Preprint, Institute of Mathematics, AS CR, Prague. 2007-12-21

Convergence to equilibria of solutions to a conserved phase-field system with memory

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Abstract

We show that the trajectories of a conserved phase-field model with memory are compact in the space of continuous functions and, for an exponential relaxation kernel, we establish the convergence of solutions to a single stationary state as time goes to infinity. In the latter case, we also estimate rate of decay to equilibrium.

1 Introduction

This paper is devoted to the study of asymptotic properties and convergence to equilibria of conserved phase-field systems of Caginalp type, where the classical Fourier law $\mathbf{q} = -k_0 \nabla \vartheta$ ($k_0 > 0$ is a so called instantaneous heat conductivity coefficient) is replaced by the following nonlocal condition

$$\mathbf{q}(\mathbf{t}, \mathbf{x}) = -\int_{-\infty}^{t} k(t-s) \nabla \vartheta(s, x). \tag{1.1}$$

The relation (1.1) states that the heat flux depends only on the temporal history of the temperature gradient; this turns out to be compatible with classical thermodynamical laws, and entails that ϑ propagates with a finite speed, cf. Gurtin and Pipkin [17], Joseph and Preziosi [20]. We will study a fourth order conserved system, which reads as follows:

$$\partial_t \Big(\vartheta + \lambda(\chi) \Big) + \operatorname{div} \mathbf{q} = f, \tag{1.2}$$

$$\tau \partial_t \chi = -\xi^2 \Delta (\xi^2 \Delta \chi - W'(\chi) + \lambda'(\chi) \vartheta). \tag{1.3}$$

Here ϑ and χ designate the (relative) temperature and the order parameter (phase variable) respectively, W is typically a double-well potential, λ' represents the latent

*The paper was written while the second author was visiting Ohio University, whose kind hospitality is gratefully acknowledged. Her work was also partially supported by Grant IAA100190606 of GA AV ČR

heat, f is a heat source, $\tau > 0$ and $\xi > 0$ stand for a relaxation time and correlation length, respectively, and the heat flux $\mathbf{q}$ is given by (1.1).

The material occupies a bounded domain $\Omega \subset \mathbb{R}^3$, with a sufficiently smooth boundary $\partial\Omega$, and the system (1.1) - (1.3) is complemented by a homogeneous Neumann boundary condition for both χ, ϑ, and also for the so called chemical potential $-\xi^2\Delta\chi + W'(\chi) - \lambda'(\chi)\vartheta$, which can be expressed by

$$\nabla\chi \cdot \mathbf{n}|_{\partial\Omega} = \nabla\vartheta \cdot \mathbf{n}|_{\partial\Omega} = \nabla(\Delta\chi) \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{1.4}$$

with $\mathbf{n}$, the outer normal vector. For the sake of simplicity, we set the constants and the measure of the set Ω equal to 1:

$$\tau = \xi = 1, \quad |\Omega| = 1. \tag{1.5}$$

Systems of the same or comparable type, conserved or nonconserved, with or without memory terms, have been studied by many authors. See [1-5, 8, 10-14, 20, 23]. The questions of well-posedness and existence of finite dimension attractors were considered in [14], [15], and the dissipativity of the system was studied in [24]. In particular, the long-time behavior of solutions seems to be well understood and the equilibrium (stationary) solutions have been identified as the only candidates to belong to the $\omega-$limit set of each individual trajectory (cf. [10, Theorem 2.2]). More specifically, for $W'(\chi) = \chi^3 - \chi$, $\lambda'(\chi) = const = \lambda_0$, the functions $(\vartheta_\infty, \chi_\infty)$ in the ω-limit set $\omega(\vartheta_0, \chi_0)$ satisfy the following equations:

$$\vartheta_\infty = \int_\Omega \vartheta(0)\ \mathrm{d}x + \int_0^\infty \int_\Omega g\ \mathrm{d}x\ \mathrm{d}t, \tag{1.6}$$

$$-\Delta\chi_\infty + W'(\chi_\infty) - \lambda_0\vartheta_\infty = w_\infty, \quad \mathbf{n} \cdot \nabla\chi_\infty\Big|_{\partial\Omega} = 0, \tag{1.7}$$

$$\int_\Omega \chi_\infty\ \mathrm{d}x = \int_\Omega \chi(0)\ \mathrm{d}x, \quad w_\infty = \int_\Omega W'(\chi_\infty)\ \mathrm{d}x - \lambda_0\vartheta_\infty, \tag{1.8}$$

where $g(t,x) = f(t,x) + \int_{-\infty}^0 k(t-s)\Delta\vartheta(s)\ \mathrm{d}s$.

If the ω-limit set consists of only a finite number of solutions, then the compactness of trajectories implies that any solution $(\chi(t), \vartheta(t))$ converges, as $t \to \infty$, to a single stationary state. See, e.g., [1] for such a result for a non-conserved system in the one-dimensional case. However, the structure of the set of stationary solutions for a general domain may be quite complicated; in particular, the set in question may contain a continuum of nonradial solutions if Ω is a ball or an annulus. If this is the case, it seems highly nontrivial to decide whether or not the solutions converge to a single stationary state. It is well-known that nonconvergent trajectories may occur even in finite-dimensional dynamical systems (cf. Aulbach [6]). Similar examples for semilinear parabolic equations were derived by Poláčik and Rybakowski [22]. Positive convergence results for the phase-field system (1.2), (1.3) and its nonconserved variant, with the heat flux given by

$$\mathbf{q} = -k_I\nabla\vartheta - \int_0^\infty k(s)\nabla\vartheta(t-s)\ \ \mathrm{d}s$$

were proved in [3], [4], [5] and [2], respectively. The positive instantaneous heat conductivity coefficient k_I was crucial in the proofs. The compactness of trajectories and the $\omega-$limit sets for a heat law of the form (1.1), with a positive type kernel were studied by Colli and Laurençot [11] in the non-conserved case, and Colli et al. [10] for (1.1)-(1.3). Convergence for the Cattaneo-Maxwel heat conduction law with an additional inertial term χ_{tt} and a strong dissipative term $\alpha\Delta\chi_t,\ \alpha > 0$, but without a heat source was proved in [16].

