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SOME NEW SCALES OF WEIGHT CHARACTERIZATIONS OF THE CLASS B_p

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ABSTRACT. We present an equivalence theorem, which includes all known characterizations of the class B_p, i.e., the weight class of Ariño and Muckenhoupt, and also some new equivalent characterizations. We also give equivalent characterizations for the classes B_p^*, B_∞^* and RB_p, and prove and apply a "gluing lemma" of independent interest.

1. INTRODUCTION

In their paper [2] M. Ariño and B. Muckenhoupt characterized the class of weights B_p, $1 < p < \infty$, such that the Hardy operator is bounded on $L^p(w)$ for non-negative and non-increasing functions. Such results are of interest because they can be used to characterize the mapping properties of the maximal operator M between weighted Lorentz $\Lambda_p(w)$ - spaces. According to the Ariño - Muckenhoupt result the weight w belongs to the class B_p, if and only if

$$B_{AM}(p) := \sup_{0<t<\infty} \frac{t^p}{W(t)} \int_t^\infty s^{-p} w(s)\, ds < \infty. \tag{AM}$$

(For the case $0 < p \le 1$, the class B_p was defined in [3]). Here and in the sequel $W(t) := \int_0^t w(s)\, ds$, and we assume that $W(x) < \infty$ for every $x \in (0, \infty)$.

In [12], [4], [13] and [5] the authors gave other characterizations of this condition: w belongs to the class B_p, $1 < p < \infty$, if and only if any of the following expressions is finite:

$$B_{Sa1}(p) = \sup_{0<t<\infty} \left(\int_t^\infty s^{-p} w(s)\, ds \right)^{\frac{1}{p}} \left(\int_0^t s^{p'} W(s)^{-p'} w(s)\, ds \right)^{\frac{1}{p'}}; \tag{Sa1}$$

$$B_{Sa2}(p) = \sup_{0<t<\infty} \frac{W(t)^{\frac{1}{p}}}{t} \left(\int_0^t \left(\frac{W(s)}{s} \right)^{1-p'} ds \right)^{\frac{1}{p'}}; \tag{Sa2}$$

$$B_{CS}(p) = \sup_{0<t<\infty} \frac{W(t)^{\frac{1}{p}}}{t} \left(\int_0^t \left(\frac{W(s)}{s} \right)^{-p'} w(s)\, ds \right)^{\frac{1}{p'}}; \tag{CS}$$

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$$B_{So1}(p) = \sup_{0<t<\infty} \frac{W(t)}{t^p} \int_0^t \frac{s^{p-1}}{W(s)}\, ds; \tag{So1}$$

$$B_{So2} = \sup_{0<t<\infty} \frac{t^p}{W(t)} \int_t^\infty \frac{W(s)}{s^{p+1}}\, ds; \tag{So2}$$

$$B_{So3}(p) = \sup_{0<t<\infty} \frac{W(t)^{\frac{1}{p}}}{t} \int_0^t \frac{ds}{W(s)^{\frac{1}{p}}}; \tag{So3}$$

$$B_{So4}(p) = \sup_{0<t<\infty} \frac{t}{W(t)^{\frac{1}{p}}} \int_t^\infty \frac{W(s)^{\frac{1}{p}}}{s^2}\, ds; \tag{So4}$$

$$B_{CM}(p) = \sup_{0<t<\infty} \left(\int_t^\infty s^{-p-1} W(s)\, ds \right)^{\frac{1}{p}} \left(\int_0^t s^{p'-1} W(s)^{1-p'}\, ds \right)^{\frac{1}{p'}}. \tag{CM}$$

In Section 2, we formulate a theorem from [6], on which our results are based, and prove a "gluing lemma" (Lemma 2.2) of independent interest. In Section 3 we prove an equivalence theorem, which includes all results mentioned above. In fact, our Theorem 3.1 shows that there are six scales of weight characterizations of the class B_p. The proof is elementary and mostly based on our result from paper [6]. In Section 4 we give some new characterizations of the classes B_p^*, B_∞^* and RB_p, generalize a result of Y. Sagher [11] (Proposition 4.4) and apply Lemma 2.2 to give a new proof of a result of Andersen [1] (Proposition 4.6).

