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A NOTE ON GROTHENDIECK DUALITY THEOREM

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Abstract. We prove that the canonical mapping $Y^* \otimes_\pi X \to (\mathcal{L}(X,Y),\tau)^*$ in the Grothendieck duality theorem is not always injective. This answers a question posed in the book by Defant and Floret.

An important result in the topological theory of tensor products is the theorem of Grothendieck, which gives a description the linear topological dual of the space of bounded linear operators $\mathcal{L}(X,Y)$, equipped with the topology of uniform convergence on compact sets.

Theorem 0.1. *(Grothendieck) Let X,Y be Banach spaces. By τ we denote the LCS topology on $\mathcal{L}(X,Y)$ of uniform convergence on compact sets in X. The continuous linear functionals on $(\mathcal{L}(X,Y),\tau)$ consist of all*

$$\phi(T)=\sum_{i=1}^{\infty}\langle y_i^*,Tx_i\rangle, x_i\in X, y_i^*\in Y^*, \sum_{i=1}^{\infty}\|x_i\|\|y_i^*\|<\infty \tag{1}$$

This formulation of Grothendieck theorem is taken from [LT] (Prop. 1.e.3.). Its advantage is that it uses only standard functional analytic language, and in particular it is not relying on tensor products. However, it is more natural to rephrase this result using the language of the theory of tensor products in the following way: The canonical mapping (which is described by the formula (1)) $Y^*\otimes_\pi X\to(\mathcal{L}(X,Y),\tau)^*$ is surjective. This is the formulation to be found in [DF] (Prop. 5.5). A natural question immediately follows, namely is the canonical mapping above also injective? In their book, Defant and Floret [DF] (p. 65) pose this problem explicitly, along with some partial positive solutions (see below). Of course, a positive solution would provide a useful duality pair ready for proving more isomorphisms, apart from its obvious elegance.

The main result of this note is a negative solution to this problem. This is achieved by combining some known results in the theory of tensor products. As a main ingredient we are using the existence of a Banach space X with the approximation property, such that X^* fails to have the A.P.. The construction of such a space relies of course on the fundamental result of Enflo [E], and is shown e.g. in [LT] (Thm. 1.e.7.) (using the method of [J] and [L]). Alternatively, one can use the space constructed in [FJ].

We are also using an equivalent condition describing when X^* has an A.P., which is almost certainly known to the specialists in the field, but which we have not found explicitly in the literature. Before being able to formulate our results and proofs, in order to make the note understandable to non-specialists in tensor products, we need to recall some known results. Most of these are contained in the books [DF], [Ja] and [LT]. We assume that the reader is familiar with the algebraic tensor

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product of two Banach spaces $X \otimes Y$. We recall that a couple $\langle \mathcal{L}(X, Y^*), X \otimes Y \rangle$ forms a duality pair defined as follows. For $T \in \mathcal{L}(X, Y^*)$, $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$ put

$$\langle T, z \rangle = \sum_{i=1}^{n} \langle T(x_i), y_i \rangle. \tag{2}$$

The pairing enables us to introduce a projective norm π on $X \otimes Y$ as follows.

Definition 0.2. *Let X, Y be Banach spaces. We define a projective norm $\pi(\cdot)$ on $X \otimes Y$ for $z \in X \otimes Y$ as follows:*

$$\pi(z) = \sup\{\langle T, z \rangle, \|T\| \leq 1, T \in \mathcal{L}(X, Y^*)\}. \tag{3}$$

We denote by $X \otimes_\pi Y$ the projective tensor product, that is the completion of $(X \otimes Y, \pi)$.

A more concrete description of the elements of $X \otimes_\pi Y$ uses infinite series.

