ for $r \in [t, s]$. Denote $S = H \cap R_\delta(h)$. Because we are working in the Euclidean space, S is a Euclidean circle of radius not exceeding δ. Its curvature is therefore at least $\frac{1}{\varepsilon}$, and it connects the points p, q. On the other hand, γ lies on the outside of S in H, it connects p, q and its curvature is less than $\frac{1}{2\varepsilon}$. Moreover γ is tangent to S at p, q. This is a contradiction.

□

Definition 20. *Let $I = [a, b]$, $\gamma : I \to \mathbb{R}^n$ be a ε-planar curve with radius of curvature $\rho(t) > 3\varepsilon$. We let $D_\varepsilon(t)$ to be the $(n-1)$-dimensional Euclidean ball centered at $\gamma(t)$ with radius ε and such that $\gamma'(t)$ is perpendicular to the affine hyperplane containing $D_\varepsilon(t)$. We denote by $T_\varepsilon = \cup_{t \in [a,b]} D_\varepsilon(t)$ the ε-tubus around the curve γ.*

It follows from Lemma 19 that T_ε is an open set and to each $y \in T_\varepsilon$ there corresponds a unique t_y such that $y \in D_\varepsilon(t_y)$.

Lemma 21. *Let $I = [a, b]$, $\gamma : I \to \mathbb{R}^n$ be a ε-planar curve with radius of curvature $\rho(t) > 3\varepsilon$. Let*

$$f : T_\varepsilon \to \mathbb{R}^n \ \text{ be the vector field } \ f(y) = \gamma'(t_y). \tag{13}$$

Then f is a C^∞-smooth vector field, which defines an autonomous differential equation

$$x' = f(x) \text{ in } T_\varepsilon. \tag{14}$$

Given any $p \in D_\varepsilon(0)$, the solution corresponding to the initial condition $x(0) = p$ is perpendicular to $D_\varepsilon(s)$ at the point of their intersection p_s. Then the mapping $F : I \times D_\varepsilon(0) \to \mathbb{R}^n$, $F(s, p) = p_s$ is a C^∞-smooth transport.

Proof. It follows from Lemma 19 that $D_\varepsilon(t) \cap D_\varepsilon(ts) = \emptyset$ for $t \neq s$. So f is correctly defined. Since we have not specified the mapping L, it suffices to check the transport conditions locally. First assume that $n = 2$. In this case, due to the curvature and smoothness conditions on γ it is standard to check, using tools such as C^∞-smooth inverse mapping theorem, that the mapping $(s, p) \to p_s$ is a C^∞-diffeomorphism between the respective sets. This implies conditions 1 and 3. Thus the function $g(x) = \mathrm{dist}^2(x, \hat{\gamma})$ is C^∞-smooth on the tubus T_ε. We know that $f(x) \in \mathrm{Ker} Dg(x)$, where $Dg \in X^*$ stands for the Frechet derivative of g, or in this case the total differential of the function g. Consequently, every curve corresponding to a solution of (14) preserves the value of g. In other words, all

points of the integral curve preserve their distance to the original curve γ. Note, however, that replacing γ by any other curve corresponding to a solution of (14) in the above argument leads to the same conclusion on preserving the distance. This shows that the transport mappings $p \to F(s,p)$, with a fixed s, act as linear isometries, so condition 2. is satisfied, which finishes the proof.
In the general case $X = \mathbb{R}^n$, we use that γ is locally planar. We split $\mathbb{R}^n = \mathbb{R}^2 \oplus \mathbb{R}^{n-2}$, assuming that γ lies in the first direct summand. Now using that the function f is locally independent on the perpendicular directional subspace $\mathbb{R}^{n-2}$, we again see that the mapping $F(s,\cdot)$ is a a linear isometry. □

Lemma 22. *Let $X = \mathbb{R}^n = \mathbb{R}^{n-1} \oplus \mathbb{R}$. Let $\gamma : [a,b] \to \mathbb{R}^n$ be a ε-planar curve with radius of curvature $\rho(t) > 3\varepsilon$. Let $L : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be any (linear) isometry with $det(L) = 1$. Then there is a C^∞-smooth L-transport $F : [a,b] \times D_\varepsilon(0) \to \mathbb{R}^n$, $L : D_\varepsilon(0) \to D_\varepsilon(1)$.*

Proof. The transport constructed in Lemma 21 has all the properties except the control of the endpoint isometry L. Denote $L_1(z) = F(b,z)$, $p = \gamma(a), q = \gamma(b)$, $A = D_\varepsilon(0)$. We have that $\det(L_1) = 1$. Indeed, we can C^∞-smoothly deform the given curve γ into a planar curve connecting p,q. For planar curves it is clear that the endpoint mapping is the identity. However, the sign of $\det L$ for the endpoint mapping will be constant during the curve deformation process.
Consider the transport $F_1 : [a,s] \times D_\varepsilon(0) \to \mathbb{R}^n$, $s = b - \delta$, δ small enough, which is a restriction of F. Let $B_1 = F(s,A)$. By the endpoint condition $1'$, we know that $B_1 = L_1(A)$. Choose an isometry L_2 such that $Id = L_2 \circ L_1$. It is well-known (Theorem 3.67 in [War72]) that the real orthonormal group $\mathcal{O}_n$ on $\mathbb{R}^n$ is a C^∞-smooth manifold with two components (accoding to the sign of the determinant). Thus there exists for $s < s_1 < s_2 < b$ a C^∞-smooth curve $\eta : [s_1,s_2] \to \mathcal{O}_n$ which starts at Id and ends at L_2. By reparametrizing, assume that all derivatives of η at the endpoints are equal to zero. Extend the definition of η onto $[s,b]$ by putting $\eta(t) = \eta(s_1)$ whenever $t \le s_1$, and $\eta(t) = \eta(s_2)$ whenever $t \ge s_2$. Now, we define a transport $F_2 : [s,b] \times L_1(A) \to \mathbb{R}^n$ by $F_2(r,z) = \eta(r) \circ F_1(z) + (r-s)y$. To finish the proof, set $F = F_2 \frown F_1$ and apply Lemma 15.

□

Lemma 23. *Let U be an open connected subset of a Banach space X, $p,q \in U$. Then p,q are connected via a ε-planar curve $\gamma : [0,1] \to U$, such that the ε-tubus $T_\varepsilon \subset U$.*

Proof. As U is arcwise connected, there exists a continuous curve in U connecting p and q. Approximate this curve inside U by a finite broken line $\eta : [0,1] \to U$ whose ε'-neighborhood still lies in U. There exists a finite dimensional linear space F containing $\widehat{\eta}$. By smoothing up the corners we obtain a C^∞-smooth ε-planar curve γ in F connecting p,q in U. □

Lemma 24. *Let $X = Y \oplus [y]$. Let U be an open arcwise connected of a Banach space X, $p,q \in U$, $L = Id_Y$. Then there exists $\varepsilon > 0$ so that for $A = p + \varepsilon B_Y$, $B = q + \varepsilon B_Y$ there exists a C^∞-smooth L-transport $F : [0,1] \times A$ of A to B inside U.*

Proof. By above Lemmas, there exists a finite dimensional subspace $F_1 \hookrightarrow Y$, ε'-planar curve γ connecting p,q, that lies in $F = F_1 \oplus [y]$, and such that $T_{\varepsilon'} \subset U$. Using the topological direct sum $Y = F_1 \oplus H^1$, we express every element of A in a unique way $a = p + a_1 + a_2$, where $a_1 \in F_1$, $a_2 \in H^1$, and where $\|a_1\|, \|a_2\| < \varepsilon''$. Next we equip $F_1 \oplus [y]$ with Euclidean norm, so that y is perpendicular to F_1. Let $A' = p + \varepsilon'' B_{(F_1, \|\cdot\|_2)}$, $B' = q + \varepsilon'' B_{(F_1, \|\cdot\|_2)}$. Using Lemma 21 construct an isometric transport G' of A' to B', whose tubus $T'_{\varepsilon''} \subset T_{\varepsilon'}$. In order to extend the definition of

G' the whole set A, use the decomposition of A: $G(t,p+a_1+a_2) = G'(t,p+a_1)+a_2$. Finally, choose $\varepsilon > 0$ so that $\|a_1+a_2\|_X < \varepsilon$ implies that the transport $G([0,1]\times A$ remains inside U.

□

Lemma 25. *(Size reduction) Let $X = Y \oplus [y]$, U be an open set in X. Given $\alpha,\beta,\lambda > 0$, $p, q = p+\lambda y \in X$, assume that the convex hull of the set $p+\alpha B_Y \cup p+\lambda y+\beta B_Y$ lies in U. Then there is a C^∞-smooth transport F of $A = p+\alpha B_Y$ to $q+\beta B_Y$ in U.*

Proof. Let ϕ be a C^∞-smooth monotone function on $[0,\lambda]$ such that $\phi(t) = \alpha$ in the neighbourhood of 0 and $\phi(t) = \beta$ in the neighbourhood of λ. Put $F(t,p+a) = p+ty+\phi(t)a$. □

Smooth transport can be used to define smooth vector fields with controlled behaviour near the boundary of the tubus. We keep the notation from previous lemmas.

