FREE ACTIONS ON SEMIPRIME RINGS

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Abstract. We identify some situations where mappings related to left centralizers, derivations and generalized (α,β)-derivations are free actions on semiprime rings. We show that for a left centralizer, or a derivation T, of a semiprime ring R the mapping $\psi\colon R\to R$ defined by $\psi(x)=T(x)x-xT(x)$ for all $x\in R$ is a free action. We also show that for a generalized (α,β)-derivation F of a semiprime ring R, with associated (α,β)-derivation d, a dependent element a of F is also a dependent element of $\alpha+d$. Furthermore, we prove that for a centralizer f and a derivation d of a semiprime ring R, $\psi=d\circ f$ is a free action.

Keywords: prime ring, semiprime ring, dependent element, free action, centralizer, derivation

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1. Introduction

Murray and von Neumann [14] and von Neumann [15] introduced the notion of free action on abelian von Neumann algebras and used it for the construction of certain factors (see Dixmier [9]). Kallman [12] generalized the notion of free action of automorphisms of von Neumann algebras, not necessarily abelian, by using implicitly the dependent elements of an automorphism. Choda, Kashahara and Nakamoto [7] generalized the concept of freely acting automorphisms to C^*-algebras by introducing dependent elements associated to automorphisms. Several other authors have studied dependent elements on operator algebras (see [8] and references therein). A brief account of dependent elements in W^*-algebras has also appeared in the book of Stratila [17]. It is well-known that all C^*-algebras and von Neumann algebras are semiprime rings; in particular, a von Neumann algebra is prime if and only if its center consists of scalar multiples of identity. Thus a natural extension of the notions of dependent elements of mappings and free actions on C^*-algebras and von Neumann algebras is the study of these notions in the context of semiprime rings and prime rings.

Laradji and Thaheem [13] initiated a study of dependent elements of endomorphisms of semiprime rings and generalized a number of results of [7] to semiprime rings. Recently, Vukman and Kosi-Ulbl [19] and Vukman [20] have made further study of dependent elements of various mappings related to automorphisms, derivations, (α, β)-derivations and generalized derivations of semi-prime rings. The main focus of the authors of [19], [20] has been to identify various freely acting mappings related to these mappings, on semiprime and prime rings.

The theory of centralizers (also called multipliers) of C^*-algebras and Banach algebras is well established (see [1], [2] and references therein). Recently, Zalar [22], Vukman [18] and Vukman and Kosi-Ulbl [21] have studied centralizers in the general framework of semiprime rings.

On the one hand, motivated by the work of Laradji and Thaheem [13], Vukman and Kosi-Ulbl [19] and Vukman [20] on dependent elements of mappings and free actions of semiprime rings and, on the other hand, by the work of Zalar [22], Vukman [18] and Vukman and Kosi-Ulbl [21] on centralizers of semiprime ring, we investigate some mappings related to left centralizers, centralizers, derivations, (α, β)-derivations and generalized (α, β)-derivations which are free actions on semiprime rings. We show that for a left centralizer T of a semiprime ring R, the mapping $\psi\colon R \to R$ defined by $\psi(x) = T(x)x - xT(x)$ $(x \in R)$, is a free action. We also prove that for a generalized (α, β)-derivation F of a semiprime ring R with the associated (α, β)-derivation d, a dependent element a of F is also a dependent element of $\alpha + d$.

Throughout, R will stand for associative ring with center $Z(R)$. As usual, the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D\colon R \to R$ is called a derivation provided $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. Let α be an automorphism of a ring R. An additive mapping $D\colon R \to R$ is called an α-derivation if $D(xy) = D(x)\alpha(y) + xD(y)$ holds for all $x, y \in R$. Note that the mapping, $D = \alpha - I$, where I denotes the identity mapping on R, is an α-derivation. Of course, the concept of an α-derivation generalizes the concept of a derivation, since any I-derivation is a derivation. α-derivations are further generalized as (α, β)-derivations. Let α, β be automorphisms of R, then an additive mapping $D\colon R \to R$ is called an (α, β)-derivation if $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$ holds for all pairs $x, y \in R$. α-derivations and (α, β)-derivations have been applied in various situations; in particular, in the solution of some functional equations. For more information on α-derivations and (α, β)-derivations we refer the reader to [3]–[6] and references therein.

