STEADY-STATE BUOYANCY-DRIVEN VISCOUS FLOW WITH MEASURE DATA

Tomáš Roubíček, Praha

Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. Steady-state system of equations for incompressible, possibly non-Newtonean of the p-power type, viscous flow coupled with the heat equation is considered in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, with heat sources allowed to have a natural L^1-structure and even to be measures. The existence of a distributional solution is shown by a fixed-point technique for sufficiently small data if $p > 3/2$ (for $n = 2$) or if $p > 9/5$ (for $n = 3$).

Keywords: non-Newtonean fluids, heat equation, dissipative heat, adiabatic heat

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1. Introduction, problem formulation

This paper deals with the steady-state buoyancy-driven flow of heat-conductive, possibly non-Newtonean, incompressible fluids. There are various models appearing in literature, cf. e.g. [3], [7], [14], [18] for a genesis of various possibilities. The starting point is always the complete evolutionary compressible fluid system of $n+2$ conservation laws for mass, impulse, and energy; n denotes the spatial dimension. Then, the so-called incompressible limit represents a small perturbation around a stationary homogeneous state, i.e. around constant mass density, constant temperature, and zero velocity; note that small perturbations of velocity u do not necessarily mean small ∇u, which makes it sensible to consider nonlinearity in stress τ below. This incompressible limit system of $n+1$ equations need not be thermodynamically consistent, however.

We consider Ω a bounded smooth (namely $C^{3,1}$-) domain in $\mathbb{R}^n$, $n = 2$ or 3; for Ω a $C^{0,1}$-domain see Remark 2 below. To cover various possibilities, we consider the following fairly general system of equations:

$$(1.1a)\quad (u\cdot\nabla)u-\operatorname{div}\tau(e(\nabla u))+\nabla\pi=g(1-\alpha_0\theta),\quad e(\nabla u)=\frac{1}{2}\nabla u+\frac{1}{2}(\nabla u)^{\mathrm T}$$

$$(1.1b)\qquad \operatorname{div}u=0,$$

$$(1.1c)\qquad u\cdot\nabla\theta-\kappa\Delta\theta=\alpha_1\tau(e(\nabla u))\colon e(\nabla u)+\alpha_2\theta g\cdot u+h,$$

where $[\tau_{ij}]:[e_{ij}]=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\tau_{ij}e_{ij}$, κ is the heat conductivity, α_0 is the linearized relative mass density variation with respect to temperature, α_1 reflects the dissipation effects, α_2 expresses the adiabatic heat effects, $\tau(e)$ is the viscous stress, g an external (e.g. gravitational or centrifugal) force, and $h=h(x)$ is the external heat source. For simplicity of notation, we normalize the mass density and the heat capacity to 1.

For a rigorous derivation of a system like (1.1) in the evolution case, we refer to Kagei, Růžička and Thäter [7, System (16)] who showed how the coefficient α_1 depends on Ostrach's dissipation number, while the coefficient α_2 depends also on the Reynolds and the Prandtl numbers.

The system should be completed by boundary conditions. For simplicity, we will consider a no-slip boundary condition for velocity and the Newton condition with prescribed heat flux f for temperature, i.e.

$$(1.2)\qquad u=0,\qquad \kappa\frac{\partial\theta}{\partial\nu}+b\theta=f\quad\text{on}\quad\Gamma,$$

with ν denoting the unit normal to the boundary $\partial\Omega=:\Gamma$ of Ω and b denoting the coefficient of the heat transfer through Γ.

Often a simpler, so-called Oberbeck-Boussinesq model is used for the buoyancy-driven flow of heat-conductive incompressible fluids. This model neglects both the dissipative and the adiabatic heat sources, i.e. $\alpha_1=\alpha_2=0$, and usually considers $\tau(e)=e$ which turns (1.1a,b) into the Navier-Stokes system, cf. e.g. Gebhart et al. [5] or Rajagopal et al. [18], and sometimes it is combined with other phenomena as solidification, see Rodriguez [19]. For a non-Newtonean model coupled with the heat equation we refer to Málek at al. [13] and to Rodriguez and Urbano [20] who allowed the viscosity to depend also on temperature. Temperature dependence of the viscosity tensor τ was investigated also by Baranger and Mikelić [2] for the special case $\alpha_1=1$, $\alpha_0=0$ (i.e. no buoyancy) and $\alpha_2=0$, which makes the situation quite different from the buoyancy driven flow. Besides, some buoyancy-driven models include the dissipative heat but not the adiabatic heat sources (i.e. our model (1.1) with $\alpha_1>0$ but $\alpha_2=0$), cf. Landau and Lifshitz [9, Sect. 50] or also, e.g., Kagei [6] or Moseenkov [14].

The measures as heat sources for the buoyancy-driven flow have been investigated for $b = 0$ and $f = 0$ in [16] in the evolutionary case, which differs from the steady-state case both factually (existence of a non-negative solution holds for arbitrarily large data) and technically (L^1-accretivity for the heat equation can be used instead of mere $W^{2,2}$-regularity and interpolation with transposition).

