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THE MULTIDIMENSIONAL REVERSE HARDY INEQUALITIES.

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Abstract. In this paper we characterize the validity of the inequalities

$$\|g\omega\|_{L_p(\mathbb{R}^n)} \le C \left\| v(t) \int_{B(0,t)} g(y)dy \right\|_{L_q(0,\infty)}$$

and

$$\|g\omega\|_{L_p(\mathbb{R}^n)} \le C \left\| v(t) \int_{\complement B(0,t)} g(y)dy \right\|_{L_q(0,\infty)}$$

for non-negative measurable functions on $\mathbb{R}^n$, where $0 < p \le 1$, $0 < q \le +\infty$, ω and v are a weight functions on $\mathbb{R}^n$ and $(0,\infty)$ respectively.

1. Introduction

The classical Hardy inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right) \int_0^\infty f^p(x)dx, \ 1 < p < \infty, \tag{1.1}$$

for functions $f \ge 0$ defined on $(0,\infty)$ is equivalent to the inequality

$$\int_{-\infty}^{+\infty} \left(\frac{1}{2|x|}\int_{-|x|}^{+|x|} f(t)dt\right)^p dx \le \left(\frac{p}{p-1}\right) \int_{-\infty}^{+\infty} f^p(x)dx \tag{1.2}$$

for functions $f \ge 0$ defined on $(-\infty,+\infty)$.

Now, let us denote, for $x \in \mathbb{R}^n$, by $B(x)$ the ball $\{y \in \mathbb{R}^n : |y| \le |x|\}$ and by $|B(x)|$ its volume. In M.Christ, L.Grafakos [4], it is shown that the "n-dimensional Hardy operator" H_n defined by

$$(H_n f)(x) = \frac{1}{|B(x)|}\int_{B(x)} f(y)dy, \ x \in \mathbb{R}^n,$$

satisfies

$$\int_{\mathbb{R}^n} |H_n f(x)|^p dx \le \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^n} |f(x)|^p dx, \ 1 < p < \infty,$$

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the constant $\left(\frac{p}{p-1}\right)^p$ being again the best possible. In P.Drábek, H.P.Heinig, A.Kufner [7], this "Hardy inequality" was extended to general n-dimensional weights u, v and to the whole range of parameters p, q, $1 < p < \infty$, $0 < q < \infty$. The necessary and sufficient conditions for the validity of the inequality

$$\left(\int_{\mathbb{R}^n} |H_n f(x)|^q u(x)dx\right)^{\frac{1}{q}} \le c \left(\int_{\mathbb{R}^n} |f(x)|^p v(x)dx\right)^{\frac{1}{p}}$$

are exactly analogous of the corresponding conditions for dimension one; hence, for $1 < p \le q < \infty$ this condition reads

$$\sup_{x\in\mathbb{R}^n} \left(\int_{\mathbb{R}^n\setminus B(x)} u(y)dy\right)^{\frac{1}{q}} \left(\int_{B(x)} v^{1-p'}(y)dy\right)^{\frac{1}{p'}} < \infty.$$

In [11] G.H. Hardy proved the following celebrated inequality: Let $1 < p < +\infty$ and f a non-negative measurable function on $(0, +\infty)$. Then, if $\varepsilon < 1/p' = 1 - 1/p$,

$$\int_0^{+\infty} \left(x^{\varepsilon-1} \int_0^x f(t)\,dt\right)^p dx \le c \int_0^{+\infty} \left(x^\varepsilon f(x)\right)^p dx \tag{1.3}$$

for some constant c independent of f. If $\varepsilon > 1/p'$, the inequality takes the form

$$\int_0^{+\infty} \left(x^{\varepsilon-1} \int_x^{+\infty} f(t)\,dt\right)^p dx \le c \int_0^{+\infty} \left(x^\varepsilon f(x)\right)^p dx. \tag{1.4}$$

The best possible constants c in (1.3) and (1.4) are equal and this common value was determined by E. Landau in [14] as

$$c = |\varepsilon - 1/p'|^{-p}. \tag{1.5}$$

In [3] G.A. Bliss established the inequality

$$\left(\int_0^{+\infty} \left(x^{-\frac{1}{q}-\frac{1}{p'}} \int_0^x f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \le c \left(\int_0^{+\infty} f(x)^p\,dx\right)^{\frac{1}{p}}$$

for $1 < p < q < +\infty$ and proved that the best possible constant is

$$c = \left(\frac{p' r^r}{q}\right)^{1/q} \left[B\left(\frac{1}{r}, \frac{q-1}{r}\right)\right]^{-r/q},$$

where $r = q/p - 1$ and B is the classical beta function.

During the last two decades, many authors have considered extensions of the form

$$\left(\int_a^b \left(w(x) \int_a^x f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \le c \left(\int_a^b (v(x) f(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.6}$$

and

$$\left(\int_a^b \left(w(x) \int_x^b f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \le c \left(\int_a^b (v(x) f(x))^p\,dx\right)^{\frac{1}{p}}, \tag{1.7}$$

with $-\infty \le a < b \le +\infty$, w, v weights on (a, b), $0 < q \le +\infty$, $1 \le p \le +\infty$. The weights w and v for which (1.6) and (1.7) hold for all non-negative f have been completely characterized. The solution of this problem (under different assumptions on p and q) is associated with the names M. Artola, J.S. Bradley,

V. Kokilashvili, V.G. Maz'ja, B. Muckenhoupt, A.L. Rozin, E. Sawyer, G. Sinnamon, G. Talenti, G. Tomaselli and others. We refer to [17] and [13] for a survey of results.

In [19] E. Sawyer noted that if $0 < p < 1$, then the inequalities (1.6) and (1.7) hold only for trivial weights. This observations led to a study of the so-called reverse Hardy inequalities

$$\left(\int_a^b \left(w(x)\int_a^x f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \geq c\left(\int_a^b (v(x)f(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.8}$$

and

$$\left(\int_a^b \left(w(x)\int_x^b f(t)\,dt\right)^q dx\right)^{\frac{1}{q}} \geq c\left(\int_a^b (v(x)f(x))^p\,dx\right)^{\frac{1}{p}} \tag{1.9}$$

in the case $0 < p \leq 1$.

However, these are integral forms of inequalities first considered by E.T. Copson in [5, 6] for infinite series; such reverse inequalities for infinite series were also investigated by G. Bennett [2] and K.-G. Grosse-Erdmann [10]. Conditions on the weights w, v, which are either necessary or sufficient for (1.8) and (1.9) to hold when $0 < q \leq p \leq 1$ were established by P. R. Beesack and H. P. Heinig [1]. Discrete analogues of (1.8) and (1.9) were proved in [10], where it is also remarked that the techniques used in the proofs may be applicable to the continuous versions of the inequalities, namely to (1.8) and (1.9). No estimates of the constants c are mentioned in [10].

In [8] W.D.Evans, A.Gogatishvili and B.Opic have characterized the validity of inequalities

$$\|g\|_{p,(a,b),\lambda} \leq c\left\|u(x)\int_{(a,x)} g(y)\,d\mu\right\|_{q,(a,b),\nu}$$

and

$$\|g\|_{p,(a,b),\lambda} \leq c\left\|u(x)\int_{(x,b)} g(y)\,d\mu\right\|_{q,(a,b),\nu}$$

for every non-negative Borel measurable functions g on the interval $(a,b) \subseteq \mathbb{R}$, where $0 < p \leq 1$, $0 < q \leq +\infty$, λ, μ and ν are non-negative Borel measures on (a,b), and u is a weight function on (a,b).

