NUMERICAL SOLUTION OF COMPRESSIBLE TURBULENT FLOWS USING EARSM MODEL

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Introduction

This article deals with the numerical solution of compressible turbulent flows in aerodynamics. Compressible turbulent flow is described by the system of Favre averaged Navier-Stokes equations, which are closed by the explicit algebraic Reynolds stress model (EARSM) of Wallin and Johansson. The averaged Navier-Stokes equations together with EARSM model of turbulence are discretized by the finite volume method based on HLLC Riemann solver with piecewise linear WENO reconstruction and explicit two-stage TVD Runge-Kutta method. Source terms in transport equations of turbulence model are treated by point implicit method for better stability of explicit scheme. The numerical method is validated by comparison to theoretical results for the subsonic flow around the flat plate and experimental results for the transonic flow around the RAE 2822 airfoil.

1. Governing equations

The most general model of compressible turbulent flow is system of Navier-Stokes equations [2]. Solving this system with proper initial and boundary conditions leads to the Direct Numerical Simulation (DNS). Unfortunately, for applications of our interest this is impossible due to high Reynolds numbers and consequently small scales of turbulence which must be modelled. Mathematical model used in this work is based on decomposition of instantaneous variables to the averaged part and fluctuating part:

- Classical time averaging (Reynolds averaging) for density and pressure

$$\overline{\Phi} = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \Phi(\tau) d\tau \tag{1}$$

$$\Phi = \overline{\Phi} + \Phi' \tag{2}$$

- Density weighted time averaging (Favre averaging) for components of velocity vector and total energy

$$\widetilde{\Phi} = \frac{1}{\overline{\rho}} \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \rho(\tau)\Phi(\tau) d\tau \tag{3}$$

$$\Phi = \widetilde{\Phi} + \Phi'' \tag{4}$$

Where Φ is some instantaneous variable and Φ' and Φ'' are fluctuating parts for Reynolds averaging and Favre averaging respectively. Substituting decomposed variables into the system of Navier-Stokes equations and performing the density weighted averaging, we obtain Favre averaged Navier-Stokes equations:

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{u_j})}{\partial x_j} = 0 \tag{5}$$

$$\frac{\partial(\overline{\rho}\widetilde{u_i})}{\partial t} + \frac{\partial(\overline{\rho}\widetilde{u_i}\widetilde{u_j})}{\partial x_j} + \frac{\partial \overline{p}}{\partial x_i} = \frac{\partial}{\partial x_j}\left(\overline{\tau_{ij}} - \overline{\rho u_i'' u_j''}\right) \tag{6}$$

$$\frac{\partial \widetilde{e}}{\partial t} + \frac{\partial}{\partial x_j}\left[(\widetilde{e}+\overline{p})\widetilde{u_j}\right] = \frac{\partial}{\partial x_j}\left[\left(\overline{\tau_{ij}} - \overline{\rho u_i'' u_j''}\right)\widetilde{u_i}\right] - \frac{\partial}{\partial x_j}\left[\overline{q_j} + \overline{\rho u_j'' h''} - \overline{\tau_{ij} u_i''} + \overline{\rho u_j'' \frac{1}{2} u_i'' u_i''}\right] \tag{7}$$

Where $\overline{\tau_{ij}}$ and $\overline{q_j}$ are given by:

$$\overline{\tau_{ij}} = 2\overline{\mu}\overline{S_{ij}}, \quad \overline{\mu} = \mu_{ref}\left(\frac{\rho_{ref}}{p_{ref}}\frac{\overline{p}}{\overline{\rho}}\right)^{\frac{3}{4}}, \quad \overline{S_{ij}} = \frac{1}{2}\left(\frac{\partial \widetilde{u_i}}{\partial x_j} + \frac{\partial \widetilde{u_j}}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial \widetilde{u_k}}{\partial x_k}\right) \tag{8}$$

$$\overline{q_j} = -\frac{\kappa}{\kappa - 1}\frac{\overline{\mu}}{Pr}\left(\frac{\overline{p}}{\overline{\rho}}\right) \tag{9}$$

