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Smooth dependence on data of solutions and contact regions for a Signorini problem

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Abstract

We prove that the solutions to a 2D Poisson equation with unilateral boundary conditions of Signorini type as well as their contact intervals depend smoothly on the data. The result is based on a certain local equivalence of the unilateral boundary value problem to a smooth abstract equation in a Hilbert space and on an application of the Implicit Function Theorem to that equation.

1 Introduction

Let $\ell > 0$, $\Omega := (0,1) \times (0,\ell)$, $\Gamma_D := (\{0\} \times (0,\ell)) \cup (\{1\} \times (0,\ell))$, $\Gamma_U := (\gamma_1, \gamma_2) \times \{0\} \subset ((0,1) \times \{0\})$ be an open interval and $\Gamma_N := \partial\Omega \setminus (\Gamma_D \cup \Gamma_U)$. Let h be a positive number and $f \in C^k(\overline{\Omega})$ a real function, $k \in \mathbb{N}$. We will study the Signorini boundary value problem

$$-\Delta u = f \quad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N, \tag{1.2}$$

$$u \le h, \quad \frac{\partial u}{\partial \nu} \le 0, \quad (u-h)\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_U. \tag{1.3}$$

Our goal is to prove that the data-to-solution map

$$(f,h) \in C^k(\overline{\Omega}) \times \mathbb{R} \mapsto u \in W^{1,2}(\Omega)$$

is C^k-smooth in a neighbourhood of a given (f_0, h_0) under reasonable assumptions. In other words, to show that the triplets $(u,f,h) \in W^{1,2}(\Omega) \times C^k(\overline{\Omega}) \times \mathbb{R}$ satisfying in a weak sense

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(1.1), (1.2), (1.3) form a C^k-smooth manifold in $W^{1,2}(\Omega) \times C^k(\overline{\Omega}) \times \mathbb{R}$ in a neighbourhood of (u_0, f_0, h_0). In particular, we will assume that the contact set

$$\mathcal{A}_h(u) := \{x \in [\gamma_1, \gamma_2] \,:\, u(x,0) = h\}$$

of the solution $u = u_0$ of (1.1)–(1.3) with f_0, h_0 is an interval and show that then $\mathcal{A}_h(u)$ is also an interval depending C^k-smoothly on f, h for all solutions u corresponding to f, h close to f_0, h_0 (see Theorem 2.4). We will also show that under simple additional conditions on f_0, h_0, this assumption about u_0 is really fulfilled (Proposition 2.12).

We introduce the real Hilbert space H and its closed convex subset K_h (for any $h > 0$) by

$$H := \{u \in W^{1,2}(\Omega) : u = 0 \text{ on } \Gamma_D\}, \quad K_h := \{u \in H : u \leq h \text{ on } \Gamma_U\}$$

and consider the weak formulation of (1.1)–(1.3) in terms of the variational inequality

$$f \in C^k(\overline{\Omega}), h > 0, u \in K_h : \int_\Omega \nabla u \nabla(\varphi - u) - f(\varphi - u)\, \mathrm{d}x\, \mathrm{d}y \geq 0 \text{ for all } \varphi \in K_h. \tag{1.4}$$

The known regularity properties of this variational inequality will be recalled in Remark 2.13 and Proposition 2.14.

An essential part of our considerations is related to the boundary value problem (1.1), (1.2),

$$u = h \text{ on } I_{\alpha,\beta}, \quad \partial_y u = 0 \text{ on } E_{\alpha,\beta}, \tag{1.5}$$

where

$$I_{\alpha,\beta} := \{(x,0) \in \Gamma_U : \alpha < x < \beta\} = (\alpha, \beta) \times \{0\},$$
$$E_{\alpha,\beta} := \{(x,0) \in \Gamma_U : \gamma_1 < x < \alpha \text{ or } \beta < x < \gamma_2\} = \Gamma_U \setminus \overline{I_{\alpha,\beta}},$$

$\gamma_1 < \alpha < \beta < \gamma_2$. Roughly speaking, the strategy of our proof is as follows. First, we transform by means of a smooth transformation of the space variable $x \in [0,1]$ the mixed boundary value problem (1.1), (1.2), (1.5), which has (α, β)-depending Dirichlet and Neumann boundary parts, into a mixed boundary value problem with (α, β)-independent Dirichlet and Neumann boundary parts, but (α, β)-depending coefficients in the differential equation. To this new mixed boundary value problem we add two scalar equations which ensure the regularity condition $u \in W^{2,2}(\Omega) \cap C^1(\overline{\Omega})$ in order to get a well-posed problem for determining α, β and u in terms of h and f. This new system is highly nonlinear in α and β. We transform it into an abstract equation in a certain Hilbert space and solve it locally with respect to α, β and u by means of the Implicit Function Theorem under the non-degeneracy condition (2.16). Finally, we show by using the maximum principle that the solutions of this equation satisfy also the variational inequality (1.4) under our sign assumptions about f_0 and u_0.

An exact formulation of the non-degeneracy assumption (2.16) of our main results (Theorems 2.4, 2.5) needs a rather long preliminaries leading to a definition of certain special functions $v^\delta_{\alpha,\beta}, w^\delta_{\alpha,\beta}$. In order to make this theorem understandable without reading these technical details, let us explain here briefly the sense of that condition.

It follows from [8, Theorem 1] (see also Remark 3.4) that there exist two functionals $\ell^1_{\alpha,\beta}, \ell^2_{\alpha,\beta} : L^2(\Omega) \times W^{1/2,2}(I_{\alpha,\beta}) \to \mathbb{R}$ such that a weak solution to (1.1), (1.2), (1.5) (even with an arbitrary $h \in W^{1/2,2}(I_{\alpha,\beta})$) satisfies $u \in W^{2,2}(\Omega)$ if and only if $\ell^1_{\alpha,\beta}(f,h) = \ell^2_{\alpha,\beta}(f,h) = 0$. If (u_0, f_0, h_0) is the starting solution of (1.4) which should be continued then the assumption (2.16) can be formulated also as

$$\det \begin{pmatrix} \frac{\partial}{\partial\alpha}(\ell^1_{\alpha,\beta}(f_0,h_0)) & \frac{\partial}{\partial\beta}(\ell^1_{\alpha,\beta}(f_0,h_0)) \\ \frac{\partial}{\partial\alpha}(\ell^2_{\alpha,\beta}(f_0,h_0)) & \frac{\partial}{\partial\beta}(\ell^2_{\alpha,\beta}(f_0,h_0)) \end{pmatrix}\Bigg|_{\alpha=\alpha_0,\beta=\beta_0} \neq 0.$$

In Theorem 2.4 we use a concrete representation of the funcionals $\ell^1_{\alpha,\beta}, \ell^2_{\alpha,\beta}$ in terms of functions $v^\delta_{\alpha,\beta}, w^\delta_{\alpha,\beta}$ which makes possible to verify this condition numerically in concrete situations.

Our paper is organized as follows. The formulation of our main result (Theorems 2.4, 2.5) and technical preliminaries are subject of Section 2. Proposition 2.12 in that section shows that the solutions of (1.4) for which the contact set is an interval and which can be smoothly continued by Theorem 2.4 really exist under simple additional assumptions. The proof of the main result is devided into two steps — the proof of a local equivalence of our variational inequality to a certain variational equation, and application of the Implicit Function Theorem to this equation. The formulation and the proof of the equivalence result are contained in Section 3. This proof is based on the results of P. Grisvard [8] concerning a regularity of solutions of mixed boundary value problems in a neighbourhood of contact points of Dirichlet and Neumann parts of the boundary. The considerations are related to those from V.G. Mazya, S.A. Nazarov, B.A. Plamenevskii [13], S.A. Nazarov, B.A. Plamenevskii [15], and C. Eck, S.A. Nazarov, W. Wendland [4], but they are simpler in our approach. The formulation and the proof of the continuation result for the variational equation from Section 3 are given in Section 4. The proofs of the main results then easily follow (Section 5). The Appendix is devoted to technical proofs of assertions used in previous sections and to recalling some known but essential facts.

Let us recall that a smooth continuation of solutions and contact regions for variational inequalities in one dimension (a model of a unilaterally supported beam) was proved and even an existence of smooth bifurcating branches was given in our papers [18], [6]. The same basic idea (a local equivalence with an equation and use of Implicit Funtion Theorem for it) was used there but the proofs in the present paper are much more complicated. In fact, this idea in an essentially simpler form was used also in [5] for variational inequalities in cases when certain assumptions can guarantee a local independence of the contact set on parameters.

2 Main Results

We will equip the Hilbert space H with the scalar product

$$\langle u, \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}y \qquad \text{for all } u, \varphi \in H,$$

and denote the corresponding norm by $\| \cdot \|$. Norms in other spaces X will be denoted by $\| \cdot \|_X$. By ∂_x, ∂_y, ∂_α, ∂_β and ∂_ν we denote the partial derivative with respect to x, y, α, β and outer normal derivative, respectively, and by ∂^2_{xx}, ∂^2_{yy}, ∂^2_{xy} the second partial derivatives.

In the sequel, we will consider fixed (α_0, β_0), $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$, and set $\delta_0 := \frac{1}{3}\min\{\alpha_0 - \gamma_1, \beta_0 - \alpha_0, \gamma_2 - \beta_0\}$,

$$D := \{(\alpha, \beta) \,:\, |\alpha - \alpha_0| < \delta_0,\, |\beta - \beta_0| < \delta_0\}.$$

In order to formulate the assumption (2.16) of our main results, we need to introduce special functions $v^\delta_{\alpha,\beta}$, $w^\delta_{\alpha,\beta}$. Considerations necessary for their definition will be used simultaneously in the proof of main theorems.

First, we will introduce coordinate transformations in $\overline{\Omega}$, i.e. diffeomorphisms of $\overline{\Omega}$ onto itself which map $I_{\alpha,\beta}$ onto I_{α_0,β_0} and $E_{\alpha,\beta}$ onto E_{α_0,β_0} and change the (α, β)-dependence from the boundary conditions into the coefficients in the differential equations.

For any $(\alpha, \beta) \in D$ let $\xi_{\alpha,\beta} : [0, 1] \to [0, 1]$ be a function such that

$$\text{the map } (\alpha, \beta, x) \mapsto \xi_{\alpha,\beta}(x) \text{ is } C^\infty\text{-smooth on } D \times [0, 1], \tag{2.1}$$

$$\xi_{\alpha_0,\beta_0}(x) = x \qquad \text{for all } x \in [0, 1], \tag{2.2}$$

$$\left.\begin{array}{l} \xi_{\alpha,\beta}(0) = 0, \quad \xi_{\alpha,\beta}(1) = 1, \quad \xi_{\alpha,\beta}(\alpha) = \alpha_0, \quad \xi_{\alpha,\beta}(\beta) = \beta_0, \\ \xi^{-1}_{\alpha,\beta}(x) = x + \alpha - \alpha_0 \quad \text{for } |x - \alpha_0| \le \delta_0, \\ \xi^{-1}_{\alpha,\beta}(x) = x + \beta - \beta_0 \quad \text{for } |x - \beta_0| \le \delta_0, \\ \xi'_{\alpha,\beta}(x) > 0 \quad \text{for } x \in [0, 1], \\ \xi_{\alpha,\beta} \text{ is a diffeomorphism of } [0, 1] \text{ onto } [0, 1], \end{array}\right. \tag{2.3}$$

where $\xi^{-1}_{\alpha,\beta}$ denotes the inverse function to $\xi_{\alpha,\beta}$. Let us define the mapping $\Phi_{\alpha,\beta} : L^2(\Omega) \to L^2(\Omega)$ by

$$(\Phi_{\alpha,\beta}) f(x, y) := f(\xi_{\alpha,\beta}(x), y) \qquad \text{for any } f \in L^2(\Omega).$$

Remark 2.1 *Under the assumptions (2.1)–(2.3), the mapping $\Phi_{\alpha,\beta}$ is an isomorphism from $W^{k,2}(\Omega)$ onto itself as well as from $C^k(\overline{\Omega})$ onto itself for any $k = 0, 1, \ldots$. Moreover, a function $v : \Omega \to \mathbb{R}$ satisfies the boundary conditions $v = 0$ on Γ_D, $v = h$ on I_{α_0,β_0}, $\partial_\nu v = 0$ on $\Gamma_N \cup E_{\alpha_0,\beta_0}$ if and only if $u = \Phi_{\alpha,\beta} v$ satisfies the boundary conditions $u = 0$ on Γ_D, $u = h$ on $I_{\alpha,\beta}$, $\partial_\nu u = 0$ on $\Gamma_N \cup E_{\alpha,\beta}$.*

Lemma 2.2 *For any $k = 1, 2, \dots$ and $m = 0, 1, \dots$ the map*

$$(\alpha, \beta, f) \in D \times C^{k+m}(\overline{\Omega}) \mapsto \Phi_{\alpha,\beta} f \in C^m(\overline{\Omega})$$

is C^k-smooth.

Proof will be done in Appendix.

Let us remark that $\Phi_{\alpha,\beta}$ does not depend continuously on α, β in the strong operator norm on any of the function spaces $W^{k,2}(\Omega)$ or $C^k(\overline{\Omega})$.