2. Preliminaries. The "gluing lemma"

In [6] the equivalence of four scales of integral conditions was proved. These conditions characterize the Hardy inequality and contain the usual Muckenhoupt condition as a special case. The proof was carried out by first proving the following equivalence theorem of independent interest that also will be applied in this paper (see Theorem 2.1 in [6]):

Theorem 2.1. *For $-\infty \le a < b \le \infty$, α, β and s positive numbers and f, g measurable functions positive a.e. in (a,b), denote*

$$F(x) := \int_x^b f(t)dt, \qquad G(x) := \int_a^x g(t)dt \tag{2.1}$$

and

$$
(2.2)\qquad
\begin{aligned}
B_1(x;\alpha,\beta) &:= F^{\alpha}(x)G^{\beta}(x);\\
B_2(x;\alpha,\beta,s) &:= \left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)dt\right)^{\alpha} G^{s}(x);\\
B_3(x;\alpha,\beta,s) &:= \left(\int_a^x g(t)F^{\frac{\alpha-s}{\beta}}(t)dt\right)^{\beta} F^{s}(x);\\
B_4(x;\alpha,\beta,s) &:= \left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)dt\right)^{\alpha} G^{-s}(x);\\
B_5(x;\alpha,\beta,s) &:= \left(\int_x^b g(t)F^{\frac{\alpha+s}{\beta}}(t)dt\right)^{\beta} F^{-s}(x).
\end{aligned}
$$

The numbers $B_1 := \sup\limits_{a<x<b} B_1(x;\alpha,\beta)$ *and* $B_i(s) = \sup\limits_{a<x<b} B_i(x;\alpha,\beta,s)$ $(i = 2,3,4,5)$ *are mutually equivalent. The constants in the equivalence relations can depend on* α,β *and* s.

For this paper we also need the following "gluing lemma":

Lemma 2.2. *Let* γ, α *and* β *be positive numbers and let* f *and* g *be positive measurable functions on* $(0,\infty)$. *The following two estimates*

$$
A_1 := \sup_{0<t<\infty}\left(\int_0^t g(s)\,ds\right)^{\beta}\left(\int_t^{\infty} s^{-\gamma\beta}f(s)\,ds\right)^{\alpha}<\infty \tag{2.3}
$$

and

$$
A_2 := \sup_{0<t<\infty}\left(\int_t^{\infty} s^{-\gamma\alpha}g(s)\,ds\right)^{\beta}\left(\int_0^t f(s)\,ds\right)^{\alpha}<\infty \tag{2.4}
$$

hold if and only if the (glued-up) condition

$$
\begin{aligned}
A_3 := \sup_{0<t<\infty}&\left(\int_0^t g(s)\,ds + t^{\gamma\alpha}\int_t^{\infty} s^{-\gamma\alpha}g(s)\,ds\right)^{\beta}\times\\
&\times\left(t^{-\gamma\beta}\int_0^t f(s)\,ds + \int_t^{\infty} s^{-\gamma\beta}f(s)\,ds\right)^{\alpha}<\infty
\end{aligned} \tag{2.5}
$$

holds.

Proof. The implication (2.5) $\Rightarrow$ (2.3)&(2.4) is clear. Let us now prove the reverse implication. Suppose that (2.3) and (2.4) hold. It is enough to show that

$$
I_1 = \sup_{0<t<\infty} t^{-\gamma\beta\alpha}\left(\int_0^t g(s)\,ds\right)^{\beta}\left(\int_0^t f(s)\,ds\right)^{\alpha}<\infty, \tag{2.6}
$$

and

$$
I_2 = \sup_{0<t<\infty} t^{\gamma\beta\alpha}\left(\int_t^{\infty} s^{-\gamma\alpha}g(s)\,ds\right)^{\beta}\left(\int_t^{\infty} s^{-\gamma\beta}f(s)\,ds\right)^{\alpha}<\infty. \tag{2.7}
$$