2. Distributional solution to (1.1)–(1.2)

We want to treat the system (1.1) in as much general as possible (but still physical) situations. The heat transfer (1.1c) has a natural L^1-structure, which encourages us to consider the heat sources $h \in L^1(\Omega)$ and $f \in L^1(\Gamma)$, or even as measures. Then the concept of a weak solution is no longer relevant, and one must speak in terms of distributional solutions, using transposition and $W^{2,2}$-regularity with Hilbertian-space interpolation of the adjoint to the left-hand-side linear operator in (1.1c).

We use the following standard notation for functions spaces: $L^p(\Omega;\mathbb{R}^n)$ denotes the Lebesgue space of measurable functions $\Omega \to \mathbb{R}^n$ whose p-power is integrable, $W_0^{1,p}(\Omega;\mathbb{R}^n)$ is the Sobolev space of functions whose gradient is in $L^p(\Omega;\mathbb{R}^{n\times n})$ and whose trace on Γ vanishes, $W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n) = \{v \in W^{1,p}(\Omega;\mathbb{R}^n);\ \operatorname{div} v = 0$ in the sense of distributions$\}$, and $W^{-1,p'}(\Omega;\mathbb{R}^n) \cong W_0^{1,p}(\Omega;\mathbb{R}^n)^*$ with p' denoting the conjugate exponent, i.e. $p' = p/(p-1)$. Likewise, $W^{k,p}$ indicates all kth derivatives belonging to the L^p space; for k noninteger it refers to a fractional derivative and $W^{k,p}$ then denotes the Sobolev-Slobodetskiĭ space. Let us agree to use the norm $\|u\|_{W_0^{1,p}(\Omega;\mathbb{R}^n)} := \|\nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})}$. Also, "rca" will denote the regular countably additive set functions with respect to a Borel σ-algebra in question, also called Radon measures.

We will assume the following data qualification:

$$\tau \text{ has a } C^2\text{-potential},\ \ \tau(e) : e \geqslant \zeta_1|e|^p,\ \ |\tau(e)| \leqslant c(|e|^{p-1}+1),\ \ p > \frac{3n}{n+2}, \tag{2.1a}$$

$$(\tau(e_1)-\tau(e_2)) : (e_1-e_2) \geqslant \begin{cases} \zeta_1|e_1-e_2|^p + \zeta_2|e_1-e_2|^2 & \text{if } p \geqslant 2 \\ \zeta_0(|e_1|+|e_2|)^{p-2}|e_1-e_2|^2 & \text{if } p < 2, \end{cases} \tag{2.1b}$$

$$\sum_{i,j,k,l=1}^{n} \frac{\partial \tau_{ij}}{\partial e_{kl}}\xi_{ij}\xi_{kl} \geqslant \begin{cases} \zeta_3(1+|e|^{p-2})|\xi|^2 & \text{if } p \geqslant 2 \\ \zeta_3|e|^{p-2}|\xi|^2 & \text{if } p < 2, \end{cases} \tag{2.1c}$$

$$h \in \mathrm{rca}(\overline{\Omega}),\ \ f \in \mathrm{rca}(\Gamma),\ \ g \in L^\infty(\Omega;\mathbb{R}^n),\ \ b \in C^{0,1}(\Gamma), \tag{2.1d}$$

$$\kappa > 0,\ \ \alpha_0,\alpha_1,\alpha_2 \geqslant 0,\ \ b(x) \geqslant b_0 > 0, \tag{2.1e}$$

with $\zeta_i > 0$, $i = 0, \ldots, 3$. An example of τ satisfying (2.1a–c) is $\tau(e) = (1 + |e|^{p-2})e$ (if $p \geqslant 2$) or $\tau(e) = |e|^{p-2}e$ (if $p \leqslant 2$). Let us also recall that (2.1a–c) ensures

$$\text{(2.2a)} \int_\Omega \tau(e(\nabla u)) : \nabla u \, \mathrm{d}x \geqslant c_1 \|u\|^p_{W^{1,p}_0(\Omega;\mathbb{R}^n)}$$

$$\text{(2.2b)} \int_\Omega (\tau(e(\nabla u_1)) - \tau(e(\nabla u_2))) : e(\nabla u_1 - \nabla q_2) \, \mathrm{d}x$$
$$\geqslant \begin{cases} \zeta_1 c_{1,\Omega} \|u_1 - u_2\|^p_{W^{1,p}_0(\Omega;\mathbb{R}^n)} + \zeta_2 c_{2,\Omega} \|u_1 - u_2\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)} & \text{if } p \geqslant 2 \\ \zeta_0 c_{0,\Omega}(\|u_1\|_{W^{1,p}_0(\Omega;\mathbb{R}^n)} + \|u_2\|_{W^{1,p}_0(\Omega;\mathbb{R}^n)}) \|u_1 - u_2\|^2_{W^{1,p}_0(\Omega;\mathbb{R}^n)} & \text{if } p < 2, \end{cases}$$

with some $c_{i,\Omega} > 0$ resulting from Korn's inequality, $c_{0,\Omega}(\cdot)$ decreasing; cf. [12, Sect 5.1.2]. Let us also introduce an exponent q by

$$\text{(2.3)} \qquad \frac{2p}{p-1} \leqslant q \begin{cases} < \frac{pn}{n-p} & \text{if } p < n \\ < +\infty & \text{otherwise,} \end{cases}$$

which ensures, in particular, the compact embedding $W^{1,p}_0(\Omega;\mathbb{R}^n) \subset L^q(\Omega;\mathbb{R}^n)$. By using Green's formula once for (1.1a,b) and twice for (1.1c), one gets the following definition:

Definition. We will call $(u, \theta) \in W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n) \times W^{r,2}(\Omega)$, with $r \in [0, 1]$ satisfying

$$\text{(2.4)} \qquad \frac{2n - 2p - pn}{2p} < r < \frac{4-n}{2},$$

a distributional solution to (1.1)–(1.2) if

$$\text{(2.5)} \qquad \int_\Omega ((u \cdot \nabla)u) \cdot v + \tau(e(\nabla u)) : e(\nabla v) - g \cdot v(1 - \alpha_0 \theta) \, \mathrm{d}x = 0$$

for any $v \in W^{1,p}_0(\Omega;\mathbb{R}^n)$, and

$$\text{(2.6)} \qquad \int_\Omega ((u \cdot (\nabla v + \alpha_2 g v) + \kappa \Delta v)\theta + \alpha_1 \tau(e(\nabla u)) : e(\nabla u) v \, \mathrm{d}x + \int_{\bar\Omega} v h(\,\mathrm{d}x) + \int_\Gamma v f(\,\mathrm{d}S) = 0$$

for any v smooth with $\kappa \frac{\partial}{\partial \nu} v + bv = 0$ on Γ.

Note that (2.1) ensures that $r \in [0, 1]$ satisfying (2.4) does exist (recall that $n \leqslant 3$); in other words, (2.4) brings no restriction on p if $n \leqslant 3$, as assumed. Let us remark that the inequalities in (2.4) imply respectively $W^{r,2}(\Omega) \subset L^{q'}(\Omega)$ and $W^{2-r,2}(\Omega) \subset$

$C(\overline{\Omega})$; of course, $q' := q/(q-1)$. Also, (2.1) implies that all integrals in (2.2)–(2.6) have good sense. Also note that (2.1a) indeed enables us to choose q such that $p^{-1} + 2q^{-1} \leqslant 1$, see (2.3), which implies that, e.g., the expression like $|v|^2\nabla v$ is integrable for any $v \in W^{1,p}(\Omega)$.

3. Existence of the distributional solution

We will prove the existence nonconstructively by using the Schauder fixed point theorem. First, we define the mapping

$$\mathcal{A}\colon\ \vartheta \mapsto u\colon\ L^{q'}(\Omega) \to W_0^{1,p}(\Omega;\mathbb{R}^n) \tag{3.1}$$

by u being the weak solution to

$$(u\cdot\nabla)u - \operatorname{div}\tau(e(\nabla u)) + \nabla\pi = g(1-\alpha_0\vartheta), \quad \operatorname{div} u = 0, \quad u|_\Gamma = 0. \tag{3.2}$$

For $q < pn/(n-p)$, let us agree to denote by $N_q^{1,p}$ the norm of the embedding $W_0^{1,p}(\Omega;\mathbb{R}^n) \subset L^q(\Omega;\mathbb{R}^n)$.

Lemma 1. *Assume* (2.1). *Then there is* $R = R(p,\Omega,c,\zeta_0,\dots,\zeta_2) > 0$ *such that* $\mathcal{A}$ *is single-valued and (weak,norm)-continuous with respect to the topologies indicated in* (3.1) *on the set*

$$S_R := \{\vartheta \in L^{q'}(\Omega);\ \|g(1-\alpha_0\vartheta)\|_{L^{q'}(\Omega;\mathbb{R}^n)} < R\}. \tag{3.3}$$

Proof. Take $\vartheta^k \rightharpoonup \vartheta$ in $L^{q'}(\Omega)$, which implies $\vartheta^k \to \vartheta$ in $W^{-1,p'}(\Omega)$ because $L^{q'}(\Omega) \subset W^{-1,p'}(\Omega)$ compactly, cf. (2.3). Then denote by u^k the weak solution to (3.2) corresponding to ϑ^k in place of ϑ; for the existence of u^k we refer to Lions [10, Ch. II, Remark 5.5] after a modification to τ depending on $e(\nabla u)$ instead of ∇u or, even for $p \geqslant 2n/(n+1)$, also Frehse, Málek and Steinhauer [4] or Růžička [22]. By testing with u^k, we get in a standard way the a-priori estimate

$$\|u^k\|^{p-1}_{W_0^{1,p}(\Omega;\mathbb{R}^n)} \leqslant \frac{N_q^{1,p}}{\zeta_1}\|g(1-\alpha_0\vartheta^k)\|_{L^{q'}(\Omega;\mathbb{R}^n)} < \frac{N_q^{1,p}R}{\zeta_1} =: R_0^{p-1}. \tag{3.4}$$

Taking a weakly convergent subsequence in $W_0^{1,p}(\Omega;\mathbb{R}^n)$, it is a standard procedure to show that its limit, denote it by u, is a weak solution to (3.2), cf. again [4], [10], [22].