Proposition 0.3. *(*[DF]*, Chap. 3) Let X, Y be Banach spaces. Every element $z \in X \otimes_\pi Y$ admits a representation $z = \sum_{i=1}^\infty x_i \otimes y_i$, such that $\sum_{i=1}^\infty \|x_i\| \|y_i\| < \infty$ (WLOG $(\|x_i\|) \in c_0$ and $(\|y_i\|) \in \ell_1$) and*

$$\pi(z) = \inf\{\sum_{i=1}^{\infty} \|x_i\| \|y_i\| : z = \sum_{i=1}^{\infty} x_i \otimes y_i\} \tag{4}$$

Moreover, we have

Proposition 0.4. *(*[DF]*, Chap. 3) Let X, Y be Banach spaces. Then the canonical dual pairing gives a topological linear duality*

$$(X \otimes_\pi Y)^* = \mathcal{L}(X, Y^*). \tag{5}$$

Closely connected to the projective tensor product $X^* \otimes_\pi Y$ is the notion of nuclear operator.

Definition 0.5. *Let X, Y be Banach spaces. An operator $T : X \to Y$ is called nuclear if there exists a couple of sequences $\{x_i^*\}_{i=1}^\infty$ in X^*, and $\{y_i\}_{i=1}^\infty$ in Y, such that $\sum_{i=1}^\infty \|x_i^*\| \|y_i\| < \infty$ and $Tx = \sum_{i=1}^\infty \langle x_i^*, x \rangle y_i$. We introduce the nuclear norm*

$$N(T) = \inf\{\sum_{i=1}^{\infty} \|x_i^*\| \|y_i\| : Tx = \sum_{i=1}^{\infty} \langle x_i^*, x \rangle y_i\}. \tag{6}$$

By $\mathcal{N}(X, Y)$ we denote the space of all nuclear operators, with the nuclear norm.

It is well-known that $\mathcal{N}(X, Y)$ is a Banach space. Let $J : \sum_{i=1}^\infty x_i^* \otimes y_i \to \sum_{i=1}^\infty x_i^* \otimes y_i$ be the formal identity mapping defined for all pairs of sequences $\{x_i^*, \}_{i=1}^\infty \in X^*$, $\{y_i\}_{i=1}^\infty \in Y$ such that $\sum_{i=1}^\infty \|x_i^*\| \|y_i\| < \infty$. As we have seen above such series represent all elements of $X^* \otimes_\pi Y$ as well as of $\mathcal{N}(X, Y)$.

Proposition 0.6. *(*[DF]*, Chap. 3)*
The formal identity J is a well-defined quotient mapping $J : X^* \otimes_\pi Y \to \mathcal{N}(X, Y)$. More precisely, let $z = \sum_{i=1}^\infty x_i^* \otimes y_i$ be a representation of $z \in X^* \otimes_\pi Y$, then the nuclear operator T represented by the same sum $J(z)$

$$T = J(\sum_{i=1}^{\infty} x_i^* \otimes y_i), T(x) = \sum_{i=1}^{\infty} x_i^*(x) y_i, \tag{7}$$

is independent of the concrete representation of the tensor z.

We pass now to a description of the topology τ, and its close relationship to the approximation property.

Definition 0.7. *Let X, Y be Banach spaces. By τ we denote the LCS topology on $\mathcal{L}(X,Y)$ of uniform convergence on compact sets in X, generated by seminorms $\|T\|_K$, $K \subset X$ norm compact set.*

Definition 0.8. *We say that a Banach space X has the approximation property (A.P. for short), if*

$$Id \in \overline{\mathcal{F}}^{\tau}(X)$$

Next theorem contains a list of conditions characterizing the A.P. for a Banach space X. It combines some conditions from Thm. 1.e.4. of [LT] and Theorem 5.6 of [DF].

Theorem 0.9. *(Grothendieck)*
Let X be a Banach space. The following conditions are equivalent:
1. X has the A.P..
2. For every Banach space Y, $\overline{\mathcal{F}}^{\tau}(X,Y) = \mathcal{L}(X,Y)$.
3. For every Banach space Y, $\overline{\mathcal{F}}^{\tau}(Y,X) = \mathcal{L}(Y,X)$.
4. $J : X^* \otimes_\pi X \to \mathcal{N}(X)$ is injective, or equivalently it is an isometry.
5. For every Banach space Y, $J : Y^* \otimes_\pi X \to \mathcal{N}(Y,X)$ is injective, or equivalently it is an isometry.