Let us fix $t \in (0,\infty)$ and define the point $y(t) \in (0,t)$ so that

$$\int_0^{y(t)} f(s)\,ds = \int_{y(t)}^t f(s)\,ds. \tag{2.8}$$

Using (2.8) we obtain that

$$\begin{aligned}
I_1 &\le 2^{\max(\beta,1)-1}\left[t^{-\gamma\beta\alpha}\left(\int_{y(t)}^t g(s)\,ds\right)^\beta\left(\int_0^t f(s)\,ds\right)^\alpha\right.\\
&\qquad\left.+\,t^{-\gamma\beta\alpha}\left(\int_0^{y(t)} g(s)\,ds\right)^\beta\left(\int_0^t f(s)\,ds\right)^\alpha\right]\\
&\le 2^{\max(\beta,1)+\alpha-1}\left[\left(\int_{y(t)}^t s^{-\gamma\alpha}g(s)\,ds\right)^\beta\left(\int_0^{y(t)} f(s)\,ds\right)^\alpha\right.\\
&\qquad\left.+\,t^{-\gamma\beta\alpha}\left(\int_0^{y(t)} g(s)\,ds\right)^\beta\left(\int_{y(t)}^t f(s)\,ds\right)^\alpha\right]\\
&\le 2^{\max(\beta,1)+\alpha-1}\left[\left(\int_{y(t)}^\infty s^{-\gamma\alpha}g(s)\,ds\right)^\beta\left(\int_0^{y(t)} f(s)\,ds\right)^\alpha\right.\\
&\qquad\left.+\left(\int_0^{y(t)} g(s)\,ds\right)^\beta\left(\int_{y(t)}^t s^{-\gamma\beta}f(s)\,ds\right)^\alpha\right]\\
&\le 2^{\max(\beta,1)+\alpha-1}\left[\left(\int_{y(t)}^\infty s^{-\gamma\alpha}g(s)\,ds\right)^\beta\left(\int_0^{y(t)} f(s)\,ds\right)^\alpha\right.\\
&\qquad\left.+\left(\int_0^{y(t)} g(s)\,ds\right)^\beta\left(\int_{y(t)}^\infty s^{-\gamma\beta}f(s)\,ds\right)^\alpha\right]\\
&\le 2^{\max(\beta,1)+\alpha-1}(A_1+A_2)\\
&< \infty.
\end{aligned}$$

Similarly we can show that

$$I_2 \le 2^{\max(\beta,1)+\alpha-1}(A_1+A_2) < \infty.$$

Using the estimate

$$A_3 \le 2^{\max(\beta,1)+\max(\alpha,1)-2}\,(A_1+A_2+I_1+I_2)$$

we obtain that $A_3 < \infty$. The proof is complete. □

Remark 2.3. In the proof of Lemma 2.2 in fact we have shown that

$$A_3 \approx A_1 + A_2.$$

For α, $\beta < 1$ and $\beta \leq 1-\alpha$, Lemma 2.2 was proved by Andersen in [1] using the Hardy inequality and the Stieltjes transformation. Our proof is direct and allows us to consider all parameters.

3. The Equivalence Theorem

Our main result in this section reads:

Theorem 3.1. *Let p, ε, α and β be positive numbers, and denote*

$$B_1(p,\varepsilon,t) := \left(\frac{t^p}{W(t)}\right)^{\varepsilon} \int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1-\varepsilon} s^{-p} w(s)\, ds;$$

$$B_2(p,\varepsilon,t) := \left(\frac{t^p}{W(t)}\right)^{\varepsilon} \int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1-\varepsilon} s^{-p-1} W(s)\, ds;$$

$$B_3(p,\varepsilon,t) := \left(\frac{t^p}{W(t)}\right)^{-\varepsilon} \int_0^t \left(\frac{s^p}{W(s)}\right)^{1+\varepsilon} s^{-p} w(s)\, ds;$$

$$B_4(p,\varepsilon,t) := \left(\frac{t^p}{W(t)}\right)^{-\varepsilon} \int_0^t \left(\frac{s^p}{W(s)}\right)^{1+\varepsilon} s^{-p-1} W(s)\, ds;$$

$$B_5(p,\alpha,\beta,t) := \left(\int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1-\alpha} s^{-p} w(s)\, ds\right)^{\beta} \left(\int_0^t \left(\frac{s^p}{W(s)}\right)^{1+\beta} s^{-p} w(s)\, ds\right)^{\alpha};$$

$$B_6(p,\alpha,\beta,t) := \left(\int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1-\alpha} s^{-p-1} W(s)\, ds\right)^{\beta} \left(\int_0^t \left(\frac{s^p}{W(s)}\right)^{1+\beta} s^{-p-1} W(s)\, ds\right)^{\alpha}.$$

Then the weight w belongs to the class B_p if and only if any of the numbers $B_i(p,\varepsilon) := \sup_{0<t<\infty} B_i(p,\varepsilon,t)$ $(i = 1,2,3,4)$ and $B_i(p,\alpha,\beta) := \sup_{0<t<\infty} B_i(p,\alpha,\beta,t)$ $(i = 5,6)$ is finite.