Let us denote by $\Phi^*_{\alpha,\beta} \in \mathcal{L}(L^2(\Omega))$ the adjoint mapping to $\Phi_{\alpha,\beta}$ in $L^2(\Omega)$. Introducing a new variable $\bar{x} = \xi_{\alpha,\beta}(x)$ and finally renaming $\bar{x}$ again to x, we get

$$\begin{aligned} \int_\Omega (\Phi^*_{\alpha,\beta} f)(x,y) g(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega f(x,y) g(\xi_{\alpha,\beta}(x), y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega f(\xi^{-1}_{\alpha,\beta}(x), y) g(x,y) (\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x)))^{-1}\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

i.e.

$$(\Phi^*_{\alpha,\beta} f)(x,y) = f\left(\xi^{-1}_{\alpha,\beta}(x), y\right)\left(\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))\right)^{-1}. \tag{2.4}$$

We have $\partial_x(\Phi_{\alpha,\beta} u)(x,y) = \partial_x u(\xi_{\alpha,\beta}(x), y)\xi'_{\alpha,\beta}(x)$ and

$$\partial^2_{xx}(\Phi_{\alpha,\beta} u)(x,y) = \partial^2_{xx} u(\xi_{\alpha,\beta}(x), y)(\xi'_{\alpha,\beta}(x))^2 + \partial_x u(\xi_{\alpha,\beta}(x), y)\xi''_{\alpha,\beta}(x).$$

Denoting $\Delta_{\alpha,\beta} := \Phi^*_{\alpha,\beta} \Delta \Phi_{\alpha,\beta}$, we get

$$\Delta_{\alpha,\beta} = \partial_x(\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))\partial_x) + (\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x)))^{-1}\partial^2_{yy}.$$

Therefore, denoting $\nabla_{\alpha,\beta} := (\sqrt{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))}\partial_x, \frac{\partial_y}{\sqrt{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))}})$ and $\partial^{\alpha,\beta}_\nu := (\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x)))^{-1}\partial_\nu$, we get that

$$\begin{aligned} &\int_\Omega \nabla_{\alpha,\beta} u \nabla_{\alpha,\beta} v = \int_\Omega \nabla \Phi_{\alpha,\beta} u \nabla \Phi_{\alpha,\beta} v \\ &\qquad \text{for all } u \in W^{1,p}(\Omega), v \in W^{1,q}(\Omega) \text{ with } p > 1,\ 1/p + 1/q = 1, \\ &\int_\Omega u \Delta_{\alpha,\beta} v = \int_\Omega \Phi_{\alpha,\beta} u \Delta \Phi_{\alpha,\beta} v \quad \text{for all } u \in L^2(\Omega), v \in W^{2,2}(\Omega). \end{aligned} \tag{2.5}$$

Let us define for $\delta > 0$ and $z \in (0,1)$ the sets

$$\begin{aligned} A_\delta(z) &:= \{(x,y) \in \Omega : \delta^2/4 < (x-z)^2 + y^2 < \delta^2\}, \\ B_\delta(z) &:= \{(x,y) \in \Omega : (x-z)^2 + y^2 < \delta^2\}. \end{aligned}$$

It follows from the choice of δ_0 and (2.3) that

$$\Delta_{\alpha,\beta} = \Delta, \quad \nabla_{\alpha,\beta} = \nabla \text{ in } B_\delta(\alpha_0) \cup B_\delta(\beta_0) \text{ for } \delta \in (0, \delta_0). \tag{2.6}$$

For any $\delta \in (0, \delta_0)$ let us introduce a C^∞-smooth function $\chi_\delta : [0, \infty) \to [0,1]$ such that

$$\begin{aligned} &\chi_\delta(r) = 1 \text{ for } 0 \le r \le \delta/2, \\ &\chi_\delta(r) = 0 \text{ for } r \ge \delta. \end{aligned} \tag{2.7}$$

Let us define functions

$$X^\delta(x,y) = X^\delta(\alpha_0 + r\cos\omega, r\sin\omega) := \chi_\delta(r) r^{-1/2}\sin\tfrac{\omega}{2},$$
$$Y^\delta(x,y) = Y^\delta(\beta_0 + r\cos\omega, r\sin\omega) := \chi_\delta(r) r^{-1/2}\sin\tfrac{\omega}{2}, \tag{2.8}$$

where r is the distance of $(x,y)\in\overline{\Omega}$ from $(\alpha_0,0)$, ω is the angle measured anticlockwise from the segment $\overline{(x,y),(\alpha_0,0)}$ to I_{α_0,β_0} in the definition of X^δ, and r is the distance of $(x,y)\in\overline{\Omega}$ from $(\beta_0,0)$, ω is the angle measured clockwise from the segment $\overline{(x,y),(\beta_0,0)}$ to I_{α_0,β_0} in the definition of Y^δ, respectively.

Lemma 2.3 *The following conditions hold for any $\delta\in(0,\delta_0)$.*

(i) $X^\delta, Y^\delta \in L^q(\Omega)$ for all $1\le q<4$, $X^\delta, Y^\delta\in W^{1,q}(\Omega)$ for all $1\le q<\frac{4}{3}$, $X^\delta, Y^\delta\in C^\infty(\overline{\Omega'})$ for any subdomain Ω' such that $\overline{\Omega'}\subset\overline{\Omega}\setminus\{(\alpha_0,0),(\beta_0,0)\}$.

(ii) $X^\delta = Y^\delta = 0$ on I_{α_0,β_0}, $X^\delta = 0$ in $\Omega\setminus B_\delta(\alpha_0)$, $Y^\delta = 0$ in $\Omega\setminus B_\delta(\beta_0)$, $\Delta X^\delta = 0$ in $\Omega\setminus A_\delta(\alpha_0)$, $\Delta Y^\delta = 0$ in $\Omega\setminus A_\delta(\beta_0)$ and $\Delta X^\delta, \Delta Y^\delta\in C^\infty(\overline{\Omega})$.

(iii) For any $(\alpha,\beta)\in D$, $\partial_\nu X^\delta = \partial_\nu^{\alpha,\beta}X^\delta = \partial_\nu Y^\delta = \partial_\nu^{\alpha,\beta} Y^\delta = 0$ on $\Gamma_N\cup E_{\alpha_0,\beta_0}$,

$$\int_\Omega \Delta_{\alpha,\beta}X^\delta\varphi = -\int_\Omega \nabla_{\alpha,\beta}X^\delta\cdot\nabla_{\alpha,\beta}\varphi,\quad \int_\Omega \Delta_{\alpha,\beta}Y^\delta\varphi = -\int_\Omega \nabla_{\alpha,\beta}Y^\delta\cdot\nabla_{\alpha,\beta}\varphi$$
$$\text{for all } \varphi\in W^{1,p}(\Omega) \text{ with } \varphi = 0 \text{ on } \Gamma_D\cup I_{\alpha_0,\beta_0} \text{ and } p>4. \tag{2.9}$$

(iv) For any $(\alpha,\beta)\in D$ we have

$$\nabla_{\alpha,\beta}X^\delta = \nabla X^\delta,\ \Delta_{\alpha,\beta}X^\delta = \Delta X^\delta,\ \nabla_{\alpha,\beta}Y^\delta = \nabla Y^\delta,\ \Delta_{\alpha,\beta}Y^\delta = \Delta Y^\delta \text{ in } \Omega.$$

Proof. The assertions (i) and (ii) follow from the definition of X^δ and Y^δ and calculus by using polar coordinates. In particular, explicit expression of the Laplace operator in polar coordinates gives that ΔX^δ and ΔY^δ vanish in the sets where the corresponding function χ_δ is constant (let us remark that $\partial^2_{xx}X^\delta$, $\partial^2_{yy}X^\delta$ and $\partial^2_{xx}Y^\delta$, $\partial^2_{yy}Y^\delta$ have singularities of opposite signs in $(\alpha_0,0)$ and $(\beta_0,0)$, respectively).

It follows directly from (2.8) that $\partial_y X^\delta = \partial_y Y^\delta = \partial_\nu^{\alpha,\beta}Y^\delta = 0$ on $\Gamma_N\cup E_{\alpha_0,\beta_0}$. Hence, due to (i) the Green Formula implies that for any smooth function ψ with a compact support in $\Omega\cup\Gamma_N\cup E_{\alpha_0,\beta_0}$ (i.e. $\Phi_{\alpha,\beta}\psi$ has a compact support in $\Omega\cup\Gamma_N\cup E_{\alpha,\beta}$) we have

$$\int_\Omega \Delta\Phi_{\alpha,\beta}X^\delta\cdot\Phi_{\alpha,\beta}\psi + \int_\Omega \nabla\Phi_{\alpha,\beta}X^\delta\cdot\nabla\Phi_{\alpha,\beta}\psi = \int_{\partial\Omega}\partial_\nu\Phi_{\alpha,\beta}X^\delta\cdot\Phi_{\alpha,\beta}\psi = 0$$

where all derivatives exist in the classical sense in $\operatorname{supp}\psi$. Similarly for Y^δ. Such functions ψ are dense in $\{\varphi\in W^{1,p}(\Omega) : \varphi = 0 \text{ on } \Gamma_D\cup I_{\alpha_0,\beta_0}\}$ and (iii) follows by using the limiting process and the formulas (2.5) for $\Delta_{\alpha,\beta}$ and $\nabla_{\alpha,\beta}$.

The assertion (iv) is a consequence of (ii) and (2.6). ■

Let $X^\delta_{\alpha,\beta}, Y^\delta_{\alpha,\beta} \in W^{1,2}(\Omega)$ satisfy

$$\begin{gathered} X^\delta_{\alpha,\beta} = Y^\delta_{\alpha,\beta} = 0 \text{ on } \Gamma_D \cup I_{\alpha_0,\beta_0}, \\ \int_\Omega \nabla_{\alpha,\beta} X^\delta_{\alpha,\beta} \cdot \nabla_{\alpha,\beta}\varphi - \Delta X^\delta \varphi = \int_\Omega \nabla_{\alpha,\beta} Y^\delta_{\alpha,\beta} \cdot \nabla_{\alpha,\beta}\varphi - \Delta Y^\delta \varphi = 0 \\ \text{for all } \varphi \in W^{1,2}(\Omega) \text{ with } \varphi = 0 \text{ on } \Gamma_D \cup I_{\alpha_0,\beta_0}, \end{gathered} \tag{2.10}$$

i.e. $X^\delta_{\alpha,\beta}$ and $Y^\delta_{\alpha,\beta}$ is the weak solution to the boundary value problem $-\Delta_{\alpha,\beta} u = g$ in Ω with $g = \Delta X^\delta$ and $g = \Delta Y^\delta$, respectively, and with the boundary conditions $u = 0$ on $\Gamma_D \cup I_{\alpha_0,\beta_0}$, $\partial^{\alpha,\beta}_\nu u = 0$ on $\Gamma_N \cup E_{\alpha_0,\beta_0}$. Let us remark that $X^\delta_{\alpha,\beta}, Y^\delta_{\alpha,\beta} \in W^{1,2}(\Omega)$, but $X^\delta, Y^\delta \notin W^{1,2}(\Omega)$. Finally, denote

$$v^\delta_{\alpha,\beta} := \Phi_{\alpha,\beta}(X^\delta_{\alpha,\beta} + X^\delta), \quad w^\delta_{\alpha,\beta} := \Phi_{\alpha,\beta}(Y^\delta_{\alpha,\beta} + Y^\delta). \tag{2.11}$$

The last functions $v^\delta_{\alpha,\beta}$, $w^\delta_{\alpha,\beta}$ are of the key importance for the formulation of our main result.

For any $d \in (0,\ell)$ let us denote

$$\begin{aligned} \Omega_d &:= (0,1) \times (0,d), \\ \Gamma_d &:= (0,1) \times \{d\}. \end{aligned}$$

Theorem 2.4 *Let $k \in \mathbb{N}$, let (u_0, h_0, f_0) satisfy (1.4), $\mathcal{A}_{h_0}(u_0) = [\alpha_0, \beta_0]$ with $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$. Let us assume that there are $d > 0$, $\varepsilon > 0$ such that f_0 and u_0 satisfy the conditions*

$$f_0 \geq \varepsilon > 0 \quad \text{on } I_{\alpha_0,\beta_0}, \tag{2.12}$$

$$\partial_y f_0 \geq \varepsilon > 0 \quad \text{on } \Omega_d, \tag{2.13}$$

$$\partial_y u_0 > 0 \quad \text{on } \Gamma_d, \tag{2.14}$$

$$\partial^2_{xy} u_0(0,d) > 0 > \partial^2_{xy} u_0(1,d), \tag{2.15}$$

$$\det\begin{pmatrix} \frac{\partial}{\partial\alpha}\left(\int_\Omega f_0 v^\delta_{\alpha,\beta} - \nabla\Phi_{\alpha,\beta} u_0 \cdot \nabla v^\delta_{\alpha,\beta}\, \mathrm{d}x\, \mathrm{d}y\right) & \frac{\partial}{\partial\beta}\left(\int_\Omega f_0 v^\delta_{\alpha,\beta} - \nabla\Phi_{\alpha,\beta} u_0 \cdot \nabla v^\delta_{\alpha,\beta}\, \mathrm{d}x\, \mathrm{d}y\right) \\ \frac{\partial}{\partial\alpha}\left(\int_\Omega f_0 w^\delta_{\alpha,\beta} - \nabla\Phi_{\alpha,\beta} u_0 \cdot \nabla w^\delta_{\alpha,\beta}\, \mathrm{d}x\, \mathrm{d}y\right) & \frac{\partial}{\partial\beta}\left(\int_\Omega f_0 w^\delta_{\alpha,\beta} - \nabla\Phi_{\alpha,\beta} u_0 \cdot \nabla w^\delta_{\alpha,\beta}\, \mathrm{d}x\, \mathrm{d}y\right) \end{pmatrix}\Bigg|_{\substack{\alpha=\alpha_0 \\ \beta=\beta_0}} \neq 0 \tag{2.16}$$

for some $\delta \in (0,\delta_0)$. Then there exist neighbourhoods $V \subset C^k(\overline{\Omega}) \times \mathbb{R}$ and $W \subset H$ of (f_0, h_0) and of u_0, respectively, and a C^k-mapping $\hat{u} : V \to W$ such that $\hat{u}(f_0, h_0) = u_0$ and that $(f, h, u) \in V \times W$ satisfies (1.4) if and only if $u = \hat{u}(f, h)$. Moreover, there exist C^k-functions $\hat{\alpha}, \hat{\beta} : V \to \mathbb{R}$ such that $\hat{\alpha}(f_0, h_0) = \alpha_0$, $\hat{\beta}(f_0, h_0) = \beta_0$ and

$$\mathcal{A}_h\left(\hat{u}(f,h)\right) = \left[\hat{\alpha}(f,h), \hat{\beta}(f,h)\right] \text{ for all } (f,h) \in V.$$

In the following theorem we replace the assumption (2.13) by the weaker condition (2.17) but we obtain a result of a different type. We get only smoothness of the data-to-solution map with respect to solutions corresponding to right-hand sides satisfying (2.17), that means solutions of (1.4) do not form a smooth manifold in a neighbourhood of (u_0, f_0, h_0).

Theorem 2.5 *Let us consider the assumptions of Theorem 2.4 but replace (2.13) by*

$$\partial_y f_0 \geq 0 \quad \text{on } \Omega_d. \tag{2.17}$$

Then there exist neighbourhoods $V \subset C^k(\overline{\Omega}) \times \mathbb{R}$ and $W \subset H$ of (f_0, h_0) and of u_0, respectively, and a C^k-mapping $\hat{u} : V \to W$ such that $\hat{u}(f_0, h_0) = u_0$ and that $(f, h, u) \in V \times W$ with $\partial_y f \geq 0$ on Ω_d satisfies (1.4) if and only if $u = \hat{u}(f, h)$. Moreover, there exist C^k-functions $\hat{\alpha}, \hat{\beta} : V \to \mathbb{R}$ such that $\hat{\alpha}(f_0, h_0) = \alpha_0$, $\hat{\beta}(f_0, h_0) = \beta_0$ and

$$\mathcal{A}_h\left(\hat{u}(f, h)\right) = \left[\hat{\alpha}(f, h), \hat{\beta}(f, h)\right] \text{ for all } (f, h) \in V \text{ with } \partial_y f \geq 0 \text{ on } \Omega_d.$$

Remark 2.6 *The existence of $d > 0$ satisfying (2.13) is guaranteed if we assume $\partial_y f_0 \geq \varepsilon > 0$ on Γ_0. However, we need (2.13) with the same $d > 0$ for which (2.14) is fulfilled, and this assumption cannot be replaced by an analogous condition on Γ_0 because $\partial_y u = 0$ on a part of Γ_0 (u satisfies (1.2) and (2.24) by Proposition 2.14 below).*

Remark 2.7 *The condition (2.15) is used only for the proof that (2.14) remains valid even for all solutions of (1.1), (1.2), (1.5) sufficiently close to $(u_0, \alpha_0, \beta_0, f_0, h_0)$, which is essential for the proof of Theorem 2.4 (see the proof of Lemma 3.10). Therefore it will be seen from Proposition 2.10 below that if we assume global conditions for f_0 considered there then the assumption (2.15) is not needed.*

The conditions (2.14), (2.15) for a starting solution u_0 as well as the conditions (2.12), (2.13) for a starting f_0 will be used for the proof of an equivalence of our variational inequality to a certain operator equation on a neighbourhood of (u_0, f_0, h_0), which is the main tool of the proof of Theorem 2.4. The precise formulation of this local equivalence is a subject of Theorem 3.1 below.