By motivation of study Morrey-type spaces it is very important to get multidimensional analogous of reverse Hardy inequalities. In this paper we make a comprehensive study of general inequalities of the form

$$\|g\omega\|_{L_p(\mathbb{R}^n)} \leq C\left\|v(t)\int_{B(0,t)} g(y)dy\right\|_{L_q(0,\infty)}$$

and

$$\|g\omega\|_{L_p(\mathbb{R}^n)} \leq C\left\|v(t)\int_{\complement_{B(0,t)}} g(y)dy\right\|_{L_q(0,\infty)}$$

with complete proofs and estimates for c, using technique from [8].

2. Notation and Preliminaries

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R}^n : |x-y| \le r\}$ be the closed ball centered at x of radius r and ${}^{\complement}B(x,r) := \mathbb{R}^n \backslash B(x,r)$.

Let μ be a non-negative Borel measure on $\mathbb{R}^n$ and ν be a non-negative Borel measure on $(0,+\infty)$. We denote by $\mathfrak{B}^+(0,\infty)$ and $\mathfrak{B}^+(\mathbb{R}^n)$ the set of all non-negative Borel measurable function on $(0,\infty)$ and $\mathbb{R}^n$ accordingly. If E is a nonempty Borel measurable subset on $\mathbb{R}^n$ and f is a Borel measurable function on E, then we put

$$\|g\|_{L_p(E,\mu)} := \left(\int_E |f(y)|^p d\mu\right)^{\frac{1}{p}}, \ 0 < p < +\infty,$$

$$\|f\|_{L_\infty(E,\mu)} := \sup\{\alpha : \mu\{y \in E : |f(y)| \ge \alpha\} > 0\}.$$

If I a nonempty Borel measurable subset on $(0,+\infty)$ and g is a Borel measurable function on I, then we define $\|g\|_{L_p(I,\nu)}$ and $\|g\|_{L_\infty(I,\nu)}$ correspondingly. In the notation $\|f\|_{L_p(E,\mu)}$, $0 < p \le +\infty$, we omit the symbol μ if μ is the Lebesgue measure.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B *are equivalent*.

We put

$$p' := \begin{cases} \frac{p}{1-p} & \text{if} \quad 0 < p < 1, \\ +\infty & \text{if} \quad p = 1, \\ \frac{p}{p-1} & \text{if} \quad 1 < p < +\infty, \\ 1 & \text{if} \quad p = +\infty, \end{cases}$$

and $1/(+\infty) = 0$, $0/0 = 0$, $0 \cdot (\pm\infty) = 0$ and $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, +\infty\}$.

Definition 2.1. Let $N, M \in \overline{\mathbb{Z}}$, $N < M$. A positive non-increasing sequence $\{\tau_k\}_{k=N}^M$ is called *almost geometrically decreasing* if there are $\alpha \in (1,+\infty)$ and $L \in \mathbb{N}$ such that

$$\tau_k \le \frac{1}{\alpha}\tau_{k-L} \quad \text{for all} \quad k \in \{N+L, \ldots, M\}.$$

A positive non-decreasing sequence $\{\sigma_k\}_{k=N}^M$ is called *almost geometrically increasing* if there are $\alpha \in (1,+\infty)$ and $L \in \mathbb{N}$ such that

$$\sigma_k \ge \alpha\sigma_{k-L} \quad \text{for all} \quad k \in \{N+L, \ldots, M\}.$$

Remark 2.2. Definition 2.1 implies that if $0 < q < +\infty$, then following three statements are equivalent:

(i) $\{\tau_k\}_{k=N}^M$ is an almost geometrically decreasing sequence;
(ii) $\{\tau_k^q\}_{k=N}^M$ is an almost geometrically decreasing sequence;
(iii) $\{\tau_k^{-q}\}_{k=N}^M$ is an almost geometrically increasing sequence.

Let $\emptyset \ne \mathcal{Z} \subseteq \overline{\mathbb{Z}}$, $0 < q \le +\infty$ and let $\{w_k\} = \{w_k\}_{k\in\mathcal{Z}}$ be a sequence of positive numbers. We denote by $\ell^q(\{w_k\}, \mathcal{Z})$ the following discrete analogue of a weighted Lebesgue space: if $0 < q < +\infty$, then

$$\ell^q(\{w_k\}, \mathcal{Z}) = \Big\{\{a_k\}_{k\in\mathcal{Z}} : \quad \|a_k\|_{\ell^q(\{w_k\},\mathcal{Z})} := \Big(\sum_{k\in\mathcal{Z}} |a_k w_k|^q\Big)^{\frac{1}{q}} < +\infty\Big\}$$

and

$$\ell^\infty(\{w_k\}, \mathcal{Z}) = \Big\{\{a_k\}_{k\in\mathcal{Z}} : \quad \|a_k\|_{\ell^\infty(\{w_k\},\mathcal{Z})} := \sup_{k\in\mathcal{Z}} |a_k w_k| < +\infty\Big\}.$$

If $w_k = 1$ for all $k \in \mathcal{Z}$, we write simply $\ell^q(\mathcal{Z})$ instead of $\ell^q(\{w_k\}, \mathcal{Z})$.

We quote some known results. Proofs can be found in [15] and [16].

Lemma 2.3. *Let $N, M \in \overline{\mathbb{Z}}$, $N \le M$. Then, for any positive sequence $\{\tau_k\}_{k=N}^M$ and all $m \in \overline{\mathbb{Z}}$ satisfying $N < m < M$,*

$$\sum_{k=m}^{M} \tau_k \lesssim \tau_m \tag{2.1}$$

or

$$\sum_{k=N}^{m} \tau_k \lesssim \tau_m \tag{2.2}$$

if and only if the sequence $\{\tau_k\}_{k=N}^M$ is almost geometrically decreasing or increasing, respectively.

Lemma 2.4. *Let $q \in (0, +\infty]$, $N, M \in \overline{\mathbb{Z}}$, $N \le M$, $\mathcal{Z} = \{N, N+1, \dots, M-1, M\}$ and let $\{\tau_k\}_{k=N}^M$ be an almost geometrically decreasing sequence. Then*

$$\left\|\tau_k \sum_{m=N}^{k} a_m\right\|_{\ell^q(\mathcal{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathcal{Z})} \tag{2.3}$$

and

$$\|\tau_k \sup_{N\le m\le k} a_m\|_{\ell^q(\mathcal{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathcal{Z})} \tag{2.4}$$

for all non-negative sequences $\{a_k\}_{k=N}^M$.

Lemma 2.5. *Let $q \in (0, +\infty]$, $N \le M$, $N, M \in \overline{\mathbb{Z}}$, $\mathcal{Z} = \{N, N+1, \dots, M-1, M\}$ and let $\{\sigma_k\}_{k=N}^M$ be an almost geometrically increasing sequence. Then*

$$\left\|\sigma_k \sum_{m=k}^{M} a_m\right\|_{\ell^q(\mathcal{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathcal{Z})} \tag{2.5}$$

and

$$\|\sigma_k \sup_{k\le m\le M} a_m\|_{\ell^q(\mathcal{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathcal{Z})} \tag{2.6}$$

for all non-negative sequences $\{a_k\}_{k=N}^M$.

The following two lemmas are discrete version of the classical Landau resonance theorems. Proofs can be found, for example, in [9].