Remark 3.2. Let us point out that Theorem 3.1 contains all results mentioned in the introduction since

$$B_{AM}(p) = B_1(p,1);$$

$$B_{Sa1}(p) = B_5(p,1,p'-1)^{\frac{1}{p'}};$$

$$B_{Sa2}(p) = B_4(p,p'-1)^{\frac{1}{p'}};$$

$$B_{CS}(p) = B_3(p,p'-1)^{\frac{1}{p'}};$$

$$B_{So1}(p) = B_4(p,1);$$

$$B_{So2}(p) = B_2(p,1);$$

$$B_{So3}(p) = B_4(p,\frac{1}{p});$$

$$B_{So4}(p) = B_2(p,\frac{1}{p});$$

$$B_{CM}(p) = B_6(p, 1, p' - 1)^{\frac{1}{p'}}.$$

Proof of Theorem 3.1. It is easy to see that

$$B_2(p, \varepsilon, t) = \frac{1}{p} B_1(p, \varepsilon, t) + \frac{1}{p\varepsilon}; \tag{3.1}$$

$$B_4(p, \varepsilon, t) = \frac{1}{p} B_3(p, \varepsilon, t) + \frac{1}{p\varepsilon}; \tag{3.2}$$

$$B_5(p, \alpha, \beta, t) = B_1(p, \alpha, t)^{\beta} B_3(p, \beta, t)^{\alpha}; \tag{3.3}$$

$$B_6(p, \alpha, \beta, t) = B_2(p, \alpha, t)^{\beta} B_4(p, \beta, t)^{\alpha}. \tag{3.4}$$

Using (3.1), (3.2) and (3.3) we obtain from (3.4) that

$$B_6(p, \alpha, \beta, t) \approx B_5(p, \alpha, \beta, t) + B_1(p, \alpha, t)^{\beta} + B_3(p, \beta, t)^{\alpha} + 1. \tag{3.5}$$

Therefore, we have the following equivalences:

$$B_1(p, \varepsilon) < \infty \Leftrightarrow B_2(p, \varepsilon) < \infty;$$
$$B_3(p, \varepsilon) < \infty \Leftrightarrow B_4(p, \varepsilon) < \infty.$$

If $B_i(p, \varepsilon) < \infty$ for some ε, $i = 2, 4$ it is not difficult to see that the function $\frac{t^p}{W(t)}$ is equivalent to increasing function. Now assume, that $B_i(p, \varepsilon) < \infty$ with $i = 1, 3$. Then we have also that $B_i(p, \varepsilon) < \infty$, $i = 2, 4$, and, hence, the function $\frac{t^p}{W(t)}$ is equivalent to increasing function. Using Theorem 2.1 we obtain that $B_{AM} \approx B_i(p, \varepsilon)$, $i = 1, 3$, and $B_{So1} \approx B_i(p, \varepsilon)$, $i = 2, 4$. Since $B_1(p, 1) = B_{AM}$, we have proved the following equivalence:

$$B_i(p, \varepsilon) < \infty, \quad \text{for some} \quad \varepsilon \Leftrightarrow B_{AM} < \infty, \quad i = 1, 2, 3, 4.$$

According to (3.3) and (3.4) we have the following implication:

$$B_{AM} < \infty \Rightarrow B_i(p, \alpha, \beta) < \infty, \quad i = 5, 6.$$

Moreover, by (3.5) we have that

$$B_6(p, \alpha, \beta) < \infty \Rightarrow B_{AM} < \infty.$$

To finish the proof we need to prove the implication

$$B_5(p, \alpha, \beta) < \infty \Rightarrow B_{AM} < \infty.$$

It is sufficient to prove that

$$B_5(p, \alpha, \beta) < \infty \Rightarrow B_6(p, \alpha, \beta) < \infty.$$

Let $B_5(p, \alpha, \beta) < \infty$. Then

$$\left(\int_0^t W(s)^{\alpha - 1} w(s)\, ds \right)^{\beta} \left(\int_t^{\infty} W(s)^{-1-\beta} w(s)\, ds \right)^{\alpha} \le \alpha^{-\beta} \beta^{-\alpha} < \infty,$$