An additive mapping F of a ring R into itself is called a generalized derivation, with the associated derivation d, if there exists a derivation d of R such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. The concept of a generalized derivation covers both the concepts of a derivation and of a left centralizer provided $F = d$ and $d = 0$, respectively (see [11] and references therein). An additive mapping $f\colon R \to R$ is called centralizing (commuting) if $[f(x), x] \in Z(R)$ ($[f(x), x] = 0$) for all $x \in R$. By Zalar [22], an additive mapping $T\colon R \to R$ is called a left (right) centralizer if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) for all $x, y \in R$. If $a \in R$, then $L_a(x) = ax$ and $R_a(x) = xa$ ($x \in R$) define a left centralizer and a right centralizer of R, respectively. An additive mapping $T\colon R \to R$ is called a centralizer if $T(xy) = T(x)y = xT(y)$ for all $x, y \in R$. Following [13], an element $a \in R$ is called a dependent element of a mapping $F\colon R \to R$ if $F(x)a = ax$ holds for all $x \in R$. A mapping $F\colon R \to R$ is called a free action if zero is the only dependent element of F. It is shown in [13] that in a semiprime ring R there are no nonzero nilpotent dependent elements of a mapping $F\colon R \to R$. We shall use this fact without any specific reference. For a mapping $F\colon R \to R$, $D(F)$ denotes the collection of all dependent elements of F. For other ring theoretic notions used but not defined here we refer the reader to [10].

2. Results

In order to prove our results we first give the proof of our earlier theorem [16, Theorem 2.1] for completeness. The first part of this result is a special case of Theorem 4 in [19].

Theorem 2.1. *Let R be a semiprime ring and T a left centralizer of R. Then $a \in D(T)$ if and only if $a \in Z(R)$ and $T(a) = a$.*

Proof. Let $a \in D(T)$. Then

$$T(x)a = ax \tag{1}$$

Replacing x by xy in (1), we get $T(xy)a = axy$. That is,

$$T(x)ya = axy. \tag{2}$$

Multiplying (2) by z on the right, we get

$$T(x)yaz = axyz. \tag{3}$$

Replacing y by yz in (2), we get

$$T(x)yza = axyz. \tag{4}$$

Subtracting (4) from (3), we get $T(x)y(az - za) = T(x)y[a, z] = 0$. Replacing y by ay and then using semiprimeness of R, we get $T(x)a[a, z] = 0$. That is, $ax[a, z] = 0$, which, by semiprimeness of R, implies $a[a, z] = 0$ for all $a \in R$. Now using Lemma 1.1.4 [10], we get $a \in Z(R)$.

Since $a \in Z(R)$, we have $ay = ya$. Thus $T(ay) = T(ya)$. That is, $T(a)y = T(y)a = ay$. So $(T(a) - a)y = 0$, which, by semiprimeness of R, implies $T(a) - a = 0$. Thus $T(a) = a$.

Conversely, let $T(a) = a$ and $a \in Z(R)$. Then $T(x)a = T(xa) = T(ax) = T(a)x = ax$. Thus $a \in D(T)$.

Theorem 2.2. *Let R be a prime ring and $T \neq I$ a left centralizer of R. Then T is a free action on R.*

Proof. Let $a \in D(T)$. Then $T(x)a = ax$. Moreover, $a \in Z(R)$ by Theorem 2.1. Thus $T(x)a = xa$. That is,

$$(T(x) - x)a = 0. \tag{5}$$

Since $a \in Z(R)$, from (5) we get $(T(x) - x)za = 0$ for all $z \in R$. Since $T \neq I$ and R is prime, we have $a = 0$. So T is a free action. □

Theorem 2.3. *Let R be a semiprime ring and T an injective left centralizer of R. Then $\psi = T + I$ is a free action on R.*

Proof. Obviously $T + I$ is a left centralizer of R. Let $a \in D(T + I)$. Then by Theorem 2.1, $a \in Z(R)$ and $(T + I)(a) = a$. Thus $T(a) = 0$. So $a \in \operatorname{Ker}(T)$. Since T is injective, we have $a = 0$. Hence T is a free action. □

Theorem 2.4. *Let T be a left centralizer of a semiprime ring R. Then $\psi\colon R \to R$, defined by $\psi(x) = [T(x), x]$ for all $x \in R$, is a free action.*