In the present paper, we extend the asymptotic compactness result obtained in [10] to a broader class of nonlinearities, and show that the trajectories of the order parameter are precompact even in the space of continuous functions. Moreover, for a constant latent heat (which implies that limit temperature is uniquely defined), and an exponential kernel k, we prove the convergence of the whole trajectory to a single stationary state.

The paper is organized as follows. In Section 2, we list our assumptions and state the main results. In Section 3, we derive some *a priori* estimates and prove the compactness result. The decay of the temperature and the heat flux, together with uniform bounds of solutions are established in Section 4. Finally, the convergence of the order parameter is proved in Section 5, and its convergence rate is estimated in Section 6.

2 Preliminaries and main results

Let $H = L^2(\Omega)$, $\mathbf{H} = L^2(\Omega)^3$, $V = W^{1,2}(\Omega)$, $\mathbf{V} = \mathbf{W^{1,2}(\Omega)^3}$, $U = W^{2,2}(\Omega)$. We denote by $(\cdot,\cdot)$ the inner product in H, by $\langle\cdot,\cdot\rangle$ the duality pairing between V and its dual V', and by $\|\cdot\|$ the norm in H or $\mathbf{H}$. We will also use V_0 to designate the subspace of all functions $v \in V$ with null average, i.e., $\int_\Omega v\ \mathrm{d}x = 0$.

We start with the homogeneous Neumann problem associated with the Laplace equation. For all $1 < q < \infty$, define a linear operator A_q on the Banach space $L^q(\Omega)$ by

$$\mathcal{D}(A_q) = \{v \in W^{2,q}(\Omega) \mid \nabla v\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\ A_q v = -\Delta v. \tag{2.1}$$

The unique solution v of the problem

$$\begin{cases} -\Delta v = l \text{ in } \Omega, \\ \nabla v\cdot\vec{n} = 0 \text{ on } \partial\Omega,\ \int_\Omega v\ \mathrm{d}x = 0, \end{cases} \tag{2.2}$$

for $l \in L^q(\Omega)$, $\int_\Omega l = 0$ will be denoted by $v = A_q^{-1}[l]$. Note that (after a standard extension), we may write

$$A_q^{-1}A_q v = v - \int_\Omega v \quad\text{for any}\quad v \in L^q(\Omega). \tag{2.3}$$

For simplicity, we will omit the subscript 2, and write $A_2 = A,\ A_2^{-1} = A^{-1}$. We also have

$$\int_\Omega \nabla A^{-1}[v]\cdot\nabla w \le c\|v\|_{V'}\ \|w\|_V \text{ for all } v \in V',\quad w \in V_0, \tag{2.4}$$

$$\int_\Omega |\nabla A^{-1}[v]|^2 = \langle v, A^{-1}[v]\rangle \text{ is equivalent to } \|v\|_{V'}^2 \text{ for all } v \in V' \text{ satisfying } \langle v,1\rangle = 0. \tag{2.5}$$

We will assume that the past history of the temperature, $\chi(0)$, and the heat source are given such that

$$\vartheta(0) \in H,\quad \chi(0) \in V, \tag{2.6}$$

$$h(t) = \int_{-\infty}^{0} k(t-s)\Delta\vartheta(s)\ \mathrm{d}s, \quad g = h + f \in L^1(0, +\infty; H), \tag{2.7}$$

and the kernel k satisfies

$$k \in L^1(0, \infty), \quad \int_0^\infty k(s)\ \mathrm{d}s \neq 0, \tag{2.8}$$

k is of positive type, i.e.,

$$\int_0^T \int_0^t k(t-s)v(s)\ \mathrm{d}s\ v(t)\ \mathrm{d}t \geq 0 \quad \text{for all } v \in L^2(0,T) \text{ and all } T > 0. \tag{2.9}$$

The free energy function $W : \mathbb{R} \mapsto \mathbb{R}$ and the function λ will be supposed to satisfy the following hypotheses:

$$W(z) \geq 0 \text{ for all } z \geq 0, \tag{2.10}$$

$$W'(z)z > 0 \quad \text{for} \quad |z| > R > 0, \tag{2.11}$$

$$W'(z)z \geq c_1 W(z) - c_2, \quad z \in \mathbb{R}, \tag{2.12}$$

$$W''(z) \geq -c_3, \tag{2.13}$$

$$W \in C^{3+\mu}(\mathbb{R}), \ |W''(z)| \leq c_4(1 + |z|^p), \ 1 \leq p < 4, \tag{2.14}$$

$$\lambda \in C^1(\mathbb{R}), \quad |\lambda'(z)| \leq c_5, \tag{2.15}$$

where $c_j, \ j = 1, ...5$ denote positive constants. Remark that the classical double-well potential

$$W(z) = (z^2 - 1)^2/4 \tag{2.16}$$

satisfies (2.10)-(2.14). The existence and uniqueness of global solutions in the class

$$\vartheta \in H^1(0,T;U') \cap L^\infty(0,T;H), \quad \chi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;U) \tag{2.17}$$

was proved in [21], [10] for W as in (2.16) and $\lambda'(z) = const$ using Faedo-Galerkin approximations. It will be clear from our *a priori* estimates that these results can also be achieved for the potentials and nonlinear latent heat satisfying (2.10)-(2.15). In what follows, we use (ϑ, χ) to designate a solution of (1.2)-(1.4) that satisfies (2.17).