Let us now prove uniqueness of u provided $\vartheta \in S_R$ from (3.3) with R small enough. Take two weak solutions u^1, u^2 of (3.2), and test the difference of the weak formulation of (3.2) by $u^{12} := u^1 - u^2$. This gives

$$
\begin{aligned}
(3.5)\quad c\|u^{12}\|^2_{W_0^{1,\min(2,p)}(\Omega;\mathbb{R}^n)} &\leqslant \int_\Omega (\tau(e(\nabla u^1)) - \tau(e(\nabla u^2)) : e(\nabla u^{12})\,\mathrm{d}x \\
&= \int_\Omega ((u^2\cdot\nabla)u^2 - (u^1\cdot\nabla)u^1)\cdot u^{12}\,\mathrm{d}x \\
&= -\int_\Omega ((u^{12}\cdot\nabla)u^2)\cdot u^{12}\,\mathrm{d}x - \int_\Omega ((u^1\cdot\nabla)u^{12})\cdot u^{12}\,\mathrm{d}x \\
&\leqslant \|\nabla u^2\|_{L^p(\Omega;\mathbb{R}^{n\times n})}\|u^{12}\|^2_{L^{2p'}(\Omega;\mathbb{R}^n)} \leqslant R_0\|u^{12}\|^2_{L^{2p'}(\Omega;\mathbb{R}^n)}
\end{aligned}
$$

with $c = \zeta_2 c_{2,\Omega}$ (if $p \geqslant 2$) or $c = \zeta_0 c_{0,\Omega}(2R_0)$ (if $p < 2$). Then, if R is small enough so that, by (3.4), $R_0 < c(N_{2p'}^{1,\min(2,p)})^{-2}$, we get $u^{12} = 0$. This, together with (3.4), gives the bound in (3.3).

Having the uniqueness of u, we can conclude that even the whole sequence $\{u^k\}$ converges weakly to u. Let us prove the strong convergence: subtracting (3.2) with u and u^k, testing by $u^k - u$, and using Korn's inequality (2.2), we get

$$
\begin{aligned}
\varepsilon\|u^k - u\|^{\max(2,p)}_{W_0^{1,p}(\Omega;\mathbb{R}^n)} &\leqslant \int_\Omega (\tau(e(\nabla u^k)) - \tau(e(\nabla u)) : e(\nabla u^k - \nabla u)\,\mathrm{d}x \\
&= \int_\Omega ((u^k\cdot\nabla)u^k - (u\cdot\nabla)u)\cdot(u^k - u) + \alpha_0(\vartheta^k - \vartheta)g\cdot(u^k - u)\,\mathrm{d}x \\
&=: I_{1k} + I_{2k}
\end{aligned}
\tag{3.6}
$$

with $\varepsilon = \zeta_1 c_{1,\Omega}$ (if $p \geqslant 2$) or $\varepsilon = \zeta_0 c_{0,\Omega}(2R_0)$ (if $p < 2$). By using $\operatorname{div} u^k = 0 = \operatorname{div} u$ and Green's formula, we can calculate

$$
\begin{aligned}
I_{1k} &= \int_\Omega \sum_{j=1}^n \Big(\Big(\sum_{i=1}^n u_i^k \frac{\partial}{\partial x_i}\Big)u_j^k - \Big(\sum_{i=1}^n u_i \frac{\partial}{\partial x_i}\Big)u_j\Big)(u_j^k - u_j)\,\mathrm{d}x \\
&= \int_\Omega \sum_{i,j=1}^n \frac{\partial}{\partial x_i}(u_i^k u_j^k - u_i u_j)(u_j^k - u_j)\,\mathrm{d}x \\
&= -\int_\Omega \sum_{i,j=1}^n (u_i^k u_j^k - u_i u_j)\frac{\partial}{\partial x_i}(u_j^k - u_j)\,\mathrm{d}x \\
&= -\int_\Omega (u^k\otimes u^k - u\otimes u) : \nabla(u^k - u)\,\mathrm{d}x.
\end{aligned}
\tag{3.7}
$$

Due to the boundedness of $\nabla(u^k - u)$ in $L^p(\Omega;\mathbb{R}^{n\times n})$, the compact embedding $W^{1,p}(\Omega) \subset L^q(\Omega)$ with $p^{-1} + 2q^{-1} \leqslant 1$, and the continuity of the Nemytskiĭ mapping

$u \mapsto u \otimes u\colon\ L^q(\Omega;\mathbb{R}^n) \to L^{q/2}(\Omega;\mathbb{R}^{n\times n})$, we have $u^k \otimes u^k \to u \otimes u$ in $L^{q/2}(\Omega;\mathbb{R}^{n\times n})$, and eventually we get $I_{1k} \to 0$.