To close this system is also necessary to specify equation of state. Assuming a calorically perfect gas and following the same procedure as in the case of Navier-Stokes equation, we derive relation:

$$\overline{p} = (\kappa - 1)\left[\widetilde{e} - \frac{1}{2}\overline{\rho}\widetilde{u_j}\widetilde{u_j} - \overline{\rho}k\right] \tag{10}$$

Where k is turbulent kinetic energy defined as

$$k = \frac{1}{2}\overline{u_i'' u_i''} \tag{11}$$

2. Model of turbulence and closure approximations

Favre averaged Navier-Stokes equations form an open set of partial differential equations. To close this system, we must introduce some suitable approximations of terms with fluctuations. For Reynolds stress tensor $\tau_{ij}^t = -\overline{\rho u_i'' u_j''}$ can be derived transport equation [7]. An alternative to using Reynolds stress transport equation is to reformulate the equation in terms of the Reynolds stress anisotropy and the turbulent kinetic energy [6]. After this reformulation we obtain transport equation for the Reynolds stress anisotropy in following symbolic form

$$TR(a_{ij}) = f_{ij}(a_{kl}, \Omega^*_{kl}, S^*_{kl}) \tag{12}$$

Where $TR(a_{ij})$ represents the advection and diffusion of the Reynolds stress anisotropy and f_{ij} the production, dissipation and redistribution terms [6]. In flows where the anisotropy varies slowly in time and space, the transport equation is reduced to an implicit algebraic relation[1] in form

$$0 = f_{ij}(a_{kl}, \Omega^*_{kl}, S^*_{kl}) \tag{13}$$

[1] In many flows of engineering interest the flow is steady and advection and diffusion can be neglected.

Where Ω^*_{kl} and S^*_{kl} are normalized tensors of rotation and strain-rate defined as

$$\Omega^*_{kl} = \tau\overline{\Omega_{kl}}, \quad S^*_{kl} = \tau\overline{S_{kl}} \tag{14}$$

and τ is turbulent time scale. System of algebraic equation (13) not include any diffusion or damping and therefore, it is very difficult to solve numerically. Computational efforts has been found to be so excessively large, that advantages of using algebraic Reynolds stress model instead of differential Reynolds stress transport model are lost. For that reason, the work in the area of algebraic Reynolds stress modelling has been focused on finding explicit expressions. In our case, we had chosen explicit algebraic Reynolds stress model (EARSM) of Wallin and Johansson [6]. Reynolds stress tensor is given by

$$\tau^t_{ij} = -\overline{\rho u''_i u''_j} = 2\mu_T\overline{S_{ij}} - \frac{2}{3}\delta_{ij}\overline{\rho}k - \overline{\rho}k a^{(ex)}_{ij} \tag{15}$$

Where turbulent viscosity μ_T is defined as

$$\mu_T = -\frac{1}{2}\beta_1\overline{\rho}k\tau \tag{16}$$

and extra anisotropy $a^{(ex)}_{ij}$ as:

$$a^{(ex)}_{ij} = \beta_4(S^*_{ik}\Omega^*_{kj} - \Omega^*_{ik}S^*_{kj}) \tag{17}$$

Coefficients β_1 and β_4 are given by relations

$$\beta_1 = -\frac{6}{5}\frac{N}{N^2 - 2II_\Omega}, \quad \beta_4 = -\frac{6}{5}\frac{1}{N^2 - II_\Omega} \tag{18}$$

Parameter N is defined as:

$$N = \begin{cases} \frac{C'_1}{3} + \left(P_1 + \sqrt{P_2}\right)^{\frac{1}{3}} + \mathrm{sign}\left(P_1 - \sqrt{P_2}\right)\left|(P_1 - \sqrt{P_2}\right|^{\frac{1}{3}} & \text{pro} \quad P_2 \geq 0 \\ \frac{C'_1}{3} + 2\left(P_1^2 - P_2\right)^{\frac{1}{6}}\cos\left[\frac{1}{3}\arccos\left(\frac{P_1}{\sqrt{P_1^2 - P_2}}\right)\right] & \text{pro} \quad P_2 < 0 \end{cases} \tag{19}$$

Where P_1 and P_2 are given by relations:

$$P_1 = \left(\frac{C'^2_1}{27} + \frac{9}{20}II_S - \frac{2}{3}II_\Omega\right)C'_1 \tag{20}$$

$$P_1 = P_1^2 - \left(\frac{C'^2_1}{9} + \frac{9}{10}II_S + \frac{2}{3}II_\Omega\right)^3 \tag{21}$$