The compactness of trajectories of the order parameter χ in the space $C(\overline{\Omega})$ is stated below:

Theorem 2.1 *Let (2.6)-(2.15) be satisfied. Then, the trajectory* $\cup_{t \geq 1}\chi(t)$ *is precompact in the space* $C(\overline{\Omega}) \cap V$*, and* $\cup_{t \geq 1}\vartheta(t)$ *is precompact in* $L^2_{weak}(\Omega)$*.*

The proof of this theorem is given in Section 3.

To derive the convergence of the order parameter to a single stationary state, we need an exponential kernel, a constant latent heat (which we set to be 1 for simplicity), and a stronger assumption on the forcing term f, namely:

$$k(t) = e^{-at}, \quad a > 0, \tag{2.18}$$

$$\lambda(\chi) = \chi \tag{2.19}$$

$$\sup_{t\geq T} t^{1+\delta}\int_t^\infty \|f(s)\|^2 \ \mathrm{d}s < \infty \ \text{ for some } \ T>0 \text{ and } \delta>0. \tag{2.20}$$

In addition, we require $f \in L^2_{loc}([0,\infty);V)$ with

$$\int_t^{t+1} \|f(s)\|_V^2 \ \mathrm{d}s \leq c_f, \ t>0. \tag{2.21}$$

Now, we can formulate the main result of our paper:

Theorem 2.2 *Let assumptions (2.7), (2.10)-(2.14), (2.18)-(2.21) be satisfied. Moreover, suppose that W is real analytic, $\vartheta(0)\in V$, $\chi(0)\in U$, $\mathbf{q}(0)\in \mathbf{V}$ and $\mathbf{q}(0)\cdot \mathbf{n}=0$ on $\partial\Omega$. Then $\mathbf{q}(t)\to 0$ in $\mathbf{H}$, and*

$$\vartheta(t)\to\vartheta_\infty \ \text{ in } H, \tag{2.22}$$

$$\chi(t)\to\chi_\infty \ \text{ in } C(\overline{\Omega}), \tag{2.23}$$

as $t\to\infty$, where ϑ_∞, χ_∞ satisfy (1.6)-(1.8).

The proof of this result is carried out in Sections 4 and 5.

3 A priori estimates. Asymptotic compactness

In this section we prove Theorem 2.1. From now on, C will denote a generic positive constant, which may vary from line to line. As the integral mean of χ is a conserved quantity, we normalize the initial value $\chi(0)$ such that $\int_\Omega \chi(0)\ \mathrm{d}x = 0$. Then

$$\int_\Omega \chi(t)\ \mathrm{d}x = 0 \ \text{ for all } \ t\geq 0. \tag{3.1}$$

Multiplying equation (1.3) by $A^{-1}[\chi_t]$ and integrating the resulting expression by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{1}{2}|\nabla\chi|^2 + W(\chi)\ \mathrm{d}x\Big) + \int_\Omega |A^{-\frac{1}{2}}[\chi_t]|^2\ \mathrm{d}x - \int_\Omega \vartheta\lambda(\chi)_t\ \mathrm{d}x = 0. \tag{3.2}$$

Then we test (1.2) by ϑ, to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\vartheta\|^2_{L^2(\Omega)} + \int_\Omega\int_0^t k(s)\nabla\vartheta(t-s)\ \mathrm{d}s\nabla\vartheta(t)\ \mathrm{d}x + \int_\Omega \vartheta\lambda(\chi)_t\ \mathrm{d}x = \int_\Omega g\vartheta\ \mathrm{d}x, \tag{3.3}$$

where $g=f+h$. If we add (3.2) and (3.3), integrate with respect to t and take (2.9) into account, we obtain the energy inequality

$$\begin{aligned}&\frac{1}{2}\|\nabla\chi(t)\|^2 + \frac{1}{2}\|\vartheta(t)\|^2 + \int_\Omega W(\chi(t)) + \int_0^t \|\chi_t(s)\ \mathrm{d}s\|^2_{V'}\\ \leq &\int_0^t\int_\Omega g(s)\vartheta(s)\ \mathrm{d}s + \frac{1}{2}\|\vartheta(0)\|^2 + \frac{1}{2}\|\nabla\chi(0)\|^2 + \int_\Omega W(\chi(0)).\end{aligned} \tag{3.4}$$

Now, applying a suitable version of Gronwall's Lemma ([7, Lemme A4]) we arrive at:

Lemma 3.1 *Let W, λ, k and g satisfy (2.7)-(2.15). Then, there exists E_0 depending only on the quantities*

$$\|\nabla\chi(0)\|,\ \|\vartheta(0)\|,\ \|g\|_{L^1(0,\infty;H)}$$

such that

$$\sup_{t>0}\|\vartheta(t)\| + \sup_{t>0}\|\nabla\chi(t)\| \le E_0, \tag{3.5}$$

$$\int_0^\infty \|\chi_t(t)\|_{V'}^2\ dt \le E_0. \tag{3.6}$$

Next, we multiply (1.4) by χ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\chi\|^2 + \|\Delta\chi\|^2 + \int_\Omega W''(\chi)|\nabla\chi|^2\ \mathrm{d}x = -\int_\Omega \lambda'(\chi)\vartheta\Delta\chi\ \mathrm{d}x.$$

Consequently, (3.5),(2.13),(2.15), and the Poincaré and Young inequalities imply

$$\int_t^{t+1}\|\Delta\chi\|^2\ \mathrm{d}\tau \le E_0 \ \text{ for any } t \ge 0. \tag{3.7}$$

To improve estimates on χ, we write (1.3) as an evolutionary equation

$$\frac{\partial\chi}{\partial t} + \Delta^2\chi = \Delta[W'(\chi)] - \Delta[\lambda'(\chi)\vartheta], \tag{3.8}$$

and use maximal regularity and a bootstrap type argument, as in [4]. We sketch the reasoning here for the reader's convenience.