Lemma 26. *Let X be a separable Banach space admitting a C^k-smooth bump function b. Let V be an open set in X, $f : V \to X$ be a C^k-smooth vector field. Then there exists a C^k-smooth and 1-Lipschitz vector field $g : X \to X$, such that $g(z) = \lambda(z)f(z), \lambda(z) > 0$ for every $z \in V$, and $g(z) = 0$ for $z \notin V$.*

Proof. WLOG $b \geq 0$. Let $G = \mathrm{int}(\mathrm{supp}(b))$, $0 \in G$. Let $K > 0$ be such that $\|D^i b\| < K, 0 \leq i \leq k$ on X. Since f is locally Lipschitz, by using the Lindeloff property of V, there exists a countable collection of sets $\{B_n\}_{n=1}^\infty$, $B_n = x_n + \rho_n G, \rho_n < 1$, $\mathrm{dist}(B_n, X\setminus V) = d_n > 0$ with the following properties. $V = \cup_n B_n$, $f\restriction_{B_n}$ is L_n-Lipschitz, and $\|D^i f\| < L_n, 0 \leq i \leq k$. Choose a suitable positive sequence $\varepsilon_n \searrow 0$ so that:
1. $\sum_{n=1}^\infty \varepsilon_n K L_n (1+\frac{1}{\rho_n}) < 1$
2. $\sum_{n=1}^\infty \frac{\varepsilon_n (k+1) K L_n}{\rho_n^k} < 1$
Then $g(z) = \sum_{n=1}^\infty \varepsilon_n b(\frac{1}{\rho_n} z - x_n) f(z)$ is well-defined on V, and has the required properties. The 1-Lipschitz condition follows directly from condition 1 above. Combining condition 2 and the Leibnitz formula (7) we see that $\sum_{n=m}^\infty \varepsilon_n b(\frac{1}{\rho_n} z - x_n) f(z)$ has derivatives uniformly tending to zero. The support of the initial sum $\sum_{n=1}^m \varepsilon_n b(\frac{1}{\rho_n} z - x_n) f(a)$ is on the other hand of positive distance to $X \setminus V$. □

Lemma 27. *Let X be a Banach space of dimension at least $n+1$. Given bounded open convex sets $C_1, \dots, C_n$, let $U = \cup_{i=1}^n C_i$, $V = \partial U$. If the set U is connected, then given any $\delta > 0$, $W = \overline{U} + \delta B_X$, we have that $W \setminus \overline{U}$ is arcwise connected.*

Proof. Let us assume first that $X = \mathbb{R}^{n+1}$. Note the important fact that $X \setminus V$ has exactly two components. Indeed, one component U is connected by assumptions. On the other hand, given any $p \notin U \cup V$, by the Hahn-Banach separation theorem there exist functionals $\phi_i \in X^*$ and $\alpha_i \in \mathbb{R}$ such that $\phi_i(p) > \alpha_i$, and $\phi_i(C_i) < \alpha_i$. Since the linear dimension of X is $n+1$, there exists some $0 \neq h \in \cap_i \mathrm{Ker}\phi_i$. The line $p+th, t \in \mathbb{R}$ is clearly disjoint from $U \cup V$. It is now obvious that p, q are connected by a curve γ that is disjoint from $U \cup V$, whenever $p, q \notin U \cup V$. So $X \setminus (U \cup V)$ is connected. Choose some $p_i \in C_i$ and let $M_i : X \to \mathbb{R}$ be the function $M_i(x) = (\sup\{\lambda : \lambda(x-p_i) \in C_i\})^{-1}$. In other words, M_i is the Minkowski functional of the convex set $C_i - p_i$, shifted back to the point p_i. The function $F(x) = \min_i M_i(x)$ is continuous. We have $U = \{x : F(x) < 1\}$, $V = \{x : F(x) = 1\}$. Given any $\delta > 0$ and a C^∞-smooth function F_δ, $0 \leq F_\delta - F < \delta$, we may WLOG assume that the graph of F_δ in $X \oplus \mathbb{R}$ is transversal to the set $\{(x,1) : x \in X\}$ and $F_\delta^{-1}(1) \subset (V + \delta B_X)$ (i.e. the derivative F_δ' has rank one at this set). Indeed this follows from the well-known theorem of Sard (Theorem

10.2.1 in [DFN85]), which claims that the range of all points where the rank is zero has Lebesgue measure zero. Thus, $V_\delta = F_\delta^{-1}(1)$ is a C^∞-smooth n-dimensional C^∞-smooth manifold $\tilde{M}_\delta$ embedded into X. (Theorem III.5.8 in [Boo86]). $\tilde{M}_\delta$ has finitely many connected components, $\tilde{M}_\delta = \cup_{j=1}^l \tilde{M}_\delta^j$. Each of its components is of course a compact connected C^∞-smooth n-manifold embedded into $X = \mathbb{R}^{n+1}$. It follows ([Dol80] p. 269) that $X \setminus \tilde{M}_\delta^j$ has exactly two components U_j, V_j, where U_j unbounded. Clearly, given $i \neq j$, we have either $\tilde{M}_\delta^i \subset U_j$ or $\tilde{M}_\delta^i \subset V_j$. So we can select a subset $J \subset \{1, \dots, l\}$ so that for every $i \in \{1, \dots, l\}$ there exists exactly one $j \in J$ such that $\tilde{M}_\delta^i \subset V_j$. Since $U \cap F_\delta^{-1}(1) = \emptyset$, we see that there exists some $j \in J$ for which $U \subset V_j$. Denote $M = \tilde{M}_\delta^j$. Since M is a compact connected manifold, it is also arcwise connected. Consequently, for any pair of points $p, q \in M$ there exists a curve γ connecting them in M. Note that $\gamma \subset V + \delta B_X$, so it lies within distance δ of V and inside the unbounded component of $X \setminus V$. So given $\delta > 0$, $W = \overline{U} + \delta B_X$, we have that $W \setminus \overline{U}$ is arcwise connected. In the general space X, choose a suitable $(n+1)$-dimensional linear subspace X' such that the set $\cup_{i=1}^n (X' \cap C_i)$ is connected. Applying the previous result we obtain the conclusion. □

The next statement is probably known, but we have not found a reference in the literature. We include the proof for completeness.

Lemma 28. *Let X be a Banach space of dimension at least $n+1$. Given bounded open convex sets $C_1, \dots, C_n$, let $U = \cup_{i=1}^n C_i$, $V = \partial U$. If the set U is connected, then V is arcwise connected.*

Proof. First, we claim that $\overline{U}$ is an absolute neighborhood retract (ANR). Let us recall the property (Δ) in [Bo67] p.163: A topological space Y has the property (Δ) if for every $y \in Y$ and every neighborhood U of y, there is a neighborhood $V \subset U$ of y, such that every compact $A \subset V$ is contractible to a point in a subset of U having dimension $\leq \dim A + 1$. Clearly, a closed convex and bounded set in $\mathbb{R}^{n+1}$ has (Δ). Moreover, every finite dimensional compact with (Δ) is ANR [Bo67] p.163. By Theorem 4.1 in [Bo67] p.167, given $Y_1, Y_2, Y_1 \cap Y_2$ that are closed ANR with (Δ) then $Y_1 \cup Y_2$ also has (Δ). Use this theorem inductively to prove that the closure of $\cup_{i=1}^n C_i$ has (Δ) (and thus it is also ANR). For $n = 1$ it is clear. Also $n = 2$ is clear, as $C_1 \cap C_2$ is convex. Inductive step. Let $Y_1 = \overline{\cup_{i=1}^{n-1} C_i}$, $Y_2 = \overline{\cup_{i=2}^{n} C_i}$, $Y_3 = \overline{\cup_{i=2}^{n-1} C_i}$. By assumption both Y_1, Y_2 have (Δ). Now $Y_1 \cap Y_2 = Y_3 \cup \overline{(C_1 \cap C_n)}$. By inductive hypothesis the last set again has (Δ). This completes the proof. By Corollary 3.3 in [Bo67] p. 104, there exists an open neighbourhood O of $\overline{U}$ in $\mathbb{R}^{n+1}$ and a homotopy $h_t : [0,1] \times O \to O$ such that $h_0 = id_O$ and h_1 is a retraction of O onto $\overline{U}$. Such homotopy serves to conclude that there exists a retraction $r : O \to \overline{U}$ with the additional property that $r(O \setminus U) \subset V$. Indeed, we put

$$r(o) = h_t(o), \quad \text{where} \quad t = \min\{s \in [0,1], h_s(o) \in \overline{U}\}. \tag{15}$$

Combining this statement and Lemma 27 with standard arguments we get our desired conclusion. □

3. Type I

In this section we give the proof of Theorem 3. Let us assume first that X is separable, $X = Y \oplus [y]$, $A = B_Y$. Let U be open and connected, $S \subset \partial U$ be Polish. Let $\{s_n\}_{n=1}^\infty$ be a dense subset of S, and $\{r_n\}_{n=1}^\infty$ be dense in X, with $\rho_n = \operatorname{dist}(r_n, S) > 0$. Let $\{n_i\}_{i=1}^\infty$ be a sequence of natural numbers that contains every integer infinitely many times.

We are going to construct inductively a sequence $\{t_i\}_{i=1}^\infty \subset X$, $\varepsilon_i \searrow 0$, $A_i = t_i + \varepsilon_i A$, an increasing sequence $\{U_i\}_{i=1}^\infty$ of open, arcwise connected sets $U = U_1 \subset U_2 \subset \dots$

such that $S \subset \partial U_i$, $J_i = [a_i, a_{i+1}]$ and transports $F_i : J_i \times A_i \to U_i$, $F_i(\{a_{i+1}\} \times A_i) = A_{i+1}$ so that the following conditions hold:
1. $\|t_{i+1} - s_{n_{i+1}}\| < \varepsilon_i$
2. $T_{F_i} \cap T_{F_{i+1}} = A_{i+1}$, $T_{F_i} \cap T_{F_j} = \emptyset$ unless $|i-j| \le 1$.
3. if $j > i$, then $T_{F_j} \cap B(r_i, \frac{\rho_i}{2}) = \emptyset$
Inductive step from n to $n+1$. (The initial step is similar): We let $U_{n+1} = U_n \cup B(r_n, \frac{2\rho_n}{3})$ provided that $U_n \cap B(r_n, \frac{2\rho_n}{3})$ has non-empty interior. In this way, U_{n+1} being a union of two arcwise connected open sets with nonempty interior is again an arcwise connected open set, and moreover S is a subset of the topological boundary of U_{n+1}. In the alternative case $U_n \cap B(r_n, \frac{2\rho_n}{3}) = \emptyset$ we let $U_{n+1} = U_n$. Denote $P = \{i \le n+1 : B(r_i, \frac{2\rho_i}{3}) \subset U_{n+1}\}$. By Lemma 15 $F = F_{n-1}^\frown \dots {}^\frown F_1$ is a transport from A_1 to A_n in U_n. Still, by the inductive assumption there exists a transport $\tilde{F}_n : J_n \times A_n$ in U_n with final set $\tilde{A}_{n+1} = t_{n+1} + \tilde{\varepsilon}_{n+1} A$, $\tilde{A}_{n+1} = \tilde{F}_n(\{a_{n+1}\}, A_n)$. Put

$$C = T_{\tilde{F}_n} \cup \cup_{i=1}^{n-1} T_{F_i} \cup \cup_{i \in P} B(r_i, \frac{\rho_i}{2}) \tag{16}$$