Also, the term I_{2k} converges to zero because $\vartheta^k \to \vartheta$ in $W^{-1,p'}(\Omega)$ and $u^k \rightharpoonup u$ in $W_0^{1,p}(\Omega;\mathbb{R}^n)$. $\square$

Furthermore, let us consider the Nemytskiĭ-type mapping $\mathcal{N}\colon\ W_0^{1,p}(\Omega;\mathbb{R}^n) \times L^{q'}(\Omega) \to \mathrm{rca}(\overline{\Omega})$ defined by

$$\mathcal{N}\colon\ (u,\vartheta) \mapsto h_1 = \alpha_1 \tau(e(\nabla u))\colon\ e(\nabla u) + \alpha_2 g \cdot u\vartheta + h, \tag{3.8}$$

and, for $u \in W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$, the linear operator

$$\mathcal{B}_u\colon\ (h_1, f) \mapsto \theta\colon\ \mathrm{rca}(\overline{\Omega}) \times \mathrm{rca}(\Gamma) \to W^{r,2}(\Omega) \tag{3.9}$$

with θ being the distributional solution to

$$u \cdot \nabla\theta - \kappa\Delta\theta = h_1 \ \text{ on } \Omega, \quad \kappa\frac{\partial\theta}{\partial\nu} + b\theta = f \ \text{ on } \Gamma, \tag{3.10}$$

i.e. $\theta \in W^{r,2}(\Omega)$ satisfies the identity

$$\int_\Omega (u \cdot \nabla v + \kappa\Delta v)\theta \,\mathrm{d}x + \int_{\overline{\Omega}} v h_1(\,\mathrm{d}x) + \int_\Gamma v f(\,\mathrm{d}S) = 0 \tag{3.11}$$

for any v smooth with $\kappa\frac{\partial}{\partial\nu}v + bv = 0$ on Γ.

Lemma 2. *Let* (2.1) *be valid. Then the mappings $\mathcal{N}$ and $\mathcal{B}_u$ are well defined and both $\mathcal{N}\colon\ W_0^{1,p}(\Omega;\mathbb{R}^n) \times L^{q'}(\Omega) \to \mathrm{rca}(\overline{\Omega})$ and $(u,h_1) \mapsto \mathcal{B}_u(h_1,f)\colon\ W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n) \times \mathrm{rca}(\overline{\Omega}) \to L^{q'}(\Omega)$ are (norm×weak*,weak*)-continuous.*

P r o o f. By the classical result about Nemytskiĭ mappings, $\mathcal{N}_0\colon\ (\xi,\vartheta) \mapsto \alpha_1\tau(e(\xi))\colon\ e(\xi) + \alpha_2 g \cdot u\vartheta\colon\ L^p(\Omega;\mathbb{R}^{n\times n}) \times L^{q'}(\Omega) \to L^1(\Omega)$ is continuous, so that $\mathcal{N} = (\mathcal{N}_0 \circ \nabla) + h$ is continuous, as claimed.

Let us consider the weak solution to the auxiliary linear problem

$$-u \cdot \nabla v - \kappa\Delta v = \xi \ \text{ on } \Omega, \quad \kappa\frac{\partial v}{\partial\nu} + bv = 0 \ \text{ on } \Gamma. \tag{3.12}$$

The existence of v can be proved by the standard energy method by testing (3.12) by v; note that

$$\int_\Omega (u \cdot \nabla v)v \,\mathrm{d}x = \frac{1}{2}\int_\Omega u \cdot \nabla v^2 \,\mathrm{d}x = -\frac{1}{2}\int_\Omega (\operatorname{div} u)v^2 \,\mathrm{d}x = 0 \tag{3.13}$$

so that we have the estimate $\|v\|_{W^{1,2}(\Omega)} \leqslant K_1\|\xi\|_{W^{1,2}(\Omega)^*}$ independent of u. Moreover, we have also the estimate

$$\begin{aligned}\int_\Omega (u\cdot\nabla v)\Delta v\,\mathrm{d}x &\leqslant \|u\|_{L^q(\Omega;\mathbb{R}^n)}\|\nabla v\|_{L^{2q/(q-2)}(\Omega;\mathbb{R}^n)}\|\Delta v\|_{L^2(\Omega)}\\ &\leqslant \|u\|_{L^q(\Omega;\mathbb{R}^n)}\|\nabla v\|^{\lambda}_{L^2(\Omega;\mathbb{R}^n)}\|\nabla v\|^{1-\lambda}_{L^6(\Omega;\mathbb{R}^n)}\|\Delta v\|_{L^2(\Omega)}\\ &\leqslant N_q^{1,p}\|\nabla u\|_{L^p(\Omega;\mathbb{R}^{n\times n})}K_1^{\lambda}(N_2^{1,2})^{\lambda}\|\xi\|^{\lambda}_{L^2(\Omega)}(K_0N_6^{1,2})^{1-\lambda}\|\Delta v\|^{2-\lambda}_{L^2(\Omega)}\end{aligned} \tag{3.14}$$

for $\lambda\in(0,1)$ such that $\lambda\frac{1}{2}+(1-\lambda)\frac{1}{6}=\frac{q-2}{2q}$ which certainly does exist for $p>3/2$, and where the constant K_0 comes from the standard Laplace-operator regularity $\|v\|_{W^{2,2}(\Omega)}\leqslant K_0\|\Delta v\|_{L^2(\Omega)}$ with the boundary condition $\kappa\frac{\partial v}{\partial\nu}+bv=0$ with $b\in C^{0,1}(\Gamma)$ on the $C^{3,1}$-domain Ω; see Nečas [15]. Then, multiplying (3.12) by Δv and integrating over Ω, we get the estimate