Let $p \in [1,4)$ and let W satisfy (2.14). We prove first that

$$\chi \in L^r(t,t+1;W^{2,q_1}(\Omega)),\ t \ge 0,\ \text{for any } 1 \le r < \infty,\ q_1 = \min\{2, 6/p\}.$$

For this, we rewrite (3.8) in the abstract form

$$\chi_t + A_q^2\chi = l;\quad l = l_1 + l_2,$$

where

$$l_1 = \Delta[W'(\chi)];\quad l_2 = -\Delta(\lambda'(\chi)\vartheta).$$

From (3.5) we know that l_2 is bounded in $L^\infty(t,t+1;\mathcal{D}(A^{-1}))$, uniformly for all $t \ge 0$. On the other hand, using (3.5) and the Sobolev imbedding $V \subset L^6(\Omega)$, we have $\chi \in L^\infty(0,\tau;L^6(\Omega))$ for all $\tau > 0$. From (2.14) we get $W'(\chi) \in L^\infty(0,\tau;L^{\frac{6}{p}}(\Omega))$.

Hence, for $q_1 = \min\{2, \frac{6}{p}\}$, we have, recalling (2.3),

$$\begin{aligned}\|l\|_{\mathcal{D}(A_{q_1}^{-1})} = \|A_{q_1}^{-1}[l]\|_{L^{q_1}(\Omega)} &= \|[W'(\chi) - \lambda'(\chi)\vartheta] - \frac{1}{|\Omega|}\int_\Omega [W'(\chi) - \lambda'(\chi)\vartheta]\ \mathrm{d}x\|_{L^{q_1}(\Omega)}\\ &\le C\Big(\|W'(\chi)\|_{L^{q_1}(\Omega)} + \|\vartheta\|_{L^{q_1}(\Omega)}\Big).\end{aligned}$$

This implies that $\chi \in L^r(t,t+1;W^{2,q_1}(\Omega)),\ r \ge 1$. Consequently, by the Sobolev embedding theorem,

$$\chi \in L^r(t,t+1;L^{q_2}(\Omega)) \ \text{ with } \ q_2 = \frac{3q_1}{3-2q_1} \ \text{if } 2q_1 < 3,\ q_2 = \infty \ \text{otherwise}.$$

Next we argue by induction (bootstrap argument). We deduce from (2.14) that

$$W'(\chi) \in L^{\frac{r}{p}}(t, t+1; L^{\frac{q_2}{p}}(\Omega)).$$

Remark that we have

$$\frac{q_2}{p} - q_1 = \frac{6}{p(p-4)} - \frac{6}{p} > 0$$

if $p \in (4,5)$, $q_2 = \infty$ if $p \leq 4$. Hence, after a finite number of steps we find that $\chi \in L^r(t, t+1; W^{2,2}(\Omega)) = L^r(t, t+1; U)$. Consequently,

$$\chi \in L^r(t, t+1; U) \subset L^r(t, t+1; L^\infty(\Omega)),\ t \geq 0, \text{ for any } 1 \leq r < \infty. \tag{3.9}$$

Also, by (3.8), $\chi_t \in L^r(t, t+1; U')$ which implies

$$\chi \in C\Big([t, t+1]; \Big(U, U'\Big)_\theta\Big), \quad \text{with } \theta \text{ satisfying } \theta(1 - \frac{1}{r}) > \frac{1-\theta}{r},$$

(that is, $\theta > \frac{1}{r}$), where $(.,.)_\theta$ denotes the interpolation space (see, e.g., [23, Corollary 8, page 90]). As $r > 1$ is arbitrary, we can choose θ small enough such that $\Big(U, U'\Big)_\theta \hookrightarrow C(\overline{\Omega})$, and the embedding is compact. Therefore

$$\sup_{t>0} \|\chi(t)\|_{C(\overline{\Omega})} \leq C_\infty. \tag{3.10}$$

This implies that $W''(\chi)$, $W'''(\chi)$ are bounded, and $\nabla\chi$ is bounded in $L^r(t, t+1; L^6(\Omega))$ for all r, independently of $t > 0$. Then (cf. also (3.7)),

$$\int_t^{t+1} \|\Delta W'(\chi(s))\|^2 \, \mathrm{d}s < C \text{ for all } t > 0. \tag{3.11}$$

The conclusion of Theorem 2.1 now follows on account of (3.9)-(3.11) and Lemma 3.1.

4 Decay and uniform bounds of solutions

To prove the convergence of solutions to problem (1.1)-(1.4), we take an exponential kernel k and also consider a linear latent heat $\lambda(\chi) = \lambda_0\chi$ where, for simplicity, we set $\lambda_0 = 1$, i.e.,

$$k(t) = e^{-at} \ \ (a > 0), \ \ \lambda(\chi) = \chi. \tag{4.1}$$

The choice of a linear latent heat implies that the stationary temperature ϑ_∞ is uniquely defined by the initial datum and the heat source.