Invariants II_S a II_Ω are given by relations

$$II_S = tr\{S^*_{ik}S^*_{kj}\}, \quad II_\Omega = tr\{\Omega^*_{ik}\Omega^*_{kj}\} \tag{22}$$

Turbulent time scale τ is defined as

$$\tau = \max\left(\frac{1}{\beta^*\omega}, \; C_\tau\sqrt{\frac{\overline{\mu}}{\beta^*\overline{\rho}k\omega}}\right) \tag{23}$$

Constants of turbulence model are $C_1' = 1.8$ a $C_\tau = 6$. This version of EARSM is based on transport equations of Kok's TNT $k - \omega$ model of turbulence:

$$\frac{\partial(\overline{\rho}k)}{\partial t} + \frac{\partial(\overline{\rho}k\widetilde{u}_j)}{\partial x_j} = P - \beta^*\overline{\rho}k\omega + \frac{\partial}{\partial x_j}\left[\left(\overline{\mu} + \sigma^*\mu_T\right)\frac{\partial k}{\partial x_j}\right] \tag{24}$$

$$\frac{\partial(\overline{\rho}\omega)}{\partial t} + \frac{\partial(\overline{\rho}\omega\widetilde{u}_j)}{\partial x_j} = \alpha\frac{\omega}{k}P - \beta\overline{\rho}\omega^2 + \frac{\partial}{\partial x_j}\left[\left(\overline{\mu} + \sigma\mu_T\right)\frac{\partial \omega}{\partial x_j}\right] + C_D \tag{25}$$

Where production P and cross diffusion C_D are defined as

$$P = -\overline{\rho u_i'' u_j''}\frac{\partial\widetilde{u}_i}{\partial x_j} = \tau_{ij}^t\frac{\partial\widetilde{u}_i}{\partial x_j}, \quad C_D = \frac{1}{2}\frac{\overline{\rho}}{\omega}\max\left(\frac{\partial k}{\partial x_j}\frac{\partial \omega}{\partial x_j},\ 0\right) \tag{26}$$

Constants of TNT model of turbulence are:

$$\alpha = 0.553, \quad \beta^* = 0.09, \quad \beta = 0.075, \quad \sigma^* = \frac{2}{3}, \quad \sigma = 0.5 \tag{27}$$

It is also necessary to approximate rest of the unknown correlations in equation (7). Correlation between u_j'' and h'' is turbulent transport of heat

$$q_j^t = \overline{\rho u_j'' h''} = \frac{\mu_T}{Pr_t}\frac{Pr}{\overline{\mu}}\overline{q_j} \tag{28}$$

Where Pr_t is turbulent Prandtl number, which can be chosen as $Pr_t = 0.91$. Finally, last two correlations in equation (7) are turbulent transport and molecular diffusion of turbulent energy. This terms are approximated as

$$d_j^t = \overline{\tau_{ij}u_i''} - \overline{\rho u_j''\frac{1}{2}u_i''u_i''} = \left(\overline{\mu} + \sigma^*\mu_T\right)\frac{\partial k}{\partial x_j} \tag{29}$$

3. Numerical solution

We consider only two-dimensional flow in this work. System of Favre averaged Navier-Stokes equations with transport equations for turbulent kinetic energy k (24) and specific dissipation rate ω (25) can be rewritten in a vector form:

$$\frac{\partial W}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \frac{\partial R}{\partial x} + \frac{\partial S}{\partial y} + Q \tag{30}$$

$$W = (\rho, \rho u, \rho v, e, \rho k, \rho\omega)^T$$

$$F = \left(\rho u, \rho u^2 + p, \rho u v, (e+p)u, \rho u k, \rho u \omega\right)^T$$

$$G = \big(\rho v, \rho uv, \rho v^2 + p, (e+p)v, \rho vk, \rho v\omega\big)^T$$

$$R = \big(0, \tau_{xx} + \tau^t_{xx}, \tau_{xy} + \tau^t_{xy}, u(\tau_{xx} + \tau^t_{xx}) + v(\tau_{xy} + \tau^t_{xy}) - q_x - q^t_x + d^t_x, d^t_x, o^t_x\big)^T$$
$$S = \big(0, \tau_{yx} + \tau^t_{yx}, \tau_{yy} + \tau^t_{yy}, u(\tau_{yx} + \tau^t_{yx}) + v(\tau_{yy} + \tau^t_{yy}) - q_y - q^t_y + d^t_y, d^t_y, o^t_y\big)^T$$

$$Q = (0, 0, 0, 0, P - \beta^* \rho k\omega, \alpha\frac{\omega}{k}P - \beta\rho\omega^2 + C_D)^T$$