By Lemma 16 together with Lemma 28 or 27, $U_{n+1} \setminus C$ is arcwise connected. Pick $t_{n+2} \in U \setminus C$, satisfying condition 1 above. By Lemma 24 there exists some $\varepsilon_{n+2} > 0$, $\varepsilon_{n+2} < \frac{\tilde{\varepsilon}_{n+1}}{2}$, such that $t_{n+1} + \varepsilon_{n+2} A$ is connected with $t_{n+2} + \varepsilon_{n+2} A$ inside $U_{n+1} \setminus (C + \varepsilon_{n+2} B_X)$ by means of transport $\tilde{F}_{n+1} : J_{n+1} \times (t_{n+1} + \varepsilon_{n+2} A)$. In order to be able to properly connect the successive transports, we now invoke the size-reduction Lemma 25 to replace the transport $\tilde{F}_n$ by $F_n : J_n \times A_n$ that satisfies $F_n(a_{n+1} \times A_n) = A_{n+1} = t_{n+1} + \varepsilon_{n+2} A$. This verifies condition 2 above. This finishes the inductive step.
The process yields an infinite "tubus" $T = \cup_{i=1}^{\infty} T_{F_i} \subset X$, in whose interior we have a C^∞-smooth vector field f that is tangent to the transport curves by Lemma 26. Now, assuming that X has a C^k-smooth bump function we invoke Lemma 26 in order to obtain a C^k-smooth vector field g, that can be extended into a C^k-smooth vector field on the whole X. It is clear that the solution to the autonomous ordinary differential equation $x' = g(x)$, with the initial condition $x(0) = t_1$ satisfies the conditions of the theorem. Indeed, it " follows the tubus" T and so by condition 1 its ω-limit sets contains S. On the other hand, condition 3 guarantees that the solution eventually keeps away from every point $p \notin S$. Our construction has the property that the tubus is getting progressively "thinner". It is therefore clear that the separability assumption on X can be dropped. Given general X, arcwise connected $U \subset X$ with a subset $S \subset \partial U$, choose a separable subspace $Y \hookrightarrow X$, a component of $\tilde{U} \subset U \cap Y$ and $S \subset \partial \tilde{Y}$ satisfying our conditions. Then the method of the above proof gives a solution.
The opposite implication. Let x be a solution to (1) with $\Omega(x) = S$. In the first case, $S \cap \{x(t) : t \in [0, \infty)\} = \emptyset$. In this case, choose a component U of $X \setminus S$ which contains the solution x. U is arcwise connected and $S \subset \partial U$.
The same proof immediately yields the next result.

Corollary 29. *Let X be a Banach space, $S \subset X$ be a separable and closed subset with empty interior whose complement $X \setminus S$ is connected. Then there exists an autonomous differential equation $x' = f(x)$, where $f : X \to X$ is a Lipschitz mapping, and its solution $x(t)$, with the property $S = \omega(x)$. Moreover, if X admits a C^k-smooth bump function then f may be chosen C^k-smooth as well.*

On the other hand, it is easy to give examples of closed sets with empty interior satisfying the assumptions of Theorem 3 but not of Corollary 29. Take e.g. $\cup_{i=1}^{m} (2me + S_X)$, where $\|e\| = 1$.
In fact a nonessential modification of the above proof gives the following result.

Theorem 30. *Let X be a Banach space, $S_n \subset X$ be separable and closed subsets, such that there exists an open set $U \subset X$ with the properties:*
1. U is connected.
2. $S_n \subset \partial U$.
Then given any sequence of distinct points $z_n \in U$, there exists an autonomous differential equation $x' = f(x)$, where $f : X \to X$ is a Lipschitz mapping, and its solution $x_n(t)$, $x_n(0) = z_n$ has the property $S_n = \omega(x_n)$ for all $n \in \mathbb{N}$. Moreover, if X admits a C^k-smooth renorming then f may be chosen C^k-smooth as well.

4. Type II

We begin the proof of Theorem 4 with a renorming lemma.

Lemma 31. *Let Y be a separable Banach space with a C^k-smooth equivalent norm $|||\cdot|||$. Then there is an equivalent C^k-smooth norm $\|\cdot\|$ and a set $\{\tilde{u}_n\}_{n=1}^\infty$ on the sphere of S_Y, functionals $\{\phi_n\}_{n=1}^\infty \in S_{Y^*}$, $\phi_n(\tilde{u}_n) = 1$, $\varepsilon_n > 0$, and $\tilde{V}_n = \{z \in B_Y : \phi_n^{-1}(z) \geq 1 - \varepsilon_n\}$ such that letting $H_n = \phi_n^{-1}(1-\varepsilon_n)$ to be affine hyperplanes in Y we have:*
1. $(\cup_{n=1}^\infty H_n) \cap \{\tilde{x}_i\}_{i=1}^\infty = \emptyset$
2. dist$(\tilde{V}_n, \tilde{V}_m) > \frac{1}{2}$ whenever $n \neq m$.

Proof. We distinguish the following cases. If Y is separable, then it has a LUR norm $\|\cdot\|$ by Theorem II.2.6 in [DGZ93]. In the C^1-smooth case, by the same theorem the space Y admits a norm $\|\cdot\|$ that is simultaneously LUR and C^1-smooth. So in these cases every point on the unit sphere of $S_{\|\cdot\|}$ is strongly exposed. We may choose as our $\{\tilde{u}_i\}_{i=1}^\infty$ an arbitrary $\frac{2}{3}$-separated subset of the sphere together with suitable norming functionals $\{\phi_n\}_{n=1}^\infty$ so that letting $\varepsilon_n \searrow 0$ small enough it holds that diam$\tilde{V}_n < \frac{1}{8}$. It remains to deal with C^k-smooth case, $k > 1$. By Theorem V.3.4 in [DGZ93] we know that Y is either superreflexive or it contains a copy of c_0. In the former case, Y is in particular an RNP space by Proposition 2.4.1 in [Bou83], so its unit sphere $S_{|||\cdot|||}$ is dentable by Corollary 2.3.7 in [Bou83] which finishes the argument in a similar way as in the previous case. Indeed it suffices to pick the points $\tilde{x}_i$ inside the suitable slices of small enough diameter. In the remaining case we have $c_0 \hookrightarrow Y$. By Sobczyk's theorem (Theorem 5.14 in [F~]) c_0 is complemented in Y by means of a projection P, so it holds $Y = c_0 \oplus Z$. We are first going to re-norm c_0 by the norm $\|\cdot\|_1$ whose unit ball is $\overline{\text{conv}}(\frac{2}{3}B_{(c_0,\|\cdot\|_\infty)} \cup \{\pm e_n\}_{n=1}^\infty)$ where $\{e_n\}_{n=1}^\infty$ is the canonical unit basis of c_0. It is easy to verify that every e_n is strongly exposed by its dual functional $\phi_n \in \ell_1$.. In fact, $\phi_n(y) > 1 - \xi$ and $\|y\|_1 \leq 1$ imply that $\|y - e_n\|_1 \leq \frac{2\xi}{3}$. By [DFH98] there exists a C^∞-smooth norm $\|\cdot\|_2$ on c_0 approximating $\|\cdot\|_1$ so that its unit ball B_2 satisfies $B_2 \subset B_{\|\cdot\|_1} \subset (1+\frac{\xi}{2})B_2$. Thus $\phi_n(y) > 1 - \xi$ and $\|y\|_2 \leq 1$ imply that $\|y - e_n\|_1 \leq \frac{2\xi}{3}$. So choosing suitable ε_n and $\tilde{\mu}_n = \lambda_n e_n$, $\|\tilde{\mu}_n\|_2 = 1$, $H_n = \phi_n^{-1}(1-\varepsilon_n)$ and $\tilde{V}_n = \{z \in B_2 : \phi_n(z) > 1 - \varepsilon_n\}$ we have diam$\tilde{V}_n < \frac{4}{3}\varepsilon_n$ and $\cup H_n \cap \{P\tilde{x}_i\} = \emptyset$. To finish, we re-norm Y by $\|y\| = |||(Id - P)y||| + \|Py\|_2$, and extend the functionals ϕ_n onto Y canonically. The separation of these slices follows from the separation of their projected images in c_0.

□

The norm $\|\cdot\|$ from Lemma 31 will be used in our proof below. Moreover, we need another simple lemma.

Lemma 32. *Let $\psi : \mathbb{R} \to \mathbb{R}^+$ be a nondecreasing C^∞-smooth such that $\psi(t) = 0$ iff $t \leq \frac{\lambda}{9}$, $\psi(t) = \frac{\varepsilon}{2}$ for $t \geq \frac{\lambda}{4} - \frac{\lambda}{9}$. We define a vector field f in $\mathbb{R}^2$ as follows.*

$$f(x,y)=\begin{cases}0 & \textit{for } y\le \psi(x)\\ (1,\psi'(x)) & \textit{otherwise}\end{cases}\tag{17}$$

Then there exists a C^∞-smooth field g on $\mathbb{R}^2$ parallel to f. This field also defines a transport of the segment $(0,0)(0,\frac{\varepsilon}{2})$ onto $(\frac{\lambda}{4},\frac{\varepsilon}{2})(\frac{\lambda}{4},\varepsilon)$.

By a parallel field we mean that $g(x,y)=\eta(x,y)f(x,y)$ where $\eta(x,y)>0$ whenever $f(x,y)\neq 0$.

Proof. The field f is C^∞-smooth on $O=\{[x,y]:y>\psi(x)\}$. Let $\eta:\mathbb{R}^2\to\mathbb{R}$ be C^∞-smooth, $\operatorname{supp}(\eta)=\overline{O}$ and all derivatives $D^i\eta=0$ on $\mathbb{R}^2\setminus O$ for all $i=0,1\dots$. It suffices to put $g(x,y)=\eta(x,y)f(x,y)$. The transport $F:[0,\frac{\lambda}{4}]\times(0,0)(0,\frac{\varepsilon}{2})\to\mathbb{R}^2$ is given by formula $F(t,(0,y))=(t,\psi(t))$. □

