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GLOBAL BIFURCATION FOR A REACTION-DIFFUSION SYSTEM WITH INCLUSIONS

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Abstract. We consider a reaction-diffusion system exhibiting diffusion driven instability if supplemented by Dirichlet-Neumann boundary conditions. We impose unilateral conditions given by inclusions on this system and prove that global bifurcation of spatially non-homogeneous stationary solutions occurs in the domain of parameters where bifurcation is excluded for the original mixed boundary value problem. Inclusions can be considered in one of the equations itself as well as in boundary conditions. The proof is based on the degree theory for multivalued mappings (jump of the degree implies bifurcation). We show how the degree for a class of multivalued maps including those corresponding to a weak formulation of our problem can be calculated.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary, and let measurable (possibly empty) subsets $\Omega_0 \subseteq \Omega$ and $\Gamma_0, \Gamma \subseteq \partial\Omega$ be fixed with $\operatorname{mes}(\Gamma_0 \cap \Gamma) = 0$. We will always assume that

$$\operatorname{mes}\Gamma_0 > 0,$$
$$\operatorname{mes}\Omega_0 > 0 \text{ or } \operatorname{mes}\Gamma > 0 \text{ (or both) and } \overline{\Omega}_0 \cap \overline{\Gamma} = \varnothing.$$

We are interested in stationary solutions of the reaction-diffusion system

$$\begin{aligned} &u_t = d_1\Delta u + b_{11}u + b_{12}v + f_1(d_1, d_2, x, u, v, \nabla u, \nabla v) \qquad \text{on } \Omega,\\ &v_t \in d_2\Delta v + b_{21}u + b_{22}v + f_2(d_1, d_2, x, u, v, \nabla u, \nabla v) + \begin{cases} \{0\} & \text{on } \Omega \setminus \Omega_0,\\ \omega_0(d_1, d_2, x, u, v, \nabla u, \nabla v) & \text{on } \Omega_0, \end{cases} \end{aligned} \tag{1.1}$$

with the boundary conditions

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0,\\ \frac{\partial u}{\partial n} = f_3(d_1, d_2, x, u, v) & \text{on } \partial\Omega \setminus \Gamma_0,\\ \frac{\partial v}{\partial n} = f_4(d_1, d_2, x, u, v) & \text{on } \partial\Omega \setminus (\Gamma_0 \cup \Gamma),\\ \frac{\partial v}{\partial n} \in f_4(d_1, d_2, x, u, v) + \omega_1(d_1, d_2, x, u, v) & \text{on } \Gamma. \end{cases} \tag{1.2}$$

Here $d_1, d_2 > 0$ are bifurcation parameters, the nonlinearities f_i are small at $(u, v) = 0$, and ω_i are multivalued functions specified later. They can desribe a certain unilateral regulation

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in Ω_0 and Γ. The coefficients b_{ij} are assumed to satisfy

$$\begin{aligned} &b_{11} > 0,\, b_{12} < 0,\, b_{21} > 0,\, b_{22} < 0, \\ &b_{11} + b_{22} < 0,\ \det := b_{11}b_{22} - b_{12}b_{21} > 0. \end{aligned} \tag{1.3}$$

The first line in (1.3) means that the system (1.1) is of an activator-inhibitor type. The second line in (1.3) ensures that if we understand (1.1) for $d_1 = d_2 = 0$ (no diffusion) and $\omega_0 \equiv 0$ as an ODE system, then 0 is a stable solution. Furthermore, under suitable assumptions about f_i, the conditions (1.3) guarantee Turing's effect [23] of "diffusion driven instability" for the corresponding classical system

$$\begin{aligned} u_t &= d_1\Delta u + b_{11}u + b_{12}v + f_1(d_1, d_2, x, u, v, \nabla u, \nabla v) \\ v_t &= d_2\Delta v + b_{21}u + b_{22}v + f_2(d_1, d_2, x, u, v, \nabla u, \nabla v) \end{aligned} \quad \text{on } \Omega \tag{1.4}$$

with classical boundary conditions

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_0. \end{cases} \tag{1.5}$$

That means that the trivial solution of (1.4), (1.5) is stable only for (d_1, d_2) from a certain subdomain D_S of $\mathbb{R}^2_+$ (domain of stability) and unstable for $(d_1, d_2) \in \mathbb{R}^2_+ \setminus \overline{D_S}$ (domain of instability). Bifurcation of stationary spatially nonhomogeneous solutions of (1.4), (1.5) (spatial patterns) occurs at the border between the domain of stability and instability under certain assumptions (see e.g. [18]).

Our goal is to prove the existence of a global bifurcation of stationary nontrivial solutions of the multivalued problem (1.1), (1.2) in D_S, where bifurcation is excluded for the classical problem (1.4), (1.5). Clearly, all nontrivial solutions are spatially non-constant, i.e. we get bifurcation of spatial patterns. In fact, we will consider diffusion coefficients changing along a curve $\sigma(s)$, s being a real bifurcation parameter, which can describe the size of the domain Ω for particular curves.

The proof will be based on the degree theory for multivalued mappings. In Section 2 we show how the degree for a class of multivalued maps including those corresponding to the weak formulation of our problem can be calculated. This can be understood as the second main result of this paper. A jump of the degree will be proved which implies a global bifurcation. We note that a similar technique concerning the degree for multivalued maps was employed in [9] for a single equation, but in contrast to [9], we are now treating the case of nonsymmetric operators. Moreover, we introduce some new ideas which apply even in the symmetric case and strengthen the results in [9].

Let us now describe the domain D_S in detail and formulate a particular case of our bifurcation theorem which will be given in whole generality in Section 3.3 (Theorem 3.2).

1.1. **Description of the Domain of Stability for** (1.4)**,** (1.5)**.**

Let κ_n, $n = 1, 2, \ldots$, denote the eigenvalues of $-\Delta$ with (1.5), $0 < \kappa_1 \le \kappa_2 \le \cdots$. With each κ_n we associate the hyperbola

$$C_n := \left\{ (d_1, d_2) \in \mathbb{R}^2_+ : d_2 = \frac{b_{12}b_{21}/\kappa_n^2}{d_1 - b_{11}/\kappa_n} + \frac{b_{22}}{\kappa_n} \right\}, \tag{1.6}$$

see Figure 1. In Figure 1, one can also see the vertical asymptotes $d_1 \equiv \frac{b_{11}}{\kappa_n}$ of the hyperbolas C_n and the line passing through 0 with the slope

$$S := \frac{-b_{12}b_{21} + \det + 2\sqrt{-b_{12}b_{21}\det}}{b_{11}^2} > 1 \tag{1.7}$$

which is tangent to these hyperbolas.

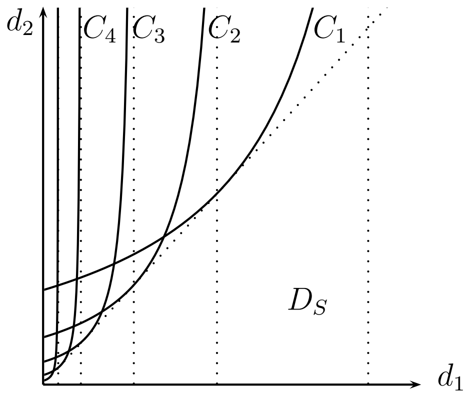

FIGURE 1. Hyperbolas (1.6) determining D_S, their vertical asymptotes, and the common tangential line with slope (1.7)

It is known (and we will re-prove it in Section 3.2) that these hyperbolas consist exactly of those points (d_1, d_2) for which nontrivial solutions of the linearized classical system

$$\begin{aligned} d_1\Delta u + b_{11}u + b_{12}v = 0 \\ d_2\Delta v + b_{21}u + b_{22}v = 0 \end{aligned} \quad \text{on } \Omega \tag{1.8}$$

with (1.5) exist. Set

$$D_S := \bigcap_n \left\{ (d_1, d_2) \in \mathbb{R}_+^2 : d_1 > \frac{b_{12}b_{21}/\kappa_n^2}{d_2 - b_{22}/\kappa_n} + \frac{b_{11}}{\kappa_n} \right\},$$

i.e. D_S is the set of all $(d_1, d_2) \in \mathbb{R}_+^2$ lying to the right from the envelope of all hyperbolas C_n. For $(d_1, d_2) \in D_S$, all eigenvalues of the corresponding eigenvalue problem deciding about the stability of the trivial solution of the classical problem (1.4), (1.5) have negative real parts, for (d_1, d_2) lying to the left from the n-th hyperbola C_n, the n-th eigenvalue of this eigenvalue problem is positive. Hence, the trivial solution of (1.4), (1.5) is linearly stable for $(d_1, d_2) \in D_S$ and linearly unstable for $(d_1, d_2) \in \mathbb{R}_2^+ \setminus \overline{D_S}$. See e.g. [22, Chapter 11].

1.2. **Formulation of a Particular Case of our Bifurcation Result.** We will consider the stationary system corresponding to (1.1), i.e.

$$\begin{aligned} &d_1\Delta u + b_{11}u + b_{12}v + f_1(d_1, d_2, x, u, v, \nabla u, \nabla v) = 0 \qquad \text{on } \Omega, \\ &d_2\Delta v + b_{21}u + b_{22}v + f_2(d_1, d_2, x, u, v, \nabla u, \nabla v) \in \begin{cases} \{0\} & \text{on } \Omega \setminus \Omega_0, \\ -\omega_0(d_1, d_2, x, u, v, \nabla u, \nabla v) & \text{on } \Omega_0, \end{cases} \end{aligned} \tag{1.9}$$

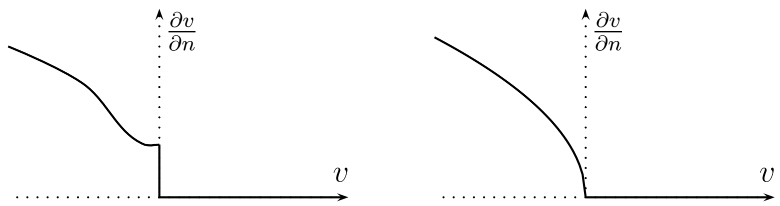

FIGURE 2. Typical graphs for ω_i (and m_i of Section 3)

with d_1, d_2 changing along a continuous curve $\sigma = (\sigma_1, \sigma_2)\colon I \to \mathbb{R}^2_+$ with some closed interval I, i.e. we will deal with the system

$$\begin{aligned}
&\sigma_1(s)\Delta u + b_{11}u + b_{12}v + f_1(\sigma(s), x, u, v, \nabla u, \nabla v) = 0 \qquad \text{on } \Omega,\\
&\sigma_2(s)\Delta v + b_{21}u + b_{22}v + f_2(\sigma(s), x, u, v, \nabla u, \nabla v) \in \begin{cases} \{0\} & \text{on } \Omega \setminus \Omega_0,\\ -\omega_0(\sigma(s), x, u, v, \nabla u, \nabla v) & \text{on } \Omega_0, \end{cases}
\end{aligned} \tag{1.10}$$

with the real bifurcation parameter $s \in I$.

Let us note that if d_1, d_2 are fixed and $\sigma_1(s) = d_1 s^2$, $\sigma_2(s) = d_2 s^2$ then a simple substitution $x' = s^{-1}x$ yields that the problem (1.10)/(1.2) in Ω for a given $s > 0$ is equivalent to the problem (1.9)/(1.2) but on the domain $s^{-1} \cdot \Omega$ of the same shape but "of the size s^{-1}". Hence, the decrease of the parameter s can describe the growth of the domain, which has a natural interpretation in models in biology.

Let us assume that f_1, f_2 satisfy standard growth conditions such that weak solutions can be introduced and that f_1, f_2 are small perturbations at 0 (see Section 3, the assumptions (3.6) e.g. with $\Lambda_0 = \mathbb{R}^2_+$). Furthermore, we need to impose certain unilateral conditions about ω_0, ω_1, e.g. we can assume that they depend only on v (not on d_1, d_2, x, u, ∇u, and ∇v) and their graphs look like in Figure 2. See Section 3.1, assumptions (3.8) and (3.9) for the general case. Set

$$\mathbb{H} := \left\{(u, v) \in W^{1,2}(\Omega, \mathbb{R}^2) : (u, v) = 0 \text{ on } \Gamma_0 \text{ in the sense of traces}\right\}.$$

The following theorem states that in the situation just described along each curve σ in $\mathbb{R}^2_+$ intersecting the asymptote $d_1 = \frac{b_{11}}{\kappa_1}$ to the first hyperbola C_1 and passing in D_S closely enough to some point $d \in \partial D_S \cap \bigcup_n C_n$ satisfying certain assumptions, a global bifurcation branch of nontrivial solutions must occur.

Theorem 1.1. *Let in the above situation $d \in \partial D_S \cap \bigcup_n C_n$ be such that there is a linear combination e of eigenfunctions e_j corresponding to eigenvalues κ_j of $-\Delta$ with* (1.5) *for which $d \in C_j$ satisfying*

$$e \geq \varepsilon \text{ in } \Omega_0 \cup \Gamma \text{ with some } \varepsilon > 0. \tag{1.11}$$

Then there exists a neighborhood W_0 of d with the following property. For any curve $\sigma = (\sigma_1, \sigma_2)\colon I \to \mathbb{R}^2_+$ with some closed interval I, satisfying $\sigma(s_0) \in W_0 \cap D_S$ for some $s_0 \in I$ and $\sigma_1(s_1) > \frac{b_{11}}{\kappa_1}$ for some $s_1 \in I$ (i.e. intersecting the asymptote to C_1), there exists a bifurcation point $s_B \in (s_0, s_1)$ of (1.10)/(1.2) *with $\sigma_1(s_B) \leq \frac{b_{11}}{\kappa_1}$ (see Figure 3).*

More precisely, there is a connected set $B \subset I \times (\mathbb{H} \setminus \{0\})$ of nontrivial solutions of (1.10)/(1.2) *with $(s_B, 0, 0) \in \overline{B}$ satisfying at least one of the following conditions:*

(1) *B is unbounded or reaches the end of σ, i.e., it contains a point from $\partial I \times \mathbb{H}$.*
(2) *$\overline{B}$ contains a point of the type $(s_2, 0, 0)$, $s_2 \notin [s_0, s_1]$, $\sigma_1(s_2) \leq \frac{b_{11}}{\kappa_1}$.*

In addition, using the result of [15], we will see that if $\Omega_0 = \varnothing$ and Γ is a smooth manifold with boundary in $\partial\Omega$ then the condition (1.11) can be replaced by

$$e > 0 \text{ on } \Gamma. \tag{1.12}$$

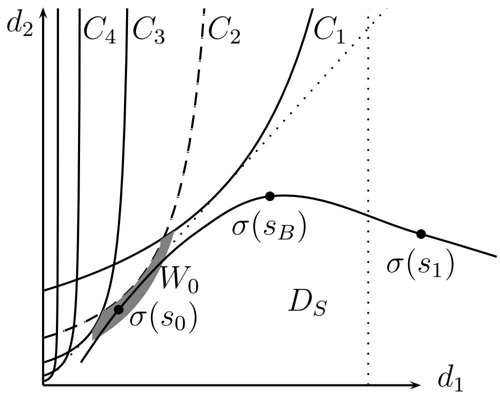

FIGURE 3. Illustration of Theorem 1.1 when each $d \in C_2$ satisfies the hypothesis

The first very particular result guaranteeing local bifurcation in D_S for the system (1.10) with $\omega_0 \equiv 0$ and with unilateral boundary conditions described by variational inequalities was given in [6]. However, only a one-dimensional domain, f_i depending only on u and v, and a particular curve $d_2 = const$ with the bifurcation parameter d_1 was considered. The method of the proof was based on a nonstandard use of a penalty technique combined with global bifurcation results known for equations. For a similar particular case of variational inequalities but in N-dimensional domains, such a result was proved in [20] by direct use of degree theory (jump of the degree implies bifurcation). This latter method was used also for the generalization to boundary conditions described by quasi-variational inequalities in [16] for a general curve σ.