Then, we have

$$\mathbf{q}(t, x) = -\int_{-\infty}^t e^{-a(t-s)} \nabla\vartheta(s, x) \, \mathrm{d}s,$$

and we can rewrite (1.1)-(1.3) as:

$$\vartheta_t + \chi_t + \nabla \cdot \mathbf{q} = f, \tag{4.2}$$

$$\mathbf{q}_t + a\mathbf{q} + \nabla\vartheta = 0, \tag{4.3}$$

$$\chi_t + \Delta\Big(\Delta\chi - W'(\chi) + \vartheta\Big) = 0. \tag{4.4}$$

We will also assume that the boundary conditions (1.4) hold for $t > 0$ and

$$\mathbf{q}(0)\cdot\mathbf{n} = 0 \text{ on } \partial\Omega. \tag{4.5}$$

Taking advantage of equation (4.3), we get additional estimates. Our procedure is just formal at this stage, but could be made rigorous by a density argument, e.g., via a Galerkin approximation scheme.

By (4.2), (4.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{q}\cdot\nabla\vartheta\ \mathrm{d}x = \int_\Omega \mathbf{q}_t\cdot\nabla\vartheta\ \mathrm{d}x + \int_\Omega \mathbf{q}\cdot\nabla\vartheta_t\ \mathrm{d}x$$

$$= -a\int_\Omega \mathbf{q}\cdot\nabla\vartheta\ \mathrm{d}x - \|\nabla\vartheta\|^2 - \int_\Omega \mathbf{q}\cdot\nabla\chi_t\ \mathrm{d}x + \|\nabla\cdot\mathbf{q}\|^2 + \int_\Omega \mathbf{q}\cdot\nabla f\ \mathrm{d}x.$$

An integration by parts and Young's inequality yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{q}\cdot\nabla\vartheta\ \mathrm{d}x + a\int_\Omega \mathbf{q}\cdot\nabla\vartheta\ \mathrm{d}x - 2\|\nabla\cdot\mathbf{q}\|^2 + \|\nabla\vartheta\|^2 - \frac{1}{2}\|\chi_t\|^2 - \frac{1}{2}\|f\|^2 \le 0. \tag{4.6}$$

Now, we test (4.2) by $(-\Delta\vartheta)$, (4.3) by $(-\nabla\nabla\cdot\mathbf{q})$, and (4.4) by χ_t to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\vartheta\|^2 - \int_\Omega \chi_t\Delta\vartheta\ \mathrm{d}x - \int_\Omega \nabla\cdot\mathbf{q}\Delta\vartheta\ \mathrm{d}x = \int_\Omega \nabla\vartheta\cdot\nabla f\ \mathrm{d}x, \tag{4.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\nabla\cdot\mathbf{q}\|^2 + a\|\nabla\cdot\mathbf{q}\|^2 + \int_\Omega \nabla\cdot\mathbf{q}\Delta\vartheta\ \mathrm{d}x = 0, \tag{4.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|\Delta\chi\|^2 + \|\chi_t\|^2 - \int_\Omega \Delta W'(\chi)\chi_t\ \mathrm{d}x + \int_\Omega \chi_t\Delta\vartheta\ \mathrm{d}x = 0. \tag{4.9}$$

Next, we chose α such that $\alpha < \min(\frac{1}{2}, a-2)$, multiply (4.6) by α, and add the result to the sum of (4.7), (4.8) and (4.9) to get the following estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla\vartheta\|^2+\|\nabla\cdot\mathbf{q}\|^2+\|\Delta\chi\|^2+\alpha\int_\Omega \mathbf{q}\cdot\nabla\vartheta\ \mathrm{d}x\Big)+C_1\Big(\|\nabla\vartheta\|^2+\|\nabla\cdot\mathbf{q}\|^2+\|\Delta\chi\|^2+\int_\Omega \mathbf{q}\cdot\nabla\vartheta\ \mathrm{d}x\Big)$$

$$\le \|\Delta W'(\chi)\|^2 + \|\Delta\chi\|^2 + C_2\|f\|_V^2, \tag{4.10}$$

where $C_1,\ C_2$ are suitable constants. On account to (3.7), (3.11), and (2.21) we derive the boundedness of the function F, given by

$$F(t) = \|\nabla\vartheta(t)\|^2 + \|\nabla\cdot\mathbf{q}(t)\|^2 + \|\Delta\chi(t)\|^2 + 2\alpha\int_\Omega \mathbf{q}(t)\cdot\nabla\vartheta(t)\ \mathrm{d}x.$$

Noting that

$$\frac{1}{2}\Big(\|\nabla\vartheta(t)\|^2 + \|\nabla\cdot\mathbf{q}(t)\|^2 + \|\Delta\chi(t)\|^2\Big) \le F(t),$$

we arrive at

Lemma 4.1 *Let the assumptions of Theorem 2 be satisfied. Moreover, let the initial data satisfy* $\nabla\vartheta(0) \in H,\ \nabla\cdot\mathbf{q}(0) \in H,\ \mathbf{q}(0)\cdot\mathbf{n} = 0,\ \Delta\chi(0) \in H$. *Then there exists a constant* E_1 *depending only on the quantities* $\|\nabla\vartheta(0)\|,\ \|\nabla\cdot\mathbf{q}(0)\|$ *and* $\|\Delta\chi(0)\|$ *such that*

$$\sup_{t>0} \|\nabla\vartheta(t)\| \le E_1, \tag{4.11}$$

$$\sup_{t>0} \|\nabla\cdot\mathbf{q}(t)\| \le E_1, \tag{4.12}$$

$$\sup_{t>0} \|\Delta\chi(t)\| \le E_1. \tag{4.13}$$

Now, we can use the result proved in [10, Theorem 2.2], namely that

$$\vartheta(t) \to \vartheta_\infty \ \text{ weakly in H and strongly in } V' \ \text{ as } t \to \infty, \tag{4.14}$$

where

$$v_\infty = \int_\Omega \vartheta(0) + \int_0^\infty \int_\Omega g. \tag{4.15}$$