$$\begin{aligned}\kappa\int_\Omega|\Delta v|^2\,\mathrm{d}x = -\int_\Omega(\xi+u\cdot\nabla v)\Delta v\,\mathrm{d}x &\leqslant \|\xi\|_{L^2(\Omega)}\|\Delta v\|_{L^2(\Omega)}\\ &+\ N_q^{1,p}\|u\|_{W_0^{1,p}(\Omega;\mathbb{R}^n)}K_1^{\lambda}(N_2^{1,2})^{\lambda}\|\xi\|^{\lambda}_{L^2(\Omega)}(K_0N_6^{1,2})^{1-\lambda}\|\Delta v\|^{2-\lambda}_{L^2(\Omega)}.\end{aligned} \tag{3.15}$$

Thus we can see that, if $\xi\in L^2(\Omega)$, Δv is bounded in $L^2(\Omega)$. Then, using again the Laplace-operator regularity, we get $\|v\|_{W^{2,2}(\Omega)}\leqslant K_u\|\xi\|_{L^2(\Omega)}$ with $K_u>0$ depending on $\|u\|_{W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)}$ continuously and increasingly. It is important that this regularity estimate holds uniformly for u ranging over bounded sets in $W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$.

The interpolation between the linear mappings $\xi\mapsto v\colon W^{1,2}(\Omega)^*\to W^{1,2}(\Omega)$ and $L^2(\Omega)\to W^{2,2}(\Omega)$ gives a mapping $W^{r,2}(\Omega)^*\to W^{2-r,2}(\Omega)$ and an estimate $\|v\|_{W^{2-r,2}(\Omega)}\leqslant K_u^{1-r}K_1^r\|\xi\|_{W^{r,2}(\Omega)^*}$.

Let us rewrite the identity (3.11) into the form $\langle B_uv,\theta\rangle+\langle F,v\rangle=0$ where $B_u\colon W^{2-r,2}(\Omega)\to W^{r,2}(\Omega)^*$ and $F\in W^{2-r,2}(\Omega)^*$ are defined by

$$B_uv:=u\cdot\nabla v+\kappa\Delta v,\quad \langle F,v\rangle=\int_{\bar\Omega}v\,h_1(\,\mathrm{d}x)+\int_\Gamma v\,f(\,\mathrm{d}S), \tag{3.16}$$

respectively. Then $\theta=-(B_u^*)^{-1}F=-F\circ B_u^{-1}\in W^{r,2}(\Omega)^{**}\cong W^{r,2}(\Omega)$ is a solution to $\langle B_uv,\theta\rangle+\langle F,v\rangle=0$. Moreover, because of surjectivity of B_u, this solution must be unique. Also, we have the estimate $\|\theta\|_{W^{r,2}(\Omega)}\leqslant K_u^{1-r}K_1^r\|F\|_{W^{2-r,2}(\Omega)^*}$ independent of u.

Then we choose $0\leqslant r\leqslant 1$ so small that $W^{2-r,2}(\Omega)\subset C(\bar\Omega)$, i.e. $r<(4-n)/2$, cf. (2.4). This eventually gives the estimate

$$\|\theta\|_{W^{r,2}(\Omega)}\leqslant K_u^{1-r}K_1^r\|F\|_{W^{2-r,2}(\Omega)^*}\leqslant N_\infty^{2-r,2}K_u^{1-r}K_1^r\|(h_1,f)\|_{\mathrm{rca}(\bar\Omega)\times\mathrm{rca}(\Gamma)} \tag{3.17}$$

with $N_\infty^{2-r,2}$ the norm of the embedding $W^{2-r,2}(\Omega) \subset L^\infty(\Omega)$; note that $(h_1, f) \mapsto F\colon \operatorname{rca}(\bar\Omega) \times \operatorname{rca}(\Gamma) \to W^{2-r,2}(\Omega)^*$ defined by (3.16) is the adjoint mapping to $v \mapsto (v, v|_\Gamma)\colon W^{2-r,2}(\Omega) \to C(\bar\Omega) \times C(\Gamma)$.

To prove continuity of $(u, h_1) \mapsto \mathcal{B}_u(h_1, f)$, let us take $h_{1,k} \to h_1$ in $\operatorname{rca}(\bar\Omega)$ weakly* and $u^k \to u$ in $W^{1,p}(\Omega; \mathbb{R}^n)$, and denote by θ^k the distributional solution to (3.10) corresponding to u^k and $h_{1,k}$ in place of u and h_1, respectively. We showed that θ^k does exist and is bounded in $W^{r,2}(\Omega)$; realize that $\{\nabla u^k\}$ is bounded in $L^p(\Omega; \mathbb{R}^{n\times n})$. Then, by Banach-Alaoglu-Bourbaki theorem, we can assume that, possibly up to a subsequence,

$$\theta^k \to \theta \qquad \text{weakly in } W^{r,2}(\Omega). \tag{3.18}$$

Then we can make the limit passage in the integral identity (3.11), which reads here

$$\int_\Omega (u^k \cdot \nabla v + \kappa \Delta v)\theta^k \,\mathrm{d}x + \int_{\bar\Omega} v h_{1,k}(\,\mathrm{d}x) + \int_\Gamma v f(\,\mathrm{d}S) = 0. \tag{3.19}$$