and, by applying Lemma 2.2 with the functions $g(s) = W(s)^{\alpha-1}w(s)$ and $f(s) = W(s)^{-1-\beta}w(s)$, we obtain that

$$\sup_{0<t<\infty}\left(\int_0^t W(s)^{\alpha-1}w(s)\,ds + t^{p\alpha}\int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1-\alpha} s^{-p}w(s)\,ds\right)^\beta \times$$
$$\times\left(t^{-p\beta}\int_0^t \left(\frac{s^p}{W(s)}\right)^{1+\beta} s^{-p}w(s)\,ds + \int_t^\infty W(s)^{-1-\beta}w(s)\,ds\right)^\alpha < \infty.$$

But this is precisely the estimate $B_6(p,\alpha,\beta) < \infty$. The proof is complete.

4. Further results

The technique we have developed in this paper can be used in many other cases. Here we just give some examples.

Characterization of the class B_p^*.

The weight w belongs to the class B_p^* (introduced for $p \geq 1$ by Neugebauer in [9]) if and only if

$$B_p^* := \sup_{0<t<\infty}\frac{t^p}{W(t)}\int_0^t s^{-p}w(s)\,ds < \infty.$$

The following result is analogous to Theorem 3.1 and shows that also the class B_p^* in fact can be characterized by infinitely many conditions, namely by six scales of equivalent conditions.

Theorem 4.1. *Let p, ε, α and β be positive numbers, and denote*

$$B_1^*(p,\varepsilon,t) := \left(\frac{t^p}{W(t)}\right)^{-\varepsilon}\int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1+\varepsilon} s^{-p}w(s)\,ds;$$
$$B_2^*(p,\varepsilon,t) := \left(\frac{t^p}{W(t)}\right)^{-\varepsilon}\int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1+\varepsilon} s^{-p-1}W(s)\,ds;$$
$$B_3^*(p,\varepsilon,t) := \left(\frac{t^p}{W(t)}\right)^{\varepsilon}\int_0^t \left(\frac{s^p}{W(s)}\right)^{1-\varepsilon} s^{-p}w(s)\,ds;$$
$$B_4^*(p,\varepsilon,t) := \left(\frac{t^p}{W(t)}\right)^{\varepsilon}\int_0^t \left(\frac{s^p}{W(s)}\right)^{1-\varepsilon} s^{-p-1}W(s)\,ds;$$
$$B_5^*(p,\alpha,\beta,t) := \left(\int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1+\alpha} s^{-p}w(s)\,ds\right)^\beta \left(\int_0^t \left(\frac{s^p}{W(s)}\right)^{1-\beta} s^{-p}w(s)\,ds\right)^\alpha;$$
$$B_6^*(p,\alpha,\beta,t) := \left(\int_t^\infty \left(\frac{s^p}{W(s)}\right)^{1+\alpha} s^{-p-1}W(s)\,ds\right)^\beta \left(\int_0^t \left(\frac{s^p}{W(s)}\right)^{1-\beta} s^{-p-1}W(s)\,ds\right)^\alpha.$$

Then the weight w belongs to the class B_p^* if and only if any of the numbers $B_i^*(p,\varepsilon) := \sup_{0<t<\infty} B_i^*(p,\varepsilon,t)$ $(i=1,2,3,4)$ and $B_i^*(p,\alpha,\beta) := \sup_{0<t<\infty} B_i^*(p,\alpha,\beta,t)$ $(i=5,6)$ is finite.

Characterization of the class B_∞^*.