However, although such a problem with variational inequalities might be considered in some sense as a sort of "linearization" of our problem (at least if $\Omega_0 = \varnothing$), these known results about the degree cannot be applied in our situation, since in the abstract formulation of the problem, we will have to calculate the degree for *multivalued* maps which, moreover, have the property that their "linearization" is not of a class for which a degree theory is available (the degrees calculated in the above mentioned references [16], [20] are for single-valued maps which stem from a different reformulation of the problem in case of inequalities; such type of reformulation appears impossible in our case).

In our case of boundary conditions given by inclusions so far only the above mentioned penalty technique was applied. For general curves σ, see e.g. [14]; for a brief survey, see [8]. However, that method gives only local bifurcation without any information about the connectedness of the bifurcating branch. Hence, the result of the present paper is an essential generalization of previous results. We obtain the existence of a global connected bifurcating branch, we consider more general f_i and the conditions given by inclusions are imposed also in the interior of Ω, not only in the boundary conditions. Moreover, we weaken the assumption (1.11) to (1.12) (cf. [15]) and formulate it in a completely abstract form.

2. Calculation of Degrees in Hilbert space

In this abstract section, let $\mathbb{H}$ be a real Hilbert space and Λ a metric space. We are interested in the inclusion problem

$$\lambda \in \Lambda,\ u \in \mathbb{H},\ u - A(\lambda)u \in G(\lambda, u) + M(\lambda, u) \tag{2.1}$$

under the following general hypotheses:

(A) $A\colon \Lambda \times \mathbb{H} \to \mathbb{H}$ is continuous and compact, and $A(\lambda) := A(\lambda, \cdot\,)\colon \mathbb{H} \to \mathbb{H}$ is linear for each $\lambda \in \Lambda$.

(B) The multivalued maps $G, M\colon \Lambda \times \mathbb{H} \multimap \mathbb{H}$ have nonempty compact convex values, are upper semicontinuous and compact.

The main motivation for the following results is the case $A(\lambda) = \lambda A_0$ with some nondegenerate interval $\Lambda \subseteq \mathbb{R}$ (although we will later have $\Lambda \subseteq \mathbb{R}^2$).

We use the notation

$$B_r := \{u \in \mathbb{H} : \|u\| < r\}\,.$$

In particular, under our hypotheses, if (2.1) has no solution on the boundary of B_r, we have a natural definition of a degree

$$\deg(id - A(\lambda) - G(\lambda, \cdot\,) - M(\lambda, \cdot\,), B_r, 0)$$

for the multivalued mapping $id - A(\lambda) - G(\lambda, \cdot\,) - M(\lambda, \cdot\,)$ on B_r with respect to 0. Since all definitions of a multivalued degree coincide in the convex-valued case, it is not important which particular definition of the degree we choose. We understand the degree e.g. in the sense of Ma [17] (for other definitions, see also [1], [3]–[5], [10]).

Let now C_A be the set of all $\lambda \in \Lambda$ such that $A(\lambda)$ has an eigenvalue 1, and for $\lambda \in C_A$, let $E_A(\lambda)$ denote the corresponding eigenspace

$$E_A(\lambda) := \{u \in \mathbb{H} : A(\lambda)u = u\}\,.$$

In the case $A(\lambda) = \lambda A_0$, C_A is the set of characteristic values of A_0, and $E_A(\lambda)$ is the corresponding eigenspace.

The general idea in the sequel is that λ is "close" to a particular value $\lambda_0 \in C_A$, and that for $\lambda \to \lambda_0$ we have a "linearization" to (2.1) which is a variational inequality with a certain cone $K \subseteq \mathbb{H}$.

More precisely, we assume that there is a closed cone $K \subseteq \mathbb{H}$ with its vertex at the origin, i.e. K is convex and closed and $0 \in K + K \subseteq K$, with the following property (where $\Lambda_0 \subseteq \Lambda$ and $\widetilde{\lambda} \in \Lambda$ will be specified later in particular situations):

> If $\Lambda_0 \ni \lambda_n \to \widetilde{\lambda}$ and $u_n, y_n \in \mathbb{H}$ with $0 < \|u_n\| \to 0$ are such that $w_n := u_n / \|u_n\| \rightharpoonup w$, $y_n \to y$ and
> $$w_n - y_n \in \frac{M(\lambda_n, u_n)}{\|u_n\|}, \tag{2.2}$$
> then $w_n \to w$, and w is a solution of the variational inequality
> $$w \in K,\ \langle w - y, v - w\rangle \geq 0 \text{ for all } v \in K.$$

Concerning the (possibly multivalued) nonlinearity G, we consider a uniform linearization type hypothesis where $\Lambda_0 \subseteq \Lambda$ will be specified later:

$$\lim_{\|u\|\to 0} \sup_{\lambda\in\Lambda_0} \frac{\sup\{\|y\| : y \in G(\lambda, u)\}}{\|u\|} = 0. \tag{2.3}$$

Occasionally, pointwise convergence will be sufficient:

$$\lim_{\|u\|\to 0} \frac{\sup\{\|y\| : y \in G(\lambda, u)\}}{\|u\|} = 0 \qquad \text{for each } \lambda \in \Lambda_0. \tag{2.4}$$

The main hypothesis for λ_0 is related to the variational inequality

$$u \in K,\ \langle u - A(\lambda_0)u, v - u\rangle \geq 0 \text{ for all } v \in K. \tag{2.5}$$

It will be convenient to denote the set of solutions of this inequality by $E_{A,K}(\lambda_0)$. Similarly to all former papers concerning bifurcation for some unilateral problems in D_S (variational inequalities or inclusions) the condition

$$E_{A,K}(\lambda_0) = E_A(\lambda_0) \cap K \tag{2.6}$$

will play a basic role. In the past, this condition was always verified by using the assumption that λ_0 is an "interior value", which means in our situation that $E_A(\lambda_0)$ contains an element from the interior (or pseudo-interior) of K. This notion was introduced in [12] (for interior) and in [21] (for pseudo-interior).

In fact, we could directly consider points λ_0 satisfying (2.6), but we will give here a definition of (K, A)-interior values which is a certain generalization of "pseudo-interior values" [21] but is simultaneously equivalent to the condition (2.6). Condition (2.6) will be verified by proving that λ_0 is a (K, A)-interior value.

Let A^* denote the family of adjoint operators, i.e. $A^*(\lambda) := A(\lambda)^*$.

Definition 2.1. A point $\lambda_0 \in C_A$ is (K, A)-*interior* if there is $v^* \in E_{A^*}(\lambda_0)$ such that the closure of the subspace

$$D_K(v^*) := \{w \in \mathbb{H} : \text{there is } \varepsilon > 0 \text{ with } v^* \pm \varepsilon w \in K\}$$

satisfies

$$\overline{D_K(v^*)} \supseteq \{u - A(\lambda_0)u : u \in E_{A,K}(\lambda_0)\}.$$

Lemma 2.1. *A point $\lambda_0 \in C_A$ is (K, A)-interior if and only if* (2.6) *holds.*

Proof. If (2.6) holds, then one may choose $v^* = 0$, and for each $u \in E_{A,K}(\lambda_0)$ the hypothesis implies $u - A(\lambda_0)u = 0 \in D_K(v^*) \subseteq \overline{D_K(v^*)}$.

Conversely, let $\lambda_0 \in C_A$ be (K, A)-interior, and let $u \in E_{A,K}(\lambda_0)$. We have to show that $z := u - A(\lambda_0)u$ vanishes. Since by hypothesis $z \in \overline{D_K(v^*)}$, it suffices to show that $z \in (\overline{D_K(v^*)})^\perp = D_K(v^*)^\perp$. Thus, let $w \in D_K(v^*)$. Using (2.5) with $v := u + (v^* \pm \varepsilon w) \in K + K \subseteq K$, we find

$$0 \leq \langle u - A(\lambda_0)u, v^* \pm \varepsilon w\rangle = \langle u, (id - A(\lambda_0)^*)v^*\rangle \pm \varepsilon \langle z, w\rangle = \pm\varepsilon \langle z, w\rangle,$$

and so $\langle z, w\rangle = 0$, i.e. $z \in D_K(v^*)^\perp$, as required. □

Some notes are in order. If v^* is an interior point of K, then even $D_K(v^*) = \mathbb{H}$. In particular, if there is $v^* \in E_{A^*}(\lambda_0)$ in the interior of K, then λ_0 is (K, A)-interior. A hypothesis of this type was introduced in [12] to obtain results for variational inequalities, cf. also [13]. For cones K with empty interior (as in our application), the hypothesis that v^* is an interior point of K was relaxed to $\overline{D_K(v^*)} = \mathbb{H}$ (i.e. that there is $v^* \in E_{A^*}(\lambda_0)$ which belongs to the so-called *pseudo-interior* of K) in [21] for variational inequalities.

Definition 2.2. Let $K_0 \subseteq \mathbb{H}$ be a cone satisfying $K_0 \subseteq K$. We say that $\lambda_0 \in C_A$ satisfies the (K, A, K_0)*-sign-condition on* $\Lambda_0 \subseteq \Lambda$, $\lambda_0 \in \overline{\Lambda}_0$, if for each $u \in E_A(\lambda_0) \cap K$ with $\|u\| = 1$ there is some $u^* \in K_0$ such that, for some $\delta > 0$,

$$\langle (id - A(\lambda))u, u^* \rangle \leq -\delta \, \|(id - A(\lambda)^*)u^*\| < 0 \quad \text{for all } \lambda \in \Lambda_0 \setminus \{\lambda_0\} \text{ close to } \lambda_0. \tag{2.7}$$

In the sequel, the choice of $\Lambda_0 \subseteq \Lambda$ will depend on λ_0. For instance, in the case of a scalar parameter set, a typical choice of Λ_0 would be an interval to the right or to the left from λ_0, or in the case of a two-dimensional parameter set a certain portion of the plane which is bordered by some curve passing through λ_0 (the portion being essentially determined by the requirement (2.7)).

We will see that Definition 2.2 with general K_0 is natural for the calculation of the degree. However, in our applications, we will always choose $K_0 = E_{A^*}(\lambda_0) \cap K$. In particular, this is the case in the natural situation when $A(\lambda) = \lambda A_0$. In this case, the above hypothesis (2.7) becomes essentially the sign condition which is also imposed in [21]:

Proposition 2.1. *Suppose that $A(\lambda) = \lambda A_0$ with a compact operator A_0, and that $\lambda_0 \in C_A$ is a real number. Then the (K, A, K_0)-sign-condition holds with $K_0 = E_{A^*}(\lambda_0) \cap K$ (and* (2.7) *holds even for all $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$) if one of the following conditions is true.*

(1) $\lambda_0 \geq 0$, $\Lambda_0 \subseteq [\lambda_0, \infty)$, *and for each $u \in E_A(\lambda_0) \cap K \setminus \{0\}$ there is $u^* \in E_{A^*}(\lambda_0) \cap K$ such that $\langle u, u^* \rangle > 0$.*
(2) $\lambda_0 \geq 0$, $\Lambda_0 \subseteq (-\infty, \lambda_0]$, *and for each $u \in E_A(\lambda_0) \cap K \setminus \{0\}$ there is $u^* \in E_{A^*}(\lambda_0) \cap K$ such that $\langle u, u^* \rangle < 0$.*
(3) $\lambda_0 \leq 0$, $\Lambda_0 \subseteq [\lambda_0, \infty)$, *and for each $u \in E_A(\lambda_0) \cap K \setminus \{0\}$ there is $u^* \in E_{A^*}(\lambda_0) \cap K$ such that $\langle u, u^* \rangle < 0$.*
(4) $\lambda_0 \leq 0$, $\Lambda_0 \subseteq (-\infty, \lambda_0]$, *and for each $u \in E_A(\lambda_0) \cap K \setminus \{0\}$ there is $u^* \in E_{A^*}(\lambda_0) \cap K$ such that $\langle u, u^* \rangle > 0$.*

Proof. If $u \in E_A(\lambda_0) \cap K \setminus \{0\}$, then $\lambda_0 A_0 u = u \neq 0$, and so $\lambda_0 \neq 0$. If $u^* \in E_{A^*}(\lambda_0) \cap K$ is the corresponding point from (2.7) then $u^* \neq 0$, $u^* = A(\lambda_0)^* u^* = \lambda_0 A_0^* u^*$, and so we have for each $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ that

$$(id - A(\lambda)^*)u^* = u^* - \lambda A_0^* u^* = u^* - \lambda \lambda_0^{-1} u^* = c_\lambda u^* \neq 0$$

because $c_\lambda := \lambda_0^{-1}(\lambda_0 - \lambda) \neq 0$. For $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ this implies $\|(id - A(\lambda)^*)u^*\| \neq 0$ and

$$\langle (id - A(\lambda))u, u^* \rangle = \langle u, (id - A(\lambda)^*)u^* \rangle = c_\lambda \langle u, u^* \rangle = \langle u, u^* \rangle \frac{\|(id - A(\lambda)^*)u^*\|}{\|u^*\|} \operatorname{sgn} c_\lambda.$$

Hence (2.7) holds with $\delta := \|u^*\|^{-1} \langle u, u^* \rangle \operatorname{sgn}(-c_\lambda) = \|u^*\|^{-1} |\langle u, u^* \rangle| > 0$ in all the above cases (1)–(4). ☐

Lemma 2.2. *Let* (A) *hold and let $\lambda_0 \in C_A$ be (K, A)-interior and satisfy the (K, A, K_0)-sign condition on $\Lambda_0 \subseteq \Lambda$. Then for each $u \in E_A(\lambda_0) \cap K \setminus \{0\}$ there are $u^* \in K_0$ and $\delta > 0$ such that all $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ close to λ_0 and all $w \in \mathbb{H}$ with $\|w - u\| < \delta$ are subject to the strict inequality*

$$\langle w - A(\lambda)w, u^* \rangle < 0.$$

Remark 2.1. We note for later use that u^*, δ and the closeness in Lemma 2.2 are in fact those from Definition 2.2 (and depend on u).