Proof. Let $a \in D(\psi)$. Then

$$[T(x), x]a = ax \quad \text{for all } x \in R. \tag{6}$$

Linearizing (6) and using (6) after linerization, we get

$$[T(x), y]a + [T(y), x]a = 0. \tag{7}$$

Replacing y by ay in (7), we get

$$\begin{aligned} 0 &= [T(x), ay]a + [T(ay), x]a = a[T(x), y]a + [T(x), a]ya + [T(a)y, x]a \\ &= a[T(x), y]a + [T(x), a]ya + T(a)[y, x]a + [T(a), x]ya. \end{aligned}$$

That is,

$$a[T(x),y]a+[T(x),a]ya+T(a)[y,x]a+[T(a),x]ya=0. \tag{8}$$

Using [7], from (8) we get $-a[T(y),x]a+[T(x),a]ya+T(a)[y,x]a+[T(a),x]ya=0$, which implies

$$-a[T(a),a]a+[T(a),a]a^2+[T(a),a]a^2=0. \tag{9}$$

Replacing y and x by a in (6) and using (6), from (9) we get $-a^3+a^3+a^3=0$. That is, $a^3=0$, which implies $a=0$. Hence ψ is a free action.

Theorem 2.5. *Let R be a semiprime ring and $d\colon R\to R$ a derivation. Then the mapping $\psi\colon R\to R$, defined by $\psi(x)=[d(x),x]$ for all $x\in R$, is a free action.*

Proof. Let $a\in D(\psi)$. Then

$$\psi(x)a=[d(x),x]a=ax. \tag{10}$$

Linearizing (10) and using (10) after linearization, we get

$$[d(x),y]a+[d(y),x]a=0 \quad \text{for all } x,y\in R. \tag{11}$$

Replacing y by x in (11), we get

$$2[d(x),x]a=0 \quad \text{for all } x\in R. \tag{12}$$

Replacing y by xy in (11), we get

$$\begin{aligned} 0 &= [d(x),xy]a+[d(xy),x]a \\ &= x[d(x),y]a+[d(x),x]ya+[d(x)y+xd(y),x]a \\ &= x[d(x),y]a+[d(x),x]ya+d(x)[y,x]a+[d(x),x]ya+x[d(y),x]a. \end{aligned}$$

That is,

$$0=x\{[d(x),y]a+[d(y),x]a\}+2[d(x),x]ya+d(x)[y,x]a. \tag{13}$$

Using (11), from (13) we get

$$2[d(x),x]ya+d(x)[y,x]a=0 \quad \text{for all } x,y\in R. \tag{14}$$

Replacing y by ya in (14), we get

$$\begin{aligned} 0 &= 2[d(x), x]ya^2 + d(x)[ya, x]a \\ &= 2[d(x), x]ya^2 + d(x)[y, x]a^2 + d(x)y[a, x]a. \end{aligned}$$

That is,

$$(2[d(x), x]ya + d(x)[y, x]a)a + d(x)y[a, x]a = 0. \tag{15}$$

Using (14), from (15) we get

$$d(x)y[a, x]a = 0. \tag{16}$$

Replacing y by xy in (16), we get

$$d(x)xy[a, x]a = 0. \tag{17}$$

Multiplying (16) by x on the left, we get

$$xd(x)y[a, x]a = 0. \tag{18}$$

Subtracting (18) from (17), we get $[d(x), x]y[a, x]a = 0$. Replacing y by ay in the last identity and then using (10), we get

$$axy[a, x]a = 0. \tag{19}$$

Replacing y by a^2y in (19), we get

$$axa^2y[a, x]a = 0. \tag{20}$$

Multiplying (19) on the left by a and replacing y by ay in (19), we get

$$a^2xay[a, x]a = 0. \tag{21}$$

Subtracting (20) from (21), we get

$$a(ax - xa)ay[a, x]a = 0. \tag{22}$$

Replacing y by ya in (22), we get $a[a, x]aya[a, x]a = 0$, which, by semiprimeness of R, implies that $a[a, x]a = 0$. In particular, $a[d(a), a]a = 0$. This, by (10), implies that $a^3 = 0$. Hence $a = 0$, which implies that $\psi(x) = [d(x), x]$ is a free action on R.

We now define a generalized (α, β)-derivation of a ring R.