Lemma 2.6. *Let $0 < p \le q \le +\infty$, $\emptyset \neq \mathcal{Z} \subseteq \overline{\mathbb{Z}}$ and let $\{v_k\}_{k\in\mathcal{Z}}$ and $\{w_k\}_{k\in\mathcal{Z}}$ be two sequences of positive numbers. Assume that*

$$\ell^p(\{v_k\}, \mathcal{Z}) \hookrightarrow \ell^q(\{w_k\}, \mathcal{Z}). \tag{2.7}$$

Then

$$\|\{w_k v_k^{-1}\}\|_{\ell^\infty(\mathcal{Z})} \le C, \tag{2.8}$$

where C stands for the norm of the embedding (2.7)*.*

Lemma 2.7. *Let $0 < q < p \le +\infty$, $\emptyset \neq \mathcal{Z} \subseteq \overline{\mathbb{Z}}$ and let $\{v_k\}_{k\in\mathcal{Z}}$ and $\{w_k\}_{k\in\mathcal{Z}}$ be two sequences of positive numbers. Assume that* (2.7) *holds. Then*

$$\|\{w_k v_k^{-1}\}\|_{\ell^r(\mathcal{Z})} \le C, \tag{2.9}$$

where $1/r := 1/q - 1/p$ and C stands for the norm of the embedding (2.7)*.*

3. Discretization of function norms

In this section we define a discretizing sequence for a non-negative, non-decreasing, finite and right-continuous function φ on $(a,b) \subseteq \mathbb{R}$. We use this sequence to discretize function norms, more precisely, we find discrete norms equivalent to the original ones.

If φ is a non-negative and monotone function on (a,b), then by $\varphi(a)$ and $\varphi(b)$ we mean the values $\varphi(a+) := \lim_{t\to a+}\varphi(t)$ and $\varphi(b-) := \lim_{t\to b-}\varphi(t)$, respectively.

Lemma 3.1. ([8]) *Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). There is a strictly increasing sequence $\{x_k\}_{k=N}^{M+1}$, $-\infty \le N \le M \le +\infty$, with elements from the closure of the interval (a,b), such that:*

(i) *if $N > -\infty$, then $\varphi(x_N) > 0$ and $\varphi(x) = 0$ for every $x \in (a, x_N)$; if $M < +\infty$, then $x_{M+1} = b$;*
(ii) $\varphi(x_{k+1}-) \le 2\varphi(x_k)$ *if* $N \le k \le M$;
(iii) $2\varphi(x_k-) \le \varphi(x_{k+1})$ *if* $N < k < M$.

Definition 3.2. ([8]) Let φ be a non-negative, non-decreasing, finite and right-continuous function on (a,b). A strictly increasing sequence $\{x_k\}_{k=N}^{M+1}$, $-\infty \le N < M \le +\infty$, is said to be *a discretizing sequence of the function* φ if it satisfies the conditions (i) – (iii) of Lemma 3.1.

Remark 3.3. ([8]) We shall use the following *convention*: if $N = -\infty$, then we put $x_N = \lim_{k\to-\infty} x_k$. It is clear that if $N = -\infty$ and $x_N > a$, then $\varphi(x) = 0$ for all $x \in (a, x_N)$ (cf. condition (i) of Lemma 3.1).

Theorem 3.4. ([8]) *Let ν be a non-negative Borel measure on $I = (a,b)$ such that the function $\varphi(t) = \nu(a,t]$ is finite on I. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function φ, then*

$$\int_{(a,b)} h(t)\,d\nu(t) \approx \sum_{k=N}^{M} h(x_k)\nu(a, x_k] \tag{3.1}$$

for all non-negative and non-increasing functions h on I.

We shall need an analogue of Theorem 3.4, where $L^1(\nu)$-norm is replaced by a weighted $L^\infty(\nu)$-norm. But there is a substantial difference between these two cases. While the function $\varphi(t) := \nu(a,t]$, $t \in I$, corresponding to the former case is right-continuous on I, the function

$$\varphi(t) := \|u\|_{L_\infty((a,t],\nu)}, \quad t \in I, \quad \text{with} \quad u \in B^+(I), \tag{3.2}$$

cannot be right-continuous on I. (To see it, let $I = (0,2)$, $u = \chi_{(0,1]} + 2\chi_{(1,2)}$ and let ν be the Lebesgue measure on I. Then $\varphi(1) = 1$ but $\varphi(1+) = 2$). Therefore, in the following theorem we consider the function φ defined by

$$\varphi(t) = \|u\|_{L_\infty((a,t+],\nu)} := \lim_{s\to t+} \|u\|_{L_\infty((a,s],\nu)}, \quad t \in I, \tag{3.3}$$

instead of φ given by (3.2). Note also that the assumptions on h are more restrictive there.

Theorem 3.5. ([8]) *Let ν be a non-negative Borel measure on $I = (a,b)$ and let $u \in B^+(I)$ be such that the function $\|u\|_{\infty,(a,t],\nu} < +\infty$ for all $t \in I$. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|u\|_{L_\infty((a,t+],\nu)}$, $t \in I$, then*

$$\|hu\|_{L_\infty((a,b),\nu)} \approx \sup_{N\le k\le M} h(x_k)\|u\|_{L_\infty((a,x_k+],\nu)} \tag{3.4}$$

for all non-negative, non-increasing and right-continuous functions h on I.

Let φ be a non-negative, non-decreasing, finite and right-continuous function on $(0,\infty)$. Using a discretizing sequence $\{x_k\}_{k=N}^{M+1}$ of φ, we define the sequence $\{J_k\}_{k=N}^{M}$ and $\{S_k\}_{k=N}^{M}$ as follows:

$$J_i = (x_i, x_{i+1}], \quad \text{if} \quad N \le i < M, \quad \text{and} \quad J_M = (x_M, \infty) \quad \text{if} \quad M < +\infty. \tag{3.5}$$

$$\begin{gathered} S_i = B(0,x_{i+1})\backslash B(0,x_i), \quad \text{if} \quad N \le i < M, \quad \text{and} \\ S_M = \mathbb{R}^n\backslash B(0,x_M) \quad \text{if} \quad M < +\infty. \end{gathered} \tag{3.6}$$

Corollary 3.6. *Let $0 < q < +\infty$. Suppose that v be a weight function on $(0,\infty)$. Let v be such that the function $\varphi(t) = \|v\|^q_{L_q(0,t)}$ is finite on $(0,\infty)$. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of φ, then*

$$\left\| v(t) \int_{\complement B(0,t)} g(y)\,dy \right\|_{L_q(0,\infty)} \approx \left(\sum_{k=N}^{M} \left(\int_{S_k} g(y)\,dy \right)^q \|v\|^q_{L_q(0,x_k)} \right)^{\frac{1}{q}} \tag{3.7}$$

and

$$\left\| v(t) \|g\|_{L_\infty(\complement B(0,t))} \right\|_{L_q(0,\infty)} \approx \left(\sum_{k=N}^{M} \|g\|^q_{L_\infty(S_k)} \|v\|^q_{L_q(0,x_k)} \right)^{\frac{1}{q}} \tag{3.8}$$

for all non-negative measurable g on $\mathbb{R}^n$, where $\{S_k\}_{k=N}^{M}$ is defined by (3.6).

Proof. We prove (3.7) only (the proof of (3.8) is analogous). By Theorem 3.4,

$$\left\| v(t) \int_{\complement B(0,t)} g(y)\,dy \right\|_{L_q(0,\infty)} \approx \left(\sum_{k=N}^{M} \left(\int_{\complement B(0,x_k)} g(y)\,dy \right)^q \|v\|^q_{L_q(0,x_k)} \right)^{\frac{1}{q}}$$

$$= \left(\sum_{k=N}^{M} \left(\sum_{i=k}^{M} \int_{S_k} g(y)\,dy \right)^q \|v\|_{L_q(0,x_k)}^q \right)^{\frac{1}{q}}.$$

The condition (iii) of Lemma 3.1 implies that $\{\|v\|_{L_q(0,x_k)}^q\}_{k=N}^M$ is an almost geometrically increasing sequence. (We can take $\alpha = L = 2$ in Definition 2.1. Indeed, by the monotonicity of φ and the condition (iii) of Lemma 3.1, $2\varphi(x_{k-1}) \le 2\varphi(x_k-) \le \varphi(x_{k+1})$ if $N < k < M$, and, on putting $k-1 = m-2$, we arrive at $2\varphi(x_{m-2}) \le \varphi(x_m)$ if $N+2 \le m \le M$.) Thus $\{\|v\|_{L_q(0,x_k)}\}_{k=N}^M$ is also an almost geometrically increasing sequence and (3.7) follows on applying Lemma 2.5. $\square$