Proof. Choosing u^* and δ corresponding to u according to (2.7), we obtain

$$\begin{aligned}\langle w - A(\lambda)w, u^*\rangle &= \langle w, (id - A(\lambda)^*)u^*\rangle \\ &= \langle u, (id - A(\lambda)^*)u^*\rangle + \langle w - u, (id - A(\lambda)^*)u^*\rangle \\ &\le \langle (id - A(\lambda))u, u^*\rangle + \|w - u\| \, \|(id - A(\lambda)^*)u^*\| \\ &\le (\|w - u\| - \delta) \, \|(id - A(\lambda)^*)u^*\| \,.\end{aligned}$$

The last term is strictly negative for $\|w - u\| < \delta$. □

Lemma 2.3. *Let* (A) *hold and let* $\lambda_0 \in C_A$ *be* (K, A)*-interior and satisfy the* (K, A, K_0)*-sign-condition on* Λ_0*. Then for all* $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ *which are sufficiently close to* λ_0*, the variational inequality*

$$u \in K, \ \langle u - A(\lambda)u, v - u\rangle \ge 0 \text{ for all } v \in K \tag{2.8}$$

has only the trivial solution $u = 0$*. The closeness depends only on* δ *and on the closeness in* (2.7) *if those are independent of* $u \in E_A(\lambda_0) \cap K$*.*

Let us note that in view of Proposition 2.1, Lemma 2.3 contains the first assertion of [21, Theorem 2(i)].

Proof. Assume by contradiction that there are $\lambda_n \to \lambda_0$ with $\lambda_n \ne \lambda_0$ such that for each n there are solutions u_n of

$$u_n \in K, \ \langle u_n - A(\lambda_n)u_n, v - u_n\rangle \ge 0 \text{ for all } v \in K \tag{2.9}$$

with $\|u_n\| = 1$. Then $u_n = P_K A(\lambda_n)u_n$ where P_K denotes the canonical projection onto the cone K (see e.g. [11, Section 1.2]). In particular, the sequence u_n is contained in a compact set and thus has a convergent subsequence. Passing to this subsequence, we may assume $u_n \to u$. Hence $\|u\| = 1$, and passing to the limit in (2.9) we obtain that u is a solution of (2.5). Thus Lemma 2.1 implies $u \in E_A(\lambda_0) \cap K \setminus \{0\}$. Choose $u^* \in K_0$ according to Lemma 2.2. Putting $v := u_n + u^* \in K + K$ in (2.9), we find

$$0 \le \langle u_n - A(\lambda_n)u_n, u^*\rangle \,.$$

This is not possible for all n by our choice of u^* (according to Lemma 2.2). The statement about the closeness follows in view of Remark 2.1. □

For the case that $A(\lambda) = \lambda A_0$ with a symmetric A_0 and we can verify the hypothesis in Proposition 2.1 with $u^* = u$, the following Theorem 2.1 reduces to a special case of [9, Theorem 2.1]. However, now we have the nonsymmetric case with a multi-dimensional parameter set, which is important for reaction-diffusion systems.

Theorem 2.1. *Let* (A) *and* (B) *hold and let* $\Lambda_1 \subset \Lambda$ *be such that the variational inequality* (2.8) *has only the trivial solution for all* $\lambda \in \Lambda_1$*. Assume that* (2.3) *holds with some* Λ_0*. Then for each* $\widetilde{\lambda} \in \Lambda_0 \cap \Lambda_1$ *satisfying* (2.2) *there are* $r = r(\widetilde{\lambda}, \Lambda_0) > 0$ *and a neighborhood* $\Lambda_2(\widetilde{\lambda})$ *of* $\widetilde{\lambda}$ *such that*

$$\deg(id - A(\lambda) - G(\lambda, \cdot\,) - M(\lambda, \cdot\,), B_\rho, 0) = \deg(id - A(\lambda) - M(\lambda, \cdot\,), B_\rho, 0) \tag{2.10}$$

for all $\rho \in (0,r)$ *and* $\lambda \in \Lambda_0 \cap \Lambda_1 \cap \Lambda_2(\widetilde{\lambda})$. *In particular, the degrees in* (2.10) *are defined and independent of* $\rho \in (0,r)$. *Furthermore, if* $\widetilde{\Lambda} \subset \Lambda_0 \cap \Lambda_1$ *is connected then the degree in* (2.10) *is independent of* $\lambda \in \widetilde{\Lambda}$, $\rho \in (0, r(\lambda))$, *with some* $r(\lambda) > 0$ *dependent on* λ *(i.e. the index is independent of* $\lambda \in \widetilde{\Lambda}$*).*

Note that Λ_0 is usually not a neighborhood of λ_0. In our applications, the (K, A, K_0)-sign-condition restricts Λ_0 to a certain "sector" (which depends on λ_0). Thus, Λ_0 will be a neighborhood of λ_0 within such a sector. (See Figures 5 and 6.)

Remark 2.2. If $\lambda_0 \in C_A$ is (K, A)-interior and satisfies the (K, A, K_0)-sign-condition on some Λ_0 then the set Λ_1 of all $\lambda \in \Lambda_0 \setminus \{\lambda_0\}$ sufficiently close to λ_0 satisfies the assumption from Theorem 2.1 by virtue of Lemma 2.3.

Corollary 2.1. *If* $\Lambda_0 \cap \Lambda_1$ *in Theorem 2.1 is an open set and* (2.2) *is satisfied for all* $\widetilde{\lambda} \in \Lambda_0 \cap \Lambda_1$ *then this set contains no bifurcation point of the inclusion* (2.1).

Indeed, if $\widetilde{\lambda} \in \Lambda_0 \cap \Lambda_1$ then we can take $\Lambda_2(\widetilde{\lambda}) \subseteq (\Lambda_0 \cap \Lambda_1)$ and (2.1) has no nontrivial solution $(\lambda, u) \in \Lambda_2(\widetilde{\lambda}) \times B_{r(\widetilde{\lambda},\Lambda_0)}$, because the degrees (2.10) are defined for all $\rho \in (0, r(\widetilde{\lambda}, \Lambda_0))$.

Proof of Theorem 2.1. In view of the homotopy invariance and the excision property of the degree, it suffices to show that there is $r > 0$ such that for all $\lambda \in \Lambda_0$ which are sufficiently close to $\widetilde{\lambda}$ the multivalued homotopy

$$H_\lambda(t, u) := u - A(\lambda, u) - tG(\lambda, u) - M(\lambda, u)$$

contains no zero on $[0, 1] \times (B_r \setminus \{0\})$.

Thus, assume by contradiction that there is a sequence $(\lambda_n, t_n, u_n) \in \Lambda_0 \times [0, 1] \times \mathbb{H}$ with $\lambda_n \to \widetilde{\lambda}$, $0 < \|u_n\| \to 0$ and $0 \in H_{\lambda_n}(t_n, u_n)$. Dividing this inclusion by $\|u_n\|$ and setting $w_n := u_n / \|u_n\|$, we obtain that

$$w_n - A(\lambda_n) w_n \in t_n \frac{G(\lambda_n, u_n)}{\|u_n\|} + \frac{M(\lambda_n, u_n)}{\|u_n\|},$$

i.e. there are

$$y_n \in A(\lambda_n) w_n + t_n \frac{G(\lambda_n, u_n)}{\|u_n\|}$$

with

$$w_n - y_n \in \frac{M(\lambda_n, u_n)}{\|u_n\|}.$$

Passing to a subsequence, we may assume that $w_n \rightharpoonup w$ for some $w \in \mathbb{H}$. In view of (A) and (2.3) we have $y_n \to A(\widetilde{\lambda})w$. Hence, the hypothesis (2.2) implies that $w_n \to w$, w is a nontrivial solution of (2.8), contradicting the hypothesis that λ is chosen such that (2.8) contains only the trivial solution.

Now, let $\widetilde{\Lambda} \subseteq \Lambda_0 \cap \Lambda_1$ be connected. The homotopy invariance of the degree implies that for each $\widetilde{\lambda} \in \widetilde{\Lambda}$ the degree (2.10) is independent of $\lambda \in \widetilde{\Lambda} \cap \Lambda_2(\widetilde{\lambda})$, $\rho \in (0, r(\widetilde{\lambda}, \Lambda_0))$ and the last assertion of Theorem 2.1 easily follows. □

Now we can prove that the degree is defined and equals zero on a small ball around 0 under certain assumptions. The first result of this type (for a variational inequality under the hypothesis of the existence of an eigenvector in the interior of the cone) was established in [19].

Theorem 2.2. *Let* (A) *and* (B) *hold, and let* $\lambda_0 \in C_A$ *be* (K, A)*-interior. Let* $\Lambda_0 \subseteq \Lambda \setminus \{\lambda_0\}$ *be such that* $\lambda_0 \in \overline{\Lambda}_0$*,* λ_0 *satisfies the* (K, A, K_0)*-sign-condition on* Λ_0*,* (2.4) *holds, and* (2.2) *holds with* $\widetilde{\lambda} = \lambda_0$ *and with all values* $\widetilde{\lambda} \in \Lambda_0$ *which are sufficiently close to* λ_0*. Suppose that there are* $u_0^* \in K \cap E_{A^*}(\lambda_0)$ *and* $u_0 \in \mathbb{H}$ *with* $\langle u_0, u_0^* \rangle > 0$ *and*

$$\langle u_0, u^* \rangle \geq 0 \qquad \text{for all } u^* \in K_0. \tag{2.11}$$

Suppose that M *satisfies the sign condition*

$$\langle z, u^* \rangle \geq 0 \quad \text{for all } z \in M(\Lambda_0 \times B_r) \text{ and all } u^* \in K_0 \cup \{u_0^*\} \tag{2.12}$$

for some $r > 0$*. Then for each* $\lambda \in \Lambda_0$ *sufficiently close to* λ_0 *there is* $r = r(\lambda) > 0$ *such that*

$$\deg(id - A(\lambda) - G(\lambda, \cdot\,) - M(\lambda, \cdot\,), B_\rho, 0) = 0 \quad \text{for all } \rho \in (0, r).$$

In particular, the degree is defined.

Remark 2.3. If $K_0 = K \cap E_{A^*}(\lambda_0)$ and $K_0 \neq -K_0$, the existence of u_0 and u_0^* satisfying $\langle u_0, u_0^* \rangle > 0$ and (2.11) is automatic (and one can choose $u_0 \in K_0$). Indeed, since K_0 is a cone which is not a subspace, this follows from [21, Lemma 2] (in [21, Lemma 2] it is only claimed that there is $u_0 \in K$ with (2.11) and $u_0 \neq 0$, but actually the relation $\langle u_0, u_0^* \rangle > 0$ for an appropriate $u_0^* \in K_0$ is shown in the proof of that result).

Remark 2.4. In view of Remark 2.3, Theorem 2.2 is stronger than [9, Theorem 2.2], even in the case $A(\lambda) = \lambda A_0$ with a symmetric operator A_0 and $K_0 := K \cap E_{A^*}(\lambda_0) = K \cap E_A(\lambda_0)$. In fact, Theorem 2.2 shows essentially that, in contrast to [9, Theorem 2.2], one does not have to require strict inequality in (2.11) (and so the existence of u_0 is even automatic in view of Remark 2.3). This means that in the main results of [9] (i.e. in Theorems 4.1 and 4.2 there) the hypothesis (2.6) in [9] is actually superfluous.

Proof of Theorem 2.2. Applying Remark 2.2 and Theorem 2.1 (only for this moment with $\widetilde{\lambda} := \lambda$ and $\Lambda_0 := \{\lambda\}$ in order to have (2.3) by (2.4)) we find that for all $\lambda \in \Lambda_0$ (now Λ_0 from the assumptions of Theorem 2.2) sufficiently close to λ_0 the degree is defined and has the same value as when $G = 0$. Hence, we may assume without loss of generality that $G = 0$. Put

$$F_\lambda(u) := u - A(\lambda)u - M(\lambda, u), \qquad H(t, u) := F_\lambda(u) - tu_0.$$

We will show that

$$tu_0 \notin F_\lambda(\overline{B}_r) \text{ for all } \lambda \in \Lambda_0 \text{ close to } \lambda_0,\ r \text{ small enough},\ t \in (0, 1]. \tag{2.13}$$

In particular, it will follow that $\deg(H(1, \cdot\,), B_\rho, 0) = 0$ for all $\rho \in (0, r)$, and because we know that $0 \notin F_\lambda(\overline{B}_r \setminus \{0\})$ (the degree in Theorem 2.1 is defined), we will have $0 \notin H(t, \overline{B}_r \setminus \{0\})$ for all $t \in [0, 1]$. The assertion of Theorem 2.2 will then follow by using the homotopy invariance of the degree.