We proceed with the proof of Theorem 4. We have $X=[e]\oplus_{\ell_2}Y$, $\|e\|=1$, and assume that the norm of Y and X is C^k-smooth. Let $B=B_Y$ be the closed unit ball of Y, B^0_Y be its interior. Choose $v_1,v_2\in U$, such that $v_2=v_1+\lambda e$ and $\delta>0$ so that $v_1v_2+3\delta B_X\subset U$. Let $J_1=[-\lambda,0]$. We have that $F'_1:J_1\times(v_1+\delta B)\to U$, $F'_1(t,p)=p+(t+\lambda)e$ is a C^∞-smooth Id-transport of $v_1+\delta B$ to $v_2+\delta B$ in U. By applying Lemma 24 (and the restriction to convex subsets) there exists $0<\varepsilon<\delta$, so that denoting $A_1=v_1+\varepsilon B$, $A_2=v_2+\varepsilon B$ we have C^∞-smooth Id-transports in U $F_1:J_1\times A_1\to U$ sending A_1 to A_2, F_1 is a restriction of $\tilde{F}'_1$, and $F_2:J_2\times A_2\to U$ sending A_2 to A_1. WLOG assume that $J_2=[0,1]$. Moreover, $T_2\cap\overline{v_1v_2+3\delta B_X}=A_1\cup A_2$. This means that we have created a loop of transports that can be connected and $F_2^\frown F_1$ acts as an identity operator on A_1. In particular, applying Lemma 26 at this point (note that if a Banach space admits a C^k-smooth norm, then it also admits a C^k-smooth bump) we have obtained a C^k-smooth and Lipschitz autonomous equation on X with the property that all of its non-trivial solutions are periodic and live inside the tubus $T=T_1\cup T_2$. Our next step will lead to an equation with T as its ω-limit set. In order to achieve this goal, we apply a result from operator theory from [An97] or [Sa95]. Namely, on every separable Banach space (in particular on Y) there exists a hypercyclic operator, i.e. a bounded linear operator $T:Y\to Y$ and $y_h\in Y$ such that $\{T^n(y_h)\}_{n=1}^\infty$ is dense in Y. Moreover, and this is equally important for us, we may assume that $T=Id+K$ where K is a compact operator with norm $\|K\|<\frac{1}{2}$. So the spectrum of $Id+\xi K$, $|\xi|\le 1$ is contained in $\{1+z:|z|\le\frac{1}{2}\}$. does not contain zero, and so such operator is a linear isomorphism ([F˜] p. 210). Let $\zeta:(-\infty,1)\to\mathbb{R}$ be a C^∞-smooth non-decreasing function such that $\zeta(t)=0$ iff $t\le 0$, $\zeta'(t)>0$ for $t>0$ and $\lim_{t\to 1}\zeta(t)=\infty$. Let $\Phi:B^0_Y\to Y$ be a C^k-diffeomorphism given by $\Phi(y)=\zeta(\|y\|)y$, and denote $z=\Phi^{-1}(y_h)$. Fix two non-negative C^∞-smooth functions $\alpha,\beta:[0,1]\to[0,1]$, such that $\alpha(t)+\beta(t)=1$, $\alpha(t)=1$ for $t\in[0,\frac{1}{9})$ and $\alpha(t)=0$ for $t\in(1-\frac{1}{9},1]$. Let $\Psi(\theta):B^0_Y\to B^0_Y$,

$$\Psi(\theta)=\Phi^{-1}\circ(\alpha(\theta)Id+\beta(\theta)T)\circ\Phi\tag{18}$$

be a C^k-smooth diffeomorphism for every $\theta\in[0,1]$. Indeed, the special form of T guarantees that $\alpha(\theta)Id+\beta(\theta)T=Id+\xi K$, $|\xi|\le 1$ is an isomorphism on Y for every θ. Moreover, it is clear that $\tilde{x}_n=\{\Psi(1)^n(z)\}_{n=1}^\infty$ is a dense set in B^0_Y, because $\{\tilde{x}_n\}_{n=1}^\infty=\{\Phi^{-1}\circ T^n(y_h)\}_{n=1}^\infty$. Define a new Ψ-transport $\tilde{F}_2:J_2\times A_2\to U$, $\tilde{F}_2(\theta,v_2+z)=F_2(\theta,v_1+\Psi(\theta)(z))$. It is standard to check that $\tilde{F}_2$ is C^∞-smooth and has the same tubus as F_2. Thus it can be connected into a "loop transport" $F=\tilde{F}_2^\frown F_1$, for which $F(\{1\}\times A_1)$ goes back into A_1 and acts as an operator $\Psi=\Psi\circ Id$. Let $x_n=v_1+\tilde{x}_n=F^n(1,z)\in A_1$ and $y_n=F_1(1,x_n)\in A_2$. Note that $y_n=x_n+\lambda y$. Let $V_n=p+\varepsilon\tilde{V}_n$ and $W_n=q+\varepsilon\tilde{V}_n$. Denote also $\tilde{A}_1=A_1\setminus\cup V_n$,

$\tilde{A}_2 = A_2 \setminus \cup W_n$. The restriction $\tilde{F}_1 = F_1 \restriction_{J_1 \times \tilde{A}_1}$ is again a transport sending $\tilde{A}_1$ onto $\tilde{A}_2$. Denote R_1 and R_2 the respective tubuses of $\tilde{F}_1, \tilde{F}_2$. We introduce trunks from V_n or W_n using ψ from Lemma 32. Denote $V_n^1 = p + \frac{\lambda}{4}e + \psi(\frac{\lambda}{4})\frac{v_n}{\|v_n\|} + \varepsilon\tilde{V}_n$ and $W_n^1 = q - \frac{\lambda}{4}e + \psi(\frac{\lambda}{4})\frac{v_n}{\|v_n\|} + \varepsilon\tilde{V}_n$.

Lemma 33. *(Trunk lemma)*
There exist C^∞-smooth transports R_n from V_n to V_n^1 with tubuses R_n^1 satisfying $R_n^1 \cap R_2 = V_n$, $R_n^1 \cap R_1 \subset \partial R_1$. Also, there exists a C^∞-smooth transports R_n^2 from W_n^1 to W_n with tubuses R_n^2 satisfying $R_n^2 \cap R_2 = V_n$, $R_n^2 \cap R_1 \subset \partial R_1$. Moreover, $R_n^i \cap R_m^j = \emptyset$ unless $i = j, n = m$.

Proof. In case of V_n^1 the transport formula is the following: $F_n^1 : [0, \frac{\lambda}{4}] \times V_n \to U$, $F_n^1(t, z) = z + \psi(t)u_n + te$. In case of W_n^1 the transport formula is: $F_n^2 : [\frac{3\lambda}{4}, \lambda] \times W_n^1 \to U$, $F_n^2(t, z) = z - \psi(\lambda - t)u_n + te$. Since these transports remain within the set $\overline{v_1 v_2 + 3\delta B_X}$ we get $R_n^1 \cap R_2 = V_n$, $R_n^2 \cap R_2 = W_n$. □

Next, we define the transport G_1 from $A_1 \setminus \cup V_n$ onto $A_2 \setminus \cup W_n$ as a restriction of F_1 and $\tilde{R}_1$ be its tubus.
It is clear that the closed set $Q = \tilde{R}_1 \cup R_2 \cup \cup_{n=1}^\infty (R_n^1 \cup R_n^2)$ has an arcwise connected complement in U. Indeed, the trunks R_n^i are $\frac{1}{2}$-separated, so the localized version of Lemma 27 will give us the conclusion. Since the rest of the proof is very similar to that of Theorem 3, we proceed at a faster pace. Choose a dense sequence $\{s_n\}_{n=1}^\infty$ in $U \setminus Q$.
We proceed by induction now. Let $Q^0 = Q$, $s_n^0 = s_n$. In the first step, using Lemma 25, Lemma 24, and Fact 14 find ε_1 such that the $s_1 + \varepsilon_1 A$ is small enough and construct a transport G_1^1 from V_1^1 onto $s_1 + \varepsilon_1 A$ inside $U \setminus Q$, and connect it with the transport G_1^2 from $s_1 + \varepsilon_1 A$ onto U_1^1. These transports result from ε_1-planar curves connecting $\tilde{u}_1$ to s_1 and from s_1 to $\tilde{v}_1$, and the tubus remains disjoint from Q. Moreover we may WLOG assume that $T_{G_1^1} \cup T_{G_1^2}$ has distance at least ε_1 from $\tilde{R}_1 \cup R_2 \cup \cup_{n=2}^\infty (R_n^1 \cup R_n^2)$. Let $Q^1 = Q^0 \cup T_{G_1^1} \cup T_{G_1^2}$.
Inductive step. We have already constructed for $i = 1, \ldots, n$: ε_i and transports G_1^i from V_i^1 onto $\tilde{s}_i + \varepsilon_i A$ inside $U \setminus Q^{i-1}$, and connected it with transports G_i^2 from $\tilde{s}_i + \varepsilon_i A$ onto U_i^1. These transports result from ε_i-planar curves connecting $\tilde{u}_i$ to $\tilde{s}_i$ and from $\tilde{s}_i$ to $\tilde{v}_i$, and the tubus remains disjoint from Q^{i-1}. Moreover we may WLOG assume that $T_{G_i^1} \cup T_{G_i^2}$ had distance at least ε_i from $\tilde{R}_1 \cup R_2 \cup \cup_{n=i+1}^\infty (R_n^1 \cup R_n^2)$. Choose the first available $s_i \notin Q^n$ and relabel it as $\tilde{s}_{n+1}$. Then repeat the first step of the inductive argument, with the obvious changes in the notation.
The inductive argument yields a family of disjoint transports $G_n^2 {}^\frown G_n^1$ from $\tilde{V}_n$ onto $\tilde{U}_n$ in U, such that $\cup_{i=1}^\infty Q^i$ is dense in U. Apply Lemma 26 to define a C^k-smooth and Lipschitz vector field g on the whole X, that is nonzero and parallel to the above constructed system of transports at each interior point of $\cup Q^i$. Let us now trace the trajectory of the solution of $x' = g(x)$, $x(0) = z$. By condition 1 in Lemma 31, $g(x) \neq 0$ for the whole solution x. By construction, this solution passes through every point $\tilde{x}_n \in A_1$, and its trajectory is a dense set in $\cup_{i=0}^\infty Q^i$. This finishes the proof.

5. Type III

Let X be a separable Banach space with a Schauder basis. Our goal is to construct by induction a C^∞-smooth and Lipschitz on bounded sets mapping $f : X \to X$ and a solution x of (1), such that $\Omega(x)$ is of type III. We renorm X so that it has a normalized Schauder basis $\{e_i, \phi_i\}_{i=1}^\infty$ ([F$^\sim$] Lemma 6.4). Let $P_n : X \to X$ be the initial projections $P_n(\sum_{i=1}^\infty z_i e_i) = \sum_{i=1}^n z_i e_i$. These projections are 1-bounded linear operators, and $P_k \circ P_n = P_{\min\{k,n\}}$. By $X_n = [e_i : 1 \leq i \leq n]$ we denote the n-dimensional subspaces of X. We have $X = \overline{\cup_{i=1}^{\infty} X_n}$. Fix a continuous mapping

$$G : X \to X^*, G(z) = -\phi_2(z)\phi_1 + \phi_1(z)\phi_2. \tag{19}$$

Definition 34. *Let Y be a linear subspace of X. We say that a mapping $f : Y \to X$ is a G-positive function (or a field) if it is C^∞-smooth, Lipschitz, and the following conditions hold:*

$$\langle G(z), f(z)\rangle > 0 \quad \textit{whenever } P_2(z) \neq 0. \tag{20}$$

$$P_2 \circ f(\sum_{i=1}^{\infty} z_i e_i) = -z_2 e_1 + z_1 e_2 \quad \textit{whenever } \|P_2(z)\| < \xi, \quad \textit{for some } \xi > 0. \tag{21}$$

This auxiliary notion will be crucial for keeping the fields f_n and the sought field f nonzero, while making the necessary perturbations that create some periodic orbits. For convenience, we will use the notation $X^2 = \{z \in X : P_2(z) \neq 0\}$, $X_n^2 = X_n \cap X^2$. Note, in particular, that (21) implies that every solution y with the initial condition from X^2 remains inside X^2. We sketch proofs of some deformation lemmas for smooth fields on X_n.