This, together with Lemma 4.1 and (3.5), yields the strong convergence of ϑ in H. Taking into account that

$$\mathbf{q}(t,x) = -\int_0^t e^{-(t-s)}\nabla\vartheta(s,x)\ \mathrm{d}s - e^{-t}\mathbf{q}(0), \tag{4.16}$$

and applying (4.12), we get the strong convergence of $\mathbf{q}$ in $\mathbf{H}$ as well. Also, (4.13) enables us to show that $\chi \in L^r(t, t+1; W^{3,2}(\Omega))$ for all $t \ge 1$, with the norm bounded independently of $t > 1$, and, as above, to conclude that the trajectory of χ is precompact in U.

Lemma 4.2 *Let the assumptions of Lemma 4.1 be satisfied. Then*

$$\vartheta \to \vartheta_\infty \ \textit{ in } H, \ \ \mathbf{q} \to 0 \ \textit{ in } \mathbf{H} \ \textit{ as } \ t \to \infty, \tag{4.17}$$

$$\cup_{t>1}\chi(t) \ \textit{ is precompact in } U. \tag{4.18}$$

5 Convergence of the order parameter

In this section, we show that the time derivative of χ is integrable on some interval $(T, +\infty)$, which will imply the convergence stated in Theorem 2.2. To this end, we derive an energy inequality and apply a version of the Łojasiewicz inequality.

Denoting the integral mean of a function z by $\overline{z} = \int_\Omega z\ \mathrm{d}x$, we can write (4.2) as:

$$(\vartheta - \overline{\vartheta})_t + \chi_t + \nabla\mathbf{q} = f - \overline{f} \tag{5.1}$$

Reasoning as in the proof of the inequality (4.6), we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbf{q}\cdot\nabla A^{-1}[\vartheta - \overline{\vartheta}]\ \mathrm{d}x$$

$$+a\int_\Omega \mathbf{q}\cdot\nabla A^{-1}[\vartheta - \overline{\vartheta}]\ \mathrm{d}x - 2\|\mathbf{q}\|^2 + \|\vartheta - \overline{\vartheta}\|^2 - \frac{1}{2}\|\chi_t\|^2_{V'} - \frac{1}{2}\|f - \overline{f}\|^2 \le 0. \tag{5.2}$$

Now, we multiply (5.1) by $\vartheta - \overline{\vartheta}$, (4.3) by $\mathbf{q}$, (4.4) by $A^{-1}[\chi_t]$, (5.2) by a suitable small constant α, and add the results to obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_\Omega \frac{1}{2}|\nabla\chi|^2 + W(\chi)\ \mathrm{d}x + \frac{1}{2}(\|\vartheta-\overline{\vartheta}\|^2 + \|\mathbf{q}\|^2) - \alpha\int_\Omega \mathbf{q}\cdot\nabla A^{-1}[\vartheta-\overline{\vartheta}]\ \mathrm{d}x\Big)$$

$$+C_1\Big(\|\vartheta-\overline{\vartheta}\|^2 + \|\mathbf{q}\|^2 + \|\chi_t\|^2_{V'}\Big) \le C_2\|f-\overline{f}\|^2, \tag{5.3}$$

with some constants $C_1,\ C_2$. At this point, we denote by $I(z)$ the functional

$$I(z) = \int_\Omega \frac{1}{2}|\nabla z(x)|^2 + W(z)\ \mathrm{d}x, \quad z\in V_0,$$

take α small enough and, taking into account (4.17) and (2.20), we infer that $I(\chi(t)) \to I_\infty$, and the limit $I_\infty = I(\chi_\infty)$ for any element χ_∞ in the $\omega-$limit set of χ.

Integrating (5.3) from t to infinity, and employing Young's inequality and Lemma 4.2 yields

$$\int_t^\infty \|\vartheta(s)-\overline{\vartheta}(s)\|^2 + \|\mathbf{q}(s)\|^2 + \|\chi_t(s)\|^2_{V'}\ \mathrm{d}s$$

$$\le C\Big(I(\chi(t)) - I(\chi_\infty) + \|\vartheta(t)-\overline{\vartheta}(t)\|^2 + \|\mathbf{q}(t)\|^2) + \int_t^\infty \|f(s)-\overline{f}(s)\|^2\ \mathrm{d}s\Big). \tag{5.4}$$

Set

$$\mathcal{M}_1 = \{t>T; \|\chi_t(t)\|^\gamma_{V'} \le \int_t^\infty \|f(s)-\overline{f}(s)\|^2\ \mathrm{d}s\},\ \mathcal{M}_2 = (T,\infty)\backslash\mathcal{M}_1 \tag{5.5}$$

for some

$$1<\gamma<\min\{2, 1+\delta\} \tag{5.6}$$

and $T,\ \delta$ as in (2.20). Then

$$\int_{\mathcal{M}_1} \|\chi_t(t)\|_{V'}\ \mathrm{d}t \le \int_{\mathcal{M}_1}\Big(2\int_t^\infty \|f(s)\|^2\ \mathrm{d}s\Big)^{\frac{1}{\gamma}}\ \mathrm{d}t \le C\int_{\mathcal{M}_1} t^{-\frac{1+\delta}{\gamma}}\ \mathrm{d}t < \infty. \tag{5.7}$$