The condition (2.16) is generically fulfilled. For given f_0, h_0, u_0, it has to be verified numerically. The assumption (2.16) will guarantee that a certain mapping is an isomorphism which will enable us to use Implicit Function Theorem to prove the existence of a unique local smooth branch of solutions of the operator equation mentioned above (see the proof of Theorem 4.1).

Remark 2.8 *The condition (2.16) can be in fact replaced by assuming that there exist functions $h^e_{\alpha,\beta} \in H \cap W^{2,2}(\Omega)$ such that $h^e_{\alpha,\beta} = h_0$ on $I_{\alpha,\beta}$, that the mapping*

$$(\alpha, \beta) \in D \mapsto \left(\int_\Omega \nabla h^e_{\alpha,\beta} \cdot \nabla v^\delta_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y, \int_\Omega \nabla h^e_{\alpha,\beta} \cdot \nabla w^\delta_{\alpha,\beta} \, \mathrm{d}x \, \mathrm{d}y\right) \in \mathbb{R}^2 \tag{2.18}$$

is smooth and that

$$\det\begin{pmatrix} \frac{\partial}{\partial\alpha}\left(\int_\Omega f_0 v^\delta_{\alpha,\beta} - \nabla h^e_{\alpha,\beta}\cdot\nabla v^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\right) & \frac{\partial}{\partial\beta}\left(\int_\Omega f_0 v^\delta_{\alpha,\beta} - \nabla h^e_{\alpha,\beta}\cdot\nabla v^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\right) \\ \frac{\partial}{\partial\alpha}\left(\int_\Omega f_0 w^\delta_{\alpha,\beta} - \nabla h^e_{\alpha,\beta}\cdot\nabla w^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\right) & \frac{\partial}{\partial\beta}\left(\int_\Omega f_0 w^\delta_{\alpha,\beta} - \nabla h^e_{\alpha,\beta}\cdot\nabla w^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\right) \end{pmatrix}\Bigg|_{\substack{\alpha=\alpha_0\\ \beta=\beta_0}} \neq 0 \tag{2.19}$$

for some $\delta \in (0,\delta_0)$. Indeed, we will see in Lemma 3.6 that the integrals in (2.19) are independent of the choice of such extension $h^e_{\alpha,\beta}$ of the constant h_0 from $I_{\alpha,\beta}$ onto the whole Ω, and $\Phi_{\alpha,\beta}u_0$ is such extension (see also Lemma 4.2). Let us note that $\Phi_{\alpha,\beta}u_0 \in W^{2,2}(\Omega)$ due to Proposition 2.14 below and Remark 2.1.

Remark 2.9 *If $f_0(x,y) = f_0(1-x,y)$ for all $(x,y) \in \overline{\Omega}$, then for all $(f,h) \in V$ with f satisfying the same symmetry condition, the functions $\hat{u}(f,h)$ from Theorem 2.4 are symmetric in x and therefore $\hat{\alpha}(f,h) = 1 - \hat{\beta}(f,h)$. This follows immediately from the unicity assertion of Theorem 2.4.*

The following Proposition 2.10 together with Proposition 2.14 show that the local assumption (2.14) concerning u_0 is automatically fulfilled even in a global form and even for all solutions of (1.1), (1.2), (1.5) sufficiently close to $(u_0,\alpha_0,\beta_0,f_0,h_0)$ under suitable global assumptions about f_0. We do not need to discuss the assumption (2.15) because it is not necessary in this situation (see also Remark 2.7 above). Furthermore, Proposition 2.12 gurantees that the assumption of the same type on f guarantee that the contact set of the solution of (1.4) is really an interval for h small enough as it is assumed in Theorem 2.4.

Proposition 2.10 *Let $k=1$, let $(u_0,\alpha_0,\beta_0,f_0,h_0)$ be a solution of (1.4), $\mathcal{A}_{h_0}(u_0) = [\alpha_0,\beta_0]$ and let (2.12) and*

$$\partial_y f > \varepsilon > 0 \quad \text{in } \Omega \tag{2.20}$$

hold for $f = f_0$. If $(u,\alpha,\beta,f,h) \in C^1(\overline{\Omega}) \times D \times C^1(\overline{\Omega}) \times \mathbb{R}_+$ satisfy (1.1), (1.2), (1.5) and $\|u-u_0\| + |\alpha-\alpha_0| + |\beta-\beta_0| + \|f-f_0\|_{C^1(\overline{\Omega})}$ is small enough then

$$\partial_y u > 0 \quad \text{in } \Omega \cup I_{\alpha,\beta}. \tag{2.21}$$

Proof will be done in Section 5.

Corollary 2.11 *The assertion of Theorem 2.4 remains valid if we replace (2.13) by (2.20) and omit the assumptions (2.14), (2.15). This will be seen from the proof of Theorem 2.4 and from Proposition 2.10.*

Proposition 2.12 *Let*

$$f \geq 0 \text{ on } \Gamma_U, \quad \partial_y f \geq 0 \quad \text{in } \Omega. \tag{2.22}$$

If (u, f, h) *satisfies (1.4),* $u(\gamma_1, 0) < h$, $u(\gamma_2, 0) < h$, $u(x_1, 0) = u(x_2, 0) = h$ *for some* $x_1, x_2 \in [\gamma_1, \gamma_2]$, $x_1 < x_2$, *then there are* $(\alpha, \beta) \in D$ *satisfying* $\mathcal{A}_h(u) = [\alpha, \beta]$. *Let, moreover,*

$$f \geq 0 \quad \text{in } \Omega. \tag{2.23}$$

There is $h_f > 0$ *such that if* $h < h_f$, (u, f, h) *satisfies (1.4), then there are* $(\alpha, \beta) \in D$ *satisfying* $\mathcal{A}_h(u) = [\alpha, \beta]$, *and if* $h > h_f$, (u, f, h) *satisfies (1.4), then we have* $u(x, 0) < h$ *for all* $x \in [\gamma_1, \gamma_2]$.

Proof will be done in Section 5.

We will devote the rest of this section to recalling some basic properties of our variational inequality we use.

Remark 2.13 *It is well known that for any* $f \in C^1(\overline{\Omega})$, $h > 0$ *there is unique* $u \in H$ *such that (1.4) holds. Moreover,* $u \in C^1(\Omega \cup \Gamma_U)$. *See e.g. [10, Theorem 1.2].*

For any $u \in H \cap C(\Gamma_U)$ let us set

$$E_h(u) := \{(x, 0) \in \Gamma_U;\ u(x, 0) \neq h\}, \quad I_h(u) := \Gamma_U \setminus \overline{E_h(u)}.$$

Hence, if $\mathcal{A}_h(u) = [\alpha, \beta]$ for some $(\alpha, \beta) \in D$ then $I_h(u) = I_{\alpha,\beta}$, $E_h(u) = E_{\alpha,\beta}$.

Proposition 2.14 *A triplet* $(u, f, h) \in H \times C^1(\overline{\Omega}) \times \mathbb{R}^+$ *satisfies (1.4) with* $k = 1$ *and* $\overline{I_h(u)} \subset \Gamma_U$ *if and only if* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, u *is a solution of the problem (1.1), (1.2),*

$$u = h \text{ on } I_h(u) \quad \partial_y u = -\partial_\nu u = 0 \text{ on } E_h(u) \tag{2.24}$$

and if in addition

$$\partial_y u = -\partial_\nu u \geq 0 \text{ on } I_h(u), \tag{2.25}$$

$$u < h \text{ on } E_h(u), \tag{2.26}$$

$u(\gamma_1, 0) < h$, $u(\gamma_2, 0) < h$. *Any such solution satisfies in fact*

$$u \in W^{2,p}(\Omega) \text{ for all } p \in (1, 4). \tag{2.27}$$

Due to embedding theorems (e.g. [16], [7]), (2.27) implies

$$u \in C^{1+\gamma}(\overline{\Omega}) \text{ for all } \gamma \in \left(0, \frac{1}{2}\right). \tag{2.28}$$

Let us emphasize that under the assumptions on f considered we will prove the sharp inequality in (2.25) (see Lemma 3.10). The assumption $\overline{I_h(u)} \subset \Gamma_U$ is essential. If $(\gamma_1, 0) \in \overline{I_h(u)}$ or $(\gamma_2, 0) \in \overline{I_h(u)}$ then u can have a jump in the first derivatives at $(\gamma_1, 0)$ or $(\gamma_2, 0)$, respectively.

We will have in fact $\mathcal{A}_h(u) = [\alpha, \beta]$ in our considerations and then (2.24) reads as (1.5).

Proof is standard and will be done in Appendix for the completeness.

3 Equivalence of the Variational Inequality to an Operator Equation

The main idea of the proof of Theorem 2.4 is to show that our variational inequality (1.4) is equivalent in a neighbourhood of (u_0, f_0, h_0) to an operator equation, and to apply Implicit Function Theorem to this equation. In order to formulate the equivalence result, let us define the mapping $F : H \times C^1(\overline{\Omega}) \to H$ by

$$\langle F(u,f),\varphi\rangle := -\int_\Omega \nabla u \nabla\varphi - f\varphi \,\mathrm{d}x\,\mathrm{d}y \quad \text{for all } \varphi \in H. \tag{3.1}$$

Then (1.4) can be written as

$$f \in C^k(\overline{\Omega}), h > 0, u \in K_h : \quad \langle F(u,f), \varphi - u\rangle \le 0 \text{ for all } \varphi \in K_h. \tag{3.2}$$

Let us introduce the closed subspace H_0 of H by

$$H_0 := \{u \in H :\ u = 0 \text{ in } I_{\alpha_0,\beta_0}\}.$$

The following theorem describes precisely the local equivalence of the variational inequality (1.4) and the operator equation mentioned above. We use the notation from Section 2. In particular, the functions $v_{\alpha,\beta}$, $w_{\alpha,\beta}$ are from (2.11).

Let us note that we consider $k = 1$ in Theorem 3.1 because it automatically implies its validity with any $k \in \mathbb{N}$.

Theorem 3.1 *Let $k = 1$ and let (u_0, f_0, h_0) satisfy (1.4), (2.16) for some $\delta \in (0, \delta_0)$ and $\mathcal{A}_{h_0}(u_0) = [\alpha_0, \beta_0]$ with $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$. Let us assume, moreover, that there are $d > 0$ and $\varepsilon > 0$ such that (2.12), (2.13), (2.14), (2.15) are valid. Then for any $\eta > 0$ there exists $\delta > 0$ such that the following holds:*

(i) For any solution (u, f, h) to (1.4) satisfying $\|u - u_0\| + \|f - f_0\|_{C^1(\overline{\Omega})} + |h - h_0| < \delta$ there exists $(v, \alpha, \beta) \in H_0 \times D$ with $\|v\| + |\alpha - \alpha_0| + |\beta - \beta_0| < \eta$ such that $\mathcal{A}_h(u) = [\alpha, \beta]$ and (v, α, β, f, h) satisfies

$$\begin{aligned} &f \in C^1(\overline{\Omega}), h > 0, (\alpha,\beta) \in D, v \in H_0 : \\ &\left\langle F\left(\Phi_{\alpha,\beta}\left(\tfrac{h}{h_0}u_0 + v\right), f\right), \Phi_{\alpha,\beta}\varphi\right\rangle = 0 \text{ for any } \varphi \in H_0, \end{aligned} \tag{3.3}$$

$$\int_\Omega f v^\delta_{\alpha,\beta} - \frac{h}{h_0}\nabla\Phi_{\alpha,\beta}u_0 \cdot \nabla v^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f w^\delta_{\alpha,\beta} - \frac{h}{h_0}\nabla\Phi_{\alpha,\beta}u_0 \cdot \nabla w^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y = 0, \tag{3.4}$$

$$u = \Phi_{\alpha,\beta}\left(\frac{h}{h_0}u_0 + v\right). \tag{3.5}$$

(ii) *For any* (v,α,β,f,h) *satisfying (3.3), (3.4),* $\|v\|+|\alpha-\alpha_0|+|\beta-\beta_0|+\|f-f_0\|_{C^1(\overline{\Omega})}+|h-h_0|<\delta$*, the triplet* (u,f,h) *with* u *from (3.5) satisfies (1.4),* $\|u-u_0\|<\eta$ *and* $\mathcal{A}_h(u)=[\alpha,\beta]$.

Proof of this theorem as well as of the following one will be done later in this section. We need for them additional notation and technical assertions, which are the subject of the forthcomming part of the text. In particular, we need to describe properties of the functions $v^\delta_{\alpha,\beta}$, $w^\delta_{\alpha,\beta}$ from (2.11).

Theorem 3.2 *Let us consider the assumptions of Theorem 3.1 but replace (2.13) by (2.17). Then for any* $\eta>0$ *there exists* $\delta>0$ *such that the following holds:*

(i) *For any solution* (u,f,h) *to (1.4) satisfying* $\|u-u_0\|+\|f-f_0\|_{C^1(\overline{\Omega})}+|h-h_0|<\delta$ *and* $\partial_y f\geq 0$ *on* Ω_d *there exists* $(v,\alpha,\beta)\in H_0\times D$ *with* $\|v\|+|\alpha-\alpha_0|+|\beta-\beta_0|<\eta$ *such that* $\mathcal{A}_h(u)=[\alpha,\beta]$ *and* (v,α,β,f,h) *satisfies (3.3)–(3.5).*

(ii) *For any* (v,α,β,f,h) *satisfying (3.3), (3.4),* $\|v\|+|\alpha-\alpha_0|+|\beta-\beta_0|+\|f-f_0\|_{C^1(\overline{\Omega})}+|h-h_0|<\delta$ *and* $\partial_y f\geq 0$ *on* Ω_d *the triplet* (u,f,h) *with* u *from (3.5) satisfies (1.4),* $\|u-u_0\|<\eta$ *and* $\mathcal{A}_h(u)=[\alpha,\beta]$.