Corollary 3.7. *Suppose that v be a weight function on $(0,\infty)$. Let v be such that the function $\varphi(t) = \|v\|_{L_\infty(0,t)}$ is finite on $(0,\infty)$. If $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|v\|_{L_\infty(a,t+)} = \lim_{s\to t+}\|v\|_{L_\infty(a,s)}$, $t \in (0,\infty)$, then*

$$\left\| v(t) \int_{\complement B(0,t)} g(y)\,dy \right\|_{L_\infty(0,\infty)} \approx \sup_{N\le k\le M} \left(\int_{S_k} g(y)\,dy \right) \|v\|_{L_\infty(0,x_k+)} \tag{3.9}$$

and

$$\left\| v(t) \|g\|_{L_\infty(\complement B(0,t))} \right\|_{L_\infty(0,\infty)} \approx \sup_{N\le k\le M} \|g\|_{L_\infty(S_k)} \|v\|_{L_\infty(0,x_k+)} \tag{3.10}$$

for all non-negative measurable g on $\mathbb{R}^n$, where $\{S_k\}_{k=N}^M$ is defined by (3.6).

Proof. This follows from Theorem 3.5 and Lemma 2.5. $\square$

4. The multidimensional reverse Hardy inequality

In this section we characterize the validity of the inequality

$$\|gw\|_{L_p(\mathbb{R}^n)} \le c \left\| v(t) \int_{\complement B(0,t)} g(y)\,dy \right\|_{L_q(0,\infty)}. \tag{4.1}$$

for all non-negative measurable g on $\mathbb{R}^n$. Our first result concerns the case when $0 < q \le p \le 1$.

Theorem 4.1. *Assume that $0 < q \le p \le 1$. Let ω and v be a weight functions on $\mathbb{R}^n$ and $(0,\infty)$ respectively. Let $\|v\|_{L_q(0,t)} < +\infty$ for all $t \in (0,\infty)$. Then the inequality (4.1) holds for all non-negative measurable g if and only if*

$$A_1 := \sup_{t\in(0,\infty)} \|w\|_{L_{p'}(B(0,t))} \|v\|_{L_q(0,t)}^{-1} < +\infty.$$

The best possible constant c in (4.1) satisfies $c \approx A_1$.

Proof. Let $0 < q \le 1$. By Corollary 3.6,

$$\left\| v(t) \int_{\complement B(0,t)} g(y)\,dy \right\|_{L_q(0,\infty)} \approx \left(\sum_{k=N}^{M} \left(\int_{S_k} g(y)\,dy \right)^q \|v\|_{L_q(0,x_k)}^q \right)^{\frac{1}{q}} \tag{4.2}$$

for all non-negative measurable g on $\mathbb{R}^n$, where $\{x_k\}_{k=N}^{M+1}$ is a discretizing sequence of the function $\varphi(t) = \|v\|^q_{L_q(0,t)}$, $t \in (0,\infty)$, and $\{S_k\}_{k=N}^{M}$ is defined by (3.6). By Lemma 3.1 (cf. also Remark 3.3),

$$\begin{aligned}&\text{if} \quad x_N > 0, \quad \text{then} \quad \|v\|_{L_q(0,x_N)} = 0;\\ &\text{if} \quad M < +\infty, \quad \text{then} \quad x_{M+1} = \infty;\end{aligned} \tag{4.3}$$

$$\|v\|^q_{L_q(0,x_{k+1})} \le 2\|v\|^q_{L_q(0,x_k)} \quad \text{if} \quad N \le k \le M; \tag{4.4}$$

$$2\|v\|^q_{L_q(0,x_k)} \le \|v\|^q_{L_q(0,x_{k+1})} \quad \text{if} \quad N < k < M. \tag{4.5}$$

Assume that $A_1 < +\infty$. This condition and (4.3) imply that

$$\|w\|_{L'_p(B(0,x_N))} = 0 \quad \text{if} \quad x_N > 0. \tag{4.6}$$

If E is a measurable subset of $(0,\infty)$ and g is a non-negative measurable function on $(0,\infty)$, then by Hőlder's inequality (with the exponents $1/p$ and p'/p),

$$\|gw\|^p_{L_p(E)} \le \|g\|^p_{L_1(E)}\|w\|^p_{Lp'(E)}. \tag{4.7}$$

Taking here $g \equiv 1$ and $E = B(0,x_N)$, we obtain from (4.6) that $\|w\|_{L_p(B(0,x_N))} = 0$ if $x_N > 0$. Therefore,

$$\|gw\|_{L_p(\mathbb{R}^n)} = \left(\sum_{k=N}^{M} \|gw\|^p_{L_p(S_k)}\right)^{\frac{1}{p}} \tag{4.8}$$

for any non-negative measurable g on $\mathbb{R}^n$.

This identity and (4.7) (with $E = S_k$, $N \le k \le M$) give

$$\begin{aligned}\|gw\|_{L_p(\mathbb{R}^n)} &\le \left(\sum_{k=N}^{M} \|g\|^p_{L_1(S_k)}\|w\|^p_{L_{p'}(S_k)}\right)^{\frac{1}{p}}\\ &\le \left(\sup_{N\le k\le M} \|w\|_{L_{p'}(S_k)}\|v\|^{-1}_{L_q(0,x_k)}\right)\left(\sum_{k=N}^{M} \|g\|^p_{L_1(S_k)}\|v\|^p_{L_q(0,x_k)}\right)^{\frac{1}{p}}.\end{aligned}$$

Moreover, using the inequality $0 < q/p \le 1$ and (4.2), we arrive at

$$\begin{aligned}&\|gw\|_{L_p(\mathbb{R}^n)}\\ &\le \left(\sup_{N\le k\le M} \|w\|_{L_{p'}(S_k)}\|v\|^{-1}_{L_q(0,x_k)}\right)\left(\sum_{k=N}^{M} \|g\|^q_{L_1(S_k)}\|v\|^q_{L_q(0,x_k)}\right)^{\frac{1}{q}}\\ &\approx \left(\sup_{N\le k\le M} \|w\|_{L_{p'}(S_k)}\|v\|^{-1}_{L_q(0,x_k)}\right)\left\|v(x)\int_{\complement B(0,x)} g(y)\,dy\right\|_{L_q(0,\infty)}.\end{aligned} \tag{4.9}$$

Applying (4.4), we get

$$\begin{aligned}&\sup_{N\le k\le M} \|w\|_{L_{p'}(S_k)}\|v\|^{-1}_{L_q(0,x_k)}\\ &\qquad\le 2^{\frac{1}{q}} \sup_{N\le k\le M} \|w\|_{Lp'(B(0,x_{k+1}))}\|v\|^{-1}_{L_q(0,x_{k+1})} \le 2^{\frac{1}{q}} A_1.\end{aligned} \tag{4.10}$$

The inequality (4.1) (with $c \lesssim A_1$) follows from (4.9) and (4.10).

We now prove necessity. The validity of the inequality (4.1) and (4.2) imply that

$$\left(\sum_{k=N}^{M} \|gw\|_{L_p(S_k)}^p \right)^{\frac{1}{p}} \lesssim c \left(\sum_{k=N}^{M} \left(\int_{S_k} g(y)\,dy \right)^q \|v\|_{L_q(0,x_k)}^q \right)^{\frac{1}{q}} \tag{4.11}$$

for all non-negative measurable g on $\mathbb{R}^n$.