For simplicity were omitted strips marking averaged variables. The numerical solution can be obtained by finite volume method in semi-discrete form

$$\frac{dW_{ij}(t)}{dt} = -\frac{1}{|\Omega_{ij}|}\sum_{k=1}^{4}\widehat{F}_k \Delta S_k + \frac{1}{|\Omega_{ij}|}\sum_{k=1}^{4}(Rn_x + Sn_y)_k \Delta S_k + Q_{ij} \tag{31}$$

$$\widehat{F} = Fn_x + Gn_y \tag{32}$$

Where $W_{ij}(t)$ is the averaged solution over a cell Ω_{ij}, $\vec{n} = (n_x, n_y)$ is the outer unit normal vector and ΔS is the size of interface. R and S are viscous numerical fluxes, which are always approximated by central differencing [3]. Q_{ij} is vector of source terms, which is evaluated directly from known values of W_{ij}. $\widehat{F}$ is the inviscid numerical flux. In our case, we use HLLC Riemann solver numerical flux:

$$\widehat{F}^{HLLC}_k = \begin{cases} F_L & \text{pro } S_L > 0 \\ F(W^*_L) & \text{pro } S_L \le 0 < S_M \\ F(W^*_R) & \text{pro } S_M \le 0 \le S_R \\ F_R & \text{pro } S_R < 0 \end{cases} \tag{33}$$

Where:

$$F_L = \begin{bmatrix} \rho_L \widetilde{u}_L \\ \rho_L u_L \widetilde{u}_L + p_L n_x \\ \rho_L v_L \widetilde{u}_L + p_L n_y \\ (e_L + p_L)\widetilde{u}_L \end{bmatrix} \qquad F_R = \begin{bmatrix} \rho_R \widetilde{u}_R \\ \rho_R u_R \widetilde{u}_R + p_R n_x \\ \rho_R v_R \widetilde{u}_R + p_R n_y \\ (e_R + p_R)\widetilde{u}_R \end{bmatrix} \tag{34}$$

$$F(W^*_L) = \begin{bmatrix} \rho^*_L S_M \\ (\rho u)^*_L S_M + p^* n_x \\ (\rho v)^*_L S_M + p^* n_y \\ (e^*_L + p^*) S_M \end{bmatrix} \qquad F(W^*_R) = \begin{bmatrix} \rho^*_R S_M \\ (\rho u)^*_R S_M + p^* n_x \\ (\rho v)^*_R S_M + p^* n_y \\ (e^*_R + p^*) S_M \end{bmatrix} \tag{35}$$

$$W^*_L = \begin{bmatrix} \rho^*_L \\ (\rho u)^*_L \\ (\rho v)^*_L \\ e^*_L \end{bmatrix} = \Omega_L \begin{bmatrix} \rho_L(S_L - \widetilde{u}_L) \\ (S_L - \widetilde{u}_L)(\rho u)_L + (p^* - P_L)n_x \\ (S_L - \widetilde{u}_L)(\rho v)_L + (p^* - P_L)n_y \\ (S_L - \widetilde{u}_L)e_L - p_L\widetilde{u}_L + p^* S_M \end{bmatrix} \tag{36}$$

$$W^*_R = \begin{bmatrix} \rho^*_R \\ (\rho u)^*_R \\ (\rho v)^*_R \\ e^*_R \end{bmatrix} = \Omega_R \begin{bmatrix} \rho_R(S_R - \widetilde{u}_R) \\ (S_R - \widetilde{u}_R)(\rho u)_R + (p^* - P_R)n_x \\ (S_R - \widetilde{u}_R)(\rho v)_R + (p^* - P_R)n_y \\ (S_R - \widetilde{u}_R)e_R - p_R\widetilde{u}_R + p^* S_M \end{bmatrix} \tag{37}$$

$$\Omega_L = \frac{1}{S_L - S_M} \qquad \Omega_R = \frac{1}{S_R - S_M} \tag{38}$$

$$p^* = \rho_L(\widetilde{u}_L - S_L)(\widetilde{u}_L - S_M) + p_L = \rho_R(\widetilde{u}_R - S_R)(\widetilde{u}_R - S_M) + p_R \tag{39}$$