Note that certainly the term $\theta^k u^k$ converges to θu (even strongly) because, as a consequence of (3.18), $\{\theta^k\}$ converges strongly in $W^{-1,p'}(\Omega)$ and $\{u^k\}$ also strongly in $W_0^{1,p}(\Omega; \mathbb{R}^n)$. Thus $\theta = \mathcal{B}_u(h, f)$ and even the whole sequence $\{\theta^k\}$ converges because of the already proved uniqueness of θ. □

Furthermore, for $\varrho > 0$, we denote the ball of the radius ϱ in $L^{q'}(\Omega)$ by

$$B_\varrho := \{\vartheta \in L^{q'}(\Omega);\ \ \|\vartheta\|_{L^{q'}(\Omega)} \leqslant \varrho\}. \tag{3.20}$$

Proposition 1. *Let* (2.1) *be fulfilled and let* $\|g\|_{L^\infty(\Omega;\mathbb{R}^n)}$ *be sufficiently small with respect to the other data* $\alpha_0, \alpha_1, \alpha_2$, $\|h\|_{\operatorname{rca}(\bar\Omega)}$ *and* $\|f\|_{\operatorname{rca}(\Gamma)}$. *Then* (1.1)–(1.2) *has at least one distributional solution* (u, θ).

P r o o f. We will investigate the mapping $\mathcal{C}\colon L^{q'}(\Omega) \to L^{q'}(\Omega)$ defined by

$$\mathcal{C}(\vartheta) := \mathcal{B}_{\mathcal{A}(\vartheta)}(\mathcal{N}(\mathcal{A}(\vartheta), \vartheta), f). \tag{3.21}$$

Note that any fixed point θ of $\mathcal{C}$ satisfies $\theta = \mathcal{B}_u(h, f)$ with $h = \mathcal{N}(u, \theta)$, where $u = \mathcal{A}(\theta)$, which just means that the pair (u, θ) is the distributional solution to (1.1)–(1.2). We will show that

$$B_\varrho \subset S_R \quad \text{and} \quad \mathcal{C}(B_\varrho) \subset B_\varrho \tag{3.22}$$

provided ϱ is chosen appropriately and g is small enough. Obviously, $(u, \theta) = (\mathcal{A}(\vartheta), \mathcal{C}(\vartheta))$ solves the decoupled system (3.2) and (3.10) with $u = \mathcal{A}(\vartheta)$ and $h_1 =$

$h_{u,\vartheta} = \mathcal{N}(u,\vartheta)$. Then, by testing (3.2) by u, we get the estimate (3.4) with the subscript k omitted.

Furthermore, using the identity $\int_\Omega \tau(e(\nabla u)) : e(\nabla u)\,\mathrm{d}x = \int_\Omega g(1-\alpha_0\vartheta)u\,\mathrm{d}x$ the source term $h_{u,\vartheta}$ in (3.10) can be estimated as

$$\begin{aligned}\|h_{u,\vartheta}\|_{\mathrm{rca}(\bar\Omega)} &\leqslant \alpha_1\|gu\|_{L^1(\Omega)} + |\alpha_0\alpha_1-\alpha_2|\|g\cdot u\vartheta\|_{L^1(\Omega)} + \|h\|_{\mathrm{rca}(\bar\Omega)} \\ &\leqslant (\alpha_1 N_1^{1,p} + |\alpha_0\alpha_1-\alpha_2|N_q^{1,p}\|\vartheta\|_{L^{q'}(\Omega)})\|g\|_{L^\infty(\Omega;\mathbb{R}^n)}\|u\|_{W^{1,p}(\Omega;\mathbb{R}^n)} \\ &\quad + \|h\|_{\mathrm{rca}(\bar\Omega)} \leqslant \gamma_1 + \gamma_2(\|g\|_{L^\infty(\Omega;\mathbb{R}^n)})\varrho^{p'},\end{aligned} \tag{3.23}$$

where we assume $\vartheta\in B_\varrho$ and take into account that $R_0 = \|g\|^{1/(p-1)}_{L^\infty(\Omega;\mathbb{R}^n)}\mathcal{O}(\|\vartheta\|^{1/(p-1)}_{L^{q'}(\Omega)})$, cf. (3.4); then $\gamma_1 = \gamma_1(\alpha_1, c, p, \|h\|_{\mathrm{rca}(\bar\Omega)})$ and $\gamma_2(\cdot)$ depends on α_0, α_2, p, and ζ_1 and moreover $\lim\limits_{a\to 0+}\gamma_2(a) = 0$.