Let g_k, $N \le k \le M$, be non-negative measurable functions that saturate Hőlder's inequality (4.7) with $E = S_k$, $N \le k \le M$, that is, functions satisfying

$$\operatorname{supp} g_k \subset S_k, \quad \|g_k\|_{L_1(S_k)} = 1 \quad \text{and} \quad \|g_k w\|_{L_p(S_k)}^p \ge \frac{1}{2} \|w\|_{L_{p'}(S_k)}^p. \tag{4.12}$$

Then we define the test function g by

$$g = \sum_{k=N}^{M} a_k\, g_k, \tag{4.13}$$

where $\{a_k\}$ is a sequence of non-negative numbers. Consequently, (4.11) yields

$$\left(\sum_{k=N}^{M} a_k^p \|w\|_{L_{p'}(S_k)}^p \right)^{\frac{1}{p}} \lesssim c \left(\sum_{k=N}^{M} a_k^q \|v\|_{L_q(0,x_k)}^q \right)^{\frac{1}{q}}, \tag{4.14}$$

and, by Lemma 2.6,

$$\sup_{N \le k \le M} \|w\|_{L_{p'}(S_k)} \|v\|_{L_q(0,x_k)}^{-1} \lesssim c. \tag{4.15}$$

Assuming that $x_N > 0$, testing (4.1) with $g = \chi_{B(0,x_N)}$ and using (4.3), we arrive at $\|w\|_{L_p(B(0,x_N))} = 0$. This implies that $|B(0,x_N)| = 0$ or $w = 0$ a.e. in $B(0,x_N)$. Consequently, (4.6) holds.

Therefore,

$$A_1 = \sup_{N \le k \le M} \sup_{x \in J_k} \|w\|_{L_{p'}(B(0,x))} \|v\|_{L_q(0,x)}^{-1}$$

and, on using (3.5), we obtain that

$$A_1 \le \sup_{N \le k \le M} \|w\|_{L_{p'}(B(0,x_{k+1}))} \|v\|_{L_q(0,x_k)}^{-1}.$$

Applying (4.6) and (3.6) again, we arrive at

$$A_1 \le \sup_{N \le k \le M} \left(\sum_{i=N}^{k} \|w\|_{L_{p'}(S_i)}^{p'} \right)^{\frac{1}{p'}} \|v\|_{L_q(0,x_k)}^{-1} \quad \text{if} \quad 0 < p < 1$$

and

$$A_1 \le \sup_{N \le k \le M} \left(\sup_{N \le i \le k} \|w\|_{L_{p'}(S_i)} \right) \|v\|_{L_q(0,x_k)}^{-1} \quad \text{if} \quad p = 1.$$

Now, the fact that $\{\|v\|_{L_q(0,x_k)}^{-1}\}_{k=N}^{M}$ is almost geometrically decreasing (cf. (4.5)) and Lemma 2.4 imply that

$$A_1 \lesssim \sup_{N \le k \le M} \|w\|_{L_{p'}(S_k)} \|v\|^{-1}_{L_q(0,x_k)},$$

which, together with (4.15), yields $A_1 \lesssim c$. $\square$

Remark 4.2. Let A_1 be the number defined in Theorem 4.1. If $p = 1$, then

$$A_1 = \left\| w(x) \|v\|^{-1}_{L_q(0,|x|)} \right\|_{L_\infty(\mathbb{R}^n)}.$$

Indeed, exchanging essential suprema, we obtain

$$\begin{aligned} A_1 &= \left\| \|w\|_{L_\infty(B(0,t))} \|v\|^{-1}_{L_q(0,t)} \right\|_{L_\infty(0,\infty)} \\ &= \left\| \left\| w(x) \|v\|^{-1}_{L_q(0,t)} \right\|_{L_\infty(B(0,t))} \right\|_{L_\infty(0,\infty)} \\ &= \left\| \left\| w(x) \chi_{B(0,t)}(x) \|v\|^{-1}_{L_q(0,t)} \right\|_{L_\infty(\mathbb{R}^n)} \right\|_{L_\infty(0,\infty)} \\ &= \left\| \left\| w(x) \|v\|^{-1}_{L_q(0,t)} \right\|_{L_\infty[|x|,\infty)} \right\|_{L\infty(\mathbb{R}^n)} \\ &= \left\| w(x) \|v\|^{-1}_{L_q(0,|x|)} \right\|_{L_\infty(\mathbb{R}^n)}. \end{aligned}$$

In the rest of the paper we shall need the Lebesgue-Stieltjes integral. To this end, we recall some basic facts.

Let φ be non-decreasing and finite function on the interval $I := (a, b) \subseteq \mathbb{R}$. We assign to φ the function λ defined on subintervals of I by

$$\lambda([\alpha, \beta]) = \varphi(\beta+) - \varphi(\alpha-), \tag{4.16}$$

$$\lambda([\alpha, \beta)) = \varphi(\beta-) - \varphi(\alpha-), \tag{4.17}$$

$$\lambda((\alpha, \beta]) = \varphi(\beta+) - \varphi(\alpha+), \tag{4.18}$$

$$\lambda((\alpha, \beta)) = \varphi(\beta-) - \varphi(\alpha+). \tag{4.19}$$

The function λ is a non-negative, additive and regular function of intervals. Thus (cf. [18]), it admits a unique extension to a non-negative Borel measure λ on I. The Lebesgue-Stieltjes integral $\int_I f\, d\varphi$ is defined as $\int_I f\, d\lambda$.

In this section the role of the function φ will be played by a function h which will be *non-decreasing* and *right-continuous* on I. Consequently, the associated Borel measure λ will be determined by (cf. (4.18))

$$\lambda((\alpha, \beta]) = h(\beta) - h(\alpha) \quad \text{for any} \quad (\alpha, \beta] \subset I \tag{4.20}$$

(since the Borel subsets of I can be generated by subintervals $(\alpha, \beta] \subset I$).

Consider now the inequality (4.1) in the case when $0 < p \le 1$, $p < q \le +\infty$ and define r by

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{4.21}$$

In such a case we shall write a condition characterizing the validity of inequality (4.1) in a compact form involving $\int_{(0,\infty)} f\, dh$, where $f(t) = \|w\|^r_{L_{p'}(B(0,t))}$

and $h(t) = -\|u\|_{L_q(0,t+)}^{-r}$, $t \in (0,\infty)$. (Hence, the Lebesgue-Stieltjes integral $\int_{(0,\infty)} f\,dh$ is defined by the non-decreasing and right-continuous function h on $(0,\infty)$). However, it can happen that $\|u\|_{L_q(0,t+)} = 0$ for all $t \in (0,c)$ with a convenient $c \in (0,\infty)$ (provided that we omit the trivial case when $u = 0$ a.e. on $(0,\infty)$). Then we have to explain what is the meaning of the Lebesgue-Stieltjes integral since in such a case the function $h = -\infty$ on $(0,c)$. To this end, we adopt the following convention.