The next theorem is almost certainly known to the specialists. As we have not found an explicit reference, we include its proof for the readers convenience.

Theorem 0.10. *Let X be a Banach space. The following conditions are equivalent:*
1. X^* has AP.
2. For every Banach space Y, $J : X^* \otimes_\pi Y \to \mathcal{N}(X,Y)$ is an isometry.

Proof. It is well-known ([Ja], p.326) that the formal transposition mapping $t : E \otimes_\pi F \to F \otimes_\pi E$, $t(\sum_{i=1}^\infty e_i \otimes f_i) = \sum_{i=1}^\infty f_i \otimes e_i$ is an isometric linear isomorphism. Next, we note that $\mathcal{N}(X, X^{**})$ and $\mathcal{N}(X^*, X^*)$ are canonically isometric, via the transposition of their elements $z = \sum_{i=1}^\infty x_i^* \otimes x_i^{**} \leftrightarrow z' = \sum_{i=1}^\infty x_i^{**} \otimes x_i^*$. Indeed, $\mathcal{N}(X, X^{**})$ is a quotient (via J) of $X^* \otimes_\pi X^{**}$, while $\mathcal{N}(X^*, X^*)$ is a quotient (via J') of the isometric transpose $t(X^* \otimes_\pi X^{**}) = X^{**} \otimes_\pi X^*$. The kernels are described as follows.

$$Ker(J) = \{z = \sum_{i=1}^\infty x_i^* \otimes x_i^{**} : \sum_{i=1}^\infty x_i^*(x) x_i^{**} = 0 \text{ for all } x \in X\}. \tag{8}$$

$$Ker(J') = \{z' = \sum_{i=1}^\infty x_i^{**} \otimes x_i^* : \sum_{i=1}^\infty x_i^{**}(x^*) x_i^* = 0 \text{ for all } x^* \in X^*\}. \tag{9}$$

Both of these conditions are indeed equivalent to a single condition $z \in Ker(J) \Leftrightarrow t(z) \in Ker(J')$, if and only if $\sum_{i=1}^\infty x_i^{**}(x^*) x_i^*(x) = 0$ for all $x \in X, x^* \in X^*$.
Substituting $Y = X^{**}$ into condition 2., and using the transposition we may transform 2. into an equivalent statement, that $J' : X^{**} \otimes_\pi X^* \to \mathcal{N}(X^*, X^*)$ is an isometry. By condition 4. of Theorem 0.9 we conclude that X^* has the A.P..
It remains to show 1. $\Rightarrow$ 2. Let $0 \neq z = \sum_{i=1}^\infty x_i^* \otimes y_i \in X^* \otimes_\pi Y$, our goal is to show that $J(z) \neq 0$. WLOG we may assume that $\sum_{i=1}^\infty \|y_i\| < \infty$ and $\lim_{i\to\infty} \|x_i^*\| = 0$. (Recall that $J(z)$ is a nuclear operator, so in particular is it also a compact operator from X to Y). We proceed by contradiction, assuming that $J(z)(y) = \sum_{i=1}^\infty x_i^*(x) y_i = 0$ for all $x \in X$. Given $\varepsilon > 0$, by condition 3. in Theorem 0.9 (it is well-known that $\{x_i^*\}_{=1}^\infty$ is contained into some compact set, [F]), there is a

$$F = \sum_{k=1}^n u_k^{**} \otimes u_k^* \in \mathcal{F}(X^*), \text{ such that } \sup_i \|F(x_i^*) - x_i^*\| < \varepsilon. \tag{10}$$