The estimate (3.17) now reads

$$\|\mathcal{B}_u(h_{u,\vartheta}, f)\|_{W^{r,2}(\Omega)} \leqslant N_\infty^{2-r,2}K_u^{1-r}K_1^r(\|h_{u,\vartheta}\|_{\mathrm{rca}(\bar\Omega)} + \|f\|_{\mathrm{rca}(\Gamma)}).$$

Altogether,

$$\begin{aligned}\|\mathcal{C}(\vartheta)\|_{L^{q'}(\Omega)} &\leqslant N_{q'}^{r,2}\|\mathcal{C}(\vartheta)\|_{W^{r,2}(\Omega)} \\ &\leqslant N_{q'}^{r,2}N_\infty^{2-r,2}K_u^{1-r}K_1^r(\gamma_1 + \gamma_2(\|g\|_{L^\infty(\Omega;\mathbb{R}^n)})\varrho^{p'} + \|f\|_{\mathrm{rca}(\Gamma)}).\end{aligned} \tag{3.24}$$

If g is small, one can find $\varrho > N_{q'}^{r,2}N_\infty^{2-r,2}K_u^{1-r}K_1^r(\gamma_1 + \|f\|_{\mathrm{rca}(\Gamma)})$ small enough so that (3.24) implies $\|\mathcal{C}(\vartheta)\|_{L^{q'}(\Omega)} \leqslant \varrho$. In other words, we have proved $\mathcal{C}(B_\varrho)\subset B_\varrho$ for such ϱ. Moreover, if g is small enough, we have also $B_\varrho \subset S_R$.

We endow B_ϱ with the weak (or, if $q' = +\infty$, weak*) topology of $L^{q'}(\Omega)$, which makes B_ϱ compact (note that, due to (2.3), always $q' > 1$). By Lemmas 1 and 2 and by (3.22), $\mathcal{C}$ maps B_ϱ (weak,weak)-continuously into itself. Then, by Schauder's theorem, it has a fixed point θ on B_ϱ. □

R e m a r k 1. The interpolation/transposition method in Hilbert-space setting was thoroughly presented by Lions and Magenes [11]. Here, however, we did not assume infinitely smooth Γ or the coefficients u and b in (3.12) and, moreover, it was important to derive the estimate (3.17) uniformly for u from bounded sets in $W^{1,p}_{0,\mathrm{DIV}}(\Omega;\mathbb{R}^n)$.

R e m a r k 2. Under a quite restrictive assumption $p > 2n$, we can alternatively use a continuous imbedding of $\mathcal{W} := \{v\in W^{1,2}(\Omega);\ \Delta v\in L^{n/2+\varepsilon}(\Omega),\ \frac{\partial}{\partial\nu}v\in L^{n-1+\varepsilon}(\Gamma)\}$ with $\varepsilon > 0$ into $C^0(\bar\Omega)$, proved by Alibert and Raymond [1] even for Lipschitz domains. Indeed, for $u\in W^{1,p}(\Omega;\mathbb{R}^n)\subset L^q(\Omega;\mathbb{R}^n)$ with q satisfying (2.3) and for $v\in W^{1,2}(\Omega)$, we have $u\cdot\nabla v\in L^{n/2+\varepsilon}(\Omega)$, which enables us to get the auxiliary mapping $\xi \mapsto v\colon L^{n/2+\varepsilon}(\Omega) \to \mathcal{W}$ in the proof of Lemma 1. Then $B_u\colon \mathcal{W} \to L^{n/2+\varepsilon}(\Omega)$ and all above considerations work equally for θ in $L^{n/(n-2)-\varepsilon}(\Omega)$ instead of $W^{r,2}(\Omega)$. Beside the $C^{0,1}$-domain Ω, this modification enables us also to consider b from $L^{4/3+\varepsilon}(\Gamma)$ (if $n=2$) or from $L^{6+\varepsilon}(\Gamma)$ (if $n=3$) because then $bv \in L^{n-1+\varepsilon}(\Gamma)$ for any $v \in W^{1,2}(\Omega)$.

Remark 3. Contrary to the evolution case (cf. [16]), if $\alpha_2 > 0$, it does not seem possible to prove $\theta \geqslant 0$ for some solution obtained in Proposition 1 even if one assumes $h \geqslant 0$ and $f \geqslant 0$. Yet, negative temperature need not be interpreted as non-physical solution because θ is a "small" deviation from some constant reference temperature rather than the absolute temperature. Nevertheless, this holds true if the adiabatic effect can be neglected, i.e. $\alpha_2 = 0$. Then, assuming $h \geqslant 0$ and $f \geqslant 0$ and regularizing (1.1c) by a term $\varepsilon\theta$ on the left-hand side, we can prove existence of the "regularized" solution $(u_\varepsilon, \theta_\varepsilon)$ again by Proposition 1 with all estimates independent of $\varepsilon > 0$ and then nonnegativity $\theta_\varepsilon \geqslant 0$ by testing $\varepsilon\theta_\varepsilon + u_\varepsilon \cdot \nabla\theta_\varepsilon - \kappa\Delta\theta_\varepsilon = h_1 \geqslant 0$ by $\operatorname{signum}(\theta_\varepsilon) - 1$ or, more rigorously, by a regularization of this test function. Then, passing with $\varepsilon \to 0$, one gets $\theta \geqslant 0$.

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Authors' address: *Tomáš Roubíček*, Mathematical Institute, Charles University, Sokolovská 83, CZ-186 75 Praha 8, e-mail: `roubicek@karlin.mff.cuni.cz`, and Institute of Information Theory and Automation, Academy of Sciences, Pod vodárenskou věží 4, CZ-182 08 Praha 8, Czech Republic.