The weight w belongs to the class B_∞^* (see again [9]) if and only if

$$B_\infty^* := \sup_{0<t<\infty} \frac{1}{W(t)} \int_0^t s^{-1} W(s)\, ds < \infty.$$

The results of Theorem 4.1 are satisfied also for $p=0$ and since $B_\infty^* = B_3(0,1)$, we obtain the following scales of characterizations of the class B_∞^*:

Theorem 4.2. *Let ε, α, β and t be positive numbers, and denote*

$$B_2^*(0,\varepsilon,t) := W^\varepsilon(t) \int_t^\infty s^{-1} W(s)^{-\varepsilon}\, ds;$$

$$B_4^*(0,\varepsilon,t) := W(t)^{-\varepsilon} \int_0^t s^{-1} W(s)^{\varepsilon}\, ds;$$

$$B_6^*(0,\alpha,\beta,t) := \left(\int_t^\infty s^{-1} W(s)^{-\alpha}\, ds\right)^\beta \left(\int_0^t s^{-1} W(s)^{\beta}\, ds\right)^\alpha.$$

Then the weight w belongs to the class B_∞^* if and only if any of the numbers $B_i^*(0,\varepsilon) := \sup_{0<t<\infty} B_i^*(0,\varepsilon,t)$ $(i=2,4)$ and $B_6^*(0,\alpha,\beta) := \sup_{0<t<\infty} B_i^*(0,\alpha,\beta,t)$ is finite.

Characterization of the class RB_p.

The weight w belongs to the class RB_p (the so called reverse B_p-class introduced by Neugebauer in [8]) if and only if

$$RB_p := \sup_{0<t<\infty} \frac{W(t)}{t^p \int_t^\infty s^{-p} w(s)\, ds} < \infty.$$

By using the estimate

$$\frac{W(t) + t^p \int_t^\infty s^{-p} w(s)\, ds}{t^p \int_t^\infty s^{-p} w(s)\, ds} \le \frac{W(t)}{t^p \int_t^\infty s^{-p} w(s)\, ds} + 1$$

we obtain that if $RB_p < \infty$, then the function $t^p \int_t^\infty s^{-p} w(s)\, ds$ is non-decreasing, and analogously as in Theorem 3.1 we get the following scales of new characterizations for the classes RB_p:

Theorem 4.3. *Let p, ε, α, β and t be positive numbers, and denote*

$$RB_1(p,\varepsilon,t) := \left(t^p \int_t^\infty s^{-p} w(s)\, ds\right)^{-\varepsilon} \int_0^t \left(s^p \int_s^\infty x^{-p} w(x)\, dx\right)^{-1+\varepsilon} w(s)\, ds;$$

$$RB_2(p,\varepsilon,t) := \left(t^p \int_t^\infty s^{-p} w(s)\,ds\right)^{-\varepsilon} \int_0^t s^{p\varepsilon-1} \left(\int_s^\infty x^{-p} w(x)\,dx\right)^{\varepsilon} ds;$$

$$RB_3(p,\varepsilon,t) := \left(t^p \int_t^\infty s^{-p} w(s)\,ds\right)^{\varepsilon} \int_t^\infty \left(s^p \int_s^\infty x^{-p} w(x)\,dx\right)^{-1-\varepsilon} w(s)\,ds;$$

$$RB_4(p,\varepsilon,t) := \left(t^p \int_t^\infty s^{-p} w(s)\,ds\right)^{\varepsilon} \int_t^\infty s^{-p\varepsilon-1} \left(\int_s^\infty x^{-p} w(x)\,dx\right)^{-\varepsilon} ds;$$

$$RB_5(p,\alpha,\beta,t) := \left(\int_0^t \left(s^p \int_s^\infty x^{-p} w(x)\,dx\right)^{-1+\alpha} w(s)\,ds\right)^{\beta} \times$$
$$\times \left(\int_t^\infty \left(s^p \int_s^\infty x^{-p} w(x)\,dx\right)^{-1-\beta} w(s)\,ds\right)^{\alpha};$$

$$RB_6(p,\alpha,\beta,t) := \left(\int_0^t s^{p\alpha-1} \left(\int_s^\infty x^{-p} w(x)\,dx\right)^{\alpha} ds\right)^{\beta} \times$$
$$\times \left(\int_t^\infty s^{-p\beta-1} \left(\int_s^\infty x^{-p} w(x)\,dx\right)^{-\beta} ds\right)^{\alpha}.$$

Then the weight w belongs to the class RB_p if and only if any of the numbers $RB_i(p,\varepsilon) := \sup_{0<t<\infty} B_i(p,\varepsilon,t)$ $(i = 1,2,3,4)$ and $RB_i(p,\alpha,\beta) := \sup_{0<t<\infty} B_i(p,\alpha,\beta,t)$ $(i = 5,6)$ is finite.