Assume by contradiction that (2.13) is not true. Then we find sequences $(\lambda_n, u_n, t_n) \in \Lambda_0 \times \mathbb{H} \times (0, 1]$ and $\widetilde{z}_n \in M(\lambda_n, u_n)$, such that $\lambda_n \to \lambda_0$, $u_n \to 0$, and

$$u_n - A(\lambda_n)u_n = \widetilde{t}_n u_0 + \widetilde{z}_n.$$

In particular, putting $\alpha := \langle u_0, u_0^* \rangle > 0$, we have by (2.12) that

$$\langle u_n - A(\lambda_n)u_n, u_0^* \rangle = \langle \widetilde{t}_n u_0 + \widetilde{z}_n, u_0^* \rangle = \alpha \widetilde{t}_n + \langle \widetilde{z}_n, u_0^* \rangle \geq \alpha \widetilde{t}_n > 0,$$

and so $u_n \neq 0$. Consequently, we may define $w_n := u_n / \|u_n\|$, $t_n := \widetilde{t}_n / \|u_n\|$, and $z_n := \widetilde{z}_n / \|u_n\|$. We obtain

$$w_n - A(\lambda_n) w_n = t_n u_0 + z_n. \tag{2.14}$$

The left-hand sides remain bounded, hence also $t_n u_0 + z_n$ are bounded. In view of

$$\langle t_n u_0 + z_n, u_0^* \rangle = \alpha t_n + \langle z_n, u_0^* \rangle \geq \alpha t_n \geq 0,$$

we conclude that also t_n are bounded. Hence, passing to a subsequence, we may assume that $t_n \to t \in [0, \infty)$ and $w_n \rightharpoonup w$. Putting

$$y_n := A(\lambda_n) w_n + t_n u_0,$$

we thus have $y_n \to A(\lambda_0) w + t u_0$. We conclude from the hypothesis (2.2) (with $\widetilde{\lambda} = \lambda_0$) in view of (2.14) that $w_n \to w$ (in particular, $w \neq 0$) and

$$w \in K, \ \langle w - A(\lambda_0) w - t u_0, v - w \rangle \geq 0 \text{ for all } v \in K. \tag{2.15}$$

Applying this estimate with $v := w + u_0^* \in K + K \subseteq K$, we obtain

$$0 \leq \langle w - A(\lambda_0) w - t u_0, u_0^* \rangle = \langle w, (id - A(\lambda_0)^*) u_0^* \rangle - \alpha t = -\alpha t.$$

Hence $t = 0$ and so (2.15) implies that w is a solution of the variational inequality (2.5). By Lemma 2.1, we have $w \in E_A(\lambda_0) \cap K$. Choose a corresponding $w^* \in K_0$ according to Lemma 2.2. Then (2.14), (2.11) and (2.12) imply

$$\langle w_n - A(\lambda_n) w_n, w^* \rangle = \langle t_n u_0 + z_n, w^* \rangle = t_n \langle u_0, w^* \rangle + \langle z_n, w^* \rangle \geq 0.$$

This is not possible for all n by our choice of w^* (according to Lemma 2.2). □

3. Application to Reaction-Diffusion Systems

3.1. **Weak Formulation.** We are now going to describe the weak formulation of (1.9) with boundary conditions (1.2). Since we will need to work with mappings of type "$id-$compact", it will be convenient to multiply the first and second equation of (1.9) by d_1^{-1} and d_2^{-1}, respectively, and to consider the system dependent on the parameters $\lambda_i = d_i^{-1}$. This will simplify many technical considerations. To this end, we assume that the given functions in (1.9)/(1.2) depend on $\lambda = (\lambda_1, \lambda_2)$. We need not assume that the functions are defined for all $\lambda \in \mathbb{R}_+^2$ but only for $\lambda \in P$ where $P \subseteq \mathbb{R}_+^2$. Actually, we could even allow $P \subseteq \mathbb{R}^2$ without any change in our main results. However, in order to compare our hypotheses with the situation decribed in Section 1, we introduce the transformation $T(d_1, d_2) := (d_1^{-1}, d_2^{-1})$ which transforms $d = (d_1, d_2)$ into $\lambda = (\lambda_1, \lambda_2)$ (and vice versa) and define functions

$$\begin{aligned}
&g_i(\lambda, x, u, v, w, z) := \lambda_i f_i(\lambda_1^{-1}, \lambda_2^{-1}, x, u, v, w, z), && i = 1, 2, \\
&g_i(\lambda, x, u, v) := f_i(\lambda_1^{-1}, \lambda_2^{-1}, x, u, v), && i = 3, 4, \\
&m_0(\lambda, x, u, v, w, z) := \lambda_2 \omega_0(\lambda_1^{-1}, \lambda_2^{-1}, x, u, v, w, z) \quad \text{for } x \in \Omega_0, \\
&m_0(\lambda, x, u, v, w, z) := \{0\} \quad \text{for } x \in \Omega \setminus \Omega_0, \\
&m_1(\lambda, x, u, v) := \omega_1(\lambda_1^{-1}, \lambda_2^{-1}, x, u, v) \quad \text{for } x \in \Gamma, \\
&m_1(\lambda, x, u, v) := \{0\} \quad \text{for } x \in \partial\Omega \setminus (\Gamma \cup \Gamma_0).
\end{aligned} \tag{3.1}$$

Then (1.9)/(1.2) is equivalent to the system

$$\begin{aligned}\Delta u + \lambda_1 b_{11} u + \lambda_1 b_{12} v &= -g_1(\lambda, x, u, v, \nabla u, \nabla v)\\ \Delta v + \lambda_2 b_{21} u + \lambda_2 b_{22} v &\in -g_2(\lambda, x, u, v, \nabla u, \nabla v) - m_0(\lambda, x, u, v, \nabla u, \nabla v)\end{aligned} \quad \text{on } \Omega \tag{3.2}$$

with boundary conditions

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0,\\ \frac{\partial u}{\partial n} = g_3(\lambda, x, u, v) & \text{on } \partial\Omega \setminus \Gamma_0,\\ \frac{\partial v}{\partial n} \in g_4(\lambda, x, u, v) + m_1(\lambda, x, u, v) & \text{on } \partial\Omega \setminus \Gamma_0. \end{cases} \tag{3.3}$$

Concerning m_i, we assume the particular structure

$$m_0(\lambda, x, u, v, w, z) := [\underline{c}_0(\lambda)\underline{m}_0(x, u, v, w, z), \overline{c}_0(\lambda)\overline{m}_0(x, u, v, w, z)]$$

and

$$m_1(\lambda, x, u, v) := [\underline{c}_1(\lambda)\underline{m}_1(\lambda, x, u, v), \overline{c}_1(\lambda)\overline{m}_1(\lambda, x, u, v)]$$

where $\underline{c}_i$, $\underline{m}_i$, $\overline{c}_i$, and $\overline{m}_i$ are singlevalued functions. Due to this agreement, we have automatically included the trivial Neumann boundary conditions in (1.2) and similarly the case of the trivial source in the interior of Ω in (1.9).

For $i = 0, 1$, we fix exponents p_i, q_i and q_i^* according to the restrictions

$$\begin{cases} p_i \in [1/2, \infty),\ 1 \le q_i^* < q_i < \infty \text{ arbitrary} & \text{if } N \le 2,\\ p_0 := \frac{N}{N-2},\ p_1 := \frac{N-1}{N-2},\ \infty > q_0 > q_0^* := \frac{2N}{N+2},\ \infty > q_1 > q_1^* := \frac{2N-2}{N} & \text{if } N > 2. \end{cases}$$

(If we doubled the value of p_i, these choices would correspond to the exponents in [24] in the Hilbert space case; since the factor 2 will in all estimates cancel with the exponent of the underlying space $W^{1,2}(\Omega, \mathbb{R}^2)$, our above choice will be more convenient in the sequel.)

Throughout, we introduce the following general requirements.

(i) $\underline{c}_i, \overline{c}_i$ are continuous, and for all respective arguments the inequalities $\underline{c}_i\underline{m}_i \le \overline{c}_i\overline{m}_i$ are true.

(ii) For each $\lambda \in P$ the following holds: The functions $g_i(\lambda, \cdot, u, v, w, z)$ $(i = 1, 2)$ are measurable and $g_i(\lambda, x, \cdot, \cdot, \cdot, \cdot)$ are continuous for almost all x. Moreover, g_i satisfy the growth estimate

$$|g_i(\lambda, x, u, v, w, z)| \le$$
$$a_{0,\lambda}(x) + b_{0,\lambda} \cdot \left((|u| + |v|)^{p_0} + \|w\| + \|z\|\right)^{2/q_0} \quad (i = 1, 2),$$

where the numbers $\|a_{0,\lambda}\|_{L_{q_0}(\Omega)}$ and $b_{0,\lambda} \in [0, \infty)$ are locally bounded with respect to λ.

(iii) For each $\lambda_0 \in P$ there is an estimate of the form

$$|g_i(\lambda, x, u, v, w, z) - g_i(\lambda_0, x, u, v, w, z)| \le$$
$$c_{0,\lambda_0}(\lambda)\left(a_{0,\lambda_0,\lambda}(x) + \left((|u| + |v|)^{p_0} + \|w\| + \|z\|\right)^{2/q_0^*}\right) \quad (i = 1, 2)$$

where $\|a_{0,\lambda_0,\lambda}\|_{L_{q_0^*}(\Omega)} \le 1$ and $c_{0,\lambda_0}(\lambda) \to 0$ as $\lambda \to \lambda_0$.

(iv) For each $\lambda \in P$ the following holds: The functions $g_i(\lambda, \cdot, u, v)$ $(i = 3, 4)$ are measurable and $g_i(\lambda, x, \cdot, \cdot)$ are continuous for almost all x. Moreover, g_i satisfy the growth estimate

$$|g_i(\lambda, x, u, v)| \le a_{1,\lambda}(x) + b_{1,\lambda} \cdot (|u| + |v|)^{2p_1/q_1} \quad (i = 3, 4), \tag{3.4}$$

where the numbers $\|a_{1,\lambda}\|_{L_{q_1}(\partial\Omega\setminus\Gamma_0)}$ and $b_{1,\lambda} \in [0, \infty)$ are locally bounded with respect to λ.

(v) For each $\lambda_0 \in P$ there is an estimate of the form

$$|g_i(\lambda, x, u, v, w, z) - g_i(\lambda_0, x, u, v, w, z)| \le$$
$$c_{1,\lambda_0}(\lambda) \left(a_{1,\lambda_0,\lambda}(x) + (|u| + |v|)^{2p_0/q_0^*} \right) \quad (i = 3, 4)$$

where $\|a_{1,\lambda_0,\lambda}\|_{L_{q_0^*}(\partial\Omega\setminus\Gamma_0)} \le 1$ and $c_{\lambda_0}(\lambda) \to 0$ as $\lambda \to \lambda_0$.

(vi) The functions $\underline{m}_0(\,\cdot\,, u, v, w, z)$ and $\overline{m}_0(\,\cdot\,, u, v, w, z)$ are measurable, $\underline{m}_0(x, \cdot, \cdot, \cdot, \cdot)$ is lower semicontinuous, $\overline{m}_0(x, \cdot, \cdot, \cdot, \cdot)$ is upper semicontinuous, and the corresponding superposition operators

$$\underline{M}_0(u, v, w, z)(x) := \underline{m}_0(x, u(x), v(x), w(x), z(x))$$

and

$$\overline{M}_0(u, v, w, z)(x) := \overline{m}_0(x, u(x), v(x), w(x), z(x))$$

send continuous (and thus measurable) functions to measurable functions (this property is discussed in [2, Chapter 1]; it is satisfied e.g. if m_i are so-called Shragin functions, i.e. measurable with respect to a certain product measure). Moreover, we require for some $a_0 \in L_{q_0}(\Omega)$ and $b_0 < \infty$ the growth estimates

$$\max\{|\underline{m}_0(x, u, v, w, z)|, |\overline{m}_0(x, u, v, w, z)|\} \le$$
$$a_0(x) + b_0 \cdot ((|u| + |v|)^{p_0} + \|w\| + \|z\|)^{2/q_0}.$$

(vii) The functions $\underline{m}_1(\,\cdot\,, u, v)$ and $\overline{m}_1(\,\cdot\,, u, v)$ are measurable, $\underline{m}_1(x, \cdot, \cdot)$ is lower semicontinuous, $\overline{m}_1(x, \cdot, \cdot)$ is upper semicontinuous, and the corresponding superposition operators

$$\underline{M}_1(u, v)(x) := \underline{m}_1(x, u(x), v(x))$$

and

$$\overline{M}_1(u, v)(x) := \overline{m}_1(x, u(x), v(x))$$

send continuous (and thus measurable) functions to measurable functions. Moreover, we require the following growth estimates for some $a_1 \in L_{q_1}(\Gamma)$ and $b_1 < \infty$:

$$\max\{|\underline{m}_1(x, u, v)|, |\overline{m}_1(x, u, v)|\} \le a_1(x) + b_1 \cdot (|u| + |v|)^{2p_1/q_1}.$$

Let $\mathbb{H}_0$ denote the subspace of all functions from $W^{1,2}(\Omega)$ which vanish on Γ_0, and let $\mathbb{H} = \mathbb{H}_0 \times \mathbb{H}_0$ be the corresponding space of functions with values in $\mathbb{R}^2$. Since we assume $\operatorname{mes}\Gamma_0 > 0$, we can equip $\mathbb{H}$ (and similarly $\mathbb{H}_0$) with the scalar product

$$\langle U, V\rangle := \int_\Omega \langle \nabla U(x), \nabla V(x)\rangle \, dx$$

which is equivalent to the usual scalar product inherited from $\mathbb{W} := W^{1,2}(\Omega, \mathbb{R}^2)$, see e.g. [26, Theorem 4.8.1]. We consider the cone

$$K := \{U = (u_1, u_2) \in \mathbb{H} : u_2|_{\Omega_0} \ge 0 \text{ and } u_2|_\Gamma \ge 0\}$$

and define operators $A(\lambda), G(\lambda,\,\cdot\,)\colon \mathbb{H} \to \mathbb{H}$ by

$$\langle A(\lambda)U, V\rangle := \int_\Omega \left\langle \begin{pmatrix} \lambda_1 b_{11} & \lambda_1 b_{12} \\ \lambda_2 b_{21} & \lambda_2 b_{22} \end{pmatrix} U(x), V(x) \right\rangle dx,$$

$$\langle G(\lambda, U), V\rangle := \int_\Omega \left\langle \begin{pmatrix} g_1(\lambda, U(x), \nabla U(x)) \\ g_2(\lambda, U(x), \nabla U(x)) \end{pmatrix}, V(x) \right\rangle dx$$

$$+ \int_{\partial\Omega\setminus\Gamma_0} \left\langle \begin{pmatrix} g_3(\lambda, U(x)) \\ g_4(\lambda, U(x)) \end{pmatrix}, V(x) \right\rangle dx,$$

and $M(\lambda,\,\cdot\,)\colon \mathbb{H} \multimap \mathbb{H}$ by

$$M(\lambda, U) := \bigcap_{V\in K} \Big\{ Z \in \mathbb{H} : \langle Z, V\rangle \in \int_{\Omega_0} \left\langle \begin{pmatrix} 0 \\ m_0(\lambda, x, U(x), \nabla U(x)) \end{pmatrix}, V(x) \right\rangle dx +$$

$$\int_\Gamma \left\langle \begin{pmatrix} 0 \\ m_1(\lambda, x, U(x)) \end{pmatrix}, V(x) \right\rangle dx \Big\} :=$$

$$\bigcap_{V=(\widetilde{v},v)\in K} \Big\{ Z = \begin{pmatrix} 0 \\ z \end{pmatrix} \in \mathbb{H} :$$

$$\int_{\Omega_0} \underline{c}_0(\lambda)\underline{m}_0(x, U(x), \nabla U(x)) v(x)\, dx + \int_\Gamma \underline{c}_1(\lambda)\underline{m}_1(x, U(x)) v(x)\, dx \le$$

$$\langle Z, V\rangle \le \int_{\Omega_0} \overline{c}_0(\lambda)\overline{m}_0(x, U(x), \nabla U(x)) v(x)\, dx + \int_\Gamma \overline{c}_1(\lambda)\overline{m}_1(x, U(x)) v(x)\, dx \Big\}.$$

Standard considerations (Green's formula, choice of suitable test functions, etc.) imply that it is natural to define weak solutions of the problem (3.2)/(3.3) as solutions of the inclusion

$$U - A(\lambda)U \in G(\lambda, U) + M(\lambda, U). \tag{3.5}$$

We assumed for simplicity that the nonlinearities g_i (and thus the map G) are single-valued. However, this was only to simplify the formulations of the hypotheses concerning g_i. Essentially, all results in this paper hold also for those multivalued g_i for which the following Proposition 3.1 can be proved.