Lemma 35. *Let $f : X_n \to X_n$ be a G-positive field, x be a solution of* (1) *passing through $q, p, r \in X_n^2$, (in this order), $\delta > 0$ and $\|p - q\| > \delta$, $\|r - q\| < \delta$. Let $\varepsilon > 0$, be such that $\|r - q\| + \varepsilon < \delta$. Moreover, assume that*

$$\langle G(z), q - r\rangle > 0 \quad \textit{for all } \ z \in pq. \tag{22}$$

Then there exists a G-positive field $g : X_n \to X_n$, $f(z) = g(z)$ whenever $z \notin rq + \varepsilon B_{X_n}$ and such that the solution of $y' = g(y), y(0) = p$ is periodic and contains q, r.

Proof. Let $\gamma \subset x$ be the solution curve from q to r. Let $\varepsilon > 0$ and $\gamma_1 \subset rq + \frac{\varepsilon}{2} B_{X_n}$ be a C^∞-smooth curve, $\gamma_1 + \varepsilon B_X \subset B(q, \delta)$ from r to q so that $\gamma_2 = \gamma^\frown\gamma_1$ is a C^∞-smooth periodic curve, and moreover $\langle G(z), \gamma_2'(z)\rangle > 0$ for all $z \in \gamma_2$. Such a curve exists by using a standard smoothnenning of the continuous periodic curve $\gamma^\frown rq$ around the points p, q. The C^∞-smooth field γ_1' can be extended (using local coordinates) from γ_2 into a C^∞-smooth field h defined on some neighborhood $U \subset rq + \varepsilon B_{X_n} \subset X_n^2$ of γ_1 in X_n, so that $h(z) = \gamma_2'(z)$ for $z \in \gamma_2 \cap U$. Moreover we may WLOG assume that $\langle G(z), h(z)\rangle > 0$ for $z \in U$. Pick a C^k-smooth partition of unity $\{\phi_1, \phi_2\}$ on X_n, $\text{supp}(\phi_1) \subset U$, $\phi_1 \restriction_{\gamma_1} = 1$, and set $g(z) = \phi_1(z)h(z) + \phi_2(z)f(z)$. It follows that γ_2 is a solution curve of $y' = g(y)$. The desired properties are clear, in particular g is G-positive. $\square$

Lemma 36. *Let $f : X_n \to X_n$ be a G-positive field with a periodic solution x passing through $p, q \in X_n^2$, $\delta > 0$ and $\|p - q\| > \delta$. Then there exists a G-positive field $g : X_n \to X_n$ such that $f = g$ outside $B(q, \delta)$ and g is constant on $B(q, \rho)$ for some $\rho > 0$. Moreover the solution of $y' = g(y)$ passing through p is periodic and passes through q as well.*

Proof. Let $\phi = G(q) \in X^*$ and choose $\eta > 0$, $0 < 3\eta < \delta$ small enough so that

$$\phi(f(z)) > 0 \ \text{ and } \ \langle G(z), f(q)\rangle > 0 \ \text{ for } \ z \in B(q, 3\eta). \tag{23}$$

Fix a constant nonzero field $g_1(z) = f(q)$ for $z \in B(q, 2\eta)$. Choose a partition of unity $\{\phi_1, \phi_2\}$ on X_n, $\text{supp}(\phi_1) \subset B(q, 2\eta)$, $\phi_1 = 1$ on $B(q, \eta)$ and set $g_2(z) = \phi_1(z)g_1(z) + \phi_2(z)f(z)$. Clearly, g_2 is nonzero on $B(q, 3\eta)$ (as $\phi(g_1(z)) > 0$ there) and C^∞-smooth, but a solution of $y' = g_1(y)$ passing through p may no longer be periodic and pass through q. To retrieve this property, we apply Lemma 35 to g_2

twice, for suitable pairs of points p_1, q_1, p_2, q_2 from $B(q, \frac{\eta}{2})$ selected in the following way. p, p_1, q, p_2 lie (in this order) on the original solution to $x' = f(x)$ passing through p, and q_1, q, q_2 lie (in this order) on the solution to the new equation $y' = g_2(y)$ passing through q. The points q_1, q_2 are close enough to q, so that $\phi(q_1 - p_1) > 0$, $\phi(p_2 - q_2) > 0$. Let $\varsigma = \min\{\|p_1 - q\|, \|p_2 - q\|, \|q_1 - q\|, \|q_2 - q\|\}$. Using Lemma 35 we connect the points p_1 and q_1 by a curve γ^1, while perturbing g_2 only on $p_1q_1 + \frac{\varsigma}{2}B_{X_n}$. We perform the same step once more connecting q_2 and p_2. Using Lemma 35 we connect the points q_2 and p_2 by a curve γ^2, while perturbing g_2 only on $q_2p_2 + \frac{\varsigma}{2}B_{X_n}$. The result of these perturbations will be a G-positive mapping g satisfying the requirements, $g = g_2$ on $B(q, \frac{\varsigma}{2})$. $\square$

Lemma 37. *Let $f : X_n \to X_n$ be a G-positive mapping on X_n with a periodic solution curve $\gamma \subset X_n^2$, $p, r \in \gamma$, $\frac{1}{4}\|p - r\| > \delta > 0$. Then there exists a $\beta > 0$, $\beta < \delta$, a G-positive mapping $g : X_n \to X_n$, $g(z) = f(z)$ for $z \notin B(p, 4\delta)$ and such that every solution y_q of $y' = g(y)$ $y(0) = q \in B(p, \beta)$ is periodic. Moreover, $r \in y_p$.*

Proof. Choose a suitable $p' \in \gamma$ with $\|p-p'\| = \delta$. By Lemma 36 there exists $\rho > 0$ a G-positive perturbation f_1 of f, $f_1(z) = f(p) = \lambda e$, ($\lambda > 0$, $\|e\| = 1$) for $z \in B(p, \rho)$, and $f_1(z) = f(z)$ for $z \notin B(p', 2\delta)$. Also, a solution curve γ_1 of $y' = f_1(y)$ passing through r is periodic and still passes through p. We have $X_n = Y \oplus [e]$, $\|e\| = 1$, and we choose a normalized Auerbach linear basis $v_1, \dots, v_{n-1}$ of Y ([F~] Theorem 5.6). (This way we know that all coordinates of vectors from B_Y have absolute value at most one). For simplicity of notation, re-lable γ_1 as γ and f_1 as f again. Choose $q = p + \frac{\rho}{2}e \in \gamma$ and a forward parametrization $\gamma : [a, b] \to X_n$, $a < b$, $\gamma(a) = q$, $\gamma(b) = p$ (recall that γ is periodic). Let $J = [a, b]$, $\gamma_0 : J \to X_n$ be the backward solution curve from p to q, $\gamma_0(t) = \gamma(b + a - t)$. For each $t \in J$ let $W_t = \gamma_0(t) + Y$ (note that $W_a = p + Y$, $W_b = q + Y$). By using the standard arguments and Lemma 9 there exists $\alpha > 0$, $\alpha < \frac{\rho}{4}$ such that for every backward solution curve $\zeta : J \to B(p, \rho)$ starting at $z \in B(p, 2n\alpha) \cap W_a$ and ending in W_b there is a unique point of intersection $z_t \in \zeta \cap W_t$, for every $t \in J$. Denote by γ_i, $i = 1, \dots, n-1$ the backward solution curves starting at $p + \alpha e_i$ and let $\beta_i(t) = \gamma_i \cap W_t$. Assuming that α is small enough, compactness of J and Lemma 9 allow us to assume that $\{\beta_i(t) - \gamma_0(t)\}_{i=1}^{n-1}$ forms a linear basis of the space W_t (with origin shifted to $\gamma_0(t)$). (This is because $z \to z_t$, $t \in J$ fixed, is as close to a linear isomorphism in z as we wish in a small neighborhood of $z = p$). So for every $z \in B(p, 2n\alpha) \cap W_a$ and $t \in J$ there is a uniquely determined tuple $(z_t^1, \dots, z_t^{n-1})$ such that

$$z_t = \gamma_0(t) + \sum_{i=1}^{n-1} z_t^i(\beta_i(t) - \gamma_0(t)) \tag{24}$$

Put $J' = [0, \frac{\rho}{2\lambda}]$, and put $\zeta_0(t) = p + t\lambda e$, $t \in J'$. Choose a small enough $\beta < \alpha$, such that $z \in p + \beta B_Y$, $z = p + \sum_{i=1}^{n-1} z_i v_i$ implies that $\sum_{i=1}^{n-1} |z_i| < \frac{1}{4}$ and a system of suitable curves $\zeta_i : J' \to B(p, \rho)$, $1 \le i \le n-1$, connecting $p + \beta e_i \in W_a$ with $(p + \beta e_i)_b \in W_b$ with the properties:

1. $\zeta_i'(t) = f(\zeta_i(t)) = \lambda e$ is constant whenever $|t| < \frac{\rho}{8\lambda}$ or $|t - \frac{\rho}{2\lambda}| < \frac{\rho}{8\lambda}$,
2. $(\zeta_i(t) - \zeta_0(t))$, $1 \le i \le n-1$ form a linear basis of $\zeta_0(t) + Y$, for every $t \in J'$.
3. For every $z \in \zeta_0(J') + 2n\beta B_Y = A$ and every $i \in \{1, \dots, n-1\}$, $t \in J'$ we have

$$|\langle G(z), \zeta_i'(t)\rangle| < \frac{1}{8n} \min_{z \in A} \langle G(z), \lambda e\rangle. \tag{25}$$

We define the mapping $F : J' \times (p + 2\beta B_Y) \to X$ by

$$F(t,z) = \zeta_0(t) + \sum_{i=1}^{n-1} z_i(\zeta_i(t) - \zeta_0(t)), \text{ for } t \in J', z = \sum z_i v_i \in B(p, 2\beta) \cap W_a \quad (26)$$