Wave speed S_M is taken form Batten [1]:

$$S_M = \frac{\rho_R\widetilde{u}_R(S_R - \widetilde{u}_R) - \rho_L\widetilde{u}_L(S_L - \widetilde{u}_L) + p_L - p_R}{\rho_R(S_R - \widetilde{u}_R) - \rho_L(S_L - \widetilde{u}_L)} \tag{40}$$

and S_L and S_R are taken from Einfeldt [1]:

$$S_L = \min\left[\lambda_1(W_L), \lambda_1(W^{Roe})\right] \tag{41}$$
$$S_R = \max\left[\lambda_m(W_R), \lambda_m(W^{Roe})\right] \tag{42}$$

Where $\lambda_1(W_L)$ and $\lambda_m(W_R)$ are the smallest and largest eigenvalues eigenvalues of the Jacobi matrix of inviscid flux and $\lambda_1(W^{Roe})$ and $\lambda_m(W^{Roe})$ are the smallest and largest eigenvalues of the Roe matrix [3]. Because basic method suffers from strong artificial dissipation, the higher order method is needed. Second order accuracy in space is archived by the WENO piecewise linear reconstruction [3].

The resulting system of ordinary differential equations is then solved by the explicit two-stage TVD Runge-Kutta method [3] with local time-step. It is a bit difficult to solve turbulence equations by the explicit method (especially with local time-step), because they are very stiff and thus they are very sensitive to small disturbances, which usually leads to instability and rapid divergence. Therefore, it is reasonable to use point implicit method for discretization of source terms. It is well known, that implicit method is unconditionally stable in linear case and in non-linear case allows to use very high values of CFL. Let us start from semi-discrete form of finite volume method

$$\frac{dW}{dt} = R(W) + Q(W), \quad W = (\rho k, \rho\omega)^T \tag{43}$$

Where $R(W)$ contains convective and viscous terms and $Q(W)$ contains source terms. Convective and viscous part is discretized by explicit method and part with source terms by implicit method

$$\frac{W^{n+1} - W^n}{\Delta t} = R(W^n) + Q(W^{n+1}) \tag{44}$$

Now we split system (44) by introducing new variable $W^*, t^n < t^* < t^{n+1}$:

$$\frac{W^{n+1} + W^* - W^* - W^n}{\Delta t} - R(W^n) = Q(W^{n+1}) \tag{45}$$

$$\frac{W^* - W^n}{\Delta t} - R(W^n) = 0 \Rightarrow \frac{W^* - W^n}{\Delta t} = R(W^n) \tag{46}$$

$$\frac{W^{n+1} - W^*}{\Delta t} = Q(W^{n+1}) \tag{47}$$

System (46) is solved by forward Euler method in form

$$W^* = W^n + \Delta t R(W^n) \tag{48}$$

In system (47) is used linearization in form

$$Q(W^{n+1}) \approx Q(W^*) + \frac{\partial Q}{\partial W}\left(W^{n+1} - W^*\right) \tag{49}$$

After substituting relation (49) to system (47) and some calculation, we obtain

$$\left(\frac{I}{\Delta t} - \frac{\partial Q}{\partial W}\right)\left(W^{n+1} - W^*\right) = Q(W^*) \tag{50}$$

Considering that we are interested in the stationary solution, we can use approximation of jacobian $\frac{\partial Q}{\partial W}$ in form of another, "similar" matrix. In our case, we use approximation by jacobian $\frac{\partial Q}{\partial W}^-$ from negative source terms, and we are neglecting off-diagonal elements. This matrix takes following form for TNT model of turbulence

$$\frac{\partial Q}{\partial W}^- = \begin{pmatrix} -\beta^*\omega & 0 \\ 0 & -2\beta\omega - C_D/\rho\omega \end{pmatrix} \tag{51}$$