Remark 3.3 *In fact, due to Lemma 3.6 below we could replace* $\Phi_{\alpha,\beta}u_0$ *in (3.4) by an arbitrary* $h^e_{\alpha,\beta}\in H\cap W^{2,2}(\Omega)$ *such that* $h^e_{\alpha,\beta}=h$ *on* $I_{\alpha,\beta}$*. Cf. also Remark 2.8, but now we do not need any smoothness of the map (2.18) mentioned there.*

Let us define for any $(\alpha,\beta)\in D$ the closed subspace $U_{\alpha,\beta}$ of $L^2(\Omega)\times\mathbb{R}$ by

$$U_{\alpha,\beta}:=\{(-\Delta u,h)\,:\,u\in W^{2,2}(\Omega),\ h\in\mathbb{R},\ u=0 \text{ on } \Gamma_D,\ u=h \text{ on } I_{\alpha,\beta},\ \partial_\nu u=0 \text{ on } \Gamma_N\cup E_{\alpha,\beta}\}.$$

Remark 3.4 *Clearly, we have*

$$U_{\alpha,\beta}=\{(f,h)\in L^2(\Omega)\times\mathbb{R}\,:\,u\in W^{2,2}(\Omega) \textit{ for the weak solution } u \textit{ of (1.1), (1.2), (1.5)}\}.$$

Since $\Phi_{\alpha,\beta}$ *is one-to-one mapping of* $\{\varphi\in W^{2,2}(\Omega)\cap H\,:\,\varphi=h \text{ in } I_{\alpha_0,\beta_0},\ \partial_\nu u=0 \text{ on } E_{\alpha_0,\beta_0}\}$ *onto* $\{\varphi\in W^{2,2}(\Omega)\cap H\,:\,\varphi=h \text{ in } I_{\alpha,\beta},\ \partial_\nu u=0 \text{ on } E_{\alpha,\beta}\}$ *(see Remark 2.1) we get also*

$$U_{\alpha,\beta}=\{(-\Delta\Phi_{\alpha,\beta}u,h)\,:\,u\in W^{2,2}(\Omega),\ h\in\mathbb{R},\ u=0 \text{ on } \Gamma_D,\ u=h \text{ on } I_{\alpha_0,\beta_0},\\ \partial_\nu u=0 \text{ on } \Gamma_N\cup E_{\alpha_0,\beta_0}\}.$$

It follows from [8, Theorem 1] that there exist two functionals $\ell^1_{\alpha,\beta},\ell^2_{\alpha,\beta}:L^2(\Omega)\times W^{1/2,2}(I_{\alpha,\beta})\to\mathbb{R}$ *such that a weak solution to (1.1), (1.2), (1.5) (even with an arbitrary* $h\in W^{1/2,2}(I_{\alpha,\beta})$*) satisfies* $u\in W^{2,2}(\Omega)$ *if and only if* $\ell^1_{\alpha,\beta}(f,h)=\ell^2_{\alpha,\beta}(f,h)=0$*. In particular,* $U_{\alpha,\beta}$ *has a codimension two in* $L^2(\Omega)\times\mathbb{R}$.

For any fixed $(\alpha,\beta)\in D$, let us define functions $X_\delta^{(1/2)}$ and $Y_\delta^{(1/2)}$ by

$$
\begin{aligned}
X_\delta^{(1/2)}(\alpha+r\cos\omega, r\sin\omega) &:= \chi_\delta(r) r^{1/2}\sin\tfrac{\omega}{2}\\
Y_\delta^{(1/2)}(\beta+r\cos\omega, r\sin\omega) &:= \chi_\delta(r) r^{1/2}\sin\tfrac{\omega}{2}
\end{aligned}
$$

where the function χ_δ is from (2.7) and r, ω are the same as in the definition of X^δ, Y^δ in (2.8).

Remark 3.5 *Let $(u,\alpha,\beta,f,h)\in H\times D\times L^2(\Omega)\times\mathbb{R}$, let u be a weak solution of (1.1), (1.2) and (1.5). It follows from [8, Theorem 2] (cf. also [12], Theorems 10.2 and 12.5 and expressions (2.8), (2.10), (2.11), (14.3) there), that we can write u as*

$$
u = u_{\alpha,\beta,2,p} + K_\alpha^\delta X_\delta^{(1/2)} + K_\beta^\delta Y_\delta^{(1/2)}, \tag{3.6}
$$

where $u_{\alpha,\beta,2,p}\in W^{2,2}(\Omega)$ and $K_\alpha^\delta, K_\beta^\delta\in\mathbb{R}$. It follows from [8, Theorem 3] that even $u_{\alpha,\beta,2,p}\in W^{2,p}(\Omega)$ for all $p\in[2,4)$, in particular $u_{\alpha,\beta,2,p}\in C^1(\overline{\Omega})$.

Let us emphasize that none of the functions $X_\delta^{(1/2)}$, $Y_\delta^{(1/2)}$ belongs neither to $W^{2,2}(\Omega)$ nor to $C^1(\overline{\Omega})$, because of the singularity in the first derivatives at $(\alpha,0)$ or $(\beta,0)$, respectively. In particular

$$
\partial_x X_\delta^{(1/2)}(\alpha-,0) = -\infty, \quad \partial_x Y_\delta^{(1/2)}(\beta+,0) = +\infty. \tag{3.7}
$$

It follows that $u\in C^1(\overline{\Omega})$ if and only if $u\in W^{2,2}(\Omega)$, and this is true if and only if $K_\alpha^\delta = K_\beta^\delta = 0$. In this case even (2.27) is fulfilled. Of course, this holds only for some couples $(\alpha,\beta)\in D$ and the corresponding h.

Lemma 3.6 *The following assertions hold for any $(\alpha,\beta)\in D$ and $\delta\in(0,\delta_0)$.*

$$
v_{\alpha,\beta}^\delta, w_{\alpha,\beta}^\delta \in W^{1,q}(\Omega) \text{ for all } q\in\left[1,\frac{4}{3}\right), \tag{3.8}
$$

$$
\begin{aligned}
&\text{the expressions } a_{\alpha,\beta}^\delta := \textstyle\int_\Omega \nabla v_{\alpha,\beta}^\delta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y, \quad b_{\alpha,\beta}^\delta := \int_\Omega \nabla w_{\alpha,\beta}^\delta\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y\\
&\quad\text{are independent of } \varphi\in W^{2,2}(\Omega) \text{ with } \varphi=0 \text{ on } \Gamma_D, \varphi=1 \text{ on } I_{\alpha,\beta},
\end{aligned} \tag{3.9}
$$

$$
U_{\alpha,\beta} = \left[\operatorname{span}\left\{(v_{\alpha,\beta}^\delta, -a_{\alpha,\beta}^\delta), (w_{\alpha,\beta}^\delta, -b_{\alpha,\beta}^\delta)\right\}\right]^\perp, \tag{3.10}
$$

where the orthogonal complement is understood in $L^2(\Omega)\times\mathbb{R}$.

In particular, if $\delta\in(0,\delta_0)$ is given then the functionals $\ell_{\alpha,\beta}^1$, $\ell_{\alpha,\beta}^2$ mentioned in Remark 3.4 can be chosen such that on $L^2(\Omega)\times\mathbb{R}$ they have a representation

$$
\begin{aligned}
\ell_{\alpha,\beta}^1(f,h) &= \textstyle\int_\Omega f v_{\alpha,\beta}^\delta - \frac{h}{h_0}\nabla\Phi_{\alpha,\beta}u_0\cdot\nabla v_{\alpha,\beta}^\delta\,\mathrm{d}x\,\mathrm{d}y,\\
\ell_{\alpha,\beta}^2(f,h) &= \textstyle\int_\Omega f w_{\alpha,\beta}^\delta - \frac{h}{h_0}\nabla\Phi_{\alpha,\beta}u_0\cdot\nabla w_{\alpha,\beta}^\delta\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Proof. The condition (3.8) follows from Lemma 2.3(i), definition of X^δ, Y^δ, $X_{\alpha,\beta}^\delta$, $Y_{\alpha,\beta}^\delta$, (2.11) and the smoothness of $\xi_{\alpha,\beta}$.

In order to prove (3.9), let us show first that if $\varphi \in W^{2,2}(\Omega)$ with $\varphi = 0$ on $\Gamma_D \cup I_{\alpha,\beta}$ then

$$\int_\Omega \nabla v^\delta_{\alpha,\beta} \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \nabla w^\delta_{\alpha,\beta} \cdot \nabla \varphi \,\mathrm{d}x\,\mathrm{d}y = 0. \tag{3.11}$$

Let us take such φ. Then using the definition of $\Phi^*_{\alpha,\beta}$, $v^\delta_{\alpha,\beta}$, $\nabla_{\alpha,\beta}$, Lemma 2.3(iii) and (iv) (realizing the imbedding $W^{2,2}(\Omega) \subset W^{1,q}(\Omega)$ for all $2 \le q < \infty$) we get

$$\begin{aligned}\textstyle\int_\Omega \nabla\varphi \cdot \nabla v^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y &= \textstyle\int_\Omega \nabla \Phi_{\alpha,\beta}\Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla\Phi_{\alpha,\beta}(X^\delta_{\alpha,\beta} + X^\delta) \,\mathrm{d}x\,\mathrm{d}y \\ &= \textstyle\int_\Omega \nabla_{\alpha,\beta}\Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla_{\alpha,\beta}X^\delta_{\alpha,\beta} - \Phi^{-1}_{\alpha,\beta}\varphi\Delta X^\delta \,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

which is zero due to (2.10) with $\Phi^{-1}_{\alpha,\beta}\varphi$ instead of φ there (let us observe that we have $\Phi^{-1}_{\alpha,\beta}\varphi \in W^{2,2}(\Omega)$ and $\Phi^{-1}_{\alpha,\beta}\varphi = 0$ on $\Gamma_D \cup I_{\alpha_0,\beta_0}$ due to Remark 2.1). Similarly one can show that $\int_\Omega \nabla\varphi \cdot \nabla w^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y = 0$.

If φ_1, φ_2 are two functions proper for (3.9) then the choice $\varphi := \varphi_1 - \varphi_2$ in (3.11) implies that the integrals in (3.9) with $\varphi = \varphi_1$ and $\varphi = \varphi_2$ are the same, and the assertion (3.9) follows.

Due to the linear independence of $v^\delta_{\alpha,\beta}$, $w^\delta_{\alpha,\beta}$ and Remark 3.4, for the proof of (3.10) it is sufficient to show that

$$\begin{aligned}\textstyle\int_\Omega f v^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y - h a^\delta_{\alpha,\beta} = 0,\\ \textstyle\int_\Omega f w^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y - h b^\delta_{\alpha,\beta} = 0\end{aligned}$$

for all $(f,h) \in U_{\alpha,\beta}$. Let us take $u \in W^{2,2}(\Omega)$, $h > 0$ with $u = 0$ on Γ_D, $u = h$ on $I_{\alpha,\beta}$, $\partial_\nu u = 0$ on $\Gamma_N \cup E_{\alpha,\beta}$ and set

$$f := -\Delta u.$$

Let us choose $\varphi := u/h$ in (3.9). Then

$$\begin{aligned}\textstyle\int_\Omega f v^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y - h a^\delta_{\alpha,\beta} &= \textstyle\int_\Omega f v^\delta_{\alpha,\beta} - h\nabla\varphi \cdot \nabla v^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y \\ &= \textstyle -\int_\Omega \Delta u v^\delta_{\alpha,\beta} + \nabla u \cdot \nabla v^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y = 0\end{aligned}$$

by the Green formula and the facts that $u \in W^{2,2}(\Omega)$, $v^\delta_{\alpha,\beta} \in W^{1,q}(\Omega)$, $q \in \left[1, \frac{4}{3}\right)$, $v^\delta_{\alpha,\beta} = 0$ on $\Gamma_D \cup I_{\alpha,\beta}$. Similarly for $w^\delta_{\alpha,\beta}$, and (3.10) is proved. The representation of $\ell^1_{\alpha,\beta}$, $\ell^2_{\alpha,\beta}$ from the last assertion now follows from (3.10) by the choice $\varphi := \Phi_{\alpha,\beta}u_0/h_0$ in (3.9). ■

Let us remark that in the following Proposition, the condition (3.4) could be again formulated as it is mentioned in Remark 3.3, cf. also Remark 2.8.

Proposition 3.7 *A point $(v, \alpha, \beta, f, h) \in H \times D \times C^1(\overline{\Omega}) \times \mathbb{R}$ satisfies (3.3) if and only if u from (3.5) is a weak solution of the boundary value problem (1.1), (1.2), (1.5). In this case, (f,h) satisfies (3.4) if and only if $u \in C^1(\overline{\Omega})$. Then in particular (1.1), (1.2), (1.5) are fulfilled in the classical sense and also (2.27) holds.*

Proof. The statement about the equivalence of (3.5) with a weak formulation of (1.1), (1.2), (1.5) follows from standard considerations and the fact that $\Phi_{\alpha,\beta}$ is one-to-one mapping of H_{α_0,β_0}

onto $H_{\alpha,\beta}$ (see Remark 2.1). If (f,h) satisfies, moreover, (3.4) then $(f,h)\in U_{\alpha,\beta}$ by Lemma 3.6 where we choose $\varphi=\Phi_{\alpha,\beta}u_0/h_0$ in the expressions for $a^\delta_{\alpha,\beta}$, $b^\delta_{\alpha,\beta}$. Remark 3.4 implies that the weak solution of (1.1), (1.2), (1.5) belongs in addition to $W^{2,2}(\Omega)$. Remark 3.5 gives that in this case $u\in C^1(\overline{\Omega})$ and even (2.27) holds.

The last part of Remark 3.5 implies that if $u\in C^1(\overline{\Omega})$ then $u\in W^{2,2}(\Omega)$. This means that $(f,h)=(-\Delta u, u|_{I_{\alpha,\beta}})\in U_{\alpha,\beta}$ by Remark 3.4, and Lemma 3.6 (again with $\varphi=\Phi_{\alpha,\beta}u_0/h_0$) implies that (3.4) holds. ■

For any $(\alpha,\beta)\in D$ and any $h>0$, let us define the subspace $H_{\alpha,\beta}$ and the affine space $H^h_{\alpha,\beta}$ in H by

$$H_{\alpha,\beta}:=\{u\in H:\ u=0 \text{ in } I_{\alpha,\beta}\},\quad H^h_{\alpha,\beta}:=\{u\in H:\ u=h \text{ in } I_{\alpha,\beta}\}.$$

Lemma 3.8 *Let $(u,\alpha,\beta,f,h),(u_n,\alpha_n,\beta_n,f_n,h_n)\in H\times D\times C^1(\overline{\Omega})\times\mathbb{R}$ satisfy (1.1), (1.2), (1.5) in the weak sense and let*

$$|\alpha_n-\alpha|+|\beta_n-\beta|+\|f_n-f\|_{C^1(\overline{\Omega})}+|h_n-h|\to 0. \tag{3.12}$$

Then $\|u_n-u\|\to 0$.

Proof. Proposition 3.7 implies that (u,α,β,f,h), $(u_n,\alpha_n,\beta_n,f_n,h_n)$ satisfy (1.1), (1.2), (1.5) if and only if (w,α,β,f,h), $(w_n,\alpha_n,\beta_n,f_n,h_n)$ with $w=\Phi^{-1}_{\alpha,\beta}(u)$, $w_n(x,y)=\Phi^{-1}_{\alpha_n,\beta_n}(u_n)$ satisfy

$$f\in C^1(\overline{\Omega}), h>0, (\alpha,\beta)\in D, w\in H^h_{\alpha_0,\beta_0}:\ \langle F\left(\Phi_{\alpha,\beta}\left(w\right),f\right),\Phi_{\alpha,\beta}\varphi\rangle=0 \quad \text{for any } \varphi\in H_0. \tag{3.13}$$

Realizing the definitions of F and $\Phi_{\alpha,\beta}$ we can write (3.13) in the form

$$\int_\Omega \nabla w(\xi_{\alpha,\beta}(x),y)\nabla\varphi(\xi_{\alpha,\beta}(x),y)-f(x,y)\varphi(\xi_{\alpha,\beta}(x),y)\,\mathrm{d}x\,\mathrm{d}y=0 \quad \text{for all } \varphi\in H_0.$$

Introducing a new variable $\bar{x}=\xi_{\alpha,\beta}(x)$ and finally renaming $\bar{x}$ again to x we get

$$\int_\Omega \partial_x w(x,y)\partial_x\varphi(x,y)\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))+\frac{\partial_y w(x,y)\partial_y\varphi(x,y)-f(\xi^{-1}_{\alpha,\beta}(x),y)\varphi(x,y)}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))}\,\mathrm{d}x\,\mathrm{d}y=0 \quad \text{for all } \varphi\in H_0.$$

It follows from (2.1) that $\xi'_{\alpha_n,\beta_n}(\xi^{-1}_{\alpha_n,\beta_n}(x))\to\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))$, $\frac{1}{\xi'_{\alpha_n,\beta_n}(\xi^{-1}_{\alpha_n,\beta_n}(x))}\to\frac{1}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta}(x))}$ in $C([0,1])$. We obtain $\|w_n-w\|\to 0$ under the assumption (3.12) from known results about dependence of weak solutions to boundary value problems on data ([16, Proposition 6.2, Theorem 6.2]). Our assertion now follows by using Remark 2.1. ■

In the following assertions we will consider often $(u,\alpha,\beta,f,h)\in C^1(\overline{\Omega})\times D\times C^1(\overline{\Omega})\times\mathbb{R}$ satisfying (1.1), (1.2), (1.5). Hence, in this case u will satisfy also (2.27) by Remark 3.5.