By our construction we see that this mapping is C^∞-smooth and $g_1 = \frac{\partial F(t,z)}{\partial t}$ is a C^∞-smooth and Lipschitz field that coincides with f on some neighborhoods of p, q. The set $U = F(J' \times (p + 2\beta B_Y))$ is a neighborhood of $\zeta_0([\frac{\rho}{9\lambda}, \frac{4\rho}{9\lambda}])$. It is standard to check that (25) and (26) together imply that

$$\langle G(z), g_1(z)\rangle \geq \langle G(z), \lambda e\rangle - \sum_{i=1}^{n-1} |z_i|(\langle G(z), \lambda e\rangle + \sum_{i=1}^{n-1} |\langle G(z), \zeta_i'(t)\rangle|) \geq \quad (27)$$

$$\geq \langle G(z), \lambda e\rangle - \frac{1}{4}(\frac{1}{8}\min_{z\in A}\langle G(z), \lambda e\rangle + \langle G(z), \lambda e\rangle) > \frac{1}{4}\min_{z\in A}\langle G(z), \lambda e\rangle, \quad (28)$$

so

$$\langle G(z), g_1(z)\rangle > 0 \text{ for } z \in F(J' \times (p + 2\beta B_Y)). \quad (29)$$

To finish, use the C^∞-smooth partitions of unity $\{\phi_1, \phi_2\}$ on X_n,

$$\mathrm{supp}(\phi_1) \subset F(J' \times (p + 2\beta B_Y)), \text{ and } \phi_1 = 1 \text{ on } F([\frac{\rho}{9\lambda}, \frac{4\rho}{9\lambda}] \times (p + \beta B_Y)) \quad (30)$$

and set $g(z) = \phi_1(z)g_1(z) + \phi_2(z)f(z)$. Clearly, g is G-positive and solutions passing through points in $B(p, \beta)$ are periodic. □

The above perturbation in $B(p, 4\delta)$ cannot be done with a good control on the derivatives of the perturbation (as the solutions are joined after having completed possibly very long trajectories). This is a source of additional technical difficulties in our proof. In order to be able to control the limit process with f_n below, we are forced to artificially modify the periodic solutions by sending them far away from the origin, so that the spots where large perturbations occur do not accumulate in one place, spoiling the convergence of the derivatives of f_n.

We are now ready to begin the proof of Theorem 5. Let $Q_n \subset X$ be the sets with the property

$$z \in Q_n \text{ iff } \max\{\phi_{6n+2}(z), \phi_{6n+4}(z), \phi_{6n+6}(z)\} > 1. \quad (31)$$

By induction we are going to construct the following system of objects.

1. A system of numerical parameters:

$$0 < \nu_n, \rho_n, \vartheta_n^1, \alpha_n, \delta_n^m < \frac{1}{2^n} \text{ where } n, m \in \mathbb{N}, n < m. \quad (32)$$

$$\vartheta_n^m \searrow \frac{1}{2}\vartheta_n^1, m \geq n. \quad (33)$$

2. Sequences $\{p_n\}_{n=1}^\infty$, $\{r_n\}_{n=1}^\infty$ of points from X, $p_n, r_n \in X_{6n}$.
3. A sequence $\{f_n\}_{n=1}^\infty$, $f_n : X \to X_{6n}$ of G-positive fields with properties:

$$\bullet f_n = f_n \circ P_{6n} \text{ on } X, \ f_n(z) = f_0(z) \text{ if } \|P^{6n}(z)\| \text{ is large enough} \quad (34)$$

$$\bullet \sup_{z\in X} \|D^i f_n(z)\| < \infty, \|D^i f_{n+1} - D^i f_n\| < \nu_n \text{ holds on } X \setminus Q_n \text{ for } i \leq n. \quad (35)$$

(so f_n converges locally uniformly to a C^∞-smooth field f).

• Every solution to $y' = f_n(y)$, $y(0) = r \in e_1 + \rho_n B_X$ is periodic with a solution curve γ_r^n. It holds $\gamma_{e_1}^{n+1}(t) = \gamma_{e_1}^n(t)$ for all $t \in [0, t_n]$ and $p_n = \gamma_{e_1}^n(t_n)$. We denote

$$R_0^n = \{z \in X : z \in \gamma_r^n, r \in e_1 + \rho_n B_X\} \subset \gamma_{e_1}^n + \alpha_n B_X \tag{36}$$

• Every solution (curve) γ_r^n to $y' = f_n(y)$, $y(0) = r \in r_n + \vartheta_n^n B_X$ is periodic. We denote $\gamma_n = \gamma_{r_n}^n$. We denote

$$R_n^n = \{z \in X : z \in \gamma_r, r \in r_n + \vartheta_n^n B_X\} \subset \gamma_n + \alpha_n B_X \tag{37}$$

For $i > n$ we introduce

$$R_n^i = \{z \in X : z \in \gamma_r, r \in r_n + \vartheta_n^i B_X\}, \tag{38}$$

$$R_n^{i,j} = \{z \in X : z \in \gamma_r, r \in r_n + (\frac{3-j}{3}\vartheta_n^i + \frac{j}{3}\vartheta_n^{i+1})B_X\}, j = 1, 2, 3 \tag{39}$$

$$R_n = \{z \in X : z \in \gamma_r, r \in r_n + \frac{1}{2}\vartheta_n^n B_X\}. \tag{40}$$

• It holds

$$\gamma_r \subset R_n + \alpha_m B_X \text{ for every } \gamma_r \subset R_n^m, \tag{41}$$

$$\operatorname{dist}(X_{6n} \setminus R_n^m, R_m^m) > \delta_n^m, \text{ and } f_m = f_n \text{ on } R_n^m \text{ for every } m > n \geq 0. \tag{42}$$

• $\cup_{n=1}^\infty R_n$ is norm dense in X and $\{p_n\}_{n=1}^\infty$ is norm dense in $X \setminus \cup_{i=1}^\infty R_i$.

An important ingredient in our inductive argument is contained in the next lemma.

Lemma 38. *Let $f : X \to X_n$ be a G-positive mapping such that $f \circ P_n = f$. Let $\rho' > 0$, $\zeta > 0$, $t_1, t_2 \geq 0$, $S_1, S \subset X$ be f-invariant sets with properties:*

$$S_1 + \zeta B_X \subset S, P_n(S) \subset S, P_{n+1}(S) \subset S, P_n(S_1) \subset S_1, \tag{43}$$

$$P_{n+1}(S_1) \subset S_1, S \subset X_n + \frac{1}{2}B_X. \tag{44}$$

Let $p, q \in X^2 \setminus S$, $\iota > 0$, $y^1 : \mathbb{R} \to X_n$ be a solution to $y' = f(y), y(0) = p$, and $y^2 : \mathbb{R} \to X_n$ be a solution to $y' = f(y), y(0) = q$. Assume that for some $\iota > 0$

$$y^1([-\iota, \iota]) \cap (y^2([-\iota, \iota]) = \emptyset, \tag{45}$$

Then there exists a G-positive perturbation $f_3 : X \to X_{n+2}$ of f such that

$$f_3 \circ P_{n+2} = f_3, \|D^i(f_3 - f)(z)\| < \rho_n' \text{ whenever } \phi_{n+2}(z) < 2, \tag{46}$$

$$f_3(z) = f(z) \text{ whenever } z \in S_1. \tag{47}$$

and a solution y_3 of $y' = f_3(y), y(0) = p$ satisfies:

$$y_3(t) = y^1(t) \text{ for } t \in [-t_1, 0], \text{ and } y_3(t_0 + t) = y^2(t) \text{ for } t \in [0, t_2]. \tag{48}$$

Moreover y_3 is disjoint from S, the set S_1 is f_3-invariant, and y_3 passes through some point $v \in X_{n+2}$, $\phi_{n+2}(v) > \frac{3}{2}$.

Proof. Denote $\tilde{\gamma}_1$ the solution curve of $y^1[-t_1,0]$ and $\tilde{\gamma}_2$ the solution curve of $y^2[0,t_2]$. We may WLOG assume that ι is small enough so that for every solution curve τ passing through $B(p,2\iota)\cup B(q,2\iota)$ it holds

$$\tau\cap(B(p,4\iota)\cup B(q,4\iota))\neq\emptyset. \tag{49}$$

Let and $K_i>0, i\le n$ be some small enough constants, whose value will be determined later. The construction depends on a C^∞-smooth non-negative function $\lambda: X\to\mathbb{R}$, $\lambda=\lambda\circ P_n$, with the properties:

$$\begin{cases}\lambda(z)=0 \ \text{ for } \ z\in y^1([-\iota,0])\\ \lambda(z)>0 \ \text{ for } \ z\in y^1((0,\iota])\end{cases} \tag{50}$$

$$\begin{cases}\lambda(z)=0 \ \text{ for } \ z\in y^2([0,\iota])\\ \lambda(z)>0 \ \text{ for } \ z\in y^1((-\iota,0))\end{cases} \tag{51}$$

$$\lambda(z)=\lambda_0>0 \ \text{ for } \ z\notin B(p,2\iota)\cup B(q,2\iota). \tag{52}$$

$$\|D^i\lambda(z)\|<K_i \ \text{ for } \ z\in X, i\le n, D^n\lambda(p)=0, D^n\lambda(q)=0, \ \text{ for } \ n\in\mathbb{N}. \tag{53}$$

Let $f_1: X\to X_{n+1}$, $f_1=f+\lambda e_{n+1}$ be a perturbation of f, and its "lifted" pair of forward solution starting at p and backward solution starting at q:

$$\begin{cases}y_1^1(t)=y^1(t)+\int_0^t\lambda(y^1(\tau))d\tau e_{n+1}, t\in[0,\infty)\\ y_1^2(t)=y^2(t)-\int_t^0\lambda(y^2(\tau))d\tau e_{n+1}, t\in(-\infty,0]\end{cases} \tag{54}$$

Note that these solutions do not intersect with S because $P_n(S)\subset S$. It is also clear from (52) and (49) that

$$\lim_{t\to\infty}\phi_{n+1}(y_1^1(t))=\infty, \ \text{ and } \ \lim_{t\to-\infty}\phi_{n+1}(y_1^2(t))=-\infty. \tag{55}$$

Let $\theta:\mathbb{R}\to[0,2]$ be an odd C^∞-smooth function $\theta\restriction_{[-1,1]}=0$, $|D^i\theta(t)|<K_i'$ for $t\in\mathbb{R}$, $\theta(t)=2$ for $t\ge T$. The values of K_i', T and further properties of θ will be described later in the course of the proof. Choose a perturbation

$$f_2(z)=f_1(z)+\theta(\phi_{n+1}(z))e_{2n+2} \tag{56}$$

Again, its "lifted" pair of forward solution starting at p and backward solution starting at q satisfy:

$$\begin{cases}y_2^1(t)=y_1^1(t)+\int_0^t\theta(\phi_{n+1}(y_1^1(\tau)))d\tau e_{n+2}, t\in[0,\infty)\\ y_2^2(t)=y_1^2(t)-\int_t^0\theta(\phi_{n+1}(y_1^2(\tau)))d\tau e_{n+2}, t\in(-\infty,0]\end{cases} \tag{57}$$

Note that these solutions do not intersect S because $P_{n+1}(S)\subset S$ and using (55)

$$\lim_{t\to\infty}\phi_{n+2}(y_2^1(t))=\infty, \ \text{ and } \ \lim_{t\to-\infty}\phi_{n+2}(y_2^2(t))=\infty. \tag{58}$$

Observe that the perturbations f_1,f_2 satisfy $P_2(f_1)=P_2(f_2)=f$ so they are still G-positive mappings, and $y_2^1,y_2^2\subset X^2$. Choose $T_1>0$ such that $\phi_{n+2}\circ y_2^1(T_1)>2$, $\phi_{n+2}\circ y_2^2(-T_1)>2$, and let $p_1=y_2^1(T_1)$, $q_1=y_2^2(-T_1)$. Clearly, $p_1,q_1\in X^2$. We also denote by $\gamma_1:[0,T_1]\to X_{n+2}$ the solution curve of y_2^1 from p to p_1, and $\gamma_3:[T_1+1,2T_1+1]\to X_{n+2}$ the reparametrized solution curve of y_2^2 from q_1 to q (considered as a forward solution). In order to connect the initial and final parts of the desired solution curve, we need the next lemma.