Next, we prove that there exists $\tau \ge T$ such that $\|\chi_t\|_{V'}$ is also integrable over $\mathcal{M}_2\cap(\tau,+\infty)$. To accomplish this, we need the following result, which is proved in [13, Lemma 7.1]:

Lemma 5.1 *Let $Z\ge 0$ be a Lebesgue measurable function on $(0,\infty)$ such that*

$$Z\in L^2(0,\infty),\ \|Z\|_{L^2(0,\infty)} \le Y,$$

and there exist $\beta\in(1,2)$, $\xi>0$ and an open set $\mathcal{M}\subset(0,\infty)$ such that

$$\Big(\int_t^\infty Z^2(s)\ ds\Big)^\beta \le \xi\ Z^2(t)\ \textit{for a.a. } t\in\mathcal{M}. \tag{5.8}$$

Then $Z\in L^1(\mathcal{M})$ and there exists a constant $c = c(\xi,\alpha,Y)$ independent of $\mathcal{M}$ such that

$$\int_{\mathcal{M}} Z(t)\ dt \le c.$$

We take

$$Z^2(s) = \|\vartheta(s) - \overline{\vartheta}(s)\|^2 + \|\mathbf{q}(s)\|^2 + \|\chi_t(s)\|_{V'}^2,$$

and show that (5.8) holds for $t \in \mathcal{M}_2 \cap (\tau, +\infty)$, for a sufficiently large τ. Since the trajectory of χ is uniformly bounded, we may suppose that W has been modified outside the interval $[-L, L]$, where $|\chi(t,x)| \le \frac{L}{2}$ in such a way that

$$|W'(z)|,\ |W''(z)| \ \text{ are uniformly bounded for } \ z \in \mathbb{R}. \tag{5.9}$$

Moreover,

$$W(z) \ \text{ is real analytic on } (-L, L). \tag{5.10}$$

The proof of following version of the Łojasiewicz inequality may be performed in the same way as in [13, Proposition 6.1]; for a general setting see [9, Corollary 3.11].

Proposition 5.1 *(Lojasiewicz inequality). Let W satisfy the hypotheses (5.9), (5.10). Let $w \in U \cap V_0, \nabla w \cdot \mathbf{n} = 0$ on $\partial\Omega$,*

$$-L/2 < w(x) < L/2 \text{ for all } x \in \Omega.$$

Then, for any $P > 0$, there exist constants $\rho \in (0, 1/2)$, $M(P)$, $\varepsilon(P)$ such that

$$|I(v) - I(w)|^{1-\rho} \le M\| - \Delta v + W'(v)\|_{V'}, \tag{5.11}$$

for any $v \in V_0$, satisfying

$$\|v - w\|_H < \varepsilon,\ |I(v) - I(w)| < P. \tag{5.12}$$

Let

$$\mathcal{M}_3 = \{t \in (0,\infty) \mid \|\chi(t) - \chi_\infty\|_H < \varepsilon\}.$$

For $t \in \mathcal{M}_2 \cap \mathcal{M}_3$ we can apply Proposition 5.1 and estimate the right-hand side of (5.4) as follows:

$$\begin{aligned} |I(\chi(t) - I(\chi_\infty)| + \|\vartheta(t) - \overline{\vartheta}(t)\|^2 + \|\mathbf{q}(t)\|^2 + \|\chi_t(t)\|_{V'}^2\Big) + \int_t^\infty \|f - \overline{f}\|^2 \,\mathrm{d}s \\ \le C\Big(\| - \Delta\chi(t) + W'(\chi(t))\|_{V'}^{\frac{1}{1-\rho}} + \|\vartheta - \overline{\vartheta}(t)\|^2 + \|\mathbf{q}\|^2 + \|\chi(t)\|_{V'}^{\gamma}\Big) \\ \le C\Big(\|A^{-1}[\chi_t(t)] - (\vartheta(t) - \overline{\vartheta}(t))\|_{V'}^{\frac{1}{1-\rho}} + \|\vartheta(t) - \overline{\vartheta}(t)\|^2 + \|\mathbf{q}(t)\|^2 + \|\chi(t)\|_{V'}^{\gamma}\Big). \end{aligned}$$

Let $\eta = \min(\gamma, \frac{1}{1-\rho})$ and $T_1 \ge T$ be such that

$$\|\vartheta(t) - \overline{\vartheta}(t)\|^2 \le \|\vartheta(t) - \overline{\vartheta}(t)\|^\eta, \quad \|\mathbf{q}(t)\|^2 \le \|\mathbf{q}(t)\|^\eta \text{ for all } t \ge T_1.$$