Lemma 3.9 *Let T be a subset of $\partial\Omega$, $\overline{T} \subset \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup E_{\alpha_0,\beta_0}$ (i.e. $\overline{T} \subset \partial\Omega \setminus \overline{I_{\alpha_0,\beta_0}}$), and Ω' a subdomain of Ω such that $\overline{\Omega'} \subset \Omega \cup T$. There is $C > 0$ such that if $(u, \alpha, \beta, f, h), (u_0, \alpha_0, \beta_0, f_0, h_0) \in C^1(\overline{\Omega}) \times D \times C^1(\overline{\Omega}) \times \mathbb{R}$ satisfy (1.1), (1.2), (1.5) and $\overline{I_{\alpha,\beta}} \cap \overline{T} = \emptyset$ then $u_0, u \in C^2(\overline{\Omega'})$,*

$$\|u - u_0\|_{C^2(\overline{\Omega'})} \leq C \left(\|u - u_0\| + \|f - f_0\|_{C^1(\overline{\Omega})} \right).$$

Proof. First, let us consider an arbitrary subdomain Ω' satisfying our assumptions with $T \subset (\{0\} \times [0, \ell)) \cup ([0, \alpha_0 - \delta) \times \{0\})$, $\delta \in (0, \alpha_0)$. Clearly there is a domain $\Omega'' \subset \Omega$ such that $\overline{\Omega'} \subset \Omega'' \cup T$ and also $\overline{\Omega''} \subset \Omega \cup T$. We will prolong the domain Ω and the functions u, u_0, f, f_0 to the left and down in the following way. Let us define

$$\begin{aligned}
&\Omega^- := \{(x, y) \in \mathbb{R}^2 \, : \, (x, -y) \in \Omega\} = (0, 1) \times (-\ell, 0), \\
&\Omega_L := \{(x, y) \in \mathbb{R}^2 \, : \, (-x, y) \in \Omega\} = (-1, 0) \times (0, \ell), \\
&\Omega_L^- := \{(x, y) \in \mathbb{R}^2 \, : \, (-x, -y) \in \Omega\} = (-1, 0) \times (-\ell, 0), \\
&\Omega_0 := (-1, 1) \times (-\ell, \ell), \\
&\Omega_1 := (0, 1) \times (-\ell, \ell), \\
&\Gamma_{0,\ell} := \{0\} \times (-\ell, \ell), \\
&I^-_{\alpha_0,\beta_0} := \{(x, y) \in \mathbb{R}^2 \, : \, (-x, y) \in I_{\alpha_0,\beta_0}\}
\end{aligned}$$

and introduce the functions $\widetilde{u}, \widetilde{u}_0, \widetilde{f}, \widetilde{f}_0 : \Omega_0 \to \mathbb{R}$ defined by

$$\widetilde{u}(x, y) := \begin{cases} u(x, y), & (x, y) \in \Omega, \\ u(x, -y), & (x, y) \in \Omega^-, \\ -u(-x, y), & (x, y) \in \Omega_L, \\ -u(-x, -y), & (x, y) \in \Omega_L^- \end{cases}$$

and analogously for $\widetilde{u}_0$, $\widetilde{f}$, $\widetilde{f}_0$. Let us choose domains Ω_0', $\Omega_0'' \subset \Omega_0$ such that $\Omega_0' \cap \Omega = \Omega'$, $\Omega_0'' \cap \Omega_0 = \Omega''$, $\overline{\Omega}_0' \subset \Omega_0''$, $\Omega_0'' \cap (\overline{I_{\alpha_0,\beta_0}} \cup \overline{I^-_{\alpha_0,\beta_0}}) = \emptyset$. (We can take e.g. symmetric prolongations to the left and down of Ω', Ω'' similarly as above.) It is easy to verify that $\widetilde{u}, \widetilde{u}_0 \in W^{1,2}(\Omega_0'') \cap C^1\left(\overline{\Omega_0''}\right)$ and $\widetilde{f}, \widetilde{f}_0 \in L^2(\Omega_0)$. (Let us note that $\widetilde{f}$, $\widetilde{f}_0$ need not be continuous on $\Gamma_{0,\ell}$.) Moreover, both $(\widetilde{u}, \widetilde{f})$ and $(\widetilde{u}_0, \widetilde{f}_0)$ satisfy in a weak sense

$$-\Delta \widetilde{u} = \widetilde{f} \quad \text{in } \Omega_0''.$$

We get from [7, Theorem 8.8] together with the embedding theorem that there are $C_1, C_2 > 0$ such that

$$\|\widetilde{u} - \widetilde{u}_0\|_{C(\overline{\Omega_0'})} \leq C_1 \|\widetilde{u} - \widetilde{u}_0\|_{W^{2,2}(\Omega_0')} \leq C_2 \left(\|\widetilde{u} - \widetilde{u}_0\|_{W^{1,2}(\Omega_0'')} + \left\|\widetilde{f} - \widetilde{f}_0\right\|_{L^2(\Omega_0'')} \right).$$

These estimates together with the definitions of reflected functions $\widetilde{u}$, $\widetilde{u}_0$, $\widetilde{f}$, $\widetilde{f}_0$ imply that

$$\|\widetilde{u} - \widetilde{u}_0\|_{C(\overline{\Omega_0'})} \leq 4C_2(\|u - u_0\| + \|f - f_0\|_{C^1(\overline{\Omega})}). \tag{3.14}$$

In particular,

$$\|u - u_0\|_{C(\overline{\Omega'})} \leq 4C_2(\|u - u_0\| + \|f - f_0\|_{C^1(\overline{\Omega})}). \tag{3.15}$$

Furthermore, let us take $\Omega_1' := \Omega_1 \cap \Omega_0'$, $\Omega_1'' := \Omega_1 \cap \Omega_0''$. Now, let us realize that the restrictions of $\widetilde{u}$, $\widetilde{u}_0$, $\widetilde{f}$, $\widetilde{f}_0$ on Ω_1 (called for the sake of simplicity again $\widetilde{u}$, $\widetilde{u}_0$, $\widetilde{f}$, $\widetilde{f}_0$) satisfy $\widetilde{u}, \widetilde{u}_0 \in W^{1,2}(\Omega_1'') \cap C^1(\overline{\Omega_1''})$, and $\widetilde{f}, \widetilde{f}_0 \in C^{0,1}(\Omega_1'')$ (the space of Lipschitz functions) and both $(\widetilde{u}, \widetilde{f})$, $(\widetilde{u}_0, \widetilde{f}_0)$ satisfy

$$-\Delta \widetilde{u} = \widetilde{f} \quad \text{in } \Omega_1'', \tag{3.16}$$

$$u = 0 \text{ on } \Gamma_{0,\ell}. \tag{3.17}$$

by the assumptions, we see that Moreover, $\widetilde{u}$ and $\widetilde{u}_0$ satisfies the boundary value problem (6.1), (6.2) with $g = \widetilde{f}$ and $g = \widetilde{f}_0$, respectively, in Ω_1'', $\varphi = \widetilde{u}$ and $\varphi = \widetilde{u}_0$, respectively, on $\partial\Omega_1''$. Remark 6.1 implies that $\widetilde{u}, \widetilde{u}_0 \in C^2(\Omega_1'')$. Using [7, Theorem 4.12] with Ω_1'' instead of Ω, $T = \Gamma_{0,\ell}$ we get the existence of $C_3 > 0$ such that the estimate

$$\|\widetilde{u} - \widetilde{u}_0\|_{C^2(\overline{\Omega_1'})} \leq C_3 \left(\|\widetilde{u} - \widetilde{u}_0\|_{C(\overline{\Omega_1''})} + \left\| \widetilde{f} - \widetilde{f}_0 \right\|_{C^{0,1}(\overline{\Omega_1''})} \right)$$

holds. Restricting back to Ω and realizing (3.14) we obtain that

$$\|u - u_0\|_{C^2(\overline{\Omega'})} \leq \|\widetilde{u} - \widetilde{u}_0\|_{C^2(\overline{\Omega'})} \leq C_4 \left(\|u - u_0\| + \|f - f_0\|_{C^1(\overline{\Omega})} \right) \tag{3.18}$$

with $C_4 = 4C_2 + 2C_3$. The assertion of Lemma 3.9 for the case $T \subset (\{0\} \times [0, \ell)) \cup ([0, \alpha_0 - \delta) \times \{0\})$ if $\delta \in (0, \alpha_0)$ is proved.

Similarly we can treat subdomains Ω' adhering to the left upper corner of Ω, i.e. the case $T \subset (\{0\} \times (0, \ell]) \cup ([0, 1) \times \{\ell\})$. Now we prolong our domain and functions to the left and above and use similar considerations to obtain the same estimate again. (Now we do not remove a neighbourhood of I_{α_0,β_0} from the domain which arises.) Finally, we can do the same with the right corners, i.e. we prolong to the right and down or up, respectively. Realizing that the union of four suitable subdomains of the four types considered cover an arbitrary domain from the assumptions of Lemma and summarizing the estimates obtained for the particular cases, we get the assertion. ■

Lemma 3.10 *Let (u_0, f_0, h_0) satisfy (1.4), $\mathcal{A}_{h_0}(u_0) = [\alpha_0, \beta_0]$ with $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$. Let us assume, moreover, that there are $d > 0$, $\varepsilon > 0$ such that (2.12), (2.13), (2.14), (2.15) hold. Then there exists $\eta > 0$ such that if $(u, \alpha, \beta, f, h) \in C^1(\overline{\Omega}) \times D \times C^1(\overline{\Omega}) \times \mathbb{R}$ satisfy (1.1), (1.2), (1.5),*

$$\|u - u_0\| + \|f - f_0\|_{C^1(\overline{\Omega})} + |h - h_0| + |\alpha - \alpha_0| + |\beta - \beta_0| < \eta \tag{3.19}$$

then

$$\partial_y u > 0 \quad \text{in } \Omega_d \cup I_{\alpha,\beta} \cup \Gamma_d, \tag{3.20}$$

$$u < h \quad \text{on } E_{\alpha,\beta}. \tag{3.21}$$

Remark 3.11 *It is easy to see from the proof below that if we replace the assumption (2.13) by (2.17) in Lemma 3.10 then the assertion holds only for all* $(u, \alpha, \beta, f, h) \in C^1(\overline{\Omega})\times D\times C^1(\overline{\Omega})\times\mathbb{R}$ *satisfying (1.1), (1.2), (1.5), (3.19) and in addition* $\partial_y f \geq 0$ *in* Ω_d.

Proof. First, it follows from (2.12), (2.13), (2.14), (2.15) and Lemma 3.9 that if $(u, \alpha, \beta, f, h) \in C^1(\overline{\Omega}) \times D \times C^1(\overline{\Omega}) \times \mathbb{R}$ satisfy (1.1), (1.2), (1.5) and (3.19) with η sufficiently small then we have

$$f > 0 \quad \text{on } I_{\alpha,\beta}, \tag{3.22}$$

$$\partial_y f \geq 0 \quad \text{in } \Omega_d, \tag{3.23}$$

$$\partial_y u > 0 \quad \text{on } \Gamma_d. \tag{3.24}$$

Let us not that $\partial_y u(0, d) = \partial_y u(1, d) = 0$ and we need both (2.14), (2.15) for the proof of (3.24). Let us prove that

$$\partial_y u > 0 \quad \text{on } I_{\alpha,\beta} \tag{3.25}$$

for all u, α, β under consideration. Let us assume for a contradiction that there are $(u_n, \alpha_n, \beta_n, f_n, h_n) \in C^1(\overline{\Omega})\times D\times C^1(\overline{\Omega})\times\mathbb{R}$ satisfying (1.1), (1.2), (1.5), (3.23), (3.22) and points $(x_n, 0) \in I_{\alpha_n,\beta_n}$ such that $\partial_y u_n(x_n, 0) \leq 0$. Let us denote $v_n := \partial_y u_n$. It follows from Remark 3.5 that $u_n \in W^{2,2}(\Omega)$, that means $v_n \in W^{1,2}(\Omega)$ and it is easy to see that v_n satisfy in the weak sence

$$-\Delta v = \partial_y f \quad \text{in } \Omega_d, \tag{3.26}$$

$$v = 0 \quad \text{on } \partial\Omega_d \cap (\Gamma_D \cup \Gamma_N \cup \overline{E_{\alpha,\beta}}) \tag{3.27}$$

with $(\alpha, \beta) = (\alpha_n, \beta_n)$, $f = f_n$. Of course, we have also $v_n(\alpha_n, 0) = v_n(\beta_n, 0) = 0$ due to $u_n \in C^1(\overline{\Omega})$. Because of (3.23) we can apply Maximum Principle for weak solutions (see [7, Theorem 8.1] for $-v_n$ in Ω_d to obtain that v_n has to attain its minimum on $\partial\Omega_d$. We have $\min v_n|_{I_{\alpha_n,\beta_n}} \leq 0$ and therefore if $\arg\min v_n|_{\overline{\Omega}_d} \notin I_{\alpha_n,\beta_n}$ for n arbitrarily large then $\min v_n|_{\overline{\Omega}_d} < 0$, and because of (3.27) there are points $(\zeta_n, d) \in \Gamma_d$ with $v_n(\zeta_n, d) < 0$. This contradicts to (3.24) for n large enough. Hence, $\arg\min v_n|_{\overline{\Omega}_d} \in I_{\alpha_n,\beta_n}$ for all n large enough. Let us denote this argument of minimum by $(z_n, 0)$.

Due to the minimality we get $\partial_y v_n(z_n, 0) \geq 0$ ($\partial_y v_n(z_n, 0)$ exists due to Remark 6.1 and $\partial_y v_n(z_n, 0) < 0$ leads immediately to the contradiction). We get (realizing (1.1) and (1.5) implying $\partial^2_{xx} u_n(z_n, 0) = 0$) with help of the assumption (3.22) and Remark 6.1 that

$$0 \leq \partial_y v_n(z_n, 0) = \partial^2_{yy} u_n(z_n, 0) = -f_n(z_n, 0) < 0.$$

This contradiction implies (3.25).