After some calculations, we obtain

$$W^{n+1} = W^* + \frac{\Delta t}{1 - \Delta t A} Q(W^*) \tag{52}$$

Where A is vector given by

$$A = (-\beta^*\omega, -2\beta\omega - C_D/\rho\omega)^T \tag{53}$$

From linear analysis resulting, that method will be positive and stable if we use explicit method in the case of positive eigenvalues of matrix (51) and implicit method in the case of negative eigenvalues of matrix (51). For that reason, relation (53) is reformulated to final form

$$W^{n+1} = W^* + \frac{\Delta t}{1 - \Delta t \min(A, 0)} Q(W^*) \tag{54}$$

Resulting scheme is two-stage explicit method:

$$\begin{aligned} W^* &= W^n + \Delta t R(W^n) \\ W^{n+1} &= W^* + \frac{\Delta t}{1 - \Delta t \min(A, 0)} Q(W^*) \end{aligned} \tag{55}$$

After substituting first step to second and some calculation, we obtain

$$\frac{dW(.,t^n)}{dt} = \frac{W^{n+1} - W^n}{\Delta t} + \mathcal{O}(\Delta t) = R(W^n) + Q(W^n + \Delta t R(W^n)) \frac{1}{1 - \Delta t \min(A, 0)} \tag{56}$$

For $\Delta t \rightarrow 0$, we obtain original equation (43). That means, that this method is first order accurate.

4. Validation and results

Validation of numerical method is done by comparison to the theoretical results for the subsonic flow around the flat plate. Here, we compare profiles of velocity and distributions of friction coefficient.
Problem was solved on computational domain (fig. 1) covered by structured H-type mesh with 110x80 cells with flow specifications $M_\infty = 0.2, \alpha_\infty = 0°, Re = 8 \cdot 10^5$.

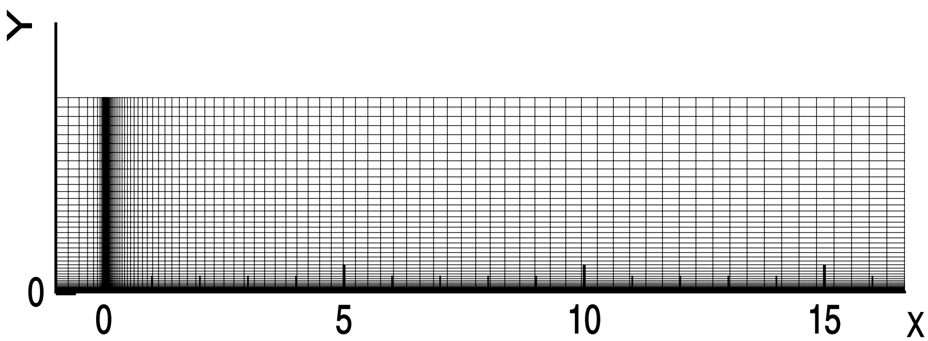
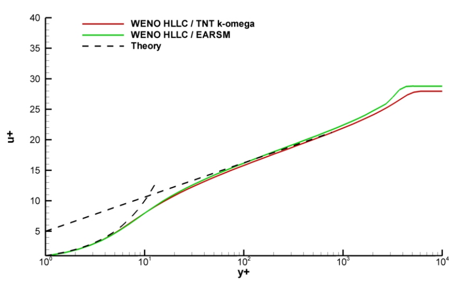
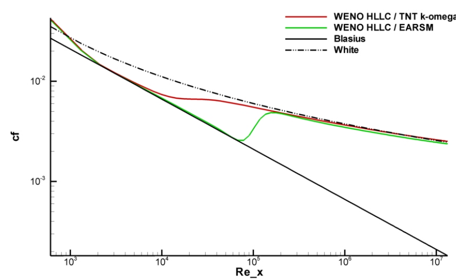