Such a T_1 exists due to (4.17). We also realize that if the Łojasiewicz inequality holds with some ρ, then it is also true with $\rho_1 < \rho$. Consequently,

$$\begin{aligned} \int_t^\infty Z^2(s)\,\mathrm{d}s = \int_t^\infty \|\vartheta(s) - \overline{\vartheta}(s)\|^2 + \|\mathbf{q}(s)\|^2 + \|\chi_t(s)\|_{V'}^2 \,\mathrm{d}s \\ \le C\Big(\|\vartheta(t) - \overline{\vartheta}(t)\|^\eta + \|\mathbf{q}(t)\|^\eta + \|\chi_t(t)\|_{V'}^\eta\Big) \end{aligned}$$

$$\leq C\Big(\|\vartheta(t)-\overline{\vartheta}(t)\|^2+\|\mathbf{q}(t)\|^2+\|\chi_t(t)\|^2_{V'}\Big)^{\frac{\eta}{2}} = CZ(t)^{\eta}, \tag{5.13}$$

and Lemma 5.1 implies the integrability of $\|\chi_t\|_{V'}$ on $\mathcal{M}_2\cap\mathcal{M}_3\cap(T_1,+\infty)$. A simple contradiction argument (cf., e.g., [3]) yields the existence of $\tau\geq T_1$ such that $\mathcal{M}_2\cap\mathcal{M}_3\cap(\tau,+\infty)=\mathcal{M}_2\cap(\tau,+\infty)$. It follows that the function Z, and therefore, $\|\chi_t\|_{V'}$ is integrable over the set $(\tau,+\infty)$, which yields the convergence of the trajectory of χ to χ_∞ in the space V'. On the other hand, the compactness of the trajectory in the space $C(\overline{\Omega})$, proved in Theorem 2.1, yields the convergence of the whole trajectory in this space, which concludes the proof of Theorem 2.2.

6 Rate of convergence

In some cases, namely if we can estimate the exponent in the Łojasiewicz inequality, we can also give the rate of convergence of the order parameter to the corresponding stationary state. Our approach follows the procedure performed in [18], where also some examples of the rate of decay to equilibria for solutions of parabolic equations can be found.

We are going to estimate the L^1- norm of $\|\chi_t\|_{V'}$. To begin, we assume that $(\tau,+\infty)\subset\mathcal{M}_2$ (where τ is as in the proof of Theorem 2.2, cf. Section 5), and deduce from Lemma 5.1 that

$$\int_t^\infty \|\chi_t\|^2_{V'} \leq Kt^{-2\mu-1} \text{ for all } t\geq\tau, \tag{6.1}$$

where

$$\mu=\frac{\eta-1}{2-\eta}. \tag{6.2}$$

In fact, according to (5.13),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\int_t^\infty Z^2(s)\ \mathrm{d}s\Big)^{1-\beta} = \frac{Z^2(t)}{\Big(\int_t^\infty Z^2(s)\ \mathrm{d}s\Big)^{\beta}} \geq C,$$

with $\beta=\frac{\eta}{2}$. Integrating with respect to t, we get

$$\int_t^\infty Z^2(s)\ \mathrm{d}s \leq \Big(\frac{1}{C(\beta-1)}\Big)^{\frac{1}{\beta-1}} t^{-\frac{1}{\beta-1}}.$$

We obtain the same estimate for general $\mathcal{M}_2$ as defined in (5.5), if we follow the proof of Lemma 7.1 in [13]. With $\beta=\frac{2}{\eta}$ and μ as in (6.2), we infer that

$$\int_t^\infty \|\chi_t\|^2_{V'} \leq \int_t^\infty |Z(t)|^2 \leq t^{-2\mu-1} \text{ for all } t\geq\tau. \tag{6.3}$$

Now, we apply the following simple result, the proof of which can be found in [18, Lemma 3.3].

Lemma 6.1 *Assume that for some $\mu>0$,*

$$\int_t^\infty \|\chi_t\|^2_{V'} \leq Ct^{-2\mu-1} \textit{ for all } t\geq\tau.$$

Then

$$\int_t^\infty \|\chi_t\|_{V'} \leq Ct^{-\mu}, \quad t\geq\tau. \tag{6.4}$$

For $t \in \mathcal{M}_1$, we have the estimate (5.7). This, together with (6.3) and (6.4) gives

$$\int_t^{\infty} \|\chi_t\|_{V'} \leq Ct^{-\omega},\ \omega = \min\Big\{\frac{\rho}{1-2\rho}, \frac{1+\delta-\gamma}{\gamma}\Big\}. \tag{6.5}$$

Hence, we can estimate the distance between $\chi(t)$ and χ_∞ in H. Using the boundedness of χ in the $V-$norm, and (6.5), we can interpolate to get:

$$\|\chi(t) - \chi_\infty\| \leq C\Big(\int_t^{\infty} \|\chi_t(s)\|_{V'}\ \mathrm{d}s\Big)^{\frac{1}{2}} \cdot \|\chi(t) - \chi_\infty\|_V^{\frac{1}{2}} \leq Ct^{-\frac{\omega}{2}}.$$

With (3.10) and (4.13) at hand, we can further interpolate between H and U to deduce:

$$\|\chi(t) - \chi_\infty\|_{H^s} \leq \|\chi(t) - \chi_\infty\|_U^{1-\theta}\|\chi(t) - \chi_\infty\|_H^{\theta},\quad s = 2(1-\theta),\quad \theta \in (0,1).$$

We thereby arrive at:

Proposition 6.1 *Let χ be a solution of (4.2)-(4.5), (1.4), and let χ_∞ be the limit solution. Then, for each $\theta \in (0,1)$, there exists a constant C depending on θ and* $\sup_{t\geq 0}\|\chi(t)\|_U$ *such that the following estimate holds:*

$$\|\chi(t) - \chi_\infty\|_{H^s} \leq Ct^{-\frac{\omega\theta}{2}},\quad t > 0, \tag{6.6}$$

where $s = 2(1-\theta)$ and ω is given by (6.5).

Remark. Taking $\theta < \frac{1}{4}$ in (6.6), we get the rate of convergence of $\chi(t)$ to χ_∞ in the space $C(\overline{\Omega})$.

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