It follows from (1.2), (1.5), (3.24), (3.25) and Maximum Principle for $v := \partial_y u$ on $\overline{\Omega}_d$ that $v \geq 0$ on $\overline{\Omega}_d$. Strong Maximum Principle for weak solutions (see [7, Theorem 8.19]) implies $v > 0$ in Ω_d. This together with (3.24) and (3.25) gives (3.20).

Let us prove (3.21). First, it follows from (2.12) that $\delta > 0$, $\eta > 0$ can be chosen so small that if $(u, \alpha, \beta, f, h) \in C^1(\overline{\Omega}) \times D \times C^1(\overline{\Omega}) \times \mathbb{R}$ satisfy (1.1), (1.2), (1.5) and (3.19) then

$$f(x, 0) > 0 \text{ for all } x \in [\alpha - \delta, \beta + \delta]. \tag{3.28}$$

Since $u_0 < h$ on E_{α_0,β_0} by the assumption $\mathcal{A}(u_0) = [\alpha_0, \beta_0]$, it follows from Lemma 3.9 that η can be chosen sufficiently small such that

$$u(x, 0) < h \text{ for all } x \in [\gamma_1, \alpha - \delta] \cup [\beta + \delta, \gamma_2] \tag{3.29}$$

for all (u, α, β, f, h) under consideration.

Further, let us prove that

$$\partial^2_{xx} u(x, 0) < 0 \text{ for all } x \in (\alpha - \delta, \alpha) \cup (\beta, \beta + \delta). \tag{3.30}$$

If this were not true then the equation (1.1), Remark 6.1 and (3.28) would give

$$\partial^2_{yy} u(x, 0) = -f(x, 0) - \partial^2_{xx} u(x, 0) < 0$$

for some $x \in (\alpha - \delta, \alpha) \cup (\beta, \beta + \delta)$. Since $\partial_y u(x, 0) = 0$ by (1.5), we would obtain $\partial_y u(x, y) < 0$ for $y > 0$ small, which contradicts (3.20) and (3.30) is proved.

Due to (1.5) and the assumption $u \in C^1(\overline{\Omega})$ we have $u(\alpha, 0) = u(\beta, 0) = \partial_x u(\alpha, 0) = \partial_x u(\beta, 0) = 0$ and therefore (3.30) implies that

$$\partial_x u(x, 0) > 0 \text{ for all } x \in (\alpha - \delta, \alpha),\ \partial_x u(x, 0) < 0 \text{ for all } x \in (\beta, \beta + \delta).$$

Hence,

$$u(x, 0) < h \text{ for all } x \in (\alpha - \delta, \alpha) \cup (\beta, \beta + \delta).$$

This together with (3.29) give (3.21) and the proof is completed. ■

Proof of Theorem 3.1. Let us have $(u_0, \alpha_0, \beta_0, f_0, h_0)$ satisfying all assumptions.

First, we will prove (ii). Let us consider $(v, \alpha, \beta, f, h) \in H_0 \times D \times C^1(\overline{\Omega}) \times \mathbb{R}$ satisfying (3.3), (3.4). The map $\Phi_{\alpha,\beta}$ is one-to-one of H_{α_0,β_0} onto $H_{\alpha,\beta}$ (see Remark 2.1) and it follows easily from (3.3) that u from (3.5) is a weak solution of (1.1), (1.2), (1.5). Proposition 3.7 implies that $u \in C^1(\overline{\Omega})$, (1.1), (1.2), (1.5) hold even in the classical sense and (2.27) is fulfilled. It follows from Remark 2.1 that for any $\eta > 0$ there is $\delta > 0$ such that if $\|v\| + |\alpha - \alpha_0| + |\beta - \beta_0| + \|f - f_0\|_{C^1(\overline{\Omega})} + |h - h_0| < \delta$ then $\|u - u_0\| + |\alpha - \alpha_0| + |\beta - \beta_0| + \|f - f_0\|_{C^1(\overline{\Omega})} + |h - h_0| < \eta$ with u from (3.5). Moreover, if η is sufficiently small then it follows from Lemma 3.10 that (3.20), (3.21) are fulfilled. In particular, $\mathcal{A}_h(u) = [\alpha, \beta]$. Now, Proposition 2.14 (where $I_h(u) = I_{\alpha,\beta}$ and $E_h(u) = E_{\alpha,\beta}$) yields that (u, f, h) satisfies (1.4), and (ii) is proved.

To prove (i), let $\eta > 0$ be given. Let $\delta > 0$ be such that (ii) already proved holds. First, let (u, f, h) satisfy (1.4) and, moreover, $\mathcal{A}_h(u) = [\alpha, \beta]$ with some $(\alpha, \beta) \in D$. Consider the corresponding v from (3.5). It follows from Proposition 2.14 that $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and (1.1), (1.2), (1.5) hold. Hence, Proposition 3.7 ensures that (3.3) and (3.4) are fulfilled. Thus, in order to finish the proof, it is sufficient to show that for any triplet (u, f, h) satisfying (1.4) and lying in a neighbourhood of (u_0, f_0, h_0) there is really some $(\alpha, \beta) \in D$ such that $\mathcal{A}_h(u) = [\alpha, \beta]$.

The following assertion follows directly from Theorem 4.1 which will be proved completely independently in Section 4. There is $\delta' > 0$ such that if $\|f - f_0\|_{C^1(\overline{\Omega})} + |h - h_0| < \delta'$ then there are $\hat{v}(f, h) \in H_0$, $\left(\hat{\alpha}(f, h),\ \hat{\beta}(f, h)\right) \in D$ satisfying (3.3), (3.4) (with $\alpha = \hat{\alpha}(f, h), \beta = \hat{\beta}(f, h)$) $\|\hat{v}(f, h)\| + |\hat{\alpha}(f, h) - \alpha_0| + \left|\hat{\beta}(f, h) - \beta_0\right| < \frac{\delta}{2}$. We can take $\delta' < \frac{\delta}{2}$ and then $\|\hat{v}(f, h)\| + |\hat{\alpha}(f, h) - \alpha_0| + \left|\hat{\beta}(f, h) - \beta_0\right| + \|f - f_0\| + |h - h_0| < \delta$. Now, by the assertion (ii) already proved, the triplet (u, f, h) with $u = \hat{u}(f, h) := \Phi_{\alpha,\beta}\left(\frac{h}{h_0} u_0 + \hat{v}(f, h)\right)$ satisfies (1.4), $\|\hat{u}(f, h) - u_0\| < \eta$, $\mathcal{A}_h\left(\hat{u}(f, h)\right) = \left[\hat{\alpha}(f, h), \hat{\beta}(f, h)\right]$. However, the variational inequality (1.4) has for any f, h a unique solution u and therefore $u = \hat{u}(f, h)$ for any (u, f, h) under consideration, and in particular $\mathcal{A}_h(u) = \left[\hat{\alpha}(f, h), \hat{\beta}(f, h)\right]$. ■

Proof of Theorem 3.2 is the same as that of Theorem 3.1 but we use Remark 3.11 instead of Lemma 3.10.

4 Application of the Implicit Function Theorem: Continuation for the Operator Equation

We will use the notation from Sections 2, 3. In particular, $v^\delta_{\alpha,\beta}$, $w^\delta_{\alpha,\beta}$ and $a^\delta_{\alpha,\beta}$, $b^\delta_{\alpha,\beta}$ are from (2.11) and (3.9) (Lemma 3.6), respectively.

Theorem 4.1 *Let $k \in \mathbb{N}$, let (u_0, f_0, h_0) satisfy (1.4), (2.16) for some $\delta \in (0, \delta_0)$, $\mathcal{A}_{h_0}(u_0) = [\alpha_0, \beta_0]$ with $\gamma_1 < \alpha_0 < \beta_0 < \gamma_2$. Then there exist neighbourhoods $V \subset C^k(\overline{\Omega}) \times \mathbb{R}$, $W_0 \subset H_0$ and $W_\alpha, W_\beta \subset \mathbb{R}$ of (f_0, h_0), 0, α_0 and β_0, respectively, and C^k-mappings $\hat{v} : V \to W_0$, $\hat{\alpha} : V \to W_\alpha$, $\hat{\beta} : V \to W_\beta$ such that $\hat{v}(f_0, h_0) = 0$, $\hat{\alpha}(f_0, h_0) = \alpha_0$, $\hat{\beta}(f_0, h_0) = \beta_0$ and that $(v, \alpha, \beta, f, h) \in W_0 \times W_\alpha \times W_\beta \times V$ satisfies (3.3), (3.4) if and only if $v = \hat{v}(f, h)$, $\alpha = \hat{\alpha}(f, h)$, $\beta = \hat{\beta}(f, h)$.*

Proof will be done later in this section.

Lemma 4.2 *For any fixed $\delta \in (0, \delta_0)$ the map $(\alpha, \beta) \in D \mapsto (a^\delta_{\alpha,\beta}, b^\delta_{\alpha,\beta}) \in \mathbb{R}^2$ is C^∞-smooth, and for any $k \in \mathbb{N}$ the map*

$$(\alpha, \beta, f) \in D \times C^k(\overline{\Omega}) \mapsto \left(\int_\Omega f v^\delta_{\alpha,\beta} \,\mathrm{d}x \,\mathrm{d}y, \int_\Omega f w^\delta_{\alpha,\beta} \,\mathrm{d}x \,\mathrm{d}y\right) \in \mathbb{R}^2$$

is C^k-smooth.

Proof. Let $\varphi \in C^\infty(\overline{\Omega})$ with $\varphi = 0$ on Γ_D, $\varphi = 1$ on $I_{\alpha,\beta}$ be fixed. Then we get by using the definition of $a^\delta_{\alpha,\beta}$ (Lemma 3.6), $\Phi_{\alpha,\beta}$, $\widetilde{v}^\delta_{\alpha,\beta}$, $\nabla_{\alpha,\beta}$, properties (2.5), (2.6) and Lemma 2.3 (ii) that

$$\begin{aligned} a^\delta_{\alpha,\beta} &= \int_\Omega \nabla\varphi \cdot \nabla v^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \nabla\varphi \cdot \nabla \Phi_{\alpha,\beta}(X^\delta_{\alpha,\beta} + X^\delta) \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \nabla_{\alpha,\beta}\Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla_{\alpha,\beta}(X^\delta_{\alpha,\beta} + X^\delta) \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \nabla_{\alpha,\beta}\Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla_{\alpha,\beta}X^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \nabla_{\alpha,\beta}\Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla_{\alpha,\beta}X^\delta \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \nabla_{\alpha,\beta}\Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla_{\alpha,\beta}X^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y + \int_{B_\delta(\alpha_0)} \nabla\Phi^{-1}_{\alpha,\beta}\varphi \cdot \nabla X^\delta \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.1}$$

Standard results about smooth dependence of weak solutions to linear elliptic boundary value problems on coefficients and right hand sides yield that the map

$$(\alpha,\beta) \in D \mapsto X^\delta_{\alpha,\beta} \in W^{1,2}(\Omega)$$

generated by (2.10) is C^∞-smooth. Moreover, because of the C^∞-smoothness of φ, realizing Lemma 2.2 (with $m = 1$, $k = \infty$) and the definition of $\nabla_{\alpha,\beta}$ we get that

$$\begin{aligned} &(\alpha,\beta) \in D \mapsto \nabla\Phi^{-1}_{\alpha,\beta}\varphi \in L^2(\Omega) \quad \text{is } C^\infty\text{-smooth}, \\ &(\alpha,\beta) \in D \mapsto \nabla_{\alpha,\beta}\Phi^{-1}_{\alpha,\beta}\varphi \in L^2(\Omega) \quad \text{is } C^\infty\text{-smooth}. \end{aligned} \tag{4.2}$$

Hence, the right hand side of (4.1) depends C^∞-smoothly on (α,β). Similarly for $b^\delta_{\alpha,\beta}$, and the first part follows.

To prove the second assertion, take $f \in C^k(\overline{\Omega})$. Then we get by using the definition of $\Phi^*_{\alpha,\beta}$, properties (2.5), (2.6) and (3.9) and Lemma 2.3(ii) that

$$\begin{aligned} \int_\Omega f v^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y &= \int_\Omega f \Phi_{\alpha,\beta}(X^\delta_{\alpha,\beta} + X^\delta) \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \Phi^*_{\alpha,\beta} f (X^\delta_{\alpha,\beta} + X^\delta) \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \Phi^*_{\alpha,\beta} f X^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \Phi^*_{\alpha,\beta} f X^\delta \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_\Omega \Phi^*_{\alpha,\beta} f X^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y + \int_{B_\delta(\alpha_0)} \Phi^*_{\alpha,\beta} f X^\delta \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.3}$$

Because of Lemma 2.2 (with $m = 0$) we get that

$$(\alpha,\beta,f) \in D \times C^k(\overline{\Omega}) \mapsto \Phi^*_{\alpha,\beta} f \in L^2(\Omega) \quad \text{is } C^k\text{-smooth}. \tag{4.4}$$

Hence, the right hand side of (4.3) depends C^k-smoothly on (α,β,f). Similarly for $w^\delta_{\alpha,\beta}$, and the second assertion is proved. ■

Proof of Theorem 4.1. First, due to Propositions 2.14, 3.7 and the fact that $\Phi_{\alpha,\beta}$ is one-to-one mapping of H_{α_0,β_0} onto $H_{\alpha,\beta}$ (see Remark 2.1), $(v, \alpha_0, \beta_0, f_0, h_0)$ with $v = 0$ satisfies (3.3), (3.4).