Lemma 39. *Let $C > 0$, $a, b \in X_n^2$. Assume that*

$$\phi_k(a) > C, \phi_k(b) > C \ \text{ and } \ \phi_k(S) < C \ \text{ for some } k \le n \tag{59}$$

Denote $I = [0,1]$. Then there exists a C^∞-smooth curve $\kappa : I \to X_n^2$ such that:

$$\kappa(0) = a, \kappa(1) = b, \phi_k(\kappa(t)) > C \ \text{ for all } \ t \in I \tag{60}$$

$$\langle G(\kappa(t)), \kappa'(t)\rangle > 0, \ \text{ for } \ t \in I \tag{61}$$

In particular $\kappa \cap S = \emptyset$.

Proof. Denote $a = (a_1, \dots, a_n)$, $b = (b_1, \dots, b_n)$. Let $\alpha, \beta \in [0, 2\pi)$ be such that

$$\begin{cases} Re(\exp(i\alpha)) = \frac{a_1}{(a_1^2+a_2^2)^{\frac{1}{2}}}, Im(\exp(i\alpha)) = \frac{a_2}{(a_1^2+a_2^2)^{\frac{1}{2}}}, \\ Re(\exp(i\beta)) = \frac{b_1}{(b_1^2+b_2^2)^{\frac{1}{2}}}, Im(\exp(i\beta)) = \frac{b_2}{(b_1^2+b_2^2)^{\frac{1}{2}}}. \end{cases} \tag{62}$$

Assume WLOG that $\beta > \alpha$. Using the coordinates we introduce κ as follows.

$$\begin{cases} \phi_1 \circ \kappa(t) = Re(\exp(i(1-t)\alpha + it\beta)((1-t)\|P_2(a)\| + t\|P_2(b)\|) \\ \phi_2 \circ \kappa(t) = Im(\exp(i(1-t)\alpha + it\beta)((1-t)\|P_2(a)\| + t\|P_2(b)\|) \\ \phi_i \circ \kappa(t) = (1-t)a_i + tb_i \ \text{ for } i \ge 3 \end{cases} \tag{63}$$

It is easy to verify that κ satisfies our conditions including (61). $\square$

We now apply Lemma 39 in X_{n+2} for the values $a = p_1$, $b = q_1$, $C = \frac{3}{2}$, $k = n+1$. By a standard argument we adjust κ in small neighborhoods of the connection points p_1, q_1 (contained in $\phi_k^{-1}([C, \infty))$) to obtain a C^∞-smooth curve γ_2 which still satisfies (60) and (61) so that the curve from p to q:

$$\tilde{\gamma} = \gamma_1^\frown \gamma_2^\frown \gamma_3 : [0, 2T_1 + 1] \to X_{n+2}. \tag{64}$$

is C^∞-smooth, $\tilde{\gamma} \cap S = \emptyset$. From (50) and (51) it follows that $\tilde{\gamma} \cap X_n = \{p, q\}$, and from (53) it follows that

$$\gamma = \tilde{\gamma}_1^\frown \tilde{\gamma}^\frown \tilde{\gamma}_3 : [-t_1, 2T_1 + 1 + t_2] \to X_{n+2}. \tag{65}$$

is C^∞-smooth and $\gamma \cap S = \emptyset$. Pick any $v \in \gamma_2$. By using some standard extension into the neighborhood theorem (e.g. [War72] Proposition 1.36), we know that γ' can be C^∞-smoothly extended into a field $\tilde{f}_2$ defined on $\gamma + \zeta B_{X_{n+2}}$ so that $\tilde{f}_2(z) = f_2(z)$ whenever $\phi_{n+2}(z) < \frac{3}{2}$, and moreover $\langle G(z), \tilde{f}_2(z)\rangle > 0$. We have that $S_1 \cap \gamma + \zeta B_{X_{n+2}} = \emptyset$. Choose the partitions of unity ψ_1, ψ_2 on X_{n+2} such that

$$\text{supp}(\psi_1) \subset \gamma[-t_1, 2T_1 + 1 + t_2] + \zeta B_X \ \text{ and } \ \psi_1 \restriction_{\gamma[-t_1, 2T_1+1+t_2]} = 1. \tag{66}$$

$$\|D^i \psi_1\|_{X_{n+2}} < E(\zeta, n) \ \text{ for } \ i \le n, \tag{67}$$

where the constants $E(\zeta, n)$ depend only on the values of $\zeta > 0$ and $n \in \mathbb{N}$, but not on the curve γ or T. In this formula the norms of the derivatives are taken with respect to X_{n+2} together with the norm inherited from X. Since X_{n+2} is finite dimensional (and so linearly isomorphic to the Euclidean space $\mathbb{R}^{n+2}$), there exist constants C_i independent of the function ψ_1, such that $\|D^i \psi_1\|_{X_{n+2}} < C_i \|D^i \psi_1\|_{\ell_2^{n+2}}$. So it suffices for us to obtain the result in the Euclidean space $\mathbb{R}^{n+2}$. That is the content of the next lemma.

Lemma 40. *Let $\delta > 0$, $\delta < \frac{1}{4}$. The there exists $D_i > 0$, $i \in \mathbb{N}$, such that for every compact set $M \subset \mathbb{R}^n$ there exists a C^∞-smooth function $\phi : \mathbb{R}^n \to [0,1]$, $supp(\phi) \subset M + \delta B_{\mathbb{R}^n}$, $\phi \restriction_M = 1$ and such that $\|D^i \phi\| < D_i$ on $\mathbb{R}^n$.*

Proof. Let $\psi : \mathbb{R}^n \to [0,\infty)$ be a C^∞-smooth convolution kernel, i.e.

$$\int\limits_{\mathbb{R}^n} \psi(z)dz = 1, \ \operatorname{supp}(\psi) \subset \frac{\delta}{4} B_{\mathbb{R}^n}. \tag{68}$$

By compactness, it holds $\|D^i\psi\|_{\mathbb{R}^n} < D_i$ on $\mathbb{R}^n$, for suitable $D_i > 0$ and for every $i \in \mathbb{N}$. Let $\tilde{M} = M + \frac{\delta}{2} B_{\mathbb{R}^n}$, and fix an indicator function $\chi = \chi \restriction_{\tilde{M}}$. Let

$$\phi(a) = \int\limits_{\mathbb{R}^n} \chi(a-z)\psi(z)dz. \tag{69}$$

It is well-known that

$$D^i\phi(a) = \int\limits_{\mathbb{R}^n} \chi(a-z)D^i\psi(z)dz. \tag{70}$$

and so

$$\|D^i\phi(a)\|_{\mathbb{R}^n} \le \int\limits_{\mathbb{R}^n} \|D^i\psi(z)\|_{\mathbb{R}^n} dz \le D_i \ \text{ hold for every } \ a \in \mathbb{R}^n \tag{71}$$

The remaining properties are obvious. □

We now introduce the G-positive mapping f_3.

$$f_3 = \psi_1 \tilde{f}_2 + \psi_2 f, \ \text{ on } \ X_{n+2}. \tag{72}$$

This formula of course yields (47). In order to obtain (46), recall that $K_i = K_i(n, \rho'_n, \zeta)$, $K'_i = K'_i(n, \rho'_n, \zeta)$ and $\rho''_i = \rho''_i(n, \rho'_n, \zeta)$ can be chosen small enough so that (by using that $\tilde{f}_2(z) = f_2(z)$ whenever $\phi_{n+2}(z) < \frac{3}{2}$) we have an estimate

$$\|D^i(\tilde{f}_2 - f)(z)\| < \frac{\rho''_n}{E(\zeta, n)} \ \text{ holds whenever } \ \phi_{n+2}(z) < \frac{3}{2}, 0 \le i \le n. \tag{73}$$

and by applying the Leibnitz formula to (72), we finally obtain that

$$\|D^i(f_3 - f)(z)\| < \rho'_n \ \text{ holds whenever } \ \phi_{n+2}(z) < \frac{3}{2}, 0 \le i \le n. \tag{74}$$

To finish, extend f_3 from X_n onto the whole X by $f_3 = f_3 \circ P_{n+2}$. □

For the purpose of the inductive argument, we introduce some more notation. We fix a dense sequence $\{s_n\}_{n=1}^\infty$ in X, $s_n \in X_n$. We denote $x^n : I_n \to X_{6n}$, $I_n = [0, t_n^1]$ the periodic solution (with period t_n^1) with $x^n(0) = e_1$. We denote $t_n < t_n^1$, $J_n = [0, t_n]$ and $p_n = x^n(t_n)$. (The initial segments $x^n : J_n \to X_{6n}$ will be used to build the sought solution $x(t) : [0,\infty) \to X$, $x^1(0) = p_1$, $x^n(t_n) = p_n$,) By the construction $R_n^i \to R_n$ is a decreasing sequence of sets, such that each point inside R_n determines a periodic solution of f, and every solution for f_m passing through a point in R_n^m is periodic and stays in R_n^m.

We begin inductive construction at $n = 1$: Let $\nu_1 = \frac{1}{4}, \rho_1 = \frac{1}{8}, \alpha_1 = \frac{1}{4}, \vartheta_1^1 = \frac{1}{8}, t_1 = 2\pi - \frac{1}{4}, \delta_0^1 = \frac{1}{8}$, and $r_1 = 2e_1$. Let $f_1 : X \to X_6$ be a G-positive field defined as follows:

$$f_1(\sum_{i=1}^\infty z_i e_i) = -z_2 e_1 + z_1 e_2 \tag{75}$$

Note that every solution y to $y' = f_1(y)$ with initial condition $y(0) \in X_2^2$ is periodic and lies in the 2-dimensional set $y_0 + X_2$. We choose a 2π-periodic solution x^1 to $y' = f_1(y)$, $y(0) = e_1 = r_1$. We have $I_1 = [0, 2\pi]$, $x^1 : I_1 \to X_2$, and let $t_1 = 2\pi - \varepsilon_1$, $J_1 = [0, t_1]$, and $p_1 = x^1(t_1)$.