Figure 1: Computational domain and mesh for the flow around the flat plate



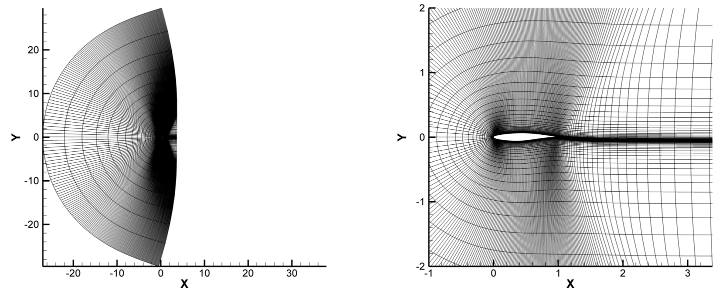
(a) Profiles of velocity at $x = 12.68$



(b) Distributions of friction coefficient

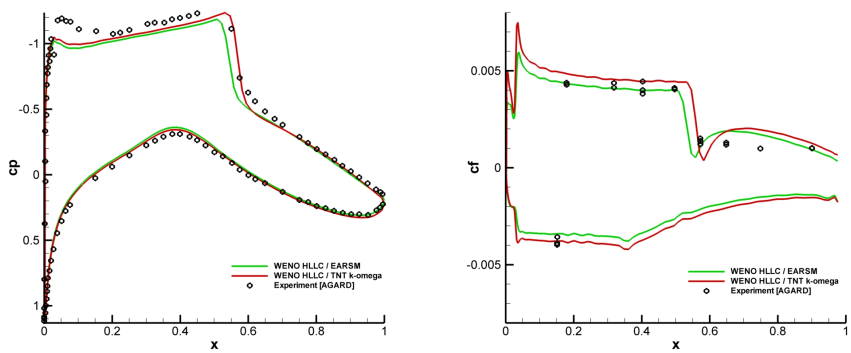
Figure 2: Subsonic turbulent flow around the flat plate

Next case is well known transonic flow around the RAE 2822 airfoil. We consider case 9 ($M_\infty = 0.734, \alpha_\infty = 2.54°, Re = 6.5 \cdot 10^6$), where no separation occurs downstream of the shockwave position and case 10 ($M_\infty = 0.754, \alpha_\infty = 2.57°, Re = 6.2 \cdot 10^6$), where shockwave-boundary-layer interaction induces massive separation of boundary layer. Problem was solved on computational domain (fig. 3) covered by structured C-type mesh with 300x70 cells with $\Delta y_1 = 5 \cdot 10^{-6}$.



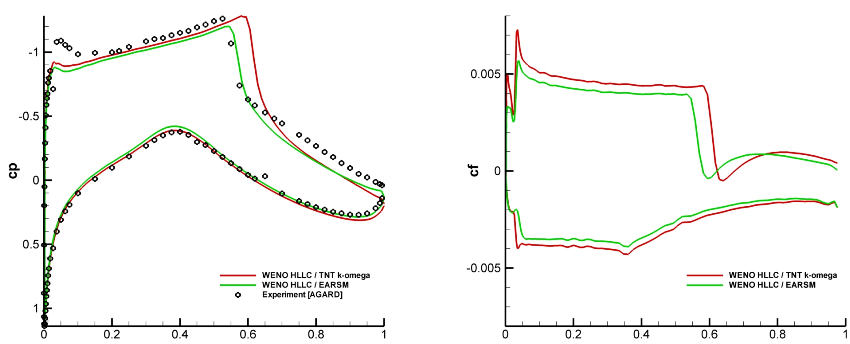
(a) Computational domain and mesh (b) Detail of mesh in the area of profile

Figure 3: Computational domain and mesh for the flow around the RAE 2822 airfoil



(a) Distributions of pressure coefficient (b) Distributions of friction coefficient

Figure 4: Transonic flow around the RAE 2822 airfoil, case 9



(a) Distributions of pressure coefficient (b) Distributions of friction coefficient

Figure 5: Transonic flow around the RAE 2822 airfoil, case 10

5. Conclusion and remarks

We described mathematical model based on Favre averaged Navier-Stokes equations and explicit algebraic Reynolds stress model of turbulence. From test cases is clear, that EARSM model improve prediction accuracy in comparison to the two equation eddy-viscosity models. EARSM model archived similar results as Kok's TNT model in case 9 (flow around the RAE 2822), but in the case 10, where separation of boundary layer occurs, EARSM model predicted much better shockwave position then TNT model. Also in case of the flow around the flat plate both models archived similar results, but EARSM model predicted much better transition between laminar and turbulent flow regime. Quality of transition position is now under investigation.

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