Definition 2.6. Let α and β be automorphisms of a ring R. An additive mapping $F\colon R \to R$ is called a generalized (α,β)-derivation, with the associated (α,β)-derivation d, if there exists an (α,β)-derivation d of R such that $F(xy) = \alpha(x)F(y) + d(x)\beta(y)$.

R e m a r k 2.7. We note that for $F = d$, F is an (α,β)-derivation and for $d = 0$ and $\alpha = I$, F is a right centralizer. So a generalized (α,β)-derivation covers both the concepts of an (α,β)-derivation and a right centralizer.

Theorem 2.8. *Let R be a semiprime ring. Let α,β be centralizing automorphisms of R and let $F\colon R \to R$ be a generalized (α,β)-derivation with the associated (α,β)-derivation d. If a is a dependent element of F, then $a \in D(\alpha + d)$.*

P r o o f. Let $a \in D(F)$. Then

$$F(x)a = ax \quad \text{for all } x \in R. \tag{23}$$

Replacing x by xy in (23), we get $F(xy)a = axy$, which implies $\alpha(x)F(y)a + d(x)\beta(y)a = axy$. That is, $\alpha(x)ay + d(x)\beta(y)a = axy = F(x)ay$. Thus

$$(F(x)a - \alpha(x)a)y = d(x)\beta(y)a. \tag{24}$$

Multiplying (24) by z on the right, we get

$$(F(x)a - \alpha(x)a)yz = d(x)\beta(y)az. \tag{25}$$

Replacing y by yz in (24), we get

$$(F(x)a - \alpha(x)a)yz = d(x)\beta(y)\beta(z)a. \tag{26}$$

Subtracting (25) from (26), we get $d(x)\beta(y)[\beta(z)a - az] = 0$, which, due to surjectivity of β, implies

$$d(x)y[\beta(z)a - az] = 0. \tag{27}$$

Since β is centralizing and R is semiprime, from (27) we get

$$d(x)[\beta(z)a - a] = 0.$$

That is,

$$d(x)\beta(z)a = d(x)az \quad \text{for all } x, z \in R. \tag{28}$$

Using (28), from (24) we get $(F(x)a - \alpha(x)a)y = d(x)ay$. That is, $(F(x)a - \alpha(x)a - d(x)a)y = 0$, which, due to semiprimeness of R, implies that

$$F(x)a - (\alpha + d)(x)a = 0. \tag{29}$$

Using (23), from (29) we get

$$(\alpha + d)(x)a = ax. \tag{30}$$

This shows that $a \in D(\alpha + d)$.

We now have the following result of Vukman and Kosi-Ulbl [19, Theorem 10] as a corollary of Theorem 2.8.

Corollary 2.9. *If F is an (α, β)-derivation of a semiprime ring R, then F is a free action.*

Proof. Let $F = d$. Then d is an (α, β)-derivation and so equation (30) gives $(\alpha + F)(x)a = ax$. That is, $\alpha(x)a + F(x)a = ax$, which implies that $\alpha(x)a + ax = ax$. Thus $\alpha(x)a = 0$ for all $x \in R$. Since α is onto, we have $xa = 0$ for all $x \in R$. Thus $axa = 0$, which implies that $a = 0$. Hence F is a free action. $\square$

Corollary 2.10. *Let R be a semiprime ring and α a centralizing automorphism of R. Let $F\colon R \to R$ be an additive mapping satisfying $F(xy) = \alpha(x)F(y)$ for all $x, y \in R$. If $a \in D(F)$, then $a \in Z(R)$.*

Proof. We take $d = 0$ in Theorem 2.8. Then $F(xy) = \alpha(x)F(y)$ and $a \in D(F)$ implies that $a \in D(\alpha)$. Since α is a centralizing automorphism, by [13, Proposition 3] we conclude that $a \in Z(R)$. $\square$

Remark 2.11. If in the above corollary we take $\alpha = I$, the identity automorphism, then F is a right centralizer. Thus all dependent elements of a right centralizer F of a semiprime ring R lie in $Z(R)$.