We let $z' = \sum_{i=1}^{\infty} F(x_i^*) \otimes y_i \in X^* \otimes_\pi Y$. Note the important fact that $z' \in X^* \otimes Y$ is actually a finite tensor. Indeed,

$$z' = \sum_{i=1}^{\infty} (\sum_{k=1}^{n} u_k^{**}(x_i^*) u_k^*) \otimes y_i = \sum_{k=1}^{n} u_k^* \otimes (\sum_{i=1}^{\infty} u_k^{**}(x_i^*) y_i). \tag{11}$$

Next, $J(z')$ satisfies the following:

$$J(z')(x) = \sum_{i=1}^{\infty} \langle F(x_i^*), x \rangle y_i = \sum_{i=1}^{\infty} \langle x_i^*, F^*(x) \rangle y_i = 0, \text{ for every } x \in X. \tag{12}$$

Hence $J(z') = 0$, as an element of $\mathcal{K}(X, Y)$, and since z' is also a finite tensor we conclude that $z' = 0$ as an element of $X^* \otimes_\pi Y$. Hence we have an estimate

$$\pi(z) = \pi(z - z') = \pi(\sum_{i=1}^{\infty} x_i^* \otimes y_i - \sum_{i=1}^{\infty} F(x_i^*) \otimes y_i) \le \varepsilon \sum_{i=1}^{\infty} \|y_i\|. \tag{13}$$

Since ε was arbitrarily small, we conclude that $\pi(z) = 0$ as desired. It is clear by the Banach open mapping theorem that J is an isomorphism. Once we know that J is an isomorphism, we obtain that it is actually an isometry for free. Indeed, we may pass from a a nuclear representation to a tensor one freely and get the norm estimate. ☐

By Proposition 0.4 (and the transposition isometry $Y^* \otimes_\pi X = t(X \otimes_\pi Y^*)$) we have $(Y^* \otimes_\pi X)^* = \mathcal{L}(X, Y^{**})$. Denote by $i : \mathcal{L}(X, Y) \to \mathcal{L}(X, Y^{**})$ the formal identity embedding. Then we have the following.

Lemma 0.11. *The mapping*

$$i : (\mathcal{L}(X, Y), \tau) \to (\mathcal{L}(X, Y^{**}), w^*) \tag{14}$$

is continuous. In particular, the dual mapping

$$i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X, Y), \tau)^* \tag{15}$$

is $w - w^*$ continuous (the topologies come from the duality pairs described above).

Proof. By Proposition 0.3 every $z \in Y^* \otimes_\pi X$ admits a representation $z = \sum_{i=1}^{\infty} y_i^* \otimes x_i$, such that $(\|x_i\|) \in c_0$ and $(\|y_i^*\|) \in \ell_1$. Let $K = \overline{\text{conv}}\{x_i\}_{i=1}^{\infty}$ be a compact and convex set in X. Let U be a τ-open set in $\mathcal{L}(X, Y)$ defined as $U = \{T : \sup_{x \in K} \|T(x)\| < 1\}$. Clearly, $T \in U$ implies $|y^*(T(x))| < \|y^*\|$ for all $y^* \in Y^*$, $x \in K$. Thus $|\langle T, \sum_{i=1}^{\infty} y_i^* \otimes x_i \rangle| \le \sum_{i=1}^{\infty} \|y_i^*\| < \infty$ for all $T \in U$, which finishes the proof. The second result follows by duality. ☐

The following is a more complete formulation of the Grohtendieck duality result in Theorem 0.1.

Theorem 0.12. *(Grothendieck,* [DF]*, Prop. 5.5)*
The mapping $i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X, Y), \tau)^*$ from (15) *is surjective. In other words, the continuous linear functionals on $(\mathcal{L}(X, Y), \tau)$ consist of all*

$$\phi(T) = \sum_{i=1}^{\infty} \langle y_i^*, T x_i \rangle, x_i \in X, y_i^* \in Y^*, \sum_{i=1}^{\infty} \|x_i\| \|y_i^*\| < \infty \tag{16}$$

In some cases, the mapping i^* is injective, leading to a perfect duality pairing. For example:

Theorem 0.13. *([DF], p. 65) Let X, Y be Banach spaces. Suppose that either X or Y^* has the A.P., or that Y is reflexive. Then the mapping $i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X,Y), \tau)^*$ from* (15) *is injective. In particular, we may write $(\mathcal{L}(X,Y), \tau)^* = Y^* \otimes_\pi X$. The pairing is canonical,*

$$\langle z, T\rangle = \sum_{i=1}^{\infty} \langle y_i^*, Tx_i\rangle, T \in \mathcal{L}(X,Y), z = \sum_{i=1}^{\infty} y_i^* \otimes x_i \in Y^* \otimes_\pi X. \tag{17}$$

Our main result is contained in the next characterization.

Theorem 0.14. *Let Y be a Banach space with the A.P.. Then the mapping i^* : $Y^* \otimes_\pi X \to (\mathcal{L}(X,Y), \tau)^*$ from* (15) *is injective if and only if Y^* has the A.P.*

Proof. We first assume the injectivity of i^*. Our goal is to establish that Y^* has the A.P.. By using Theorem 0.10, it suffices to show that $J : Y^* \otimes_\pi X \to \mathcal{N}(Y, X)$ is an isometry for every Banach space X. Recall that

$$Ker(i^*) = \{z = \sum_{i=1}^{\infty} y_i^* \otimes x_i : \langle z, S\rangle = \sum_{i=1}^{\infty} \langle y_i^*, S(x_i)\rangle = 0, \text{ for all } S \in \mathcal{L}(X,Y)\} \tag{18}$$

As Y is assumed to have the A.P., we have by condition 3. in Theorem 0.9 that for every X, $\overline{\mathcal{F}}^{\tau}(X,Y) = \mathcal{L}(X,Y)$. Thus by the bipolar and Hahn-Banach theorem (18) is equivalent to the next condition.

$$Ker(i^*) = \{z = \sum_{i=1}^{\infty} y_i^* \otimes x_i : \langle z, S\rangle = \sum_{i=1}^{\infty} \langle y_i^*, S(x_i)\rangle = 0, \text{ for all } S \in \mathcal{F}(X,Y)\} \tag{19}$$

Next, compare this condition with the condition describing the kernel of J:

$$Ker(J) = \{z = \sum_{i=1}^{\infty} y_i^* \otimes x_i : \langle T, z\rangle = \sum_{i=1}^{\infty} \langle T(y_i^*), x_i\rangle = 0, \text{ for all } T \in \mathcal{F}_{w^*}(Y^*, X^*)\} \tag{20}$$

We claim that (19) and (20) are equivalent conditions. Indeed, it suffices to note that taking the adjoints $S \to S^*$ makes an isometry from $\mathcal{F}(X,Y)$ onto $\mathcal{F}_{w^*}(Y^*, X^*)$, and thus a reformulation of (19)

$$Ker(i^*) = \{z = \sum_{i=1}^{\infty} y_i^* \otimes x_i : \langle z, S\rangle = \sum_{i=1}^{\infty} \langle S^*(y_i^*), x_i\rangle = 0, \text{ for all } S \in \mathcal{F}(X,Y)\} \tag{21}$$

is precisely (20). Since i^* is assumed to be injective, so is J. It is clear by the Banach open mapping theorem that J is an isomorphism. Once we know that J is an isomorphism, we obtain that it is actually an isometry for free. Indeed, we may pass from a a nuclear representation to a tensor one freely and get the norm estimate. This proves that Y^* has indeed the A.P.. The opposite implication follows from Theorem 0.13. □

As pointed out in the introduction, there do exist Banach spaces with A.P. whose dual fails A.P.. Therefore we obtain a negative solution to the original problem.

Corollary 0.15. *Let Y be a Banach space with the A.P., whose dual Y^* fails the A.P.. Then there exists a Banach space X such that $i^* : Y^* \otimes_\pi X \to (\mathcal{L}(X,Y), \tau)^*$ is not injective.*

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