Convention 4.3. Let $I = (a,b) \subseteq \mathbb{R}$, $f : I \to [0,+\infty]$ and $h : I \to [-\infty, 0]$. Assume that h is non-decreasing and right-continuous on I. If $h : I \to (-\infty, 0]$, then the symbol $\int_I f\,dh$ means the usual Lebesgue-Stieltjes integral. However, if $h = -\infty$ on some subinterval (a,c) with $c \in I$, then we define $\int_I f\,dh$ only if $f = 0$ on $(a,c]$ and we put

$$\int_I f\,dh = \int_{(c,b)} f\,dh.$$

In the proof of the next theorem we shall use frequently the Lebesgue-Stieltjes integral $\int_J d\varphi$, where φ is a non-decreasing, finite and right-continuous function on $I = (a,b)$ and J is a subinterval of I of the form (α,β), $[\alpha,\beta)$ or $(\alpha,\beta]$. The formulae (4.19), (4.17) and (4.18) imply that

$$\int_{(\alpha,\beta)} d\varphi = \varphi(\beta-) - \varphi(\alpha), \tag{4.22}$$

$$\int_{[\alpha,\beta)} d\varphi = \varphi(\beta-) - \varphi(\alpha-), \tag{4.23}$$

$$\int_{(\alpha,\beta]} d\varphi = \varphi(\beta) - \varphi(\alpha). \tag{4.24}$$

Theorem 4.4. *Assume that $0 < p \le 1$, $p < q \le +\infty$ and r is given by* (4.21). *Let ω and υ be a weight functions on $\mathbb{R}^n$ and $(0,\infty)$ respectively. Let υ satisfy $\|\upsilon\|_{L_q(0,t)} < +\infty$ for all $t \in (0,\infty)$ and $\upsilon \ne 0$ a.e. on $(0,\infty)$. Then the inequality* (4.1) *holds for all non-negative measurable g on $\mathbb{R}^n$ if and only if*

$$A_2 := \left(\int_{(0,\infty)} \|w\|_{L_{p'}(B(0,t))}^r \, d\left(-\|\upsilon\|_{L_q(0,t+)}^{-r} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|\upsilon\|_{L_q(0,\infty)}} < +\infty.$$

The best possible constant c in (4.1) *satisfies $c \approx A_2$.*

Remark 4.5. Let $q < +\infty$ in Theorem 4.4. Then

$$\|u\|_{L_q(0,t+)} = \|u\|_{L_q(0,t)} \quad \text{for all} \quad t \in (0,\infty),$$

which implies that

$$A_2 = \left(\int_{(0,\infty)} \|w\|_{L_{p'}(B(0,t))}^r \, d\left(-\|\upsilon\|_{L_q(0,t)}^{-r} \right) \right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|\upsilon\|_{L_q(0,\infty)}}.$$

Proof of Theorem 4.4. Let $0 < p \le 1$ and $p < q \le +\infty$.

(i) Suppose first that $q < +\infty$. Let $\{x_k\}_{k=N}^{M+1}$ be the discretizing sequence of the function $\varphi(t) = \|v\|^q_{L_q(0,t)}$, $t \in (0,\infty)$. Then (4.3)–(4.5) are satisfied. Moreover, by Corollary 3.6, (4.2) holds, where $\{S_k\}_{k=N}^{M}$ is given by (3.6).

Assume that $A_2 < +\infty$. This condition, (4.3) and Convention 4.3 imply that (4.6) holds and, as in the proof of Theorem 4.1, we arrive at (4.8). Thus, using (4.7) (with $E = S_k$, $N \le k \le M$), the discrete version of Hőlder's inequality (with the exponents q/p and r/p) and (4.2), we obtain

$$\begin{aligned}
&\|gw\|_{L_p(\mathbb{R}^n)} \\
&\quad\le \left(\sum_{k=N}^{M} \|g\|^p_{L_1(S_k)} \|w\|^p_{L_{p'}(S_k)}\right)^{\frac{1}{p}} \\
&\quad\le \left(\sum_{k=N}^{M} \|g\|^q_{L_1(S_k)} \|v\|^q_{L_q(0,x_k)}\right)^{\frac{1}{q}} \left(\sum_{k=N}^{M} \|w\|^r_{L_{p'}(S_k)} \|v\|^{-r}_{L_q(0,x_k)}\right)^{\frac{1}{r}} \\
&\quad\approx \left\| v(t) \int_{\complement B(0,t)} g(y)\,dy \right\|_{L_q(0,\infty)} \left(\sum_{k=N}^{M} \|w\|^r_{L_{p'}(S_k)} \|v\|^{-r}_{L_q(0,x_k)}\right)^{\frac{1}{r}}.
\end{aligned} \tag{4.25}$$

By (4.5),

$$2\|v\|^q_{L_q(0,x_{k+1})} \le \|v\|^q_{L_q(0,x_{k+2})} \le \|v\|^q_{L_q(0,x_{k+3})} \quad \text{if} \quad N < k+1 < M.$$

Therefore,

$$\|v\|^{-r}_{L_q(0,x_{k+3})} \le 2^{-\frac{r}{q}} \|v\|^{-r}_{L_q(0,x_{k+1})},$$

which yields

$$\|v\|^{-r}_{L_q(0,x_{k+1})} - \|v\|^{-r}_{L_q(0,x_{k+3})} \ge (1 - 2^{-\frac{r}{q}}) \|v\|^{-r}_{L_q(0,x_{k+1})} \quad \text{if} \quad N \le k \le M-2.$$

Assume that $N \le M-2$. On using (4.4) and the last estimate, we arrive at

$$\begin{aligned}
&\sum_{k=N}^{M} \|w\|^r_{L_{p'}(S_k)} \|v\|^{-r}_{L_q(0,x_k)} \\
&\qquad\lesssim \sum_{k=N}^{M} \|w\|^r_{L_{p'}(S_k)} \|v\|^{-r}_{L_q(0,x_{k+1})} \\
&\qquad\lesssim \sum_{k=N}^{M-2} \|w\|^r_{L_{p'}(S_k)} \left(\|v\|^{-r}_{L_q(0,x_{k+1})} - \|v\|^{-r}_{L_q(0,x_{k+3})}\right) \\
&\qquad+ \|w\|^r_{L_{p'}(S_{M-1})} \left(\|v\|^{-r}_{L_q(0,x_M)} - \|v\|^{-r}_{L_q(0,\infty)}\right) \\
&\qquad+ \|w\|^r_{L_{p'}(S_{M-1})} \|v\|^{-r}_{L_q(0,\infty)} + \|w\|^r_{L_{p'}(S_M)} \|v\|^{-r}_{L_q(0,\infty)}.
\end{aligned} \tag{4.26}$$

Now, by (4.23) with $\varphi(t) = -\|v\|^{-r}_{L_q(0,t)}$, $t \in (0,\infty)$, and $[\alpha,\beta) = [x_{k+1}, x_{k+3})$, $N \le k \le M-2$, or $[\alpha,\beta) = [x_M,\infty)$, we obtain that

$$\begin{aligned}
&\sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^r \|v\|_{L_q(0,x_k)}^{-r} \\
&\qquad \le \sum_{k=N}^{M-2} \|w\|_{L_{p'}(S_k)}^r \int_{[x_{k+1},x_{k+3})} d\left(-\|v\|_{L_q(0,t)}^{-r}\right) \\
&\qquad + \|w\|_{L_{p'}(S_{M-1})}^r \int_{[x_M,\infty)} d\left(-\|v\|_{L_q(0,t)}^{-r}\right) + 2\|w\|_{L_{p'}(\mathbb{R}^n)}^r \|v\|_{L_q(0,\infty)}^{-r} \\
&\qquad \le \sum_{k=N}^{M-2} \int_{[x_{k+1},x_{k+3})} \|w\|_{L_{p'}(B(0,t))}^r \, d\left(-\|v\|_{L_q(0,t)}^{-r}\right) \\
&\qquad + \int_{[x_M,\infty)} \|w\|_{L_{p'}(B(0,t))}^r \, d\left(-\|v\|_{L_q(0,t)}^{-r}\right) + 2\|w\|_{L_{p'}(\mathbb{R}^n)}^r \|v\|_{L_q(0,\infty)}^{-r} \\
&\qquad \le 2\int_{(0,\infty)} \|w\|_{L_{p'}(B(0,t))}^r \, d\left(-\|v\|_{L_q(0,t)}^{-r}\right) + 2\|w\|_{L_{p'}(\mathbb{R}^n)}^r \|v\|_{L_q(0,\infty)}^{-r} \\
&\qquad \lesssim A_2^r
\end{aligned}$$

(note that we have used (4.6) and Convention 4.3), that is,

$$\sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^r \|v\|_{L_q(0,x_k)}^{-r} \lesssim A_2^r. \tag{4.27}$$

If $N > M-2$, then (4.27) can be proved analogously. The inequality (4.1) (with $c \le A_2$) follows from (4.25) and (4.27).