We also note that by using Theorem 2.1 we obtain the following generalization of a result of Y. Sagher [11]:

Proposition 4.4. *Let $m(t)$ and $h(t)$ be positive functions and ε be a positive number. Then*

$$\int_0^r h(s)\,ds \approx m(r) \tag{4.1}$$

if and only if

$$\int_0^r m(s)^{-1+\varepsilon} h(s)\,ds \approx m(r)^{\varepsilon} \tag{4.2}$$

or

$$\int_r^\infty m(s)^{-1-\varepsilon} h(s)\,ds \approx m(r)^{-\varepsilon} \tag{4.3}$$

In [11], the equivalence of (4.1) and (4.3) for $h(s) = m(s)/s$ and $\varepsilon = 1$ was proved.

Remark 4.5. Lemma 2.2 has been used in a crucial way in the proof of Theorem 3.1. Moreover it is obvious that this lemma can be used in a number of similar situations. We finish this paper by illustrating this fact by a new proof of a result of Andersen [1].

Let S_λ be the (generalized) Stieltjes transformation, i.e.

$$S_\lambda(f)(x) := \int_0^\infty \frac{f(y)\,dy}{(x+y)^\lambda}.$$

Proposition 4.6. *Let be $\lambda \geq 0$, $1 \leq p \leq q \leq \infty$, and suppose that $U(x)$ and $V(x)$ are non-negative extended real valued functions defined on $(0,\infty)$. Then there exists a constant C independent of f such that*

$$\left(\int_0^\infty |S_\lambda(f)(x)|^q U(x)\,dx\right)^{1/q} \leq C\left(\int_0^\infty |f(x)|^p V(x)\,dx\right)^{1/p} \tag{4.4}$$

if and only if

$$K = \sup_{r>0} r^\lambda \left(\int_0^\infty \frac{U(x)}{(x+r)^{\lambda q}}\,dx\right)^{1/q}\left(\int_0^\infty \frac{V(x)^{-1/(p-1)}}{(x+r)^{\lambda p'}}\,dx\right)^{1/p'} < \infty. \tag{4.5}$$

Moreover, the smallest constant C in (4.4) satisfies $C \approx K$.

Proof. We have that

$$S_\lambda(f)(x) \approx \frac{1}{x^\lambda}\int_0^x f(y)\,dy + \int_x^\infty \frac{f(y)\,dy}{y^\lambda}$$

for all non-negative functions f. Therefore, inequality (4.4) holds if and only if the following two inequalities hold:

$$\left(\int_0^\infty \left(\frac{1}{x^\lambda}\int_0^x f(y)\,dy\right)^q U(x)\,dx\right)^{1/q} \leq C\left(\int_0^\infty (f(x))^p V(x)\,dx\right)^{1/p} \tag{4.6}$$

and

$$\left(\int_0^\infty \left(\int_x^\infty \frac{f(y)\,dy}{y^\lambda}\right)^q U(x)\,dx\right)^{1/q} \leq C\left(\int_0^\infty (f(x))^p V(x)\,dx\right)^{1/p}. \tag{4.7}$$

According to well-known results about the Hardy inequality (see e.g. [7], [10]), inequalities (4.6) and (4.7) are equivalent, respectively, to the following conditions:

$$\sup_{r>0}\left(\int_r^\infty \frac{U(x)}{x^{\lambda q}}\,dx\right)^{1/q}\left(\int_0^r V(x)^{-1/(p-1)}\,dx\right)^{1/p'} < \infty,$$

$$\sup_{r>0}\left(\int_0^r U(x)\,dx\right)^{1/q}\left(\int_r^\infty \frac{V(x)^{-1/(p-1)}}{x^{\lambda p'}}\,dx\right)^{1/p'} < \infty.$$

Finally using Lemma 2.2 with $g = U$, $f = V^{-1/(p-1)}$, $\beta = \frac{1}{q}$, $\alpha = \frac{1}{p'}$ and $\gamma = \lambda q p'$ we obtain that (4.6) and (4.7) are equivalent to (4.5) and therefore we obtain that (4.4) and (4.5) are equivalent. The proof is complete. ∎

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