Proposition 3.1. *If $\Lambda \subseteq P$ is compact, then the hypotheses* (A) *and* (B) *of Section 2 are satisfied.*

Proof. The main result of [24] states that M is upper semicontinuous and compact with nonempty convex values (an analogous result – for functions with values in $\mathbb{R}$ instead of $\mathbb{R}^2$ – has also been proved in [9]). The same result contains as a special case the continuity and compactness of G and thus, as a further special case, the continuity and compactness of A. □

Remark 3.1. Note in particular that the main result of [24] contains sufficient conditions for Proposition 3.1 also when g_i is multivalued and $g_i(\lambda, x,\,\cdot\,)$ are only upper semicontinuous. Roughly speaking, for multivalued g_i, one has to assume that the values of g_i are nonempty and compact (and in the case $N = 1$ where $\partial\Omega$ has atoms one has to assume that the values of g_3 and g_4 are intervals), and one has to replace the continuity and measurability hypotheses by the upper semicontinuity and a certain multivalued measurability hypotheses. Moreover, the dependence on λ should occur in a special form; for details, we refer to [24].

Occasionally, we will also require (for almost all $x \in \Omega$ and almost all $x \in \partial\Omega \setminus \Gamma_0$)

$$\begin{aligned}
\sup_{w,z\in\mathbb{R}^N} \sup_{\lambda\in\Lambda_0} |g_i(\lambda, x, u, v, w, z)| &\le c_{\Lambda_0} \max\big\{(|u| + |v|)^{2p_0/q_0}, |u| + |v|\big\} \quad (i = 1, 2),\\
\lim_{(u,v,w,z)\to 0} \sup_{\lambda\in\Lambda_0} \frac{|g_i(\lambda, x, u, v, w, z)|}{|u| + |v| + \|w\| + \|z\|} &= 0 \qquad (i = 1, 2),\\
\sup_{\lambda\in\Lambda_0} |g_i(\lambda, x, u, v)| &\le c_{\Lambda_0} \max\big\{(|u| + |v|)^{2p_1/q_1}, |u| + |v|\big\} \quad (i = 3, 4),\\
\lim_{(u,v)\to 0} \sup_{\lambda\in\Lambda_0} \frac{|g_i(\lambda, x, u, v)|}{|u| + |v|} &= 0 \qquad (i = 3, 4),
\end{aligned} \tag{3.6}$$

or the corresponding pointwise estimate

$$\begin{aligned}
\sup_{w,z\in\mathbb{R}^N} |g_i(\lambda, x, u, v, w, z)| &\le c_\lambda \max\big\{(|u| + |v|)^{2p_0/q_0}, |u| + |v|\big\} \quad (i = 1, 2),\\
\lim_{(u,v,w,z)\to 0} \frac{g_i(\lambda, x, u, v, w, z)}{|u| + |v| + \|w\| + \|z\|} &= 0 \quad (i = 1, 2),\\
|g_i(\lambda, x, u, v)| &\le c_\lambda \max\big\{(|u| + |v|)^{2p_1/q_1}, |u| + |v|\big\} \quad (i = 3, 4),\\
\lim_{(u,v)\to 0} \frac{g_i(\lambda, x, u, v)}{|u| + |v|} &= 0 \quad (i = 3, 4) \quad \text{for each } \lambda \in \Lambda_0,
\end{aligned} \tag{3.7}$$

where $\Lambda_0 \subseteq P$ will be specified later. (For multivalued g_i, the above assumption has to be understood uniformly for all values). Analogously to [9], we obtain:

Lemma 3.1. *The hypothesis* (3.6) *or* (3.7) *for the functions g_i implies the corresponding hypothesis* (2.3) *or* (2.4)*, respectively, for the operator G.*

Later, we will also require unilateral hypotheses (for all u, v, w, z, λ, x)

$$\begin{aligned}
0 = \underline{c}_0(\lambda)\underline{m}_0(x, u, v, w, z) &= \overline{c}_0(\lambda)\overline{m}_0(x, u, v, w, z) \quad \text{if } v > 0,\\
0 = \underline{c}_0(\lambda)\underline{m}_0(x, u, 0, w, z) &\le \overline{c}_0(\lambda)\overline{m}_0(x, u, 0, w, z),\\
0 \le \underline{c}_0(\lambda)\underline{m}_0(x, u, v, w, z) &\le \overline{c}_0(\lambda)\overline{m}_0(x, u, v, w, z) \quad \text{if } v < 0,\\
0 = \underline{c}_1(\lambda)\underline{m}_1(x, u, v) &= \overline{c}_1(\lambda)\overline{m}_1(x, u, v) \quad \text{if } v > 0,\\
0 = \underline{c}_1(\lambda)\underline{m}_1(x, u, 0) &\le \overline{c}_1(\lambda)\overline{m}_1(x, u, 0),\\
0 \le \underline{c}_1(\lambda)\underline{m}_1(x, u, v) &\le \overline{c}_1(\lambda)\overline{m}_1(x, u, v) \quad \text{if } v < 0.
\end{aligned} \tag{3.8}$$

We will also assume that at the critical level $v = 0$ we have at least a "jump in the v-derivative of the lower bounds" in the sense that

$$\begin{aligned}
\lim_{\substack{(u,v,w,z)\to 0\\ v<0}} \frac{|\underline{m}_0(x, u, v, w, z)|}{v} &= -\infty \quad \text{for almost all } x \in \Omega_0,\\
\lim_{\substack{(u,v)\to 0\\ v<0}} \frac{|\underline{m}_1(x, u, v)|}{v} &= -\infty \qquad \text{for almost all } x \in \Gamma
\end{aligned} \tag{3.9}$$

(see Figure 2). In order to avoid trivialities, we will usually supplement the previous hypothesis with the assumption

$$\operatorname{mes}\Omega_0 > 0 \text{ or } \operatorname{mes}\Gamma > 0 \text{ (or both).} \tag{3.10}$$

The following result is analogous to [9, Lemmas 4.1 and 4.2]. It is somewhat curious that for this result, the assumption (3.10) is not necessary.

Lemma 3.2. *If* (3.8) *holds, then*

$$\langle Z, V\rangle \geq 0 \qquad \text{whenever } V \in K \text{ and } Z \in M(\lambda, U), \tag{3.11}$$

and

$$\langle Z, U\rangle \leq 0 \qquad \text{whenever } Z \in M(\lambda, U). \tag{3.12}$$

If in addition (3.9) *holds, then for each* $\lambda = (\lambda_1, \lambda_2) \in P$ *with* $\underline{c}_0(\lambda) \neq 0 \neq \underline{c}_1(\lambda)$ *the statement* (2.2) *is true with* $\widetilde{\lambda} = \lambda$ *and each* $\Lambda_0 \subseteq P$.

Proof. The inequality (3.11) follows immediately from the fact that the integrals occurring in the definition of M are nonnegative. For $U = (\widetilde{u}, u) \in \mathbb{H}$ we put $U^{\pm} := (\widetilde{u}, u^{\pm})$ (with the usual notation for the positive and negative parts of a function). Note that $U^{\pm} \in K$. Choosing $V = U^+$ in the definition of M, we obtain in particular for each $Z \in M(\lambda, U)$

$$\langle Z, U^+\rangle \leq \int_{\Omega_0} \overline{c}_0(\lambda)\overline{m}_0(x, \widetilde{u}(x), u(x), \nabla U(x))u^+(x)\,dx + \int_\Gamma \overline{c}_1(\lambda)\overline{m}_1(x, \widetilde{u}(x), u(x))u^+(x)\,dx.$$

Assumptions (3.8) imply that the integrands vanish, which in view of (3.11) shows $\langle Z, U^+\rangle = 0$. Hence, in view of (3.11) we obtain that

$$\langle Z, U\rangle = \langle Z, U^+ - U^-\rangle = -\langle Z, U^-\rangle \leq 0,$$

which proves (3.12).

For the last claim, let $\lambda_n \in P$, $U_n = (\widetilde{u}_n, u_n), Y_n = (\widetilde{y}_n, y_n) \in \mathbb{H}$ satisfy $\lambda_n \to \lambda$, $0 < \|U_n\| \to 0$, $Y_n \to Y = (\widetilde{y}, y)$ and $W_n = (\widetilde{w}_n, w_n) := U_n/\|U_n\| \rightharpoonup W = (\widetilde{w}, w)$, where

$$W_n - Y_n \in M(\lambda_n, U_n)/\|U_n\|. \tag{3.13}$$

The first component of all functions in $M(\lambda_n, U_n)$ vanishes so that (3.13) implies that $\|\widetilde{w}_n - \widetilde{y}\|_{W^{1,2}(\Omega)} \to 0$. It follows from (3.13) that

$$\int_{\Omega_0} \frac{\underline{c}_0(\lambda_n)\underline{m}_0(x, U_n(x), \nabla U_n(x))}{\|U_n\|} v(x)\,dx + \int_\Gamma \frac{\underline{c}_1(\lambda_n)\underline{m}_1(x, U_n(x))}{\|U_n\|} v(x)\,dx \leq$$
$$\langle W_n - Y_n, V\rangle \leq \int_{\Omega_0} \frac{\overline{c}_0(\lambda_n)\overline{m}_0(x, U_n(x), \nabla U_n(x))}{\|U_n\|} v(x)\,dx + \int_\Gamma \frac{\overline{c}_1(\lambda_n)\overline{m}_1(x, U_n(x))}{\|U_n\|} v(x)\,dx \tag{3.14}$$

for all n and all $V = (\widetilde{v}, v) \in K$. Note that the hypothesis (3.8) implies that the right-hand side of (3.14) vanishes for $v = w_n^+$. Using (3.14) with $V = W_n^{\pm} = (\widetilde{w}_n, w_n^{\pm})$, we thus obtain

$$\begin{aligned} 1 = \langle W_n, W_n\rangle = \langle W_n - Y_n, W_n^+\rangle - \langle W_n - Y_n, W_n^-\rangle + \langle Y_n, W_n\rangle \leq \\ 0 - \int_{\Omega_0} \frac{\underline{c}_0(\lambda_n)\underline{m}_0(x, U_n(x), \nabla U_n(x))}{\|U_n\|} w_n^-(x)\,dx \\ - \int_\Gamma \frac{\underline{c}_1(\lambda_n)\underline{m}_1(x, U_n(x))}{\|U_n\|} w_n^-(x)\,dx + \langle Y_n, W_n\rangle\,. \end{aligned} \tag{3.15}$$

Since in view of (3.8) both integrands are nonnegative, we obtain $\langle Y_n, W_n\rangle \geq 1$ for all n. Moreover, since $\langle Y_n, W_n\rangle$ is bounded, also the integrals are uniformly bounded.

We claim that this implies $W \in K$. Indeed, if this is false, then either $w^-|_{\Omega_0}$ or $w^-|_\Gamma$ is not almost everywhere 0. We will show that this contradicts the boundedness of the first or second of the above integrals, respectively. Since the arguments are similar, we assume by contradiction without loss of generality that $w^- \neq 0$ on a set $E \subseteq \Omega_0$ of positive measure. Note that $U_n \to 0$ in $\mathbb{H}$ implies in particular that $(U_n, \nabla U_n)$ converges to 0 in L_2; hence, a subsequence converges to 0 almost everywhere. Passing to this subsequence, we thus find in view of (3.9) that the function sequence

$$h_n(x) := \frac{u_n(x)}{\underline{m}_0(x, U_n(x), \nabla U_n(x))}$$

converges to 0 almost everywhere. Shrinking E if necessary, we may assume by Egorov's theorem that the convergence is uniform on E, i.e. for each $k > 0$ there is n_k such that

$$\underline{m}_0(x, U_n(x), \nabla U_n(x)) \geq k\,|u_n(x)|$$

holds almost everywhere on E for all $n \geq n_k$, and so we can estimate the first integral in (3.15) from below by

$$\underline{c}_0(\lambda_n) \int_E \frac{k\,|u_n(x)|}{\|U_n\|} w_n^-(x)\,dx = \underline{c}_0(\lambda_n) \int_E k\,|w_n(x)|\,w_n^-(x)\,dx \geq k\underline{c}_0(\lambda_n) \int_E w_n^-(x)^2\,dx$$

for all $n \geq n_k$. Note now that the compactness of the Sobolev embedding implies in particular $w_n \to w$ in L_2, and so in view of $w^-|_E \neq 0$, the last integral converges to a positive number. Since $\underline{c}_0(\lambda_n) \to \underline{c}_0(\lambda) \neq 0$ and k is arbitrarily large, we find that the first integral in (3.15) is unbounded, which is the required contradiction.