We denote $\gamma_1 \subset X_{6n}$ the solution curve representing periodic solution of f_1 passing through r_1, $\operatorname{dist}(R^1_0, R^1_1) > \delta^1_0 = \frac{1}{8}$.

Inductive step from n to $n+1$: Choose $\nu_{n+1}, \alpha_{n+1} < \frac{1}{2^{n+1}}$. Choose ϑ^{n+1}_i, $i \le n$ small enough so that (41) is satisfied. Creation of R^{n+1}_{n+1}:

Choose $r_{n+1} \in X^2_{6n} \setminus (\cup^n_{i=0} R^n_i)$, so that

$$\|r_{n+1} - s_n\| \le 2\operatorname{dist}(s_n, X^2_{6n} \setminus (x^n(I_n) \cup \cup^n_{i=1} R^n_i)), \text{ in case when } s_n \notin \cup^n_{i=1} R_i \quad (76)$$

Such a choice will guarantee (at the end of the inductive process) that $\cup^\infty_{n=1} R_n$ is norm dense in X. Indeed, if $s_n \in R^n_i$, for some $1 \le i \le n$, then using (41) and (42) we find the desired r_{n+1} at a distance at most α_n of s_n.
Recall that $f_n = f_n \circ P_{6n}$, and moreover $P_{6n}(R^n_i) \subset R^n_i$ for all $0 \le i \le n$, as the Schauder basis is monotone. Choose $\rho'_n < \rho_n$ such that

$$r_{n+1} \notin \{z \in X : z \in \gamma^n_r, r \in e_1 + \rho'_n B_X\}, \tag{77}$$

and define f_n-invariant sets:

$$S = \{z \in X : z \in \gamma^n_r, r \in e_1 + \rho'_n B_X\} \cup \cup^n_{i=1} R^n_i. \tag{78}$$

$$S_1 = \{z \in X : z \in \gamma^n_r, r \in e_1 + \frac{1}{2}\rho'_n B_X\} \cup \cup^n_{i=1} R^{n,1}_i. \tag{79}$$

Find $\zeta > 0$ such that $S_1 + \zeta B_X \subset S$, and consider the equation

$$y' = f_n(y), y(0) = r_{n+1} \tag{80}$$

Find its forward solution $y^1 : [0,\infty) \to X_{6n}$ and a backward solution $y^2 : (-\infty, 0] \to X_{6n}$. Apply Lemma 38 in the setting of S, S_1, ζ, $\rho' = \frac{\nu_{n+1}}{5}$ and $f_n : X \to X_{6n}$, $p = r_{n+1}$, $q \ne p$, $q \in X^2_{6n} \setminus S$, $t_1 = t_2 = 0$, in order to obtain a G-positive C^∞-smooth perturbation $f^3_n : X \to X_{6n+2}$ with a periodic solution passing through r_{n+1}, and some point $v \in X_{6n+2}$ with $\phi_{n+2}(v) > \frac{3}{2}$. Apply Lemma 36 to $f = f^3_n$, $p = v$, $r = r_{n+1}$ to obtain another G-positive perturbation $f^4_n : X \to X_{6n+2}$ with a solution curve γ_{n+1} passing through r_{n+1}, a small enough ϑ^{n+1}_{n+1} so that the set

$$R^{n+1}_{n+1} = \{z \in X : z \in \gamma_r, r \in r_{n+1} + \vartheta^{n+1}_{n+1} B_X\} \tag{81}$$

consists of periodic orbits, and has a positive distance to S, $f^4_n = f$ on S_1.

$$\sup_z \|D^i f^4_n(z)\| < \infty, \|D^i f^4_n - D^i f_n\| < \frac{\nu_{n+1}}{5} \text{ for } i \le n \text{ holds on } X_{6n+6} \setminus Q_n, \tag{82}$$

Next we repeat a similar procedure to above, in order to extend the partial solution x^n. Let us give just the main steps. Let $S = \cup_{1 \le i \le n+1} R^{n,1}_i$, $S_1 = \cup_{1 \le i \le n+1} R^{n,2}_i$, and $\zeta > 0$ such that (43) holds. Choose $p_{n+1} \in X^2_{6n+2} \setminus (x^n(I_n) \cup S)$ following the rules: if $s_n \in X^2_{6n+2} \setminus (x^n(I_n) \cup S)$, then pick $r_{n+1} = s_n$, otherwise pick p_{n+1} so that

$$\|p_{n+1} - s_n\| \le 2\operatorname{dist}(s_n, X_{6n+2} \setminus (x^n(I_n) \cup S)). \tag{83}$$

Such choice guarantees (at the end of the inductive process) that $\{p_n\}^\infty_{n=1}$ is norm dense in $X \setminus \cup^\infty_{i=1} R_i$. Next we apply Lemma 38 to $f = f^4_n : X \to X_{6n+2}$, ζ, $\rho' = \frac{\nu_{n+1}}{5}$, $p = p_n, q = p_{n+1}$, $t_2 = 0$, $t_1 = -t_n$ (this choice guarantees that $x^{n+1}(t) = x^n(t)$ will hold for all $t \in [0, t_n]$). We obtain a G-positive perturbation $f^5_n : X \to X_{6n+4}$, $f^5_n = P_{6n+4} \circ f^5_n$ such that the forward solution $x^n_1 : [0, t_{n+1}] \to X_{6n+4}$ $(= x^n$ on $J_n)$ passing through p_n also passes through v and p_{n+1}.

Let $S = \cup_{1\le i\le n+1} R_i^{n,2}$, $S_1 = \cup_{1\le i\le n+1} R_i^{n,3}$, and $\zeta > 0$ so that (43) holds. Once more we apply Lemma 38 to $f = f_n^5 : X \to X_{6n+4}$, ζ, $\rho' = \frac{\nu_{n+1}}{5}$, $p = e_1, q = p_{n+1}$, $t_2 = 0$, $t_1 = -t_n$ (this choice guarantees that $x^{n+1}(t) = x_1^n(t)$ will hold for all $t \in [0, t_{n+1}]$). We obtain a G-positive perturbation $f_n^6 : X \to X_{6n+6}$, $f_n^6 = P_{6n+6} \circ f_n^6$ such that the forward solution x^{n+1} is periodic and ($= x^n$ on J_n) passing through p_n passes through p_{n+1}. Apply Lemma 36 to $f = f_n^6$, $p = v$, $r = e_1$ to obtain the G-positive perturbation $f_{n+1} : X \to X_{6n+6}$ with a solution curve $\gamma_{e_1}^{n+1}$, $x^{n+1}[0, t_{n+1}] \subset \gamma_{e_1}^{n+1}$ passing through p_{n+1}, a small enough ρ_{n+1} so that the set

$$R_0^{n+1} = \{z \in X : z \in \gamma_r^{n+1}, r \in e_1 + \rho_{n+1} B_X\} \tag{84}$$

consists of periodic orbits, and has a positive distance to $\cup_{1\le i\le n+1} R_i^{n+1} = S_1$. At this point we may choose the values δ_{n+1}^m, $m < n+1$, so that (42) holds. By construction, properties (34) through (42) are satisfied. This completes the inductive step. The limit $f(z) = \lim_{n\to\infty} f_n(z)$ is G-positive and has the solution $x(t) = \lim_{n\to\infty} x^n(t)$. Since f is Lipschitz on bounded sets, if X admits a C^k-smooth bump by Lemma 26 we may replace it with a Lipschitz C^k-smooth field with the same solution curves. It remains to check that $\Omega(x)$ is of type III. By construction we see that $S = \Omega(x) = X \setminus \cup_{n=1}^\infty \mathrm{int}(R_n)$, and $x \cap \partial R_n = \emptyset, n \in \mathbb{N}$. Let us see that every type III set S satisfies $X \setminus S = \cup_{n=1}^\infty U_n$ where U_n are nonempty disjoint and connected. If there were only finitely many, then $S = \cup_{i=1}^k \partial U_n$, so $x \cap \partial U_n \ne \emptyset$, which is a contradiction by applying Theorem 10 in the usual way. This ends the proof of Theorem 5.

6. Examples

Let us give an idea how to obtain a Lipschitz equation with ω-limit set S that has non-empty interior, but $S \ne \overline{\mathrm{int} S}$. For simplicity, let $X = \ell_2$, $Y = \{(\alpha_n)_{n=1}^\infty : \alpha_1 = 0\}$, $X = Y \oplus [y]$. In light of the construction in Theorem 4, it suffices to construct an equation such that the solutions with initial conditions from B_Y form a "tubus" that is squashed into a "flat stripe" and then again returns into the original tubular shape. Such a body can be used in the constructions in Theorem 4 to replace the connecting piece between the respective trunks. The resulting ω-limit set will have some "flat handles", i.e. handles with empty interior. It is also possible to compose these equations, creating ω-limit sets whose interior has an arbitrary (finite or infinite) number of open components. Let us describe now an easy example of such a flow.
Choose C^∞-smooth partitions of unity $\phi, \psi : \mathbb{R} \to [0,1]$, $\phi(t) = 1$ iff $t \in \mathbb{R} \setminus (\frac{1}{6}, \frac{5}{6})$ and $\phi(t) = 0$ iff $t \in [\frac{1}{3}, \frac{2}{3}]$. Consider the flow:

$$F_t(0, \alpha_n)_{n=1}^\infty = (t, \phi(t)\alpha_1, \psi(t)\alpha_1 + \phi(t)\alpha_2, \psi(t)\alpha_2 + \phi\alpha_3, \dots) \tag{85}$$

The corresponding C^∞-smooth and Lipschitz differential equation is X the following:

$$f(\alpha_n)_{n=0}^\infty = (1, \phi'(t)\alpha_1, \psi'(t)\alpha_1 + \phi'(t)\alpha_2, \psi'(t)\alpha_2 + \phi'\alpha_3, \dots) \tag{86}$$

It is also easy to construct elementary examples of ω-limit sets and non-ω-limit sets in a Banach space X. For example, S_X is always a type I set, and so is a union of separated spheres in X. Also, $S_X \cup \{\lambda e : \|e\| = 1, 0 \le \lambda \le 1\}$ is a type I set, but $S_X \cup \{\lambda e : \|e\| = 1, 0 \le \lambda\}$ is not an ω-limit set. Indeed, it is not of type I, but its complement has only two components.

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