Theorem 2.12. *Let R be a semiprime ring. Let f be a centralizer and d a derivation of R. Then $\psi = d \circ f$ is a free action.*

Proof. Let $a \in D(\psi)$. Then $\psi(x)a = ax$. That is,

$$d \circ f(x)a = ax \quad \text{for all } x \in R. \tag{31}$$

Replacing x by xy in (31), we get

$$axy = d \circ f(xy)a = d(f(x)y)a = d \circ f(x)ya + f(x)d(y)a.$$

That is,

$$d \circ f(x)ya + f(x)d(y)a = axy = (d \circ f)(x)ay.$$

Thus,

$$d \circ f(x)[a, y] = f(x)d(y)a \quad \text{for all } x, y \in R. \tag{32}$$

Replacing y by ay in (32), we get $d \circ f(x)[a, ay] = f(x)d(ay)a$. That is,

$$d \circ f(x)a[a, y] = f(x)d(a)ya + f(x)ad(y)a. \tag{33}$$

Using (31), from (33) we get

$$ax[a, y] = f(x)d(a)ya + f(x)ad(y)a. \tag{34}$$

Multiplying (34) on the left by z, we get

$$zax[a, y] = zf(x)d(a)ya + zf(x)ad(y)a. \tag{35}$$

Replacing x by zx in (34), we get $azx[a, y] = f(zx)d(a)ya + f(zx)ad(y)a = zf(x)d(a)ya + zf(x)ad(y)a$. That is,

$$azx[a, y] = zf(x)d(a)ya + zf(x)ad(y) \quad \text{for all } x, y, z \in R. \tag{36}$$

Subtracting (35) from (36), we get $[a, z]x[a, y] = 0$. In particular, $[a, y]x[a, y] = 0$, which, by semiprimeness of R, implies $[a, y] = 0$ for all $y \in R$. Thus $a \in Z(R)$, so from (32) we get

$$f(x)d(y)a = 0 \quad \text{for all } x, y \in R. \tag{37}$$

Replacing y by $f(y)$ in (37) and then using (31) we get $f(x)ay = 0$, which, by semiprimeness of R, implies that

$$f(x)a = 0. \tag{38}$$

Thus $d(f(x)a) = d(0) = 0$. That is

$$d \circ f(x)a + f(x)d(a) = 0,$$

which implies that

$$d \circ f(x)a^2 + f(x)d(a)a = 0. \tag{39}$$

Using (37) and (31), from (39) we get $axa = 0$. Thus $a = 0$, which implies that $d \circ f$ is a free action.

Theorem 2.13. *Let f be a left centralizer of a semiprime ring R. Let $\psi(x) = f(x)x + xf(x)$. Then ψ is a free action on R.*

Proof. Let $a \in D(\psi)$. Then $\psi(x)a = ax$. That is,

$$[f(x)x + xf(x)]a = ax. \tag{40}$$

Linearizing (40), we get

$$[f(x)y + f(y)x + yf(x) + xf(y)]a = 0. \tag{41}$$

Replacing both x and y by a in (41) and using (40), we get $0 = [f(a)a + f(a)a + af(a) + af(a)] = 2[f(a)a + af(a)]a = 2a^2$. That is,

$$2a^2 = 0. \tag{42}$$

Now replacing y by xa in (41) and using (40), we get

$$\begin{aligned} 0 &= [f(x)xa + f(xa)x + xaf(x) + xf(xa)]a \\ &= [f(x)xa + f(x)ax + xaf(x) + xf(x)a]a \\ &= (f(x)x + xf(x))a^2 + f(x)axa + xaf(x)a \\ &= axa + f(x)axa + xaf(x)a. \end{aligned}$$

That is,

$$axa + f(x)axa + xaf(x)a = 0 \quad \text{for all } x \in R. \tag{43}$$

Replacing x by a in (43) and using (40) and (42), we get $0 = a^3 + f(a)a^3 + a^2f(a)a = a^3 + f(a)a^3 - a^2f(a)a$. That is,

$$a^3 + f(a)a^3 - a^2f(a)a = 0. \tag{44}$$

Replacing x by a in (40), we get

$$f(a)a^2 + af(a)a = a^2. \tag{45}$$

Multiplying (45) by a on the left as well as on the right, we get

$$af(a)a^2 + a^2f(a)a = a^3 \tag{46}$$

and

$$f(a)a^3 + af(a)a^2 = a^3, \tag{47}$$

respectively. Subtracting (46) from (47), we get

$$f(a)a^3 - a^2f(a)a = 0. \tag{48}$$

Using (48), from (44) we get $a^3 = 0$. Thus $a = 0$, which implies that ψ is a free action.

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