For necessity we apply the same argument as in the proof of Theorem 4.1 to get (4.14). Next, by Lemma 2.7,

$$\left(\sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^r \|v\|_{L_q(0,x_k)}^{-r}\right)^{\frac{1}{r}} \lesssim c. \tag{4.28}$$

As in the necessity part of the proof of the Theorem 4.1, we can show that (4.6) holds. Together with (3.5), (4.24) and (4.22), this yields

$$\begin{aligned}
A_2^r &\approx \sum_{k=N}^{M} \int_{J_k} \|w\|_{L_{p'}(B(0,t))}^r \, d\left(-\|v\|_{L_q(0,t)}^{-r}\right) + \|w\|_{L_{p'}(\mathbb{R}^n)}^r \|v\|_{L_q(0,\infty)}^{-r} \\
&\le \sum_{k=N}^{M-1} \|w\|_{L_{p'}(B(0,x_{k+1}))}^r \int_{J_k} d\left(-\|v\|_{L_q(0,t)}^{-r}\right) \\
&+ \|w\|_{L_{p'}(\mathbb{R}^n)}^r \int_{(x_M,\infty)} d\left(-\|v\|_{L_q(0,t)}^{-r}\right) + \|w\|_{L_{p'}(\mathbb{R}^n)}^r \|v\|_{L_q(0,\infty)}^{-r} \\
&\lesssim \sum_{k=N}^{M-1} \|w\|_{L_p'(B(0,x_{k+1}))}^r \|v\|_{L_q(0,x_k)}^{-r} + \|w\|_{L_{p'}(\mathbb{R}^n)}^r \|v\|_{L_q(0,x_M)}^{-r}.
\end{aligned} \tag{4.29}$$

Thus, using (4.6) and (3.5) again, we arrive at

$$A_2^r \lesssim \sum_{k=N}^{M} \left(\sum_{i=N}^{k} \|w\|_{Lp'(S_i)}^{p'} \right)^{\frac{r}{p'}} \|v\|_{L_q(0,x_k)}^{-r} \quad \text{if} \quad 0 < p < 1$$

and

$$A_2^r \lesssim \sum_{k=N}^{M} \left(\sup_{N \le i \le k} \|w\|_{L_{p'}(S_i)} \right)^{r} \|v\|_{L_q(0,x_k)}^{-r} \quad \text{if} \quad p = 1.$$

Now, the fact that $\{\|v\|_{L_q(0,x_k)}^{-r}\}_{k=N}^{M}$ is almost geometrically decreasing (cf. (4.5)) and Lemma 2.4 imply that

$$A_2^r \lesssim \sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^{r} \|v\|_{L_q(0,x_k)}^{-r}, \tag{4.30}$$

which, together with (4.28), yields $A_2 \lesssim c$.

(ii) Suppose now that $q = +\infty$. Let $\{x_k\}_{k=N}^{M+1}$ be a discretizing sequence of the function $\varphi(t) = \|v\|_{L_\infty(0,t+)}$, $t \in (0, \infty)$. By Lemma 3.1 (cf. also Remark 3.3),

$$\text{if} \quad x_N > 0, \quad \text{then} \quad \|v\|_{L_\infty(0,x_N)} = 0; \tag{4.31}$$

$$\text{if} \quad M < +\infty, \quad \text{then} \quad x_{M+1} = \infty;$$

$$\|v\|_{L_\infty(0,x_{k+1})} \le 2\|v\|_{L_\infty(0,x_k+)} \quad \text{if} \quad N \le k \le M; \tag{4.32}$$

$$2\|v\|_{L_\infty(0,x_k)} \le \|v\|_{L_\infty(0,x_{k+1}+)} \quad \text{if} \quad N < k < M. \tag{4.33}$$

Moreover, by Corollary 3.7,

$$\left\| v(t) \int_{\complement B(0,t)} g(y)\, dy \right\|_{L_\infty(0,\infty)} \approx \sup_{N \le k \le M} \left(\int_{S_k} g(y)\, dy \right) \|v\|_{L_\infty(0,x_k+)} \tag{4.34}$$

for all non-negative measurable g on $\mathbb{R}^n$, where $\{S_k\}_{k=N}^{M}$ is given by (3.6).

Assume that $A_2 < +\infty$. This condition, (4.31) and Convention 4.3 imply that (4.6) holds, and, as in the proof of Theorem 4.1, we arrive at (4.8). Thus, using (4.7) (with $E = S_k$, $N \le k \le M$) and (4.34), we obtain

$$\begin{aligned}
&\|gw\|_{L_p(\mathbb{R}^n)} \\
&\quad \le \left(\sum_{k=N}^{M} \|g\|_{L_1(S_k)}^{p} \|w\|_{L_{p'}(S_k)}^{p} \right)^{\frac{1}{p}} \\
&\quad \le \left(\sup_{N \le k \le M} \|g\|_{L_1(S_k)} \|v\|_{L_\infty(0,x_k+)} \right) \left(\sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^{p} \|v\|_{L_\infty(0,x_k+)}^{-p} \right)^{\frac{1}{p}} \\
&\quad \approx \left\| u(t) \int_{\complement B(0,t)} g(y)\, dy \right\|_{L_\infty(0,\infty)} \left(\sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^{p} \|v\|_{L_\infty(0,x_k+)}^{-p} \right)^{\frac{1}{p}}. \qquad (4.35)
\end{aligned}$$

Analogously as in the case (i), we arrive at

$$\left(\sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^{p} \|v\|_{L_\infty(0,x_k+)}^{-p} \right)^{\frac{1}{p}} \lesssim A_2. \tag{4.36}$$

Therefore, (4.1) (with $c \lesssim A_2$) follows from (4.35) and (4.36).

Now, we prove necessity part. The validity of the inequality (4.1) and (4.34) imply that

$$\left(\sum_{k=N}^{M} \|gw\|_{L_p(S_k)}^{p} \right)^{\frac{1}{p}} \lesssim c \sup_{N \le k \le M} \left(\int_{S_k} g(y)\, dy \right) \|v\|_{L_\infty(0,x_k+)} \tag{4.37}$$

for all non-negative measurable g on $\mathbb{R}^n$. Let g_k, $N \le k \le M$, be non-negative measurable functions satisfying (4.12) and define the test function g by (4.13). Consequently, (4.37) yields

$$\left(\sum_{k=N}^{M} a_k^p \|w\|_{L_{p'}(S_k)}^{p} \right)^{\frac{1}{p}} \lesssim \sup_{N \le k \le M} a_k \|v\|_{L_\infty(0,x_k+)},$$

and, by Lemma 2.7,

$$\left(\sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^{p} \|v\|_{L_\infty(0,x_k+)}^{-p} \right)^{\frac{1}{p}} \lesssim c. \tag{4.38}$$

The same idea as that used in part (i) shows that (cf. (4.29)–(4.30))

$$A_2^p \lesssim \sum_{k=N}^{M} \|w\|_{L_{p'}(S_k)}^{p} \|v\|_{L_\infty(0,x_k+)}^{-p}, \tag{4.39}$$

which, together with (4.38), yields $A_2 \lesssim c$. $\square$

5. The reverse Hardy inequality for the dual operator

The aim of this section is to characterize the validity of the reverse Hardy inequality involving the operator H^* given by

$$(H^*g)(t) := \int_{B(0,t)} g(y)\, dy, \quad t \in (0,\infty),$$

which is the dual operator to that one given by

$$(Hg)(t) := \int_{\complement B(0,t)} g(y)\, dy, \quad x \in I,$$

where g is a non-negative measurable g on $\mathbb{R}^n$. To this end, we are going to make use of the results for the Hardy operator H proved in Section 4. Our next assertion is a counterpart of Theorem 4.1.