As announced before, we thus have shown $W \in K$. Hence, we can apply (3.14) with $V = W$ and obtain in view of the nonnegativity of the integrals that $\langle W_n - Y_n, W\rangle \geq 0$, i.e. $\langle Y_n, W\rangle \leq \langle W_n, W\rangle \to \langle W, W\rangle = \|W\|^2$, and so

$$\|W\|^2 \geq \langle Y, W\rangle = \langle Y_n, W_n\rangle + \langle Y - Y_n, W_n\rangle - \langle Y, W_n - W\rangle\,.$$

Since we have already shown that $\langle Y_n, W_n\rangle \geq 1$ and since the other two terms tend to 0, we conclude $\|W\|^2 \geq \langle Y, W\rangle \geq 1 = \|W_n\|^2$ which together with $W_n \rightharpoonup W$ implies $W_n \to W$.

As a by-result, $1 = \|W\|^2 \geq \langle Y, W\rangle \geq 1$ implies that $\langle Y - W, W\rangle = \langle Y, W\rangle - \|W\|^2 = 0$. In particular, for each $V \in K$ we have that $\langle W - Y, V - W\rangle = \langle W - Y, V\rangle \geq 0$ in view of (3.14) and the nonnegativity of the first two integrals given by (3.8). Thus, W indeed satisfies the required variational inequality. □

3.2. **(K, A)-interior values and the (K, A, K_0)-sign-condition.** Our aim is now to discuss which values $\lambda_0 = \lambda = T(d)$ have the properties of Definitions 2.1 and 2.2 for our particular situation. Let $(\mu_n, e_n) \in (0, \infty) \times \mathbb{H}_0$ be a complete orthonormal system of eigenvalues and the corresponding eigenfunctions of the operator $A_0 \colon \mathbb{H}_0 \to \mathbb{H}_0$ defined by

$$\langle A_0 u, v\rangle = \int_\Omega u(x)v(x)\,dx.$$

Since A_0 is the operator associated in the weak formulation with the harmonic equation, we can assume by an appropriate numbering that $\mu_n = 1/\kappa_n$ where κ_n is as in Section 1.1. Each $U = (u_1, u_2) \in \mathbb{H}$ has a unique representation as a series $u_i = \sum_{n=1}^\infty \langle u_i, e_n\rangle\, e_n$ $(i = 1, 2)$. In particular, if $\lambda = (\lambda_1, \lambda_2) \in P$ then $U - A(\lambda)U = 0$ (i.e. $U \in E_A(\lambda)$, which means (u_1, u_2) is a weak solution to (1.8)/(1.5) with $(d_1, d_2) = T(\lambda)$) is equivalent to the infinite system

$$(1 - \lambda_1\mu_n b_{11})\,\langle u_1, e_n\rangle - \lambda_1\mu_n b_{12}\,\langle u_2, e_n\rangle = 0,$$
$$-\lambda_2\mu_n b_{21}\,\langle u_1, e_n\rangle + (1 - \lambda_2\mu_n b_{12})\,\langle u_2, e_n\rangle = 0.$$

This has a nontrivial solution (u_1, u_2) for some n if and only if, for that n, the determinant

$$D_n := (1 - \lambda_1\mu_n b_{11})(1 - \lambda_2\mu_n b_{12}) - \lambda_1\mu_n b_{12}\lambda_2\mu_n b_{21}$$

vanishes, i.e. if $T(\lambda)$ lies on the hyperbola (1.6).

Now, let us calculate the solutions of $U - A(\lambda)U = 0$ in such a case. Let us assume for a moment that $d = T(\lambda)$ lies on only one of the hyperbolas $C_n = \cdots = C_{n+k-1}$, and so $D_j = 0$ only for $j = n, \ldots, n+k-1$, where k is the multiplicity of κ_n, i.e. $\mu_n = \cdots = \mu_{n+k-1} =: \mu$. In this case, each solution of $U = A(\lambda)U$, i.e. each $U \in E_A(\lambda)$, is given by

$$U = \begin{pmatrix} \alpha \\ 1 \end{pmatrix} e,$$

where $e \in \mathbb{H}_0$ satisfies $A_0 e = \mu e$ with $\mu = \mu_n$ and α is a solution of the (necessarily linearly dependent) system

$$\begin{pmatrix} 1 - \lambda_1 \mu b_{11} & -\lambda_1 \mu b_{12} \\ -\lambda_2 \mu b_{21} & 1 - \lambda_2 \mu b_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = 0. \tag{3.16}$$

Similarly, for the choice

$$U^* = \begin{pmatrix} \alpha^* \\ 1 \end{pmatrix} e, \tag{3.17}$$

where α^* is a solution of the "adjoint" equation

$$\begin{pmatrix} 1 - \lambda_1 \mu b_{11} & -\lambda_2 \mu b_{21} \\ -\lambda_1 \mu b_{12} & 1 - \lambda_2 \mu b_{22} \end{pmatrix} \begin{pmatrix} \alpha^* \\ 1 \end{pmatrix} = 0, \tag{3.18}$$

a straightforward calculation shows $U^* - A(\lambda)^* U^* = 0$, i.e. U^* is an eigenvector of the adjoint operator $A(\lambda)^*$ which has the same second component as U.

Now let $d = T(\lambda)$ be an intersection point of two different hyperbolas C_n and C_m, that means $D_j = 0$ only for $j = n, \ldots, n+k-1$ and $j = m, \ldots, m+\ell-1$, where k and ℓ are the multiplicities of κ_n and κ_m, respectively. We have $\kappa_n \neq \kappa_m$, $C_n = \cdots = C_{n+k-1} \neq C_m = \cdots = C_{m+\ell-1}$, $\mu_n = \cdots = \mu_{n+k-1} \neq \mu_m = \cdots = \mu_{m+\ell-1}$. In this case, all solutions of $A(\lambda)U = U$ have the form

$$U = \bar{\xi} \begin{pmatrix} \alpha_n \\ 1 \end{pmatrix} \bar{e} + \widetilde{\xi} \begin{pmatrix} \alpha_m \\ 1 \end{pmatrix} \widetilde{e}, \tag{3.19}$$

where $\bar{\xi}, \widetilde{\xi} \in \mathbb{R}$, $A_0 \bar{e} = \mu_n \bar{e}$, $A_0 \widetilde{e} = \mu_m \widetilde{e}$, and where α_n and α_m are the solutions α of (3.16) when $\mu := \mu_n$ or $\mu := \mu_m$, respectively. Note that, since A_0 is selfadjoint, the eigenfunctions $\bar{e}$ and $\widetilde{e}$ are automatically orthogonal to each other (in the space $\mathbb{H}_0$). With a function U as above we associate

$$U^* = \bar{\xi} \begin{pmatrix} \alpha_n^* \\ 1 \end{pmatrix} \bar{e} + \widetilde{\xi} \begin{pmatrix} \alpha_m^* \\ 1 \end{pmatrix} \widetilde{e}, \tag{3.20}$$

where α_n^*, α_m^* are the solutions α^* of (3.18) when $\mu := \mu_n$ or $\mu := \mu_m$, respectively. Then U^* has the same second component as U, and similarly to the above, one calculates $A(\lambda)^* U^* = U^*$,

Summarizing, we have seen that the solutions $U \in E_A(\lambda)$ and their associated functions $U^* \in E_{A^*}(\lambda)$ are uniquely determined by their second component. Moreover, a function e occurs as the second component of some (unique) $U \in E_A(\lambda)$ if and only if it is a linear combination of (one or two) eigenfunctions of A_0 to eigenvalues from $\{\mu_n : T(\lambda) \in C_n)\}$, i.e. of eigenfunctions of $-\Delta$ with (1.5) to eigenvalues from $\{\kappa_n : T(\lambda) \in C_n\}$.

Further, we will consider points $d \in \bigcup_n C_n$ such that

$$\left|\begin{array}{l} \text{a suitable linear combination } e \text{ of eigenfunctions of } -\Delta \text{ with (1.5) to the} \\ \text{eigenvalues from } \{\kappa_j : d \in C_j\} \text{ satisfies either} \\ \quad \text{(a) } e \geq \varepsilon > 0 \text{ in } \Omega_0 \text{ and on } \Gamma, \text{ or} \\ \quad \text{(b) } e > 0 \text{ on } \Gamma. \end{array}\right| \tag{3.21}$$

Let us note that if d lies only on one hyperbola (i.e. $d \in C_n$, $d \notin C_m$ for all $C_m \neq C_n$ – cf. the considerations above) and (3.21) holds then it is fulfilled for all $\widetilde{d} \in C_n$. If $d \in C_n \cap C_m$, $C_n \neq C_m$, then it can happen that d is the only point from $C_n \cup C_m$ satisfying (3.21).

Proposition 3.2. *If $d = T(\lambda)$ satisfies* (3.21)(a)*, then λ is a (K, A)-interior value.*

Proof. Let e be the linear combination satisfying (3.21)(a), and let U be the corresponding (unique) function pair of $E_A(\lambda)$ whose second component is e. Then the associated function $v^* := U^*$, defined by (3.17) or (3.20) as explained above, belongs to the pseudo-interior of K, i.e. $\overline{D_K(v^*)} = \mathbb{H}$. Indeed, since the second component e of $v^* = U^*$ satisfies (3.21)(a), the set $D_K(v^*)$ contains all smooth functions from $\mathbb{H}$. Cf. also the notes after Lemma 2.1. □

Proposition 3.3. *Let $\Omega_0 = \varnothing$ and Γ be a smooth manifold with boundary. If $d = T(\lambda)$ satisfies* (3.21)(b)*, then λ is a (K, A)-interior value.*

Proof. In [15], the property (2.6) was verified in this case. Hence, the assertion follows from Lemma 2.1. □

Remark 3.2. Probably it is true that under suitable conditions on Ω_0, Proposition 3.3 extends also to the case $\Omega_0 \neq \varnothing$, i.e. if only $e > 0$ on Γ and on Ω_0, then λ is (K, A)-interior. However, such statements depend on delicate extension results which we will not discuss here any further.

Now we will consider a fixed $\lambda_0 = (\lambda_1, \lambda_2)$ with $T(\lambda_0)$ lying on some hyperbola C_n from (1.6) and derive conditions for Λ_0 guaranteeing that λ_0 satisfies the (K, A, K_0)-sign-condition.

First, let $T(\lambda_0)$ lie only on one hyperbola and let $U = \begin{pmatrix} \alpha \\ 1 \end{pmatrix} e \in E_A(\lambda_0) \cap K$ be given (see considerations above). Denote by $U^* \in E_A(\lambda_0) \cap K$ the corresponding function from (3.17). For each point $\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2) \in \mathbb{R}^2$ for which $T(\widetilde{\lambda})$ does not lie on any hyperbola (1.6) we have that 1 is not an eigenvalue of $A(\widetilde{\lambda})$, and thus also 1 is not an eigenvalue of $A(\widetilde{\lambda})^*$. Hence,

$$\begin{aligned} 0 \neq \|(id - A(\widetilde{\lambda})^*)U^*\| &= \|(A(\lambda_0)^* - A(\widetilde{\lambda})^*)U^*\| \\ &= \left\| \begin{pmatrix} (\lambda_1 - \widetilde{\lambda}_1)\mu b_{11} & (\lambda_2 - \widetilde{\lambda}_2)\mu b_{21} \\ (\lambda_1 - \widetilde{\lambda}_1)\mu b_{12} & (\lambda_2 - \widetilde{\lambda}_2)\mu b_{22} \end{pmatrix} \begin{pmatrix} \alpha^* \\ 1 \end{pmatrix} \right\| \leq C \, \|\lambda_0 - \widetilde{\lambda}\| \end{aligned} \tag{3.22}$$

for some constant C which is independent of $\widetilde{\lambda}$. Moreover, we calculate

$$\begin{aligned} \langle (id - A(\widetilde{\lambda}))U, U^* \rangle &= \langle U, U^* \rangle - \langle A(\widetilde{\lambda})U, U^* \rangle \\ &= \left\langle \begin{pmatrix} \alpha \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha^* \\ 1 \end{pmatrix} \right\rangle - \left\langle \mu \begin{pmatrix} \widetilde{\lambda}_1 b_{11} & \widetilde{\lambda}_1 b_{12} \\ \widetilde{\lambda}_2 b_{21} & \widetilde{\lambda}_2 b_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha^* \\ 1 \end{pmatrix} \right\rangle; \end{aligned}$$

observing that (3.16) means $\lambda_1\mu(b_{11}\alpha + b_{12}) = \alpha$ and $\lambda_2\mu(b_{21}\alpha + b_{22}) = 1$, we thus obtain

$$\begin{aligned}\langle (id - A(\widetilde{\lambda}))U, U^*\rangle &= (1 + \alpha\alpha^*) - (\lambda_1^{-1}\widetilde{\lambda}_1\alpha\alpha^* + \lambda_2^{-1}\widetilde{\lambda}_2)\\ &= (1 - \lambda_1^{-1}\widetilde{\lambda}_1)\alpha\alpha^* + (1 - \lambda_2^{-1}\widetilde{\lambda}_2).\end{aligned}$$

This expression is negative if $\widetilde{\lambda}$ lies above the line L passing through λ_0 with the slope $-\lambda_1^{-1}\lambda_2\alpha\alpha^*$ and

$$\langle (id - A(\widetilde{\lambda}))U, U^*\rangle < -\widetilde{C}\,\|\lambda_0 - \widetilde{\lambda}\|$$

with some constant $\widetilde{C} > 0$ independent of $\widetilde{\lambda}$ for all $\widetilde{\lambda}$ from a circular sector as sketched in Figure 4 with positive angles of its two arms with L. This together with (3.22) imply that the hypothesis (2.7) is satisfied with $K_0 := E_{A^*}(\lambda_0) \cap K$, $\delta := \widetilde{C}C^{-1}$ and with any $\Lambda_0 \subseteq P$ whose intersection with some neighborhood of λ_0 is contained in such a circular sector. This means that λ_0 satisfies the (K, A, K_0)-sign-condition.