Furthermore, let us fix $\delta \in (0, \delta_0)$ and $k \in \mathbb{N}$ and introduce a mapping $G^\delta : H_0 \times D \times C^k(\overline{\Omega}) \times \mathbb{R} \to H_0 \times \mathbb{R}^2$, $G^\delta = (G_1, G^\delta_2, G^\delta_3)$, by

$$\begin{aligned} &\langle G_1(v,\alpha,\beta,f,h), \varphi \rangle := \left\langle -F\left(\Phi_{\alpha,\beta}\left(\tfrac{h}{h_0}u_0 + v\right), f\right), \Phi_{\alpha,\beta}\varphi \right\rangle \text{ for all } \varphi \in H_0, \\ &G^\delta_2(v,\alpha,\beta,f,h) = \int_\Omega f v^\delta_{\alpha,\beta} - h h_0^{-1} \nabla\Phi_{\alpha,\beta}u_0 \cdot \nabla v^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y, \\ &G^\delta_3(v,\alpha,\beta,f,h) = \int_\Omega f w^\delta_{\alpha,\beta} - h h_0^{-1} \nabla\Phi_{\alpha,\beta}u_0 \cdot \nabla w^\delta_{\alpha,\beta} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{4.5}$$

It follows from Lemma 2.2 and Lemma 4.2 that G_2^δ and G_3^δ are C^k-smooth. Introducing a new integration variable $\bar{x} = \xi_{\alpha,\beta}(x)$ and renaming $\bar{x}$ again to x we get

$$\begin{aligned}\langle G_1(v,\alpha,\beta,f,h),\varphi\rangle = \textstyle\int_\Omega \partial_x \left[\frac{h}{h_0}u_0(x,y)+v(x,y)\right]\partial_x\varphi(x,y)\cdot\xi'_{\alpha,\beta}\left(\xi^{-1}_{\alpha,\beta}(x)\right)\\ + \left(\partial_y\left[\frac{h}{h_0}u_0(x,y)+v(x,y)\right]\partial_y\varphi(x,y) - f\left(\xi^{-1}_{\alpha,\beta}(x),y\right)\varphi(x,y)\right)\frac{1}{\xi'_{\alpha,\beta}\left(\xi^{-1}_{\alpha,\beta}(x)\right)}\,\mathrm{d}x\,\mathrm{d}y \\ \text{for all } \varphi\in H_0.\end{aligned} \tag{4.6}$$

Hence, (also by using (2.4)) we get

$$G_1(v,\alpha,\beta,f,h) = M_1\left(\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta})\partial_x\left[\frac{h}{h_0}u_0+v\right], \frac{\partial_y\left[\frac{h}{h_0}u_0+v\right]}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta})}\right) - M_2\Phi^*_{\alpha,\beta}f,$$

where the linear bounded operators $M_1 : (L^2(\Omega))^2 \to H_0$ and $M_2 : L^2(\Omega) \to H_0$ are defined by $\langle M_1(v_1,v_2),\varphi\rangle := \int_\Omega (v_1\partial_x\varphi + v_2\partial_y\varphi)\,\mathrm{d}x\,\mathrm{d}y$ and $\langle M_2 v,\varphi\rangle := \int_\Omega v\varphi\,\mathrm{d}x\,\mathrm{d}y$ for all $\varphi\in H_0$. Using (4.4) and the C^∞-smoothness of the map

$$(\alpha,\beta,u)\in D\times W^{1,2}(\Omega)\mapsto\left(\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta})\partial_x u, \frac{\partial_y u}{\xi'_{\alpha,\beta}(\xi^{-1}_{\alpha,\beta})}\right)\in\left(L^2(\Omega)\right)^2,$$

we get the C^k-smoothness of G_1.

Now, the problem (3.3), (3.4) is equivalent to

$$G^\delta(v,\alpha,\beta,f,h) = 0. \tag{4.7}$$

Let us denote by $L^\delta : H_0\times\mathbb{R}^2 \to H_0\times\mathbb{R}^2$ the partial derivative of G^δ with respect to (v,α,β) at the point $(0,\alpha_0,\beta_0,f_0,h_0)$. Derivating (4.6) with respect to (v,α,β) at the point $(0,\alpha_0,\beta_0,f_0,h_0)$ and realizing (2.2) we get

$$\begin{aligned}\langle L_1^\delta(w,\zeta,\theta),\varphi\rangle &= \textstyle\int_\Omega \nabla w\nabla\varphi\\ &+ \zeta([\partial_x u_0\partial_x\varphi - \partial_y u_0\partial_y\varphi + f_0\varphi]\,\partial_\alpha\xi'_{\alpha,\beta}(x) + \partial_\alpha\xi_{\alpha,\beta}(x)\partial_x f_0\varphi)\\ &+ \theta([\partial_x u_0\partial_x\varphi - \partial_y u_0\partial_y\varphi + f_0\varphi]\,\partial_\beta\xi'_{\alpha,\beta}(x) + \partial_\beta\xi_{\alpha,\beta}(x)\partial_x f_0\varphi)\,\mathrm{d}x\,\mathrm{d}y\,|_{\alpha=\alpha_0,\beta=\beta_0}\\ &\qquad\text{for all }\varphi\in H_0,\\ L_2^\delta(w,\zeta,\theta) &= \zeta\partial_\alpha\left(\textstyle\int_\Omega f_0 v^\delta_{\alpha,\beta} - \nabla\Phi_{\alpha,\beta}u_0\cdot\nabla v^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\right)|_{\alpha=\alpha_0,\beta=\beta_0}\,,\\ &+ \theta\partial_\beta\left(\textstyle\int_\Omega f_0 v^\delta_{\alpha,\beta} - \nabla\Phi_{\alpha,\beta}u_0\cdot\nabla v^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\right)|_{\alpha=\alpha_0,\beta=\beta_0}\,,\\ L_3^\delta(w,\zeta,\theta) &= \zeta\partial_\alpha\left(\textstyle\int_\Omega f_0 w^\delta_{\alpha,\beta} - \nabla\Phi_{\alpha,\beta}u_0\cdot\nabla w^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\right)|_{\alpha=\alpha_0,\beta=\beta_0}\,,\\ &+ \theta\partial_\beta\left(\textstyle\int_\Omega f_0 w^\delta_{\alpha,\beta} - \nabla\Phi_{\alpha,\beta}u_0\cdot\nabla w^\delta_{\alpha,\beta}\,\mathrm{d}x\,\mathrm{d}y\right)|_{\alpha=\alpha_0,\beta=\beta_0}\,.\end{aligned}$$

The condition (2.16) for δ small gives that $L^\delta(w,\zeta,\theta) = 0$, i.e.

$$\begin{aligned}&\langle L_1^\delta(w,\zeta,\theta),\varphi\rangle = 0 \text{ for all } \varphi\in H_0,\\ &L_2^\delta(w,\zeta,\theta) = 0,\\ &L_3^\delta(w,\zeta,\theta) = 0,\end{aligned}$$

if and only if $\zeta = \theta = 0$ and w is a weak solution to the homogeneous boundary value problem

$$\left.\begin{aligned} \Delta w &= 0 && \text{in } \Omega, \\ w &= 0 && \text{on } \Gamma_D \cup I_{\alpha_0,\beta_0}, \\ \partial_\nu w &= 0 && \text{on } \Gamma_N \cup E_{\alpha_0,\beta_0}. \end{aligned}\right\}$$

This problem has only the trivial solution $w = 0$ and it follows that $L^\delta : H_0 \times \mathbb{R}^2 \to H_0 \times \mathbb{R}^2$ is injective for $\delta > 0$ small enough. Realizing the form of the operator L^δ we see that it is Fredholm, therefore its injectivity is equivalent to its surjectivity. Thus, L^δ is an isomorphism. Hence, it follows from the Implicit Function Theorem that there are neighbourhoods $V \subset C^k(\overline{\Omega}) \times \mathbb{R}$ of (f_0, h_0), $W_0 \subset H_0$ of 0, and $W_\alpha, W_\beta \subset \mathbb{R}$ of α_0 and β_0, respectively, and C^k-mappings $\hat{v} : V \to H_0$, $\hat{\alpha}, \hat{\beta} : V \to \mathbb{R}$ such that $\hat{v}(f_0, h_0) = 0$, $\hat{\alpha}(f_0, h_0) = \alpha_0$, $\hat{\beta}(f_0, h_0) = \beta_0$ and that we have $G^\delta(v, \alpha, \beta, f, h) = 0$ for $(f, h) \in V$ and $(v, \alpha, \beta) \in W_0 \times W_\alpha \times W_\beta$ if and only if $v = \hat{v}(f, h)$, $\alpha = \hat{\alpha}(f, h)$, $\beta = \hat{\beta}(f, h)$. Finally, the equivalence between (4.7) and (3.3), (3.4) mentioned above gives the assertion of the theorem. ■

5 Proof of the Main Results

Proof of Theorem 2.4 follows directly from Theorem 3.1 and Theorem 4.1. ■

Proof of Theorem 2.5 follows directly from Theorem 3.2 and Theorem 4.1. ■

Lemma 5.1 *Let $(u, \alpha, \beta, f, h) \in C^1(\overline{\Omega}) \times D \times C^1(\overline{\Omega}) \times \mathbb{R}_+$ satisfy (1.1), (1.2), (1.5),*

$$f > 0 \quad \text{on } \overline{I_{\alpha,\beta}}, \tag{5.1}$$

$$\partial_y f \geq 0 \quad \text{in } \Omega. \tag{5.2}$$

Then

$$\partial_y u > 0 \quad \text{in } \Omega \cup I_{\alpha,\beta}. \tag{5.3}$$

Proof. Set $v := \partial_y u$. We have $u \in W^{2,2}(\Omega)$ by Remark 3.5, therefore $v \in W^{1,2}(\Omega)$ and it is easy to see that v satisfies in the weak sense

$$-\Delta v = \partial_y f \quad \text{in } \Omega, \tag{5.4}$$

$$v = 0 \text{ on } \Gamma_D \cup \Gamma_N \cup \overline{E_h(u)}. \tag{5.5}$$

Because of (5.2) we can apply Maximum Principle for weak solutions to $-v$ (see [7, Theorem 8.1]). Realizing that $v \in C(\overline{\Omega})$ we obtain that v attains its minimum on $\partial\Omega$.

If there is $(x, 0) \in I_{\alpha,\beta}$ with $\partial_y u(x, 0) \leq 0$, i.e. $v(x, 0) \leq 0$, then v must attain its minimum on $I_{\alpha,\beta}$ because of (5.5). Let $(x_0, 0) \in I_{\alpha,\beta}$ be the argument of minimum. Then $v(x_0, 0) =$

$\partial_y u(x_0, 0) \leq 0$. The derivative $\partial_y v(x_0, 0)$ exists due to Remark 6.1 and clearly $\partial_y v(x_0, 0) \geq 0$. We have $\partial^2_{xx} u(x_0, 0) = 0$ by (1.5) and (1.1) together with the assumption (5.1) imply that

$$0 \leq \partial_y v(x_0, 0) = \partial^2_{yy} u(x_0, 0) = -f(x_0, 0) < 0.$$

This contradiction implies that u satisfies

$$\partial_y u > 0 \quad \text{on } I_{\alpha,\beta}. \tag{5.6}$$

The previous part of the proof implies that v is nonconstant and $v \geq 0$ in $\overline{\Omega}$. The Strong Maximum Principle for weak solutions (see [7, Theorem 8.19]) implies $\partial_y u = v > 0$ in Ω and this together with (5.6) gives our asserion. ■

Proof of Proposition 2.10. The conditions (2.12), (2.20) assumed for f_0 remain valid for f (with a smaller ε) if $\|f - f_0\|_{C^1(\overline{\Omega})}$ is small enough. Hence, the assertion follows from Lemma 5.1.

Proof of Proposition 2.12. First, let us show that (3.25) holds under the assumption (2.22). We have $u \in W^{2,2}(\Omega)$ by Proposition 2.14 and it is easy to see that the function $v = \partial_y u$ is a weak solution of the problem

$$-\Delta v = \partial_y f \text{ in } \Omega, \tag{5.7}$$

$$v = \partial_y u \text{ on } \partial\Omega. \tag{5.8}$$

Again due to Proposition 2.14 we have $\partial_y u = 0$ on $\Gamma_D \cup \Gamma_N \cup E_h(u)$ and $\partial_y u \geq 0$ on $I_h(u)$. Under the assumption (2.22), the Strong Maximum Principle for weak solutions together with the fact that $v \in C(\overline{\Omega})$ implies that $\min u|_{\overline{\Omega}} = \min u|_{\partial\Omega} = 0$ and $v > 0$ in Ω, i.e. (3.25) holds.

If the first assertion were not true then we would have also $\tilde{x}_1, \tilde{x}_2 \in [\gamma_1, \gamma_2]$ such that $u(\tilde{x}_1) = u(\tilde{x}_2) = h, u(x) < h$ for all $x \in (\tilde{x}_1, \tilde{x}_2)$. It is sufficient to prove that then it would be simultaneously $\partial^2_{xx} u(x, 0) \leq 0$ for all $x \in (\tilde{x}_1, \tilde{x}_2)$, which is impossible. If $\partial^2_{xx} u(x, 0) > 0$ for some $x \in (\tilde{x}_1, \tilde{x}_2)$ then $\partial^2_{yy} u(x, 0) = -f(x, 0) - \partial^2_{xx} u(x, 0) < 0$. We have $\partial_y u(x, 0) = 0$ by Proposition 2.14, therefore $\partial_y u(x, y) < 0$ for $y > 0$ small, which contradicts (3.25). Hence, the first assertion must hold.

For any f there is unique weak solution u_0 of the problem (1.1) with

$$u = 0 \text{ on } \Gamma_D, \tag{5.9}$$

$$\partial_\nu u = 0 \text{ on } \Gamma_N \cup \Gamma_U. \tag{5.10}$$

It follows from [8, Theorem 3] that $u_0 \in C^1(\overline{\Omega})$ because we have right angles in all corners. Hence, $u_0 \in C^2(\Omega)$ by Remark 6.1 (u_0 is a solution of (1.1), (6.2) with the C^1 right hand side $g = f$ and a continuous boundary condition $u|_{\partial\Omega}$). We assume (2.22) and the classical

Maximum Principle implies that $\min u_0|_{\overline{\Omega}} = \min u_0|_{\partial\Omega}$. This minimum cannot be attained in $(x,0)$ or (x,ℓ) $(x \in (0,1))$ because the Strong Maximum Principle would imply $\partial_y u_0(x,0) > 0$ or $\partial_y u_0(x,0) < 0$, respectively, which would contradict (5.10).

Hence, the minimum must be attained on Γ_D, that means $\min u_0|_{\overline{\Omega}} = 0$. Consequently $u_0(x,0) > 0$ for all $x \in (0,1)$. Clearly there is h_f such that if $h < h_f$ then $u_0(x_0,0) > h$ for some $x_0 \in [\gamma_1,\gamma_2]$ and if $h > h_f$ then $u(x,0) < h$ for all $x \in [\gamma_1,\gamma_2]$. In the former case, $u_0 \notin K_h$ and therefore it cannot be a solution of (1.4). The solution u_I of (1.4) must touch the obstacle h at least at two points. Indeed, if it were $u_I < h$ on Γ_U or on Γ_U with the exception of a single point then it would fulfill (1.1), (5.9), (5.10) (recall that $u_I \in C^1(\overline{\Omega})$), which would contradict the unicity of the solution of (1.1), (5.9), (5.10). Hence, the existence of α,β such that $\mathcal{A}_h(u_I) = [\alpha,\beta]$ follows from the first assertion of our Proposition. In the latter case, u_0 is simultaneously the unique solution of (1.4), and our last assertion follows. ■

6 Appendix

Proof of Lemma 2.2. Define the operator $\mathcal{F} : D \times C^{k+m}(\overline{\Omega}) \to C^{k+m}(\overline{\Omega})$ by $\mathcal{F}(\alpha,\beta,f) = \Phi_{\alpha,\beta} f$.

First, let us consider the case $m = 0$. Define $\mathcal{F}(\alpha,\beta,f) := \Phi_{\alpha,\beta} f$.