Theorem 5.1. *Assume that $0 < q \le p \le 1$. Let ω and v be a weight functions on $\mathbb{R}^n$ and $(0,\infty)$ respectively. Let $\|v\|_{L_q(t,\infty)} < +\infty$ for all $t \in (0,\infty)$. Then the inequality*

$$\|gw\|_{L_p(\mathbb{R}^n)} \le c \left\| v(t) \int_{B(0,t)} g(y)\, dy \right\|_{L_q(0,\infty)} \tag{5.1}$$

holds for all non-negative measurable g on $\mathbb{R}^n$ if and only if

$$B_1 := \sup_{t\in(0,\infty)} \|w\|_{L_{p'}(\complement B(0,t))} \|v\|^{-1}_{L_q(t,\infty)} < +\infty. \tag{5.2}$$

The best possible constant c in (5.1) satisfies $c \approx B_1$.

Proof. By writing the inequality (5.1) for $|y|^{-2n} g(\frac{y}{|y|^2})$ instead of g and using the substitutions $x = \frac{y}{|y|^2}$ on the left-hand side and $x = \frac{y}{|y|^2}$ and $\tau = \frac{1}{t}$ on the right-hand side we obtain

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} g(x)^p \left(\omega\left(\frac{x}{|x|^2}\right) |x|^{-\frac{2n}{p'}} \right)^p dx \right)^{\frac{1}{p}} \\ &\le c \left(\int_0^\infty v\left(\frac{1}{\tau}\right)^q \frac{1}{\tau^2} \left(\int_{\complement B(0,\tau)} g(x) dx \right)^q d\tau \right)^{\frac{1}{q}}. \end{aligned} \tag{5.3}$$

Consequently, the inequality (5.1) holds for all non-negative measurable g on $\mathbb{R}^n$ if and only if the inequality (5.3) holds for all non-negative measurable g on $\mathbb{R}^n$. We deduce from Theorem 4.1 that the inequality (5.1) holds for all non-negative measurable g on $\mathbb{R}^n$ if and only if

$$\sup_{t\in(0,\infty)} \left(\int_{B(0,t)} \left(\omega\left(\frac{x}{|x|^2}\right) |x|^{-\frac{2n}{p'}} \right)^{p'} dx \right)^{\frac{1}{p'}} \left(\int_0^t v\left(\frac{1}{\tau}\right)^q \frac{1}{\tau^2} d\tau \right)^{-\frac{1}{q}} < +\infty,$$

that is,

$$\sup_{t\in(0,\infty)} \|w\|_{L_{p'}(\complement B(0,t))} \|v\|^{-1}_{L_q(t,\infty)} < +\infty. \tag{5.4}$$

□

Remark 5.2. Let B_1 be the number defined in Theorem 5.1. If $p = 1$, then

$$B_1 = \left\| w(x) \|u\|^{-1}_{L_q(|x|,\infty)} \right\|_{L_\infty(\mathbb{R}^n)}.$$

Indeed, using the idea of the proof of Theorem 5.1, we obtain the result from Remark 4.2.

Consider now the inequality (5.1) in the case when $0 < p \le 1$, $p < q \le +\infty$ and define r by

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{q}. \tag{5.5}$$

As in Section 4, in such a case we shall write a condition characterizing the validity of the inequality (5.1) in a compact form involving the Lebesgue-Stieltjes integral $\int_{(a,b)} f\, dh$, say. In contrast to Section 4, now the Lebesgue-Stieltjes integral $\int_{(a,b)} f\,dh$ will be defined by a *non-decreasing* and *left-continuous function* h on I. We shall see in our next theorem that $f(t) = \|w\|^r_{L_{p'}(\complement B(0,t))}$ and $h(t) = \|u\|^{-r}_{q,(t-,b),\nu} := \lim_{s\to t-} \|u\|^{-r}_{L_q(s,\infty)}$, $t \in (0,\infty)$. However, it can happen that $\|u\|_{L_q(t-,b)} = 0$ for all $t \in (c,b)$ with some $c \in (a,b)$ (provided that we omit the trivial case when $u = 0$ a.e. on (a,b)). Then we have to explain what is the meaning of the Lebesgue-Stieltjes integral since in such a case the function $h = +\infty$ on (c,b). To this end, we adopt the following convention.

Convention 5.3. Let $I = (a,b) \subseteq \mathbb{R}$, $f : I \to [0,+\infty]$ and $h : I \to [0,+\infty]$. Assume that h is non-decreasing and left-continuous on I. If $h : I \to [0,+\infty)$, then the symbol $\int_I f\,dh$ means the usual Lebesgue-Stieltjes integral (the measure λ associated to h is given by $\lambda([\alpha,\beta)) = h(\beta) - h(\alpha)$ if $[\alpha,\beta) \subset (a,b)$ – cf. (4.17)). However, if $h = +\infty$ on some subinterval (c,b) with $c \in I$, then we define $\int_I f\,dh$ only if $f = 0$ on $[c,b)$ and we put

$$\int_I f\,dh = \int_{(a,c)} f\,dh.$$

Theorem 5.4. *Assume that* $0 < p \le 1$, $p < q \le +\infty$ *and* r *is given by* (5.5). *Let* ω *and* u *be a weight functions on* $\mathbb{R}^n$ *and* $(0,\infty)$ *respectively. Let* u *satisfy* $\|u\|_{L_q(t,\infty)} < +\infty$ *for all* $t \in (0,\infty)$ *and* $u \ne 0$ *a.e. on* $(0,\infty)$. *Then the inequality* (5.1) *holds for all non-negative measurable if and only if*

$$B_2 := \left(\int_{(0,\infty)} \|w\|^r_{L_{p'}(\complement B(0,t))}\, d\left(\|u\|^{-r}_{L_q(t-,\infty)}\right)\right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}} < +\infty.$$

The best possible constant c *in* (5.1) *satisfies* $c \approx B_2$.

Remark 5.5. Let $q < +\infty$ in Theorem 5.4. Then

$$\|u\|_{L_q(t-,\infty)} = \|u\|_{L_q(t,\infty)} \quad \text{for all} \quad t \in (0,\infty),$$

which implies that

$$B_2 = \left(\int_{(a,b)} \|w\|^r_{L_{p'}(\complement B(0,t))}\, d\left(\|u\|^{-r}_{L_q(t,\infty)}\right)\right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}}.$$

Proof of Theorem 5.4. As in the proof of Theorem 5.1, one can show that the inequality (5.1) holds if and only if the inequality (5.3) is satisfied for all non-negative measurable g on $\mathbb{R}^n$. Thus, by Theorem 4.4, the inequality (5.1) holds if and only if

$$\left(\int_{(0,\infty)} \left(\int_{B(0,t)} \left(\omega\left(\frac{x}{|x|^2}\right)|x|^{-\frac{2n}{p'}}\right)^{p'} dx\right)^{\frac{r}{p'}} d\left(-\int_{(0,t)} v\left(\frac{1}{\tau}\right)^q \frac{1}{\tau^2} d\tau\right)^{-\frac{r}{q}}\right)^{\frac{1}{r}}$$

$$+\frac{\left(\int_{\mathbb{R}^n} \left(\omega\left(\frac{x}{|x|^2}\right)|x|^{-\frac{2n}{p'}}\right)^{p'} dx\right)^{\frac{1}{p'}}}{\left(\int_0^\infty v\left(\frac{1}{\tau}\right)^q \frac{1}{\tau^2} d\tau\right)^{\frac{1}{q}}} < \infty,$$

that is,

$$\left(\int_{(0,\infty)} \|w\|^r_{L_{p'}(\complement B(0,t))}\, d\left(\|u\|^{-r}_{L_q(t-,\infty)}\right)\right)^{\frac{1}{r}} + \frac{\|w\|_{L_{p'}(\mathbb{R}^n)}}{\|u\|_{L_q(0,\infty)}} < +\infty.$$

□

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