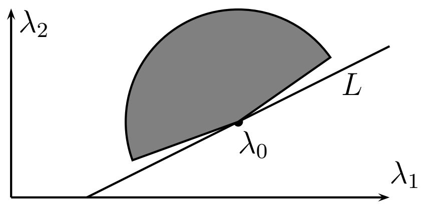

FIGURE 4. Line L and the circular sector in the (λ_1, λ_2)-plane

The line L is the graph of the function

$$L(x) := \lambda_2 - (x - \lambda_1)\lambda_1^{-1}\lambda_2\alpha\alpha^*.$$

The map T transforms this line into the graph of the function $f(x) := 1/L(1/x)$. At the point $T(\lambda_0)$, this graph has the slope

$$f'(1/\lambda_1) = L(\lambda_1)^{-2}L'(\lambda_1)\lambda_1^2 = -\frac{\lambda_1}{\lambda_2}\alpha\alpha^* = -\frac{\lambda_1}{\lambda_2}\cdot\frac{\lambda_1\lambda_2\mu^2 b_{12}b_{21}}{(1 - \lambda_1\mu b_{11})^2} = \frac{-\mu^2 b_{12}b_{21}}{(\lambda_1^{-1} - \mu b_{11})^2}.$$

This is exactly the slope of the tangent to the corresponding hyperbola (1.6) passing through $d = T(\lambda_0)$. Hence, in a sufficiently small neighborhood of λ_0, the circular sector in Figure 4 is transformed by T into a set which is contained in a circular sector as sketched in Figure 5, and vice versa. The only condition on this sector is that its arms have to form a positive angle with the tangent to the hyperbola at the point d.

Summarizing, we have proved the following result.

Lemma 3.3. *Let d be a point of a hyperbola from* (1.6) *which is not an intersection point of two such different hyperbolas (although it may lie on a family of coinciding hyperbolas). Suppose that $\Lambda_0 \subseteq P$ is such that the intersection of $T(\Lambda_0)$ with some neighborhood of d is contained in a circular sector as sketched in Figure 5 (i.e. with positive angles to the tangent). Then $\lambda_0 = T(d)$ satisfies the (K, A, K_0)-sign-condition on Λ_0 with $K_0 := E_{A^*}(\lambda_0) \cap K$.*

Now let d be an intersection point of two different hyperbolas C_n and C_m. Since the functions $\bar{e}$ and $\widetilde{e}$ in (3.19) are orthogonal to each other, a straightforward extension of the above calculation shows that the estimate (2.7) remains valid when Λ_0 is such that $T(\Lambda_0)$ is contained in the intersection of the circular sectors R_n and R_m with the vertex at $d = T(\lambda)$

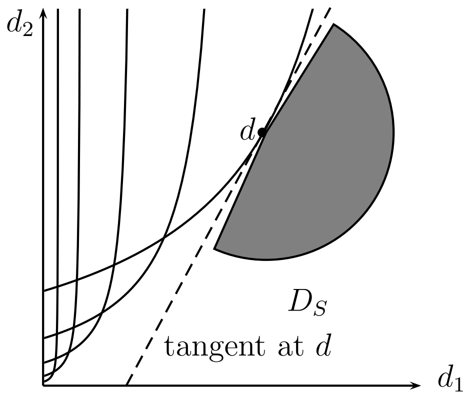

FIGURE 5. (d_1, d_2)-domain where (2.7) holds if $d = T(\lambda_0)$ belongs only to one hyperbola

such that R_j is contained strictly (except for the vertex d) in the open half-plane below the tangent to the hyperbola C_j at the point d. We thus obtain the following generalization of Lemma 3.3.

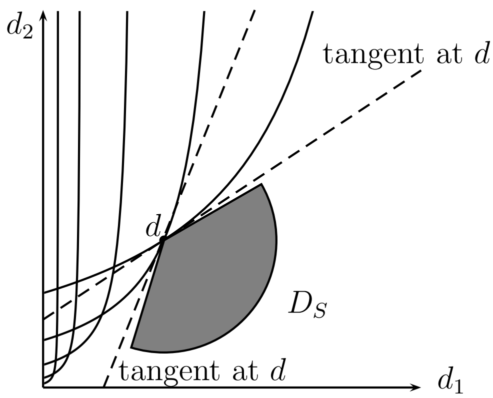

FIGURE 6. (d_1, d_2)-domain where (2.7) holds if $d = T(\lambda_0)$ is an intersection point of hyperbolas

Lemma 3.4. *Let d be a point on at least one hyperbola from* (1.6)*. Suppose that $\Lambda_0 \subseteq P$ is such that the intersection of $T(\Lambda_0)$ with some neighborhood of d is contained in a circular sector as sketched in Figure 6 (i.e. the arms form positive angles with all tangents to the hyperbolas to which d belongs). Then $\lambda_0 = T(d)$ satisfies the (K, A, K_0)-sign-condition on Λ_0 with $K_0 := E_{A^*}(\lambda_0) \cap K$.*

Using the same arguments as in [16, Lemma 2.3], only replacing in its proof the application of [16, Lemma 2.2] by (2.6), one can obtain the following result:

Proposition 3.4. *Let $C \subseteq \partial D_S$ be the family of all points $d \in \overline{D}_S \cap \bigcup_n C_n$ for which $\lambda = T(d)$ is (K, A)-interior. Then there is an open neighborhood $W \subseteq \mathbb{R}^2$ of C such that for each $d \in W \cap D_S$ the corresponding variational inequality* (2.8) *has for $\lambda = T(d)$ only the trivial solution.*

By a *local bifurcation point* of (3.2)/(3.3) we mean λ_0 such that in any neighborhood of $(\lambda_0, 0)$ in $P \times \mathbb{H}$ there is a weak solution (λ, u), $u \neq 0$, of (3.2)/(3.3).

Theorem 3.1. *Assume the unilateral sign conditions* (3.8) *and the nondegeneration hypotheses* (3.9)/(3.10). *Let* $C \subseteq \partial D_S \cap \bigcup_n C_n$ *be a set such that each* $d \in C$ *satisfies* (3.21)(a) *or, in the case when* Γ *is a smooth manifold with boundary and* $\Omega_0 = \varnothing$, *only* (3.21)(b).

Let $W \subseteq \mathbb{R}^2_+$ *contain a neighborhood (in* $\mathbb{R}^2_+$*) of* C *and be such that* $T(W \cap D_S) \subseteq P$ *and that* $\underline{c}_0$ *and* $\underline{c}_1$ *have no zero on* $T(W \cap D_S)$. *Suppose also that each point* $\lambda \in T(W \cap D_S)$ *has an open neighborhood* Λ_0 *in* P *with* (3.6).

Then there is an open set $W_0 \subseteq W$, $C \subseteq W_0$, *with the following two properties:*

(1) *No point in* $T(W_0 \cap D_S)$ *is a local bifurcation point of* (3.2)/(3.3).

(2) *For each* $\lambda \in T(W_0 \cap D_S)$ *and all* $\rho \in (0, r(\lambda))$

$$\deg(id - A(\lambda) - G(\lambda, \cdot) - M(\lambda, \cdot), B_\rho, 0) = 0. \tag{3.23}$$

In fact, Theorem 3.1 remains valid if we replace the assumption (3.21) by the hypothesis that all points $d \in C$ are (K, A)-interior values and satisfy $E_{A^*}(T(d)) \cap K \setminus (-K) \neq \varnothing$. However, the only way we know how to verify this abstract hypothesis in our situation is by using (3.21) and Proposition 3.2 or Proposition 3.3, perhaps also Remark 3.2.

Proof. We may assume that W is open. Clearly, it is sufficient to prove that each point $d \in C$ has an open neighborhood W_0 with all properties (except of $C \subseteq W_0$) required. Hence, without loss of generality, we may assume that C consists of a single point d_0.

We can assume that $T(W)$ is bounded and set $\Lambda := \overline{T(W)}$. Then Λ is compact and the hypotheses (A) and (B) are fulfilled by Proposition 3.1.

For the proof of (2) we will verify the assumptions of Theorem 2.2. The value $\lambda_0 = T(d_0)$ is (K, A)-interior by Proposition 3.2 or Proposition 3.3 under the assumption (3.21)(a) or (3.21)(b), respectively.

Set $K_0 = E_{A^*}(\lambda_0) \cap K$. Then $K_0 \neq -K_0$. Indeed, let $U \in E_A(\lambda_0)$ be the (unique) function pair with second component e with e from the assumption (3.21), and let U^* be the associated function, defined by (3.17)/(3.18) or (3.20)/(3.18) with $(\lambda_1, \lambda_2) = \lambda_0$ and $\mu = \mu_n$ or $\mu = \mu_m$, respectively. Then $U^* \in K_0$, but in view of (3.10) and the hypothesis on e we have $-U^* \notin K$, and so $U^* \notin -K_0$. Hence, the assumption about the existence of u_0, u_0^* satisfying (2.11) follows from Remark 2.3.

Due to Proposition 3.4 we can choose an open $W_0 \subseteq W$ with $d_0 \in W_0$ such that $W_0 \cap D_S$ is connected and that the variational inequality (2.8) has for all $\lambda \in T(W_0 \cap D_S)$ only the trivial solution.

In the first step, denote by Λ_0 the intersection of $T(W_0 \cap D_S)$ with a circular sector associated with $d_0 = T(\lambda_0)$ in Lemma 3.3 or Lemma 3.4, i.e. as in Figure 5 or 6. Then λ_0 satisfies the (K, A, K_0)-sign-condition on Λ_0 by Lemma 3.4, the condition (2.3) (therefore also (2.4)) follows from the assumption (3.6) by Lemma 3.1. The assumptions (2.2) and (2.12) follow from (3.8) and (3.9), respectively, by Lemma 3.2.

Hence, Theorem 2.2 implies that if $\lambda \in \Lambda_0$ then (3.23) holds for sufficiently small $\rho > 0$.

In the second step, consider $\widetilde{\lambda} \in T(W_0 \cap D_S) \setminus \Lambda_0$. Lemma 3.2 implies that any $\lambda \in T(W_0 \cap D_S)$ satisfies (2.2), now even with $\Lambda_0 = T(W_0 \cap D_S)$. The set $\Lambda_1 = T(W_0 \cap D_S)$ satisfies the assumptions of Theorem 2.1 by virtue of the properties of W_0. Since it is connected and contains $\widetilde{\lambda}$ and some λ_1 in the sector considered above, the last assertions of Theorem 2.1 and Remark 2.2 imply that (3.23) holds also for $\widetilde{\lambda}$ and all sufficiently small $\rho > 0$. Hence, (2) is proved.

For the proof of (1), let us consider an arbitrary $\widetilde{\lambda} \in T(W_0 \cap D_S)$. By the assumptions, there is an open neighborhood $\Lambda_0 \subseteq P$ of $\widetilde{\lambda}$ satisfying (3.6). Lemma 3.1 implies that (2.3) holds on this Λ_0. Hence, the set $\Lambda_0 \cap \Lambda_1$ with $\Lambda_1 := T(W_0 \cap D_S)$ satisfies the assumptions of Corollary 2.1, which guarantees that $\widetilde{\lambda}$ is not a bifurcation point, and (1) follows. □

Remark 3.3. Theorem 2.1 in [7] implies that if $\lambda = T(d)$ with $d = (d_1, d_2)$, $d_1 > \frac{b_{11}}{\kappa_1}$, then the variational inequality (2.8) has no nontrivial solution. In fact, this assertion is proved in [7] for an inclusion problem

$$u - \lambda_1(b_{11}Au + b_{12}Av) = 0,$$
$$v - \lambda_2(b_{21}Au + b_{22}Av) \in M_0(v)$$

with a certain positively homogeneous multivalued map M_0 instead of (2.8), but it is clear from the definition of M_0 that the problem is equivalent to our variational inequality (2.8). (The map M_0 is in a certain sense a linearization, or more exactly, a homogenization of M, but this is not essential for us – now we are interested just in the properties of the inequality (2.8) itself.) If we use this result for the coefficients b_{ij} replaced by tb_{ij} with some $t \in (0, 1)$, then we obtain that the variational inequality

$$W \in K,\ \langle W - tA(\lambda)W, V - W\rangle \geq 0 \text{ for all } V \in K \tag{3.24}$$

has no nontrivial solution for all $d_1 > t\frac{b_{11}}{\kappa_1}$, i.e. in particular, for all $d_1 > \frac{b_{11}}{\kappa_1}$.

Lemma 3.5. *Let $\widetilde{\lambda} = T(d)$ with $d = (d_1, d_2)$, $d_1 > \frac{b_{11}}{\kappa_1}$, and let $\underline{c}_0(\widetilde{\lambda}) \neq 0 \neq \underline{c}_1(\widetilde{\lambda})$. Suppose that* (3.8)*,* (3.9) *and* (3.6) *with some Λ_0 hold, $\widetilde{\lambda} \in \overline{\Lambda_0}$.*

Then there is $r > 0$ such that for all $\rho \in (0, r)$ and $\lambda \in \Lambda_0$ sufficiently close to $\widetilde{\lambda}$ we have

$$\deg(id - A(\lambda) - G(\lambda, \cdot) - M(\lambda, \cdot), B_\rho, 0) = 1. \tag{3.25}$$

If the set Λ_0 from the assumption (3.6) *is an open neighborhood in P of $\widetilde{\lambda}$ then $\widetilde{\lambda}$ is not a local bifurcation point of* (3.2)/(3.3)*.*

Proof. First, we will prove that there is $r > 0$ such that

$$U - tA(\lambda)U - tG(\lambda, U) \notin M(\lambda, U) \tag{3.26}$$

for all $t \in [0, 1]$, $U \in \overline{B}_r \setminus \{0\}$ and $\lambda \in \Lambda_0$ sufficiently close to $\widetilde{\lambda}$. Assume by contradiction that there are λ_n, U_n and t_n such that $0 < \|U_n\| \to 0$, $0 \leq t_n \to t \in [0, 1]$, $\lambda_n \to \widetilde{\lambda}$, $W_n := U_n/\|U_n\| \rightharpoonup W$ and

$$W_n - t_nA(\lambda_n)W_n - t_n\frac{G(\lambda_n, U_n)}{\|U_n\|} \in \frac{M(\lambda_n, U_n)}{\|U_n\|}.$$

Since the hypothesis (A) is fulfilled with any compact $\Lambda \subseteq P$ by Proposition 3.1 and (2.3) holds by Lemma 3.1, we have

$$Y_n := t_nA(\lambda_n)W_n + t_n\frac{G(\lambda_n, U_n)}{\|U_n\|} \to tA(\lambda)W.$$

Hence, (2.2) and (2.3) (which holds true by (3.8), (3.9), (3.6) and Lemmas 3.2 and 3.1) implies that $W_n \to W$ and that W satisfies the variational inequality (3.24) with $\lambda = \widetilde{\lambda}$. Since $\|W\| = 1$, this contradicts Remark 3.3 and the assertion (3.26) is proved. Consequently, by the homotopy invariance of the degree, for all $\rho \in (0, r)$ and $\lambda \in \Lambda_0$ sufficiently close to $\widetilde{\lambda}$

we have

$$\deg(id - A(\lambda) - G(\lambda,\cdot) - M(\lambda,\cdot), B_\rho, 0) = \deg(id - M(\lambda,\cdot), B_\rho, 0).$$

To see (3.25), it suffices to show that the homotopy

$$H(t,U) := U - tM(\lambda, U)$$

has no zero on $[0,1] \times (B_r \setminus \{0\})$ and to realize that $\deg(id, B_\rho, 0) = 1$. We have even $0 \notin H(t,U)$ for each $t \geq 0$ and each $U \neq 0$, because for each $Z \in M(\lambda, U)$ we have by (3.12) that

$$\langle U - tZ, U\rangle = \langle U, U\rangle - t\,\langle Z, U\rangle \geq \langle U, U\rangle > 0,$$

which in particular implies $U - tZ \neq 0$. Hence, (3.25) is proved.