If $\mathcal{F}$ is differentiable then the partial derivatives can be calculated pointwise:

$$[\partial_\alpha \mathcal{F}(\alpha,\beta,f)](x,y) = \partial_x f(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x),$$

$$[\partial_\beta \mathcal{F}(\alpha,\beta,f)](x,y) = \partial_x f(\xi_{\alpha,\beta}(x),y)\partial_\beta \xi_{\alpha,\beta}(x),$$

$$[\partial_f \mathcal{F}(\alpha,\beta,f)g](x,y) = g(\xi_{\alpha,\beta}(x),y).$$

Let us check that $\mathcal{F}$ is really partially differentiable with respect to α:

$$\begin{aligned}[\mathcal{F}(\tilde\alpha,\beta,f) - \mathcal{F}(\alpha,\beta,f)](x,y) - (\tilde\alpha-\alpha)\partial_x f(\xi_{\alpha,\beta}(x),y)\partial_\alpha\xi_{\alpha,\beta}(x) = \\ = f(\xi_{\tilde\alpha,\beta}(x),y) - f(\xi_{\alpha,\beta}(x),y) - (\tilde\alpha-\alpha)\partial_x f(\xi_{\alpha,\beta}(x),y)\partial_\alpha\xi_{\alpha,\beta}(x).\end{aligned}$$

The mean value theorem gives

$$f(\xi_{\tilde\alpha,\beta}(x),y) - f(\xi_{\alpha,\beta}(x),y) = \partial_x f(\theta,y)(\xi_{\tilde\alpha,\beta}(x)-\xi_{\alpha,\beta}(x)) \text{ and } \xi_{\tilde\alpha,\beta}(x)-\xi_{\alpha,\beta}(x) = \partial_\alpha\xi_{\eta,\beta}(x)(\tilde\alpha-\alpha)$$

with θ between $\xi_{\tilde\alpha,\beta}(x)$ and $\xi_{\alpha,\beta}(x)$ and η between $\tilde\alpha$ and α. Therefore

$$\frac{[\mathcal{F}(\tilde\alpha,\beta,f) - \mathcal{F}(\alpha,\beta,f)](x,y)}{\tilde\alpha-\alpha} - \partial_x f(\xi_{\alpha,\beta}(x),y)\partial_\alpha\xi_{\alpha,\beta}(x) \to 0$$

for $\tilde\alpha \to \alpha$ uniformly with respect to x and y because of the assumption (2.1).

Analogously one shows that $\mathcal{F}$ is partially differentiable with respect to β.

For any fixed $(\alpha,\beta) \in D$, the map $f \in C^k(\overline{\Omega}) \mapsto \mathcal{F}(\alpha,\beta,f) = \Phi_{\alpha,\beta} f \in C(\overline{\Omega})$ is linear and continuous and it follows that $\mathcal{F}$ is partially differentiable with respect to f.

In order to get the C^1-smoothness of $\mathcal{F}$ it remains to show that the maps $(\alpha,\beta,f) \in D \times C^k(\overline{\Omega}) \mapsto \partial_\alpha \mathcal{F}(\alpha,\beta,f) \in C(\overline{\Omega})$ and $(\alpha,\beta,f) \in D \times C^k(\overline{\Omega}) \mapsto \partial_\beta \mathcal{F}(\alpha,\beta,f) \in C(\overline{\Omega})$ and $(\alpha,\beta,f) \in D \times C^k(\overline{\Omega}) \mapsto \partial_f \mathcal{F}(\alpha,\beta,f) = \Phi_{\alpha,\beta} \in \mathcal{L}(C^k(\overline{\Omega}), C(\overline{\Omega}))$ are continuous. The continuity of $\partial_\alpha \mathcal{F}$ follows from

$$\begin{aligned}
&\left[\partial_\alpha \mathcal{F}(\tilde{\alpha},\tilde{\beta},\tilde{f}) - \partial_\alpha \mathcal{F}(\alpha,\beta,f)\right](x,y) = \\
&\quad = \partial_x \tilde{f}(\xi_{\tilde{\alpha},\tilde{\beta}}(x),y)\partial_\alpha \xi_{\tilde{\alpha},\tilde{\beta}}(x) - \partial_x f(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x) = \\
&\quad = \Big(\partial_x \tilde{f}(\xi_{\tilde{\alpha},\tilde{\beta}}(x),y) - \partial_x \tilde{f}(\xi_{\alpha,\beta}(x),y)\Big)\partial_\alpha \xi_{\tilde{\alpha},\tilde{\beta}}(x) + \\
&\quad +\partial_x \tilde{f}(\xi_{\alpha,\beta}(x),y)\left(\partial_\alpha \xi_{\tilde{\alpha},\tilde{\beta}}(x) - \partial_\alpha \xi_{\alpha,\beta}(x)\right) + \\
&\quad + \Big(\partial_x \tilde{f}(\xi_{\alpha,\beta}(x),y) - \partial_x f(\xi_{\alpha,\beta}(x),y)\Big)\partial_\alpha \xi_{\alpha,\beta}(x).
\end{aligned}$$

Obviously, this tends to zero uniformly with respect to x and y if $\tilde{\alpha} \to \alpha$, $\tilde{\beta} \to \beta$ and $\tilde{f} \to f$ in $C^k(\overline{\Omega})$ due to (2.1) again.

Similarly one shows the continuity of $\partial_\beta \mathcal{F}$.

Now, let us show the continuity of $\partial_f \mathcal{F} : C^k(\overline{\Omega}) \to C(\overline{\Omega})$, i.e. to show (due to the form of $\partial_f \mathcal{F}$) that

$$(\alpha,\beta) \in D \mapsto \Phi_{\alpha,\beta} \in \mathcal{L}(C^k(\overline{\Omega}), C(\overline{\Omega})) \text{ is continuous.}$$

We have $\left|\left[\Phi_{\tilde{\alpha},\tilde{\beta}} f - \Phi_{\alpha,\beta} f\right](x,y)\right| = \left|f(\xi_{\tilde{\alpha},\tilde{\beta}}(x),y) - f(\xi_{\alpha,\beta}(x),y)\right| \le \|f\|_{C^k(\overline{\Omega})} |\xi_{\tilde{\alpha},\tilde{\beta}}(x) - \xi_{\alpha,\beta}(x)|$, i.e.

$$\frac{\|\Phi_{\tilde{\alpha},\tilde{\beta}} f - \Phi_{\alpha,\beta} f\|_{C(\overline{\Omega})}}{\|f\|_{C^k(\overline{\Omega})}} \le \sup_{0 \le x \le \ell} |(\xi_{\tilde{\alpha},\tilde{\beta}}(x) - \xi_{\alpha,\beta}(x))| \to 0 \text{ for } (\tilde{\alpha},\tilde{\beta}) \to (\alpha,\beta).$$

To prove C^2-smoothness of $\mathcal{F}$ (if $k \ge 2$) we have to show that $\partial_\alpha \mathcal{F}$, $\partial_\beta \mathcal{F}$ and $\partial_f \mathcal{F}$ are C^1-smooth from $D \times C^k(\overline{\Omega})$ into $C(\overline{\Omega})$ or into $\mathcal{L}(C^k(\overline{\Omega}), C(\overline{\Omega}))$, respectively. Showing that for $\partial_\alpha \mathcal{F}$ and $\partial_\beta \mathcal{F}$ is the same procedure as above because $\partial_\alpha \mathcal{F}$ and $\partial_\beta \mathcal{F}$ map $D \times C^k(\overline{\Omega})$ into $C(\overline{\Omega})$ as $\mathcal{F}$.

Let us show that $\partial_f \mathcal{F}$ is C^1-smooth from $D \times C^k(\overline{\Omega})$ into $\mathcal{L}(C^k(\overline{\Omega}), C(\overline{\Omega}))$, i.e. that

$$(\alpha,\beta) \in D \mapsto \Phi_{\alpha,\beta} \in \mathcal{L}(C^k(\overline{\Omega}), C(\overline{\Omega})) \text{ is } C^1\text{-smooth for } k \ge 2.$$

If $\partial_f \mathcal{F}$ is differentiable then the partial derivatives can be calculated pointwise:

$$\left[\partial_\alpha \partial_f \mathcal{F}(\alpha,\beta,f) g\right](x,y) = \partial_x g(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x),$$

$$\left[\partial_\beta \partial_f \mathcal{F}(\alpha,\beta,f) g\right](x,y) = \partial_x g(\xi_{\alpha,\beta}(x),y)\partial_\beta \xi_{\alpha,\beta}(x).$$

Let us check that $\partial_f \mathcal{F}$ is really partially differentiable with respect to α. As above, using the mean value theorem, we get

$$\begin{aligned}
&[\partial_f \mathcal{F}(\tilde{\alpha},\beta,f)g - \partial_f \mathcal{F}(\alpha,\beta,f)g]\,(x,y) - (\tilde{\alpha}-\alpha)\partial_x g(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x) = \\
&\quad = g(\xi_{\tilde{\alpha},\beta}(x),y) - g(\xi_{\alpha,\beta}(x),y) - (\tilde{\alpha}-\alpha)\partial_x g(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x) = \\
&\quad = \partial_x g(\theta,y)(\xi_{\tilde{\alpha},\beta}(x) - \xi_{\alpha,\beta}(x)) - (\tilde{\alpha}-\alpha)\partial_x g(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x) = \\
&\quad = (\tilde{\alpha}-\alpha)\left(\partial_x g(\theta,y)\partial_\alpha \xi_{\eta,\beta}(x) - \partial_x g(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x)\right) = \\
&\quad = (\tilde{\alpha}-\alpha)\left(\partial_x^2 g(\zeta,y)(\theta - \xi_{\alpha,\beta}(x),y))\partial_\alpha \xi_{\eta,\beta}(x) + \partial_x g(\xi_{\alpha,\beta}(x),y)(\partial_\alpha \xi_{\eta,\beta}(x) - \partial_\alpha \xi_{\alpha,\beta}(x))\right)
\end{aligned}$$

with θ between $\xi_{\tilde{\alpha},\beta}(x)$ and $\xi_{\alpha,\beta}(x)$, η between $\tilde{\alpha}$ and α and ζ between θ and $\xi_{\alpha,\beta}(x)$. Hence

$$\|\partial_f \mathcal{F}(\tilde{\alpha},\beta,f)g - \partial_f \mathcal{F}(\alpha,\beta,f)g\|_{C(\overline{\Omega})} = o\left(|\tilde{\alpha}-\alpha|\right)\ \|g\|_{C^2(\overline{\Omega})} \text{ for } \tilde{\alpha} \to \alpha.$$

Similarly one shows that $\partial_\beta \partial_f \mathcal{F}$ exists. It remains to show that $\partial_\alpha \partial_f \mathcal{F}$ and $\partial_\beta \partial_f \mathcal{F}$ are continuous. For $\partial_\alpha \partial_f \mathcal{F}$ this follows from

$$\begin{aligned}
&\sup_{(x,y)\in\Omega} \left| \left[\partial_\alpha \partial_f \mathcal{F}(\tilde{\alpha},\tilde{\beta},\tilde{f})g - \partial_\alpha \partial_f \mathcal{F}(\alpha,\beta,f)g\right](x,y)\right| = \\
&\quad = \sup_{(x,y)\in\Omega} \left|\partial_x g(\xi_{\tilde{\alpha},\tilde{\beta}}(x),y)\partial_\alpha \xi_{\tilde{\alpha},\tilde{\beta}}(x) - \partial_x g(\xi_{\alpha,\beta}(x),y)\partial_\alpha \xi_{\alpha,\beta}(x)\right| = \\
&\quad = o\left(\|g\|_{C^2(\overline{\Omega})}\right) \text{ for } (\tilde{\alpha},\tilde{\beta}) \to (\alpha,\beta).
\end{aligned}$$

In a similar way one shows higer order smoothness (up to C^k) in the case $m = 0$.

Now, consider the case $m = 1$. Additionally to the consideratons of the case $m = 0$ we have to show that the maps

$$\mathcal{G}_1 : D \times C^{k+1}(\overline{\Omega}) \to C(\overline{\Omega}) : \mathcal{G}_1(\alpha,\beta,f)(x,y) := \partial_x\left[\mathcal{F}(\alpha,\beta,f)(x,y)\right] = \partial_x f(\xi_{\alpha,\beta}(x),y)\xi'_{\alpha,\beta}(x)$$

and

$$\mathcal{G}_2 : D \times C^{k+1}(\overline{\Omega}) \to C(\overline{\Omega}) : \mathcal{G}_2(\alpha,\beta,f)(x,y) := \partial_y\left[\mathcal{F}(\alpha,\beta,f)(x,y)\right] = \partial_y f(\xi_{\alpha,\beta}(x),y)$$

are C^k-smooth. Obviously, for $\mathcal{G}_2$ this can be done as above by replacing f by $\partial_y f$. For $\mathcal{G}_1$ there are needed some straightforward modifications because of the factor $\xi'_{\alpha,\beta}(x)$.

Similarly, if $m = 2$, then the considerations can be lead back to the case $m = 1$ etc. ■

Proof of Proposition 2.14. Let $(u,f,h) \in H \times C^1(\overline{\Omega}) \times \mathbb{R}^+$ satisfy (1.4). Then $u \in C^1(\Omega \cup \Gamma_U)$ (Remark 2.13) and standard considerations (using suitable test functions) show that u is a weak solution of the boundary value problem (1.1), (1.2), (2.24). If $\overline{I_h(u)} \subset \Gamma_U$, i.e. $\overline{I_h(u)} \subset (\tilde{\gamma}_1,\tilde{\gamma}_2)$ with some $\gamma_1 < \tilde{\gamma}_1 < \tilde{\gamma}_2 < \gamma_2$, $(\tilde{\gamma}_1,0),(\tilde{\gamma}_2,0) \in E_h(u)$, then u can be understood as a solution of the mixed boundary value problem (1.1) with the smooth Dirichlet boundary condition u on $\Gamma_D \cup ((\tilde{\gamma}_1,\tilde{\gamma}_2) \times \{0\})$ and with the zero Neumann boundary condition on the rest of $\partial\Omega$.

The only transmission points between the smooth Dirichlet boundary condition and the zero Neumann boundary condition are the points $(\tilde{\gamma}_1, 0), (\tilde{\gamma}_2, 0)$ (in their neighborhoods the C^1-regularity is already justified) and the corners of Ω. Since all corners have the right angles, it follows from [8, Theorem 1] that (2.27) holds, in particular (2.28) is true. Hence, a solution of (1.4) must satisfy also $u \in C^2(\Omega)$ because it is a solution of (1.1) with a differentiable right hand side f and continuous boundary condition $u|_{\partial\Omega}$ (see Remark 6.1 below).

Conversely, standard considerations imply that if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies (1.1), (1.2), (2.24), (2.25), (2.26) then (1.4) holds, and $u(\gamma_1, 0) < h,\ u(\gamma_2, 0) < h$ means $\overline{I_h(u)} \subset \Gamma_U$. ■

Remark 6.1 *Let us recall that a problem*

$$-\Delta u = g \quad \text{in } \Omega, \tag{6.1}$$

$$u = \varphi \quad \text{on } \partial\Omega \tag{6.2}$$

with $g \in C^\gamma(\Omega)$, $\gamma \in (0,1)$, and $\varphi \in C(\partial\Omega)$ has a unique solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ (see e.g. [7, Section 4.2]).

If u is a weak solution of (1.1), (1.2), (2.24) and in addition $u \in C(\overline{\Omega})$, then $u \in C^2(\Omega)$ (because the function u is simultaneously a solution of (6.1), (6.2) with the continuous boundary condition $\varphi = u|_{\partial\Omega}$ and with the C^1-smooth right-hand side $g = f$) and u has continuous second order derivatives up to the boundary with the exception of the points $(x, 0) \in \partial I_h(u)$ and the corners. In a neighbourhood of the points from $\Gamma_D \cup I_h(u)$, the C^2-regularity follows e.g. from [7, Theorem 4.12]. For $(x_0, y_0) \in \Gamma_N \cup E_h(u)$ it is a particular case of Lemma 3.9.

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