If Λ_0 is an open neighborhood in P of $\widetilde{\lambda}$ then (3.26) holds for $t = 1$, all $U \in \overline{B}_r \setminus \{0\}$ and λ sufficiently close to $\widetilde{\lambda}$, i.e. $\widetilde{\lambda}$ cannot be a bifurcation point. □

3.3. **Main Result.**

Theorem 3.2. *Let the conditions* (3.8) *and* (3.9)/(3.10) *be fulfilled. Let* $C \subseteq \partial D_S \cap \bigcup_n C_n$ *and* W *be such that the hypotheses of Theorem 3.1 are satisfied. Let* W_0 *be as in that theorem. Let* $\sigma = (\sigma_1, \sigma_2)\colon I \to T(P)$ *be a continuous path with some closed (not necessarily bounded) interval* I *such that*

$$\{\sigma(s) : s \in I,\ \sigma_1(s) > \frac{b_{11}}{\kappa_1}\} \subseteq W. \tag{3.27}$$

Let $s_0, s_1 \in I$*,* $s_0 < s_1$*, be such that* $\sigma(s_0) \in W_0 \cap D_S$ *and* $\sigma_1(s_1) > \frac{b_{11}}{\kappa_1}$*.*

Then there occurs a global bifurcation on (s_0, s_1) *of* (3.2)/(3.3) *along the path* $T \circ \sigma$ *in the following sense.*

There is a connected subset $B \subseteq I \times (\mathbb{H} \setminus \{0\})$ *of nontrivial solutions (i.e. for each* $(s, u, v) \in B$ *the pair of functions* $(u, v) \neq 0$ *is a weak solution of* (3.2)/(3.3) *with* $\lambda = T(\sigma(s))$*) such that the closure* $\overline{B}$ *in* $\mathbb{R} \times \mathbb{H}$ *contains a point* $(s_B, 0, 0)$ *with* $s_B \in (s_0, s_1)$*,* $\sigma_1(s_B) \leq \frac{b_{11}}{\kappa_1}$*. Moreover,* B *satisfies at least one of the following properties:*

1. *B is unbounded or reaches the end of the path σ, i.e. contains a point of the form (a, u, v) with $a \in \partial I$.*
2. *B returns to the trivial solution outside $[s_0, s_1]$, i.e. there is a point $s_2 \in I \setminus [s_0, s_1]$ such that $(s_2, 0, 0) \in \overline{B}$ and $\sigma_1(s_2) \leq \frac{b_{11}}{\kappa_1}$.*

Of course, by using the transformation T we can reformulate Theorem 3.2 for the original system (1.9)/(1.2) to describe its global bifurcation along the path σ arising between a neighborhood of some point $d \in \partial D_S \cap \bigcup_n C_n$ satisfying (3.21), and the region $d_1 > \frac{b_{11}}{\kappa_1}$. In this way we would obtain an exact form of Theorem 1.1. Let us do it in Corollary 3.1 below very briefly, without exactly reformulating all assumptions about the functions g_i, m_i to the assumptions about f_i, ω_i. Hence, consider the original system (1.9)/(1.2) with given functions f_i ($i = 1, \dots, 4$), ω_i ($i = 1, 2$) defined for $d \in Q$, $Q \subseteq \mathbb{R}^2_+$, and the corresponding functions g_i ($i = 1, \dots, 4$), m_i ($i = 1, 2$) defined by (3.1) for $\lambda \in P = T(Q)$.

Corollary 3.1. *Assume that the functions* g_i ($i = 1, \dots, 4$)*,* m_i ($i = 1, 2$) *defined by* (3.1)*, the sets* $C \subseteq \partial D_S \cap \bigcup_n C_n$*,* W *and the curve* σ *satisfy the assumptions of Theorem 3.2.*

Then there occurs a global bifurcation on (s_0, s_1) *of* (1.9)/(1.2) *along the path* σ *in the following sense.*

There is a connected subset $B \subseteq I \times (\mathbb{H} \setminus \{0\})$ *of nontrivial weak solutions of* (1.10)/(1.2) *such that the closure* $\overline{B}$ *in* $\mathbb{R} \times \mathbb{H}$ *contains a point* $(s_B, 0, 0)$ *with* $s_B \in (s_0, s_1)$ *and* $\sigma_1(s_B) \le \frac{b_{11}}{\kappa_1}$. *Moreover,* B *satisfies at least one of the following properties:*

(1) B *is unbounded or reaches the end of the path* σ, *i.e. contains a point of the form* (a, u, v) *with* $a \in \partial I$.
(2) B *returns to the trivial solution outside* $[s_0, s_1]$, *i.e. there is a point* $s_2 \in I \setminus [s_0, s_1]$ *with* $\sigma_1(s_2) \le \frac{b_{11}}{\kappa_1}$ *such that* $(s_2, 0, 0) \in \overline{B}$.

Remark 3.4. Let us assume that Ω_0 is open and Γ is relatively open in $\partial\Omega$. The eigenfunction e_1 of the Laplacian corresponding to the principal eigenvalue κ_1 is continuous in $\overline{\Omega}$, positive on $\Omega \cup (\partial\Omega \setminus \overline{\Gamma}_0)$. Hence, the assumption (3.21)(b) is always fulfilled for all $d \in C_1$, and (3.21)(a) is fulfilled for all $d \in C_1$ if $\overline{\Omega}_0 \cap \overline{\Gamma}_0 = \varnothing$ and $\overline{\Gamma} \cap \overline{\Gamma}_0 = \varnothing$.

Remark 3.5. The curve (straight line) $\sigma = (\sigma_1, \sigma_2) = (s^2 d_1, s^2 d_2)$ (with fixed (d_1, d_2)) corresponding to the growth of the domain mentioned in Section 1.2 satisfies the assumptions of Theorem 3.2 if $\frac{d_2}{d_1}$ is slightly less than the slope S from (1.7) of the common tangent to our hyperbolas. In this case Theorem 3.2 guarantees the existence of a global bifurcation for the system (3.2)/(3.3) along $T(\sigma)$, i.e. a global bifurcation for the original system (1.9)/(1.2) along σ. Simultaneously, there is no bifurcation for the corresponding classical problem (with $m_0 = m_1 = \{0\}$) because the whole σ lies in D_S and the classical system (1.9)/(1.2) with $\omega_0 = \omega_1 = \{0\}$ has no bifurcation in the domain of stability of its trivial solution.

Remark 3.6. In general, if the assumptions of Theorem 3.2 are fulfilled then there are $\widetilde{s} \in (s_0, s_1)$ and $\varepsilon > 0$ such that $\sigma_1(\widetilde{s}) = \frac{b_{11}}{\kappa_1}$, $\sigma_1(s) > \frac{b_{11}}{\kappa_1}$ for all $s \in (\widetilde{s}, \widetilde{s} + \varepsilon)$. Let $\overline{s}$ be the smallest such $\widetilde{s}$. We can take s_1 arbitrarily close to $\overline{s}$ to get in fact $s_B \in (s_0, \overline{s}]$ in the assertion of Theorem 3.2. In other words, there is a global bifurcation for the original system between a neighborhood of some point $d \in \partial D_S \cap \bigcup_n C_n$ satisfying (3.21) and the first real intersection (not only touching) point of σ with the asymptote to C_1.

Moreover, if $\sigma_1(s) > \frac{b_{11}}{\kappa_1}$ for all $s > s_1$ then in the case (2) of Theorem 3.2, we have actually $s_2 < s_0$ (because $\sigma_1(s_2) \le \frac{b_{11}}{\kappa_1}$).

The proof of Theorem 3.2 bases on a Rabinowitz type bifurcation result from [25]. For simplicity, we formulate here only a special case which we will use.

Theorem 3.3. *Let* I *be a closed interval and* $\varphi \colon I \times \mathbb{H} \multimap \mathbb{H}$ *an upper semicontinuous and compact map with nonempty closed convex values. Let*

$$\mathrm{Fix}(\varphi) := \{(s, U) \in I \times \mathbb{H} : U \in \varphi(s, U)\}$$

and let $s_0, s_1 \in I$, $s_0 < s_1$, *be such that there are* $r > 0$ *and* $\varepsilon > 0$ *satisfying*

$$\mathrm{Fix}(\varphi) \cap (([s_0 - \varepsilon, s_0] \cup [s_1, s_1 + \varepsilon]) \times (\overline{B}_r \setminus \{0\})) = \varnothing \tag{3.28}$$

and

$$\deg(id - \varphi(s_0, \cdot), B_r, 0) \ne \deg(id - \varphi(s_1, \cdot), B_r, 0). \tag{3.29}$$

Then $\mathrm{Fix}(\varphi) \setminus (I \times \{0\})$ *contains a connected set* B *such that* $\overline{B} \cap ([s_0, s_1] \times \{0\}) \ne \varnothing$ *and at least one of the following holds:*

(1) B *is unbounded or contains a point from* $(\partial I) \times \mathbb{H}$.
(2) $\overline{B}$ *contains a point of the form* $(s_2, 0)$ *with* $s_2 \notin [s_0 - \varepsilon, s_1 + \varepsilon]$.

Proof of Theorem 3.3. Apply [25, Theorem 7] with $\Omega := X := Y := \mathbb{H}$, $F := id$, the "coincidence degree" $\deg_{F,\varphi}(s,\Omega_0) := \deg(id - \varphi(s,\,\cdot\,),\Omega_0,0)$, and $\mathcal{B}_0$ being the family of all bounded subsets of $\mathbb{R}\times\mathbb{H}$. Note that for each closed set $B \in \mathcal{B}_0$ the intersection $\mathrm{Fix}(\varphi)\cap B$ is indeed compact, because φ is compact and upper semicontinuous. □

Proof of Theorem 3.2. We set

$$\varphi(s,U) := A(T(\sigma(s)),U) + G(T(\sigma(s)),U) + M(T(\sigma(s)),U)$$

and verify the assumptions of Theorem 3.3. We have $\sigma(s)\in W_0\cap D_S$ for all $s\in I\cap[s_0-\varepsilon,s_0+\varepsilon]$ and $\sigma_1(s) > \frac{b_{11}}{\kappa_1}$ for all $s\in I\cap[s_1-\varepsilon,s_1+\varepsilon]$. The assertion (1) of Theorem 3.1 and the last assertion of Lemma 3.5 (with Λ_0 an open neighbourhood of $\widetilde{\lambda} := T(\sigma(s_1))$ satisfying (3.6) ensured by the last assumption about W from Theorem 3.1) imply the existence of $r>0$ such that

$$\mathrm{Fix}(\varphi)\cap(([s_0-\varepsilon,s_0+\varepsilon]\cup[s_1-\varepsilon,s_1+\varepsilon])\times(\overline{B}_r\setminus\{0\})) = \varnothing. \tag{3.30}$$

In particular, (3.28) is fulfilled. Furthermore, the assertion (2) of Theorem 3.1 and Lemma 3.5 imply that

$$\deg(id-\varphi(\sigma(s_i),\,\cdot\,),B_\rho,0) = i, \qquad i=0,1,\ \text{for all } \rho\le r,$$

i.e. (3.29) holds. Hence, Theorem 3.3 guarantees the existence of a set B with all the properties announced in Theorem 3.2 with $s_B\in[s_0,s_1]$ instead of $s_B\in(s_0,s_1)$ and without the inequalities $\sigma_1(s_B)\le\frac{b_{11}}{\kappa_1}$, $\sigma_1(s_2)\le\frac{b_{11}}{\kappa_1}$. However, (3.30) implies that the cases $s_B=s_0$ and $s_B=s_1$ are excluded. The remaining two inequalities follow from the last assertion of Lemma 3.5, because $(s,0,0)\in\overline{B}$ implies that $\sigma(s)$ is a local bifurcation point. □

Remark 3.7. Our proof shows that in Theorem 3.2 we may replace (3.27) by

$$T(\sigma(I\cap(s_1-\varepsilon,s_1+\varepsilon)))\subseteq W \quad \text{for some } \varepsilon>0 \tag{3.31}$$

if we drop the two assertions $\sigma_1(s_B),\sigma_1(s_2)\le\frac{b_{11}}{\kappa_1}$.

Moreover, if one replaces in addition the assertion $s_B\in(s_0,s_1)$ by the weaker assertion $s_B\in(s_0,s_1]$ in Theorem 3.2, one can even relax (3.31) to the assumptions that (3.6) holds with $\Lambda_0 = T(\sigma(I\cap[s_1,s_1+\varepsilon)))$ with some $\varepsilon>0$ and that $\underline{c}_0(T(\sigma(s_1)))\ne 0\ne\underline{c}_1(T(\sigma(s_1)))$. (In the case (3.31) these assumptions were fulfilled due to the assumptions about W from Theorem 3.1.)

Remark 3.8. We point out once more that all results in this paper hold under the assumptions sketched in Remark 3.1 also for multivalued g_1,g_2,g_3,g_4 (i.e. essentially if the multivalued map $g_i(\lambda,x,\,\cdot\,)$ is only upper